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AN ESTIMATE FOR THE SECTIONAL CURVATURE OF CYLINDRICALLY BOUNDED SUBMANIFOLDS

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Abstract. We give sharp sectional curvature estimates for complete immersed cylindrically bounded m-submanifolds $\varphi\colon M^m \to N^{n-\ell} \times \mathbb{R}^\ell$, $n+\ell \leq 2m-1$, provided that either φ is proper with the norm of the second fundamental form with certain controlled growth or M has scalar curvature with strong quadratic decay. The latter gives a non-trivial extension of the Jorge-Koutrofiotis Theorem. In the particular case of hypersurfaces, that is, $m = n-1$, the growth rate of the norm of the second fundamental form is improved. Our results will be an application of a generalized Omori-Yau Maximum Principle for the Hessian of a Riemannian manifold, in its newest elaboration given by Pigola, Rigoli and Setti (2005).

1. Introduction

Given complete Riemannian manifolds M^m and N^n with dimension $m < n$, the isometric immersion problem asks whether there exists an isometric immersion $\varphi\colon M \hookrightarrow N$. When $N^n = \mathbb{R}^n$ is the Euclidean space, the isometric problem is answered by the Nash Embedding Theorem that says that there is an isometric embedding $\varphi\colon M^m \hookrightarrow \mathbb{R}^n$, provided the codimension $n-m$ is sufficiently large; see [17]. For small codimension, meaning in this paper that $n-m \leq m-1$, the answer in general depends on the geometries of M and N. For instance, the Hilbert-Efimov Theorem [9], [12] says that no complete surface M with sectional curvature $K_M \leq -\delta^2 < 0$ can be isometrically immersed in $\mathbb{R}^3$, and a classical result by C. Tompkins [24] states that a compact, flat, m-dimensional Riemannian manifold cannot be isometrically immersed in $\mathbb{R}^{2m-1}$. Tompkins' result was extended in a series of papers by Chern and Kuiper [7], Moore [16], O'Neill [19], Otsuki [20] and Stiel [23], whose results can be summarized in the following theorem (we recall that a Cartan-Hadamard manifold is a simply connected, complete, Riemannian manifold with non-positive sectional curvatures).

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Theorem 1. *Let $\varphi\colon M^m \hookrightarrow N^n$, $n \leq 2m-1$, be an isometric immersion of a compact Riemannian m-manifold M into a Cartan-Hadamard n-manifold N. Then the sectional curvatures of M and N satisfy*

$$\sup_M K_M > \inf_N K_N. \tag{1.1}$$

Theorem 1 was extended by Jorge and Koutrofiotis in [13] to bounded, complete submanifolds with scalar curvature bounded below and in the version presented by Pigola, Rigoli and Setti in [21, Theorem 1.15], with scalar curvature satisfying

$$s_M(x) \geq -B^2 \varrho_M^2(x) \cdot \prod_{j=1}^{k} \left(\log^{(j)}(\varrho_M(x))\right)^2, \ \varrho_M(x) \gg 1, \tag{1.2}$$

for some constant $B > 0$ and some integer $k \geq 1$, where ϱ_M is the distance function on M to a fixed point and $\log^{(j)}$ is the j-th iterate of the logarithm.

Theorem 2 (Jorge-Koutrofiotis, [13])**.** *Let M^m and N^n be complete Riemannian manifolds of dimensions m and n, respectively, with $n \leq 2m-1$ and let $\varphi : M \to N$ be an isometric immersion with $\varphi(M) \subset B_N(r)$, where $B_N(r)$ denotes a geodesic ball of N centered at a point $p \in N$ and radius r. Assume that the radial sectional curvature K_N^{rad} along the radial geodesics issuing from p satisfies $K_N^{\mathrm{rad}} \leq b$ in $B_N(r)$ and $0 < r < \min\{\mathrm{inj}_N(p), \pi/2\sqrt{b}\}$, where we replace $\pi/2\sqrt{b}$ by $+\infty$ if $b \leq 0$. If the scalar curvature of M satisfies* (1.2)*, then*

$$\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N, \tag{1.3}$$

where

$$C_b(t) = \begin{cases} \sqrt{b}\cot(\sqrt{b}\,t) & \text{if } b > 0 \text{ and } 0 < t < \pi/2\sqrt{b}, \\ 1/t & \text{if } b = 0 \text{ and } t > 0, \\ \sqrt{-b}\coth(\sqrt{-b}\,t) & \text{if } b < 0 \text{ and } t > 0. \end{cases}$$

Remark 3. If $N^n = \mathbb{N}^n(b)$ is the simply connected space form of constant sectional curvature b and $M = \partial B_{\mathbb{N}^n(b)}(r) \subset \mathbb{N}^n(b)$ is a geodesic sphere of radius r, then equality in (1.3) is achieved.

The purpose of this paper is to extend the Jorge-Koutrofiotis Theorem to the case of complete cylindrically bounded submanifolds of a Riemannian product $N^{n-\ell} \times \mathbb{R}^\ell$, where N is a complete Riemannian manifold of dimension $n-\ell$. In this context, an isometric immersion $\varphi : M^m \to N^{n-\ell} \times \mathbb{R}^\ell$ of a Riemannian manifold M^m is said to be *cylindrically bounded* if there exists $B_N(r)$, a geodesic ball of N centered at a point $p \in N$ with radius $r > 0$, such that $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$ (see Figure 1).

In a recent paper [1], the two first authors, jointly with Dajczer, derived an estimate for the mean curvature H of complete cylindrically bounded submanifolds into a product Riemannian manifold $N^{n-\ell} \times \mathbb{R}^\ell$. Specifically, it was proved in [1] that if the immersion $\varphi : M^m \to N^{n-\ell} \times \mathbb{R}^\ell$ is proper and $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$, then

$$\sup_M |H| \geq \frac{m-\ell}{m} C_b(r).$$

As a consequence, it follows from here that a complete hypersurface of given constant mean curvature lying inside a closed circular cylinder in Euclidean space cannot be proper if the circular bases is of sufficiently small radius (see Corollary 1 in [1]). In particular, there exists no complete minimal hypersurface properly

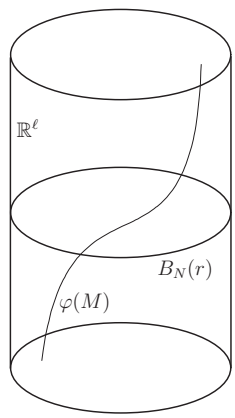

FIGURE 1. A cylindrically bounded submanifold.

immersed in $\mathbb{R}^n$ and having 2 (or more) bounded coordinates, showing that any possible counterexample to a conjecture of Calabi on complete minimal hypersurfaces [3] (see also [6]) cannot be proper.

Our main result here deals with the sectional curvature of such submanifolds, and it can be stated as follows.

Theorem 4. *Let M^m and $N^{n-\ell}$ be complete Riemannian manifolds of dimension m and $n-\ell$ respectively, with $n+\ell \leq 2m-1$. Let $\varphi : M^m \to N^{n-\ell} \times \mathbb{R}^\ell$ be a (cylindrically bounded) isometric immersion with $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$. Assume that the radial sectional curvature K_N^{rad} along the radial geodesics issuing from p satisfies $K_N^{\mathrm{rad}} \leq b$ in $B_N(r)$ and $0 < r < \min\{\mathrm{inj}_N(p), \pi/2\sqrt{b}\}$, where we replace $\pi/2\sqrt{b}$ by $+\infty$ if $b \leq 0$. If either*

(i) *the scalar curvature of M satisfies* (1.2) *or*
(ii) *φ is proper and*

$$\sup_{\varphi^{-1}(B_N(r)\times\partial B_{\mathbb{R}^\ell}(t))} \|\alpha\| \leq \sigma(t), \tag{1.4}$$

where α is the second fundamental form of the immersion and $\sigma : [0,+\infty) \to \mathbb{R}$ is a positive function satisfying $\int_0^{+\infty} 1/\sigma = +\infty$, then

$$\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N. \tag{1.5}$$

It is worth pointing out that the codimension restriction $n+\ell \leq 2m-1$ cannot be relaxed. Actually, together with the bound $m \leq n-1$, it implies that $n-\ell \geq 3$ and $m \geq \ell+2$. In particular, for $n=3$ we have that $\ell = 0$, and therefore $\varphi(M) \subset B_N(r)$. In fact, the flat cylinder $\mathbb{S}^1(r') \times \mathbb{R} \subset B_{\mathbb{R}^2}(r) \times \mathbb{R}$, for every $0 < r' < r$, shows that the restriction $2m-1 \geq n+\ell$ is necessary.

On the other hand, our estimate (1.5) is sharp. Indeed, for every $n \geq 3$ and $\ell \leq n-3$ we can consider $M = \partial B_{\mathbb{N}^{n-\ell}(b)}(r') \times \mathbb{R}^\ell$ and take $\varphi\colon M \hookrightarrow B_{\mathbb{N}^{n-\ell}(b)}(r) \times \mathbb{R}^\ell$ to be the canonical isometric immersion for every $0 < r' < r$. Therefore $\sup_M K_M$ is the constant sectional curvature of the geodesic sphere $\partial B_{\mathbb{N}^{n-\ell}(b)}(r') \subset \mathbb{N}^{n-\ell}(b)$,

which is given by

$$\sup_M K_M = K_{\partial B_{\mathbb{N}^{n-\ell}(b)}(r')} = \begin{cases} b/\sin^2(\sqrt{b}\,r') & \text{if } b>0 \text{ and } 0<r'<\pi/2\sqrt{b}, \\ 1/r'^2 & \text{if } b=0 \text{ and } r'>0, \\ -b/\sinh^2(\sqrt{-b}\,r') & \text{if } b<0 \text{ and } r'>0. \end{cases}$$

In particular, observe that

$$\sup_M K_M = K_{\partial B_{\mathbb{N}^{n-\ell}(b)}(r')} = C_b^2(r') + b.$$

Since in this case $K_{\mathbb{N}^{n-\ell}(b)} = b$, then for every $0 < r' < r$ we have

$$\sup_M K_M = C_b^2(r') + b \geq C_b^2(r) + \inf_{B_{\mathbb{N}^{n-\ell}(b)}(r)} K_{\mathbb{N}^{n-\ell}(b)},$$

which shows that our estimate (1.5) is sharp.

Let $\varphi\colon M^m \to N^{n-\ell} \times \mathbb{R}^\ell$ be an isometric immersion of a compact Riemannian m-manifold M^m, and let $\pi_N\colon N^{n-\ell} \times \mathbb{R}^\ell \to N^{n-\ell}$ be the projection onto the first factor. Denote by R_M the radius of the smallest ball of N containing $\pi_N(\varphi(M))$. We will refer to R_M as the extrinsic radius of the immersion. As a consequence of Theorem 4, we have the following versions of the Extrinsic Radius Theorem of Jorge-Xavier [14] (see also [2, Theorem 1.3]).

Corollary 5. *Let $\varphi\colon M^m \to N^{n-\ell} \times \mathbb{R}^\ell$ be an isometric immersion of a compact Riemannian m-manifold M^m into the product $N^{n-\ell} \times \mathbb{R}^\ell$ with $n + \ell \leq 2m - 1$, where $N^{n-\ell}$ is a complete Riemannian manifold with a pole and radial sectional curvature $K_N^{\mathrm{rad}} \leq b \leq 0$. Then, the extrinsic radius satisfies*

$$R_M \geq C_b^{-1}\left(\sqrt{\sup K_M - \inf K_N}\right). \tag{1.6}$$

In particular, if $N = \mathbb{R}^{n-\ell}$ we have that

$$R_M \geq \frac{1}{\sqrt{\sup K_M}}. \tag{1.7}$$

Corollary 6. *Let $\varphi\colon M^m \to \mathbb{S}^{n-\ell} \times \mathbb{R}^\ell$ be an isometric immersion of a compact Riemannian m-manifold M^m with $n + \ell \leq 2m - 1$. If $\sup K_M \leq 1$, then*

$$R_M \geq \pi/2. \tag{1.8}$$

On the other hand, it is important to remark that for hypersurfaces, the growth rate of the norm of the second fundamental form in (1.4) can be improved as follows.

Theorem 7. *Let $\varphi : M^{n-1} \to N^{n-\ell} \times \mathbb{R}^\ell$ be a properly immersed hypersurface with $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$, $n - \ell \geq 3$. Suppose that N satisfies the assumptions on the radial sectional curvatures as in Theorem 4 and the second fundamental form α satisfies*

$$\sup_{\varphi^{-1}(B_N(r)\times\partial B_{\mathbb{R}^\ell}(t))} \|\alpha\| \leq \sigma^2(t), \tag{1.9}$$

where $\sigma\colon [0,\infty) \to \mathbb{R}$ is a positive function satisfying

$$\int_0^{+\infty} \frac{1}{\sigma} = +\infty \quad \textit{and} \quad \limsup_{t\to+\infty} \frac{1}{\sigma(t)} < +\infty.$$

Then

$$\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N.$$

Remark 8. It should be remarked that Hasanis and Koutroufiotis [11] established similar sectional curvature estimates for cylindrically bounded submanifolds, with scalar curvature bounded below, of the Euclidean space $\mathbb{R}^n$. In a slightly more general situation, F. Giménez [10] established sectional curvature estimates for submanifolds with scalar curvature bounded below immersed in a tubular neighborhood of certain, (P-submanifolds), embedded submanifolds of Hadamard manifolds. Our main results, besides extending Hasanis-Koutrofiotis results to a larger class of submanifolds, can be easily adapted to reproduce Giménez's result settings.

2. Preliminaries

Our main tool to build the proof of Theorem 4 is the following (and important) version of the Omori-Yau Maximum Principle for the Hessian due to Pigola, Rigoli and Setti [21, Theorem 1.9]. Chronologically, the Omori-Yau Maximum Principle can be traced in a series of papers, starting with Omori [18] who proved the maximum principle at infinity for the class of complete Riemannian manifolds with sectional curvature bounded from below, followed by new insights given by Cheng and Yau [5], [25], who extended them to the class of complete Riemannian manifolds with Ricci curvature bounded from below, and by extensions due to C. Dias [8] (extended to the class of sectional curvature with quadratic decay) and Chen-Xin [4] (extended to the class of Ricci curvature with quadratic decay). Finally, as observed by Pigola, Rigoli and Setti in [21], the validity of the Omori-Yau Maximum Principle does not depend on curvature bounds as much as one would expect. Actually, a condition to guarantee the validity of it can be expressed in a function theoretic form by the newest elaboration stated below.

Theorem 9 ([21, Theorem 1.9])**.** *Let M^m be a Riemannian manifold and assume that there exists a non-negative C^2-function ψ satisfying the following requirements:*

(a.1) *ψ is proper, that is, $\psi(x) \to +\infty$ as $x \to \infty$;*
(a.2) *there exists a positive constant $A > 0$ such that $|grad\,\psi| \leq A\sqrt{\psi}$ outside a compact subset of M;*
(a.3) *there exists a positive constant $B > 0$ such that*

$$\operatorname{Hess}\psi \leq B\sqrt{\psi G\left(\sqrt{\psi}\right)}\langle,\rangle$$

(in the sense of quadratic forms) outside a compact subset of M,

where G is a smooth function on $[0,+\infty)$ satisfying:

$$\begin{aligned} &(i)\ G(0) > 0, && (ii)\ G'(t) \geq 0 \text{ on } [0,+\infty), \\ &(iii)\ 1/\sqrt{G(t)} \notin L^1(0,+\infty), && (iv)\ \limsup_{t\to+\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty. \end{aligned} \tag{2.1}$$

Then, given a function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$ there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset M$ such that

$$(i)\ u(x_k) > u^* - \frac{1}{k},\ (ii)\ |grad\,u(x_k)| < \frac{1}{k},\ \text{and } (iii)\ \operatorname{Hess} u(x_k) < \frac{1}{k}\langle,\rangle \tag{2.2}$$

in the sense of quadratic forms, that is,

$$\operatorname{Hess} u(x_k)(v,v) < \frac{1}{k}|v|^2 \quad \text{for all } v \in T_{x_k}M.$$

If, instead of (a.3), one replaces it by

(a.3)′ *there exists a positive constant* $B > 0$ *such that* $\Delta\gamma \leq B\sqrt{\gamma G\left(\sqrt{\gamma}\right)}$ *outside a compact subset of* M,

then one can weaken conclusion (iii) in (2.2) *to*

$$(iii)' \ \ \Delta u(x_k) < \frac{1}{k}.$$

Remark 10. It is worth pointing out that although in the statement (and in the proof) of Theorem 9 the manifold M is not required to be geodesically complete, the two assumptions (a.1) and (a.2) imply it. For the details, see [21, page 10].

For that reason, and following the terminology introduced by Pigola, Rigoli and Setti in [21], the Omori-Yau Maximum Principle for the Hessian is said to hold on a (not necessarily complete) Riemannian manifold M if, for any smooth funtion $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset M$ satisfying

$$\text{(i)} \ \ u(x_k) > u^* - \frac{1}{k}, \ \ \text{(ii)} \ \ |\operatorname{grad} u(x_k)| < \frac{1}{k}, \ \text{and (iii)} \ \ \operatorname{Hess} u(x_k) < \frac{1}{k}\langle,\rangle$$

for each $k \in \mathbb{N}$. Equivalently, for any smooth funtion $u \in C^2(M)$ with $u_* = \inf_M u > -\infty$ there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset M$ with the properties

$$\text{(i)} \ \ u(x_k) < u_* + \frac{1}{k}, \ \ \text{(ii)} \ \ |\operatorname{grad} u(x_k)| < \frac{1}{k}, \ \text{and (iii)} \ \ \operatorname{Hess} u(x_k) > -\frac{1}{k}\langle,\rangle$$

for each $k \in \mathbb{N}$. In the same way, the Omori-Yau Maximum Principle is said to hold on a (not necessarily complete) Riemannian manifold M if, for any smooth function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset M$ satisfying

$$\text{(i)} \ \ u(x_k) > u^* - \frac{1}{k}, \ \ \text{(ii)} \ \ |\operatorname{grad} u(x_k)| < \frac{1}{k}, \ \text{and (iii)}' \ \ \Delta u(x_k) < \frac{1}{k}$$

for each $k \in \mathbb{N}$.

The function theoretic approach to the Omori-Yau Maximum Principle given in Theorem 9 allows one to apply it in different situations, where the choice of ψ and G are suggested by the geometric setting. For instance, one has the following consequence (see [21, Example 1.13]).

Corollary 11. *Let* M *be a complete, non-compact, Riemannian manifold, let* $o \in M$ *be a reference point and denote by* $\varrho_M(x)$ *the Riemannian distance function from* o. *Assume that the radial sectional curvature of* M, *that is, the sectional curvature of the* 2*-planes containing* $\operatorname{grad}\varrho_M$, *satisfies*

$$K_M^{rad} \geq -G(r),$$

where G *is a smooth function on* $[0, +\infty)$ *which we assume to be even at the origin, that is,* $G^{(2k+1)}(0) = 0$ *for* $k = 0, 1, 2, \ldots$, *and satisfying the conditions listed in Theorem* 9. *Then the Omori-Yau Maximum Principle for the Hessian holds on* M.

We will need two more results. The first is known as Otsuki's Lemma (for a proof see, for instance, [15, Page 28] or [21, Lemma 1.16]).

Lemma 12. *Let $\beta\colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^q$, $q \le k-1$, be a bilinear symmetric form satisfying $\beta(X,X) \neq 0$ for $X \neq 0$. Then there exist linearly independent vectors X, Y such that*

$$\beta(X,X) = \beta(Y,Y) \quad \text{and} \quad \beta(X,Y) = 0. \tag{2.3}$$

The second is the well-known Hessian Comparison Theorem; see [22].

Theorem 13 (Hessian Comparison Theorem)**.** *Let M be a Riemannian manifold and $x_0, x_1 \in M$ be such that there is a minimizing unit speed geodesic γ joining x_0 and x_1, and let $\varrho(x) = \operatorname{dist}(x_0, x)$ be the distance function to x_0. Let $K_\gamma \le b$ be the radial sectional curvatures of M along γ. If $b > 0$ assume $\varrho(x_1) < \pi/2\sqrt{b}$. Then, we have*

$$\operatorname{Hess} \varrho(x)(\gamma', \gamma') = 0$$

and

$$\operatorname{Hess} \varrho(x)(X,X) \ge C_b(\varrho(x))|X|^2,$$

where $X \in T_xM$ is perpendicular to $\gamma'(\varrho(x))$.

3. Proof of Theorem 4

Let $\varphi\colon M^m \to \overline{M}^n$ be an isometric immersion between Riemannian manifolds. Given a function $g \in C^\infty(\overline{M})$ we set $f = g \circ \varphi \in C^\infty(M)$. Since

$$\langle \operatorname{grad}^M f, X\rangle = \langle \operatorname{grad}^{\overline{M}} g, X\rangle$$

for every vector field $X \in TM$, we obtain

$$\operatorname{grad}^{\overline{M}} g = \operatorname{grad}^M f + (\operatorname{grad}^{\overline{M}} g)^\perp$$

according to the decomposition $T\overline{M} = TM \oplus T^\perp M$. An easy computation using the Gauss formula gives the well-known relation (see e.g. [13])

$$\operatorname{Hess}^M f(X,Y) = \operatorname{Hess}^{\overline{M}} g(X,Y) + \langle \operatorname{grad}^{\overline{M}} g, \alpha(X,Y)\rangle \tag{3.1}$$

for all vector fields $X, Y \in TM$, where α stands for the second fundamental form of φ. In particular, taking traces with respect to an orthonormal frame $\{e_1, \dots, e_m\}$ in TM yields

$$\Delta_M f = \sum_{i=1}^m \operatorname{Hess}^{\overline{M}} g(e_i, e_i) + \langle \operatorname{grad}^{\overline{M}} g, \vec{H}\rangle,$$

where $\vec{H} = \sum_{i=1}^m \alpha(e_i, e_i)$.

3.1. **Proof of Theorem 4, item (i).** Let $g : N \times \mathbb{R}^\ell \to \mathbb{R}$ be given as $g(z,y) = \phi_b(\varrho_N(z))$, where

$$\phi_b(t) = \begin{cases} 1 - \cos(\sqrt{b}\,t) & \text{if } b > 0 \text{ and } 0 < t < \pi/2\sqrt{b}, \\ t^2 & \text{if } b = 0 \text{ and } t > 0, \\ \cosh(\sqrt{-b}\,t) & \text{if } b < 0 \text{ and } t > 0, \end{cases}$$

and $\varrho_N(z) = \operatorname{dist}_N(p,z)$. Consider $f : M \to \mathbb{R}$ to be the function $f = g \circ \varphi$, and let $\pi_N : N \times \mathbb{R}^\ell \to N$ be the projection on the factor N. Since $\pi_N(\varphi(M)) \subset B_N(r)$, we have that $f^* = \sup_M f \le \phi_b(r) < +\infty$. The idea of the proof is similar to the idea of Jorge-Koutrofiotis in [13]. We will need to apply the Omori-Yau Maximum Principle for the Hessian to the function f in order to control the second

fundamental form of the immersion restricted to a certain subspace and apply Otsuki's Lemma in the estimate of the sectional curvature.

To see that the Omori-Yau Maximum Principle for the Hessian holds on M, we may suppose that $\sup K_M < \infty$. Otherwise the estimate in (1.5) is trivially satisfied. In that case, since the scalar curvature is an average of sectional curvatures, it follows from (1.2) that

$$K_M^{\mathrm{rad}}(x) \geq -\widehat{B}^2 \varrho_M^2(x) \cdot \prod_{j=1}^{k} \left(\log^{(j)}(\varrho_M(x))\right)^2, \ \varrho_M(x) \gg 1, \tag{3.2}$$

for a positive constant $\widehat{B} > 0$, where K_M^{rad} denotes the radial sectional curvature of M. According to Corollary 11, this curvature decay suffices to conclude that the Omori-Yau Maximum Principle for the Hessian holds on M. Therefore, there exists a sequence of points $\{x_k\}$ in M such that

$$\text{(i) } f(x_k) > f^* - \frac{1}{k}, \ \text{(ii) } |\mathrm{grad}\, f(x_k)| < \frac{1}{k}, \text{ and (iii) } \mathrm{Hess}\, f(x_k) < \frac{1}{k}\langle , \rangle. \tag{3.3}$$

Since $f(x) = g(\varphi(x)) = \phi_b(\varrho_N(z))$, where $z = z(x) = \pi_N(\varphi(x))$, then

$$\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x)) = \mathrm{grad}\, f(x) + (\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x)))^\perp, \tag{3.4}$$

where

$$\mathrm{grad}^{N\times\mathbb{R}^\ell} g(z,y) = \phi_b'(\varrho_N(z))\mathrm{grad}^N \varrho_N(z).$$

It then follows from (3.1) that

$$\begin{aligned} \mathrm{Hess}\, f(x)(X,X) \quad = \quad & \mathrm{Hess}^N(\phi_b \circ \varrho_N)(z)(\pi_{TN}X, \pi_{TN}X) \\ & + \langle \mathrm{grad}^N(\phi_b \circ \varrho_N)(z), \alpha_x(X,X)\rangle \end{aligned} \tag{3.5}$$

for all vector fields $X \in T_xM$, where π_{TN} denotes the orthogonal projection of TM onto TN. Observe also that

$$\begin{aligned} \mathrm{Hess}^N(\phi_b \circ \varrho_N)(z)(\pi_{TN}X, \pi_{TN}X) = \phi_b''(s)\left(\frac{\partial \varrho_N}{\partial X}\right)^2 \\ + \phi_b'(s)\mathrm{Hess}^N \varrho_N(z)(\pi_{TN}X, \pi_{TN}X), \end{aligned} \tag{3.6}$$

where $s = s(x) = \varrho_N(z)$ and

$$\frac{\partial \varrho_N}{\partial X} = \langle \mathrm{grad}^N \varrho_N, X\rangle.$$

Taking into account that the function $\phi_b(t)$ satisfies the differential equation

$$\phi_b''(t) - \phi_b'(t)C_b(t) = 0,$$

it follows from (3.5) and (3.6) that

$$\begin{aligned} \mathrm{Hess}\, f(x)(X,X) = \phi_b'(s)\left(C_b(s)\left(\frac{\partial \varrho_N}{\partial X}\right)^2 + \langle \mathrm{grad}^N \varrho_N(z), \alpha_x(X,X)\rangle\right) \\ + \phi_b'(s)\mathrm{Hess}^N \varrho_N(z)(\pi_{TN}X, \pi_{TN}X). \end{aligned} \tag{3.7}$$

Since $m \geq \ell + 2$, we have for each $x \in M$ a subspace $V_x \subset T_xM \subset T_{\varphi(x)}(N \times \mathbb{R}^\ell)$ such that $V_x \perp T\mathbb{R}^\ell$ and $\dim V_x \geq m - \ell \geq 2$. Choose $\{\partial/\partial\varrho_N, \partial/\partial\theta_2, \ldots, \partial/\partial\theta_{n-\ell}\}$

orthonormal polar coordinates for TN. Then, for every $X \in V_x$ we have $\pi_{T\mathbb{R}^\ell} X = 0$ and

$$X = a_1^X \frac{\partial}{\partial \varrho_N} + \sum_{j=2}^{n-\ell} a_j^X \frac{\partial}{\partial \theta_j},$$

where $a_1^X = \partial \varrho_N / \partial X$. Therefore, using Theorem 13, we have that for every $X \in V_x$

$$\begin{aligned} \mathrm{Hess}^N \varrho_N(z)(\pi_{TN}X, \pi_{TN}X) &= \sum_{j=2}^{n-\ell} (a_j^X)^2 \mathrm{Hess}^N \varrho_N(z)(\frac{\partial}{\partial \theta_j}, \frac{\partial}{\partial \theta_j}) \\ &\geq C_b(s)\left(\langle , \rangle - d\varrho_N \otimes d\varrho_N\right)(X, X) \\ &= C_b(s)(|X|^2 - (a_1^X)^2), \end{aligned}$$

since $\pi_{TN}X = X$, so that by (3.7) we have

$$\begin{aligned} \text{(3.8)} \quad \mathrm{Hess}\, f(x)(X, X) &\geq \phi_b'(s)\left(C_b(s)|X|^2 + \langle \mathrm{grad}^N \varrho_N(z), \alpha_x(X, X)\rangle\right) \\ &\geq \phi_b'(s)\left(C_b(s)|X|^2 - |\alpha_x(X, X)|\right) \end{aligned}$$

for every $X \in V_x$. From here and (3.3) we obtain that

$$\frac{1}{k}|X|^2 \geq \mathrm{Hess}\, f(x_k)(X, X) \geq \phi_b'(s_k)\left(C_b(s_k)|X|^2 - |\alpha_{x_k}(X, X)|\right)$$

for every x_k and every $X \in V_{x_k}$, where $z_k = \pi_N(\varphi(x_k)$ and $s_k = s(x_k) = \varrho_N(z_k)$. Hence

$$|\alpha_{x_k}(X, X)| \geq \left(C_b(s_k) - \frac{1}{k\phi_b'(s_k)}\right)|X|^2$$

with

$$C_b(s_k) - \frac{1}{k\phi_b'(s_k)} > 0$$

for k sufficiently large.

Now consider $\beta_{x_k} \colon V_{x_k} \times V_{x_k} \to \mathbb{R}^{n-m}$ as the restriction of the second fundamental form α_{x_k} to V_{x_k}. We have that

$$n - m \leq m - \ell - 1 \leq \mathrm{dim} V_{x_k} - 1$$

since $2m - 1 \geq n + \ell$, and therefore we may apply Lemma 12 to β_{x_k}. The conclusion is that there are linearly independent vectors $X_k, Y_k \in V_{x_k}$ such that

$$\alpha(X_k, X_k) = \alpha(Y_k, Y_k) \quad \text{and} \quad \alpha(X_k, Y_k) = 0$$

and $|X_k| \geq |Y_k| \geq 1$. We will now compare the sectional curvature $K_M(X_k, Y_k)$ in M of the plane spanned by X_k and Y_k with the sectional curvature $K_{N\times\mathbb{R}^\ell}(X_k, Y_k)$ in $N \times \mathbb{R}^\ell$ of the same plane. Since $X_k, Y_k \in V_{x_k} \perp T\mathbb{R}^\ell$, then

$$K_{N\times\mathbb{R}^\ell}(X_k, Y_k) = K_N(X_k, Y_k).$$

Then, by the Gauss equation, we have that

$$\begin{aligned} K_M(X_k,Y_k) - K_N(X_k,Y_k) &= K_M(X_k,Y_k) - K_{N\times\mathbb{R}^\ell}(X_k,Y_k) \\ &= \frac{\langle \alpha(X_k,X_k),\alpha(Y_k,Y_k)\rangle - |\alpha(X_k,Y_k)|^2}{|X_k|^2|Y_k|^2 - \langle X_k,Y_k\rangle^2} \\ &\geq \frac{|\alpha(X_k,X_k)|^2}{|X_k|^2|Y_k|^2} \geq \left(\frac{|\alpha(X_k,X_k)|}{|X_k|^2}\right)^2 \\ &\geq \left(C_b(s_k) - \frac{1}{k\phi_b'(s_k)}\right)^2. \end{aligned}$$

Thus

$$\sup_M K_M - \inf_{B_N(r)} K_N \geq \left(C_b(s_k) - \frac{1}{k\,\phi_b'(s_k)}\right)^2. \tag{3.9}$$

Observe that $f^* = \phi_b(s^*)$, where $s^* = \sup_M s$ and $s_k \to s^* \leq r$. Therefore, letting $k \to \infty$ we have that

$$\sup_M K_M - \inf_{B_N(r)} K_N \geq C_b^2(s^*) \geq C_b^2(r).$$

This finishes the proof of item (i) in Theorem 4.

3.2. **Proof of Theorem 4, item (ii).** In this case, we cannot directly apply Theorem 9, but we may apply parts of its proof. Again consider $f : M \to \mathbb{R}$ as the function given by $f(x) = \phi_b(\varrho_N(z(x)))$, with $z(x) = \pi_N(\varphi(x))$. Since $\pi_N(\varphi(M)) \subset B_N(r)$, we have that $f^* = \sup_M f \leq \phi_b(r)$. Let $\psi : M \to [0,+\infty)$ be given by

$$\psi(x) = \exp\left(\int_0^{|y(x)|} \frac{ds}{\sigma(s)}\right),$$

where $y(x) = \pi_{\mathbb{R}^\ell}(\varphi(x))$. Since φ is proper and $\pi_N(\varphi(M)) \subset B_N(r)$, then the function $|y(x)|$ satisfies $|y(x)| \to +\infty$ as $x \to \infty$. By hypothesis we have that $\int_0^{+\infty} 1/\sigma(s)ds = +\infty$ so that $\psi(x) \to +\infty$ as $x \to \infty$.

Following the ideas of Pigola, Rigoli and Setti in the proof of [21, Theorem 1.9], we let $x_0 \in M$ with $\pi_N(\varphi(x_0)) \neq p$ and set

$$f_k(x) = \frac{f(x) - f(x_0) + 1}{\psi(x)^{1/k}}.$$

Thus $f_k(x_0) > 0$, and since $f^* \leq \phi_b(r) < +\infty$ and $\psi(x) \to +\infty$ as $x \to \infty$, we have that $\limsup_{x\to\infty} f_k(x) \leq 0$. Hence f_k attains a positive absolute maximum at a point $x_k \in M$. This procedure yields a sequence $\{x_k\}$ such that (passing to a subsequence if necessary) $f(x_k)$ converges to f^* (see page 8 of [21]).

First suppose that $x_k \to \infty$ as $k \to +\infty$. Since f_k attains a positive maximum at x_k, we have $\operatorname{grad} f_k(x_k) = 0$ and $\operatorname{Hess} f_k(x_k)(X,X) \leq 0$ for every $X \in T_{x_k}M$. This yields

$$\operatorname{grad} f(x_k) = \frac{f(x_k) - f(x_0) + 1}{k\psi(x_k)} \operatorname{grad} \psi(x_k) \tag{3.10}$$

and

$$\begin{aligned}\operatorname{Hess} f(x_k) &\leq \frac{f(x_k)-f(x_0)+1}{k\psi(x_k)}\left(\operatorname{Hess}\psi(x_k)+\left(\frac{1}{k}-1\right)\frac{1}{\psi(x_k)}d\psi\otimes d\psi\right)\\ &\leq \frac{f(x_k)-f(x_0)+1}{k\psi(x_k)}\operatorname{Hess}\psi(x_k).\end{aligned} \tag{3.11}$$

Since $\psi(x)=\zeta(y)$, where $y=y(x)$ and $\zeta(y)=\exp(\int_0^{|y|}ds/\sigma(s))$, $y\in\mathbb{R}^\ell$, from (3.1) we have that

$$\operatorname{Hess}\psi(x)(X,X)=\operatorname{Hess}^{\mathbb{R}^\ell}\zeta(y)(\pi_{T\mathbb{R}^\ell}X,\pi_{T\mathbb{R}^\ell}X)+\langle\operatorname{grad}^{\mathbb{R}^\ell}\zeta(y),\alpha_x(X,X)\rangle \tag{3.12}$$

for all vectors $X\in T_xM$, where $\pi_{T\mathbb{R}^\ell}$ denotes the orthogonal projection of TM onto $T\mathbb{R}^\ell$. Also observe that

$$\operatorname{grad}^{\mathbb{R}^\ell}\zeta(y)=\frac{\zeta(y)}{\sigma(|y|)}\operatorname{grad}^{\mathbb{R}^\ell}|y|,$$

and then

$$\operatorname{grad}\psi(x)=\frac{\psi(x)}{\sigma(|y|)}\operatorname{grad}^{\mathbb{R}^\ell}|y|. \tag{3.13}$$

Thus, for every $X\in T_xM$ such that $\pi_{T\mathbb{R}^\ell}X=0$, it follows from (3.12) that

$$\operatorname{Hess}\psi(x)(X,X)=\frac{\psi(x)}{\sigma(|y(x)|)}\langle\operatorname{grad}^{\mathbb{R}^\ell}|y|,\alpha_x(X,X)\rangle\leq\frac{\psi(x)}{\sigma(|y(x)|)}|\alpha_x(X,X)|.$$

Therefore, by (1.4) we obtain that

$$\frac{1}{\psi(x)}\operatorname{Hess}\psi(x)(X,X)\leq\frac{|\alpha_x(X,X)|}{\sigma(|y(x)|)}\leq|X|^2 \tag{3.14}$$

for every $X\in T_xM$ with $\pi_{T\mathbb{R}^\ell}X=0$.

As in the proof of item (i), since $m\geq\ell+2$, we may choose for each $x_k\in M$ a subspace $V_{x_k}\subset T_{x_k}M$ with $\dim V_{x_k}\geq m-\ell\geq 2$ and such that $V_{x_k}\perp T\mathbb{R}^\ell$. Then, $\pi_{T\mathbb{R}^\ell}X=0$ for every $X\in V_{x_k}$, and from (3.11) and (3.14) we get that

$$\operatorname{Hess} f(x_k)(X,X)\leq\frac{f(x_k)-f(x_0)+1}{k\psi(x_k)}\operatorname{Hess}\psi(x_k)(X,X)\leq\frac{\phi_b(r)+1}{k}|X|^2, \tag{3.15}$$

for every $X\in V_x$. Moreover, using Theorem 13, we also have here that

$$\operatorname{Hess} f(x)(X,X)\geq\phi_b'(s)\left(C_b(s)|X|^2-|\alpha_x(X,X)|\right) \tag{3.16}$$

for every $X\in V_x$, since $\pi_{TN}X=X$. Therefore, we obtain that

$$\frac{\phi_b(r)+1}{k}|X|^2\geq\operatorname{Hess} f(x_k)(X,X)\geq\phi_b'(s_k)\left(C_b(s_k)|X|^2-|\alpha_{x_k}(X,X)|\right)$$

for every x_k and every $X\in V_{x_k}$, where $z_k=\pi_N(\varphi(x_k))$ and $s_k=s(x_k)=\varrho_N(z_k)$. Hence

$$|\alpha_{x_k}(X,X)|\geq\left(C_b(s_k)-\frac{\phi_b(r)+1}{k\phi_b'(s_k)}\right)|X|^2$$

with

$$C_b(s_k)-\frac{\phi_b(r)+1}{k\phi_b'(s_k)}>0$$

for k sufficiently large. Reasoning now as in the last part of the proof of item (i), there exist linearly independent vectors $X_k, Y_k \in V_{x_k}$ such that, by Gauss' equation,

$$K_M(X_k, Y_k) - K_N(X_k, Y_k) = \left(\frac{|\alpha(X_k, X_k)|}{|X_k|^2}\right)^2 \geq \left(C_b(s_k) - \frac{\phi_b(r)+1}{k\phi_b'(s_k)}\right)^2.$$

We obtain from here that

$$\sup_M K_M - \inf_{B_N(r)} K_N \geq \left(C_b(s_k) - \frac{\phi_b(r)+1}{k\,\phi_b'(s_k)}\right)^2, \tag{3.17}$$

and letting $k \to \infty$ we conclude that

$$\sup_M K_M - \inf_{B_N(r)} K_N \geq C_b^2(s^*) \geq C_b^2(r),$$

where $s^* = \sup_M s$, $f^* = \phi_b(s^*)$ and $s_k \to s^* \leq r$.

To finish the proof of item (ii), we need to consider the case where the sequence $\{x_k\} \subset M$ remains in a compact set. In that case, passing to a subsequence if necessary, we may assume that $x_k \to x_\infty \in M$ and f attains its absolute maximum at x_∞. Thus $\operatorname{Hess} f(x_\infty)(X, X) \leq 0$ for all $X \in T_{x_\infty}M$. In particular, it follows from (3.16) that for every $X \in V_{x_\infty}$

$$0 \geq \operatorname{Hess} f(x_\infty)(X, X) \geq \phi_b'(s_\infty)\left(C_b(s_\infty)|X|^2 - |\alpha_{x_\infty}(X, X)|\right),$$

where $s_\infty = \varrho_N(\pi_N(\varphi(x_\infty)))$. Therefore

$$|\alpha_{x_\infty}(X, X)| \geq C_b(s_\infty)|X|^2.$$

By applying Lemma 12 to $\beta_{x_\infty} : V_{x_\infty} \times V_{x_\infty} \to \mathbb{R}^{n-m}$, the restriction of the second fundamental form α_{x_∞} to V_{x_∞}, and reasoning again as in the last part of the proof of item (i), we have that there exist linearly independent vectors $X_\infty, Y_\infty \in V_{x_\infty}$ such that, by Gauss' equation,

$$K_M(X_\infty, Y_\infty) - K_N(X_\infty, Y_\infty) = \left(\frac{|\alpha(X_\infty, X_\infty)|}{|X_\infty|^2}\right)^2 \geq C_b^2(s_\infty).$$

Thus, we conclude from here that

$$\sup_M K_M - \inf_{B_N(r)} K_N \geq C_b^2(s_\infty) \geq C_b^2(r). \tag{3.18}$$

This finishes the proof of Theorem 4.

4. Proof of Theorem 7

We proceed as in the proof of Theorem 4, item (ii), to obtain a sequence $\{x_k\}$ such that $f(x_k)$ converges to f^* and satisfying

$$\operatorname{grad} f(x_k) = \frac{f(x_k) - f(x_0) + 1}{k\psi(x_k)} \operatorname{grad} \psi(x_k) \tag{4.1}$$

and

$$\operatorname{Hess} f(x_k) \leq \frac{f(x_k) - f(x_0) + 1}{k\psi(x_k)} \operatorname{Hess} \psi(x_k). \tag{4.2}$$

Recall that (see (3.13))

$$\operatorname{grad} \psi(x) = \frac{\psi(x)}{\sigma(|y|)} \operatorname{grad}^{\mathbb{R}^\ell} |y|. \tag{4.3}$$

Let us first consider the case where $x_k \to \infty$ as $k \to +\infty$. From (4.1) and (4.3) we know that

$$|\mathrm{grad}\, f(x_k)| \leq \frac{(f^*+1)}{k}\frac{1}{\sigma(|y_k|)} \leq \frac{(\phi_b(r)+1)}{k}\frac{1}{\sigma(|y_k|)},$$

where $y_k = y(x_k)$. Since φ is proper and $\pi_N(\varphi(M)) \subset B_N(r)$, then $|y_k| \to +\infty$ as $k \to +\infty$. Therefore, taking into account that $\limsup_{t\to+\infty} 1/\sigma(t) < +\infty$ we obtain from here that

$$\lim_{k\to+\infty} |\mathrm{grad}\, f(x_k)| = 0. \tag{4.4}$$

Observe that

$$\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x)) = \phi_b'(\varrho_N(z))\mathrm{grad}^N \varrho_N(z) = \mathrm{grad}\, f(x) + (\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x)))^\perp,$$

where $z = z(x) = \pi_N(\varphi(x))$. Therefore,

$$\phi_b'(s_k)^2 = |\mathrm{grad}\, f(x_k)|^2 + |(\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x_k)))^\perp|^2, \tag{4.5}$$

with $s_k = \varrho_N(z(x_k))$, and making $k \to \infty$ here we obtain that

$$\lim_{k\to+\infty} |(\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x_k)))^\perp| = \phi_b'(s^*) > 0,$$

which implies that

$$(\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x_k)))^\perp \neq 0$$

for k sufficiently large.

As in the proof of Theorem 4, since $m = n-1 \geq \ell+2$, we may choose for each $x_k \in M$ a subspace $V_{x_k} \subset T_{x_k}M$ with $\dim V_{x_k} \geq n-1-\ell \geq 2$ and such that $V_{x_k} \perp T\mathbb{R}^\ell$. Then, using Theorem 13, we also have that

$$\mathrm{Hess}\, f(x_k)(X,X) \geq \phi_b'(s_k)\left(C_b(s_k)|X|^2 - |\alpha_{x_k}(X,X)|\right) \tag{4.6}$$

for every $X \in V_{x_k} \leq T_{x_k}M$, since $\pi_{TN}X = X$. On the other hand, we also know from (4.2) that

$$\begin{aligned} \mathrm{Hess}\, f(x_k)(X,X) &\leq \frac{(\phi_b(r)+1)}{k}\frac{\mathrm{Hess}\,\psi(x_k)(X,X)}{\psi(x_k)} \\ &= \frac{(\phi_b(r)+1)}{k}\frac{1}{\sigma(|y_k|)}\langle \mathrm{grad}^{\mathbb{R}^\ell}|y|, \alpha_{x_k}(X,X)\rangle \end{aligned} \tag{4.7}$$

for every $X \in T_{x_k}M$. Since $m = n-1$ and $(\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x_k)))^\perp \neq 0$ (for k large enough), then

$$\alpha_{x_k}(X,X) = \lambda_{x_k}(X,X)(\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x_k)))^\perp \tag{4.8}$$

for a real function λ.

Now observe that

$$\begin{aligned} \langle \mathrm{grad}^{\mathbb{R}^\ell}|y|, \alpha_{x_k}(X,X)\rangle &= \lambda_{x_k}(X,X)\langle \mathrm{grad}^{\mathbb{R}^\ell}|y|, (\mathrm{grad}^{N\times\mathbb{R}^\ell} g(\varphi(x_k)))^\perp\rangle \\ &= \lambda_{x_k}(X,X)\langle \mathrm{grad}^{\mathbb{R}^\ell}|y|, \mathrm{grad}\, f(x_k)\rangle \end{aligned}$$

because of $\langle \mathrm{grad}^{\mathbb{R}^\ell}|y|, \mathrm{grad}^N \varrho_N\rangle = 0$. Therefore,

$$\begin{aligned} \langle \mathrm{grad}^{\mathbb{R}^\ell}|y|, \alpha_{x_k}(X,X)\rangle &\leq |\lambda_{x_k}(X,X)||\mathrm{grad}\, f(x_k)| \\ &\leq |\lambda_{x_k}(X,X)|\frac{(\phi_b(r)+1)}{k}\frac{1}{\sigma(|y_k|)}. \end{aligned}$$

On the other hand, from our hypothesis (1.9) we know that

$$|\alpha_x(X,X)| \leq \sigma^2(|y(x)|)|X|^2,$$

and from (4.5) and (4.8) we have that

$$|\alpha_{x_k}(X,X)| = |\lambda_{x_k}(X,X)|\sqrt{\phi_b'(s_k)^2 - |\mathrm{grad}\, f(x_k)|^2} \leq \sigma^2(|y_k|)|X|^2.$$

That is,

$$\frac{|\lambda_{x_k}(X,X)|}{\sigma(|y_k|)} \leq \frac{\sigma(|y_k|)|X|^2}{\sqrt{\phi_b'(s_k)^2 - |\mathrm{grad}\, f(x_k)|^2}}.$$

It follows from here that

$$\langle \mathrm{grad}^{\mathbb{R}^\ell}|y|, \alpha_{x_k}(X,X)\rangle \leq \frac{(\phi_b(r)+1)}{k}\frac{\sigma(|y_k|)|X|^2}{\sqrt{\phi_b'(s_k)^2 - |\mathrm{grad}\, f(x_k)|^2}}$$

for every $X \in T_{x_k}M$, so that by (4.7) we get

$$\mathrm{Hess}\, f(x_k)(X,X) \leq \frac{(\phi_b(r)+1)^2}{k^2}\frac{|X|^2}{\sqrt{\phi_b'(s_k)^2 - |\mathrm{grad}\, f(x_k)|^2}}. \tag{4.9}$$

Therefore, from (4.6) and (4.9) we have that

$$\phi_b'(s_k)\left(C_b(s_k)|X|^2 - |\alpha_{x_k}(X,X)|\right) \leq \frac{(\phi_b(r)+1)^2}{k^2}\frac{|X|^2}{\sqrt{\phi_b'(s_k)^2 - |\mathrm{grad}\, f(x_k)|^2}} \tag{4.10}$$

for every $X \in V_{x_k}$. Hence

$$|\alpha_{x_k}(X,X)| \geq \left(C_b(s_k) - \frac{(\phi_b(r)+1)^2}{k^2\phi_b'(s_k)\sqrt{\phi_b'(s_k)^2 - |\mathrm{grad}\, f(x_k)|^2}}\right)|X|^2,$$

with

$$\lim_{k\to+\infty}\left(C_b(s_k) - \frac{(\phi_b(r)+1)^2}{k^2\phi_b'(s_k)\sqrt{\phi_b'(s_k)^2 - |\mathrm{grad}\, f(x_k)|^2}}\right) = C_b(s^*) \geq C_b(r) > 0,$$

where $s^* = \sup_M s$, $f^* = \phi_b(s^*)$ and $s_k \to s^* \leq r$. Reasoning now as in the last part of the proof of item (i), there exist linearly independent vectors $X_k, Y_k \in V_{x_k}$ such that, by Gauss' equation,

$$\begin{aligned} K_M(X_k,Y_k) - K_N(X_k,Y_k) &= \left(\frac{|\alpha(X_k,X_k)|}{|X_k|^2}\right)^2 \\ &\geq \left(C_b(s_k) - \frac{(\phi_b(r)+1)^2}{k^2\phi_b'(s_k)\sqrt{\phi_b'(s_k)^2 - |\mathrm{grad}\, f(x_k)|^2}}\right). \end{aligned}$$

We obtain from here that

$$\sup_M K_M - \inf_{B_N(r)} K_N \geq \left(C_b(s_k) - \frac{(\phi_b(r)+1)^2}{k^2\phi_b'(s_k)\sqrt{\phi_b'(s_k)^2 - |\mathrm{grad}\, f(x_k)|^2}}\right)^2, \tag{4.11}$$

and letting $k \to \infty$ we conclude that

$$\sup_M K_M - \inf_{B_N(r)} K_N \geq C_b^2(s^*) \geq C_b^2(r).$$

Finally, in the case where the sequence $\{x_k\} \subset M$ remains in a compact subset of M, and passing to a subsequence if necessary, we may assume that $x_k \to x_\infty \in M$

and f attains its absolute maximum at x_∞. Thus, $\operatorname{Hess} f(x_\infty)(X,X) \leq 0$ for all $X \in T_{x_\infty}M$. Therefore, it follows again from Theorem 13 that for every $X \in V_{x_\infty}$,

$$0 \geq \operatorname{Hess} f(x_\infty)(X,X) \geq \phi_b'(s_\infty)\left(C_b(s_\infty)|X|^2 - |\alpha_{x_\infty}(X,X)|\right),$$

where $s_\infty = \varrho_N(\pi_N(\varphi(x_\infty)))$ and $V_{x_\infty} \subset T_{x_\infty}M$ is a subspace with $\dim V_{x_\infty} \geq n-1-\ell \geq 2$ and such that $V_{x_\infty} \perp T\mathbb{R}^\ell$. The proof now finishes as at the end of item (ii) in Theorem 4.

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