AN ESTIMATE FOR THE SECTIONAL CURVATURE OF CYLINDRICALLY BOUNDED SUBMANIFOLDS

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Abstract. We give sharp sectional curvature estimates for complete immersed cylindrically bounded $m$-submanifolds $\varphi: M^m \to N^{n-\ell} \times \mathbb{R}^\ell$, $n + \ell \leq 2m - 1$, provided that either $\varphi$ is proper with the norm of the second fundamental form with certain controlled growth or $M$ has scalar curvature with strong quadratic decay. The latter gives a non-trivial extension of the Jorge-Koutrofiotis Theorem. In the particular case of hypersurfaces, that is, $m = n - 1$, the growth rate of the norm of the second fundamental form is improved. Our results will be an application of a generalized Omori-Yau Maximum Principle for the Hessian of a Riemannian manifold, in its newest elaboration given by Pigola, Rigoli and Setti (2005).

1. Introduction

Given complete Riemannian manifolds $M^m$ and $N^n$ with dimension $m < n$, the isometric immersion problem asks whether there exists an isometric immersion $\varphi: M \hookrightarrow N$. When $N^n = \mathbb{R}^n$ is the Euclidean space, the isometric problem is answered by the Nash Embedding Theorem that says that there is an isometric embedding $\varphi: M^m \hookrightarrow \mathbb{R}^n$, provided the codimension $n - m$ is sufficiently large; see [17]. For small codimension, meaning in this paper that $n - m \leq m - 1$, the answer in general depends on the geometries of $M$ and $N$. For instance, the Hilbert-Efimov Theorem [9], [12] says that no complete surface $M$ with sectional curvature $K_M \leq -\delta^2 < 0$ can be isometrically immersed in $\mathbb{R}^3$, and a classical result by C. Tompkins [24] states that a compact, flat, $m$-dimensional Riemannian manifold cannot be isometrically immersed in $\mathbb{R}^{2m-1}$. Tompkins’ result was extended in a series of papers by Chern and Kuiper [7], Moore [16], O’Neill [19], Otsuki [20] and Stiel [22], whose results can be summarized in the following theorem (we recall that a Cartan-Hadamard manifold is a simply connected, complete, Riemannian manifold with non-negative sectional curvatures).

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Theorem 1. Let \( \varphi : M^m \hookrightarrow N^n, n \leq 2m - 1, \) be an isometric immersion of a compact Riemannian \( m \)-manifold \( M \) into a Cartan-Hadamard \( n \)-manifold \( N \). Then the sectional curvatures of \( M \) and \( N \) satisfy
\[
\sup_M K_M > \inf_N K_N.
\]

Theorem 1 was extended by Jorge and Koutrofiotis in [13] to bounded, complete submanifolds with scalar curvature bounded below and in the version presented by Pigola, Rigoli and Setti in [21] Theorem 1.15], with scalar curvature satisfying
\[
s_M(x) \geq -B^2 \varrho_M^2(x) \cdot \prod_{j=1}^k \left( \log^{(j)}(\varrho_M(x)) \right)^2, \quad \varrho_M(x) \gg 1,
\]
for some constant \( B > 0 \) and some integer \( k \geq 1 \), where \( \varrho_M \) is the distance function on \( M \) to a fixed point and \( \log^{(j)} \) is the \( j \)-th iterate of the logarithm.

Theorem 2 (Jorge-Koutrofiotis, [13]). Let \( M^m \) and \( N^n \) be complete Riemannian manifolds of dimensions \( m \) and \( n \), respectively, with \( n \leq 2m - 1 \) and let \( \varphi : M \rightarrow N \) be an isometric immersion with \( \varphi(M) \subset B_N(r) \), where \( B_N(r) \) denotes a geodesic ball of \( N \) centered at a point \( p \in N \) and radius \( r \). Assume that the radial sectional curvature \( K_N^{\text{rad}} \) along the radial geodesics issuing from \( p \) satisfies \( K_N^{\text{rad}} \leq b \) in \( B_N(r) \) and \( 0 < r < \min\{\inf_N(p), \pi/2\sqrt{b}\} \), where we replace \( \pi/2\sqrt{b} \) by \( +\infty \) if \( b \leq 0 \). If the scalar curvature of \( M \) satisfies (1.2), then
\[
\sup_M K_M \geq C^2_b(r) + \inf_{B_N(r)} K_N,
\]
where
\[
C_b(t) = \begin{cases} 
\sqrt{b} \cot(\sqrt{b} t) & \text{if } b > 0 \text{ and } 0 < t < \pi/2\sqrt{b}, \\
1/t & \text{if } b = 0 \text{ and } t > 0, \\
\sqrt{-b} \coth(\sqrt{-b} t) & \text{if } b < 0 \text{ and } t > 0.
\end{cases}
\]

Remark 3. If \( N^n = \mathbb{S}^n(b) \) is the simply connected space form of constant sectional curvature \( b \) and \( M = \partial B_{\mathbb{S}^n(b)}(r) \subset \mathbb{S}^n(b) \) is a geodesic sphere of radius \( r \), then equality in (1.3) is achieved.

The purpose of this paper is to extend the Jorge-Koutrofiotis Theorem to the case of complete cylindrically bounded submanifolds of a Riemannian product \( N^{n-\ell} \times \mathbb{R}^\ell \), where \( N \) is a complete Riemannian manifold of dimension \( n-\ell \). In this context, an isometric immersion \( \varphi : M^m \rightarrow N^{n-\ell} \times \mathbb{R}^\ell \) of a Riemannian manifold \( M^m \) is said to be cylindrically bounded if there exists \( B_N(r) \), a geodesic ball of \( N \) centered at a point \( p \in N \) with radius \( r > 0 \), such that \( \varphi(M) \subset B_N(r) \times \mathbb{R}^\ell \) (see Figure 1).

In a recent paper [1], the two first authors, jointly with Dajczer, derived an estimate for the mean curvature \( H \) of complete cylindrically bounded submanifolds into a product Riemannian manifold \( N^{n-\ell} \times \mathbb{R}^\ell \). Specifically, it was proved in [1] that if the immersion \( \varphi : M^m \rightarrow N^{n-\ell} \times \mathbb{R}^\ell \) is proper and \( \varphi(M) \subset B_N(r) \times \mathbb{R}^\ell \), then
\[
\sup_M |H| \geq \frac{m-\ell}{m} C_b(r).
\]
As a consequence, it follows from here that a complete hypersurface of given constant mean curvature lying inside a closed circular cylinder in Euclidean space cannot be proper if the circular bases is of sufficiently small radius (see Corollary 1 in [1]). In particular, there exists no complete minimal hypersurface properly
immersed in $\mathbb{R}^n$ and having 2 (or more) bounded coordinates, showing that any possible counterexample to a conjecture of Calabi on complete minimal hypersurfaces \cite{3} (see also \cite{6}) cannot be proper.

Our main result here deals with the sectional curvature of such submanifolds, and it can be stated as follows.

**Theorem 4.** Let $M^m$ and $N^{n-\ell}$ be complete Riemannian manifolds of dimension $m$ and $n-\ell$ respectively, with $n+\ell \leq 2m-1$. Let $\varphi : M^m \to N^{n-\ell} \times \mathbb{R}^\ell$ be a (cylindrically bounded) isometric immersion with $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$. Assume that the radial sectional curvature $K_{N}^{rad}$ along the radial geodesics issuing from $p$ satisfies $K_{N}^{rad} \leq b$ in $B_N(r)$ and $0 < r < \min\{\text{inj}_N(p), \pi/2\sqrt{b}\}$, where we replace $\pi/2\sqrt{b}$ by $+\infty$ if $b \leq 0$. If either

(i) the scalar curvature of $M$ satisfies (1.2) or

(ii) $\varphi$ is proper and

\begin{equation}
\sup_{\varphi^{-1}(B_N(r) \times \partial B_{\ell}(t))} \|\alpha\| \leq \sigma(t),
\end{equation}

where $\alpha$ is the second fundamental form of the immersion and $\sigma : [0, +\infty) \to \mathbb{R}$ is a positive function satisfying $\int_0^{+\infty} 1/\sigma = +\infty$, then

\begin{equation}
\sup_{M} K_M \geq C_0^2(r) + \inf_{B_N(r)} K_N.
\end{equation}

It is worth pointing out that the codimension restriction $n+\ell \leq 2m-1$ cannot be relaxed. Actually, together with the bound $m \leq n-1$, it implies that $n-\ell \geq 3$ and $m \geq \ell+2$. In particular, for $n = 3$ we have that $\ell = 0$, and therefore $\varphi(M) \subset B_N(r)$. In fact, the flat cylinder $S^1(r') \times \mathbb{R} \subset B_{R^2}(r') \times \mathbb{R}$, for every $0 < r' < r$, shows that the restriction $2m-1 \geq n+\ell$ is necessary.

On the other hand, our estimate (1.3) is sharp. Indeed, for every $n \geq 3$ and $\ell \leq n-3$ we can consider $M = \partial B_{\bar{R}^{n-\ell}(0)}(r') \times \mathbb{R}^\ell$ and take $\varphi : M \to B_{\bar{R}^{n-\ell}(b)}(r') \times \mathbb{R}^\ell$ to be the canonical isometric immersion for every $0 < r' < r$. Therefore $\sup_{M} K_M$ is the constant sectional curvature of the geodesic sphere $\partial B_{\bar{R}^{n-\ell}(b)}(r') \subset N^{n-\ell}(b)$,
which is given by
\[
\sup_M K = K_{\partial B_{m-r}(r')} = \begin{cases} 
\frac{b}{\sin^2(\sqrt{b} r')} & \text{if } b > 0 \text{ and } 0 < r' < \pi/2\sqrt{b}, \\
1/r'^2 & \text{if } b = 0 \text{ and } r' > 0, \\
-b/\sinh^2(\sqrt{-b} r') & \text{if } b < 0 \text{ and } r' > 0.
\end{cases}
\]

In particular, observe that
\[
\sup_M K = K_{\partial B_{m-r}(r')} = C_b^2(r') + b.
\]

Since in this case \(K_{\partial B_{m-r}(r')} = b\), then for every \(0 < r' < r\) we have
\[
\sup_M K = C_b^2(r') + b \geq C_b^2(r) + \inf_{B_{m-r}(r)} K_{\partial B_{m-r}(r)},
\]

which shows that our estimate \(1.5\) is sharp.

Let \(\varphi: M^m \to N^{n-\ell} \times \mathbb{R}^\ell\) be an isometric immersion of a compact Riemannian \(m\)-manifold \(M^m\), and let \(\pi_N: N^{n-\ell} \times \mathbb{R}^\ell \to N^{n-\ell}\) be the projection onto the first factor. Denote by \(R_M\) the radius of the smallest ball of \(N\) containing \(\pi_N(\varphi(M))\). We will refer to \(R_M\) as the extrinsic radius of the immersion. As a consequence of Theorem 4 we have the following versions of the Extrinsic Radius Theorem of Jorge-Xavier [11] (see also [2, Theorem 1.3]).

**Corollary 5.** Let \(\varphi: M^m \to N^{n-\ell} \times \mathbb{R}^\ell\) be an isometric immersion of a compact Riemannian \(m\)-manifold \(M^m\) into the product \(N^{n-\ell} \times \mathbb{R}^\ell\) with \(n + \ell \leq 2m - 1\), where \(N^{n-\ell}\) is a complete Riemannian manifold with a pole and radial sectional curvature \(K_N^{\text{rad}} \leq b \leq 0\). Then, the extrinsic radius satisfies
\[
R_M \geq C_b^{-1} \left(\sqrt{\sup_M K} - \inf_N K\right).
\]

In particular, if \(N = \mathbb{R}^{n-\ell}\) we have that
\[
R_M \geq \frac{1}{\sqrt{\sup_M K}}.
\]

**Corollary 6.** Let \(\varphi: M^m \to S^{n-\ell} \times \mathbb{R}^\ell\) be an isometric immersion of a compact Riemannian \(m\)-manifold \(M^m\) with \(n + \ell \leq 2m - 1\). If \(\sup_M K_M \leq 1\), then
\[
R_M \geq \pi/2.
\]

On the other hand, it is important to remark that for hypersurfaces, the growth rate of the norm of the second fundamental form in \(1.4\) can be improved as follows.

**Theorem 7.** Let \(\varphi: M^{n-1} \to N^{n-\ell} \times \mathbb{R}^\ell\) be a properly immersed hypersurface with \(\varphi(M) \subset B_{N}(r) \times \mathbb{R}^\ell\), \(n - \ell \geq 3\). Suppose that \(N\) satisfies the assumptions on the radial sectional curvatures as in Theorem 4 and the second fundamental form \(\alpha\) satisfies
\[
\sup_{\varphi^{-1}(B_{N}(r) \times \partial B_\ell(t))} \|\alpha\| \leq \sigma^2(t),
\]
where \(\sigma: [0, \infty) \to \mathbb{R}\) is a positive function satisfying
\[
\int_0^{+\infty} \frac{1}{\sigma} = +\infty \quad \text{and} \quad \limsup_{t \to +\infty} \frac{1}{\sigma(t)} < +\infty.
\]
Then
\[
\sup_M K_M \geq C_b^2(r) + \inf_{B_{N}(r)} K_N.
\]
Remark 8. It should be remarked that Hasanis and Koutroufiotis [11] established similar sectional curvature estimates for cylindrically bounded submanifolds, with scalar curvature bounded below, of the Euclidean space $\mathbb{R}^n$. In a slightly more general situation, F. Giménez [10] established sectional curvature estimates for submanifolds with scalar curvature bounded below immersed in a tubular neighborhood of certain, $(P$-submanifolds), embedded submanifolds of Hadamard manifolds. Our main results, besides extending Hasanis-Koutroufiotis results to a larger class of submanifolds, can be easily adapted to reproduce Giménez’s result settings.

2. Preliminaries

Our main tool to build the proof of Theorem 4 is the following (and important) version of the Omori-Yau Maximum Principle for the Hessian due to Pigola, Rigoli and Setti [21, Theorem 1.9]. Chronologically, the Omori-Yau Maximum Principle can be traced in a series of papers, starting with Omori [18] who proved the maximum principle at infinity for the class of complete Riemannian manifolds with sectional curvature bounded from below, followed by new insights given by Cheng and Yau [5], [25], who extended them to the class of complete Riemannian manifolds with Ricci curvature bounded from below, and by extensions due to C. Dias [8] (extended to the class of sectional curvature with quadratic decay) and Chen-Xin [4] (extended to the class of Ricci curvature with quadratic decay). Finally, as observed by Pigola, Rigoli and Setti in [21], the validity of the Omori-Yau Maximum Principle does not depend on curvature bounds as much as one would expect. Actually, a condition to guarantee the validity of it can be expressed in a function theoretic form by the newest elaboration stated below.

**Theorem 9** ([21, Theorem 1.9]). Let $M^m$ be a Riemannian manifold and assume that there exists a non-negative $C^2$-function $\psi$ satisfying the following requirements:

(a.1) $\psi$ is proper, that is, $\psi(x) \to +\infty$ as $x \to \infty$;

(a.2) there exists a positive constant $A > 0$ such that $|\nabla \psi| \leq A\sqrt{\psi}$ outside a compact subset of $M$;

(a.3) there exists a positive constant $B > 0$ such that $\text{Hess} \psi \leq B\sqrt{\psi G} \langle , \rangle$ (in the sense of quadratic forms) outside a compact subset of $M$,

where $G$ is a smooth function on $[0, +\infty)$ satisfying:

$$
\begin{align*}
(i) \quad & G(0) > 0, \\
(ii) \quad & G'(t) \geq 0 \text{ on } [0, +\infty), \\
(iii) \quad & 1/\sqrt{G(t)} \notin L^1(0, +\infty), \\
(iv) \quad & \limsup_{t \to +\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty.
\end{align*}
$$

Then, given a function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$ there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset M$ such that

$$
\begin{align*}
(i) \quad & u(x_k) > u^* - \frac{1}{k}, \\
(ii) \quad & |\nabla u(x_k)| < \frac{1}{k}, \text{ and } (iii) \quad \text{Hess } u(x_k) < \frac{1}{k} \langle , \rangle
\end{align*}
$$

in the sense of quadratic forms, that is,

$$\text{Hess } u(x_k)(v, v) < \frac{1}{k} |v|^2 \quad \text{for all } v \in T_{x_k} M.$$
If, instead of (a.3), one replaces it by
\[ (a.3)' \quad \text{there exists a positive constant } B > 0 \text{ such that } \Delta \gamma \leq B \sqrt{\gamma G(\sqrt{\gamma})} \text{ outside a compact subset of } M, \]
then one can weaken conclusion (iii) in (2.2) to
\[ (iii)' \quad \Delta u(x_k) < \frac{1}{k}. \]

Remark 10. It is worth pointing out that although in the statement (and in the proof) of Theorem 9 the manifold \( M \) is not required to be geodesically complete, the two assumptions (a.1) and (a.2) imply it. For the details, see [21, page 10].

For that reason, and following the terminology introduced by Pigola, Rigoli and Setti in [21], the Omori-Yau Maximum Principle for the Hessian is said to hold on a (not necessarily complete) Riemannian manifold \( M \) if, for any smooth function \( u \in C^2(M) \) with \( u^* = \sup_M u < +\infty \), there exists a sequence of points \( \{x_k\}_{k \in \mathbb{N}} \subset M \) satisfying

\[
\begin{align*}
(i) & \quad u(x_k) > u^* - \frac{1}{k}, \\
(ii) & \quad |\nabla u(x_k)| < \frac{1}{k}, \quad \text{and (iii) } \text{Hess } u(x_k) < \frac{1}{k} \langle , \rangle
\end{align*}
\]

for each \( k \in \mathbb{N} \). Equivalently, for any smooth function \( u \in C^2(M) \) with \( u_0 = \inf_M u > -\infty \) there exists a sequence of points \( \{x_k\}_{k \in \mathbb{N}} \subset M \) with the properties

\[
\begin{align*}
(i) & \quad u(x_k) < u_0 + \frac{1}{k}, \\
(ii) & \quad |\nabla u(x_k)| < \frac{1}{k}, \quad \text{and (iii) } \text{Hess } u(x_k) > -\frac{1}{k} \langle , \rangle
\end{align*}
\]

for each \( k \in \mathbb{N} \). In the same way, the Omori-Yau Maximum Principle is said to hold on a (not necessarily complete) Riemannian manifold \( M \) if, for any smooth function \( u \in C^2(M) \) with \( u^* = \sup_M u < +\infty \), there exists a sequence of points \( \{x_k\}_{k \in \mathbb{N}} \subset M \) satisfying

\[
\begin{align*}
(i) & \quad u(x_k) > u^* - \frac{1}{k}, \\
(ii) & \quad |\nabla u(x_k)| < \frac{1}{k}, \quad \text{and (iii) } \Delta u(x_k) < \frac{1}{k}
\end{align*}
\]

for each \( k \in \mathbb{N} \).

The function theoretic approach to the Omori-Yau Maximum Principle given in Theorem 9 allows one to apply it in different situations, where the choice of \( \psi \) and \( G \) are suggested by the geometric setting. For instance, one has the following consequence (see [21, Example 1.13]).

**Corollary 11.** Let \( M \) be a complete, non-compact, Riemannian manifold, let \( o \in M \) be a reference point and denote by \( g_M(x) \) the Riemannian distance function from \( o \). Assume that the radial sectional curvature of \( M \), that is, the sectional curvature of the 2-planes containing \( \nabla g_M \), satisfies
\[ K_{r}^M \geq -G(r), \]
where \( G \) is a smooth function on \([0, +\infty)\) which we assume to be even at the origin, that is, \( G(2k+1)(0) = 0 \) for \( k = 0, 1, 2, \ldots \), and satisfying the conditions listed in Theorem 9. Then the Omori-Yau Maximum Principle for the Hessian holds on \( M \).

We will need two more results. The first is known as Otsuki’s Lemma (for a proof see, for instance, [15, Page 28] or [21, Lemma 1.16]).
Lemma 12. Let \( \beta : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^q, q \leq k - 1, \) be a bilinear symmetric form satisfying \( \beta(X, X) \neq 0 \) for \( X \neq 0. \) Then there exist linearly independent vectors \( X, Y \) such that
\[
\beta(X, X) = \beta(Y, Y) \quad \text{and} \quad \beta(X, Y) = 0.
\]

The second is the well-known Hessian Comparison Theorem; see [22].

Theorem 13 (Hessian Comparison Theorem). Let \( M \) be a Riemannian manifold and \( x_0, x_1 \in M \) be such that there is a minimizing unit speed geodesic \( \gamma \) joining \( x_0 \) and \( x_1, \) and let \( \rho(x) = \text{dist}(x_0, x) \) be the distance function to \( x_0. \) Let \( K_\gamma \leq b \) be the radial sectional curvatures of \( M \) along \( \gamma. \) If \( b > 0 \) assume \( \rho(x_1) < \pi/2\sqrt{b}. \) Then, we have
\[
\text{Hess} \rho(x)(\gamma', \gamma') = 0
\]
and
\[
\text{Hess} \rho(x)(X, X) \geq C_b(\rho(x))|X|^2,
\]
where \( X \in T_xM \) is perpendicular to \( \gamma'(\rho(x)). \)


Let \( \varphi : M^m \to \overline{M}^m \) be an isometric immersion between Riemannian manifolds. Given a function \( g \in C^\infty(\overline{M}) \) we set \( f = g \circ \varphi \in C^\infty(M). \) Since
\[
\langle \text{grad} M f, X \rangle = \langle \text{grad} \overline{M} g, X \rangle
\]
for every vector field \( X \in TM, \) we obtain
\[
\text{grad} \overline{M} g = \text{grad} M f + (\text{grad} \overline{M} g)\perp
\]
according to the decomposition \( T\overline{M} = TM \oplus T^\perp M. \) An easy computation using the Gauss formula gives the well-known relation (see e.g. [13])
\[
(3.1) \quad \text{Hess} M f(X, Y) = \text{Hess} \overline{M} g(X, Y) + \langle \text{grad} \overline{M} g, \alpha(X, Y) \rangle
\]
for all vector fields \( X, Y \in TM, \) where \( \alpha \) stands for the second fundamental form of \( \varphi. \) In particular, taking traces with respect to an orthonormal frame \( \{e_1, \ldots, e_m\} \) in \( TM \) yields
\[
\Delta M f = \sum_{i=1}^m \text{Hess} \overline{M} g(e_i, e_i) + \langle \text{grad} \overline{M} g, \overrightarrow{H} \rangle,
\]
where \( \overrightarrow{H} = \sum_{i=1}^m \alpha(e_i, e_i). \)

3.1. Proof of Theorem [4] item (i). Let \( g : N \times \mathbb{R}^\ell \to \mathbb{R} \) be given as \( g(z, y) = \phi_b(\rho_N(z)), \) where
\[
\phi_b(t) = \begin{cases} 
1 - \cos(\sqrt{b} t) & \text{if } b > 0 \text{ and } 0 < t < \pi/2\sqrt{b}, \\
t^2 & \text{if } b = 0 \text{ and } t > 0, \\
\cosh(\sqrt{b} t) & \text{if } b < 0 \text{ and } t > 0,
\end{cases}
\]
and \( \rho_N(z) = \text{dist}_N(p, z). \) Consider \( f : M \to \mathbb{R} \) to be the function \( f = g \circ \varphi, \) and let \( \pi_N : N \times \mathbb{R}^\ell \to N \) be the projection on the factor \( N. \) Since \( \pi_N(\varphi(M)) \subset B_N(r), \) we have that \( f^* = \sup M f \leq \phi_b(r) < +\infty. \) The idea of the proof is similar to the idea of Jorge-Koutrofiotis in [13]. We will need to apply the Omori-Yau Maximum Principle for the Hessian to the function \( f \) in order to control the second
K\textsuperscript{\text{rad}}(x) \geq -\hat{B}^2 \varrho_M(x) \cdot \prod_{j=1}^{k} \left( \log^{(j)}(\varrho_M(x)) \right)^2, \quad \varrho_M(x) \gg 1,

(3.2)

for a positive constant \( \hat{B} > 0 \), where \( K_M^{\text{rad}} \) denotes the radial sectional curvature of \( M \). According to Corollary 11, this curvature decay suffices to conclude that the Omori-Yau Maximum Principle for the Hessian holds on \( M \). Therefore, there exists a sequence of points \( (x_k) \) in \( M \) such that

(3.3) (i) \( f(x_k) > f^* - \frac{1}{k} \), (ii) \( |\text{grad} f(x_k)| < \frac{1}{k} \), and (iii) \( \text{Hess} f(x_k) < \frac{1}{k} \).

Since \( f(x) = g(\varphi(x)) = \phi_b(\varrho_N(z)) \), where \( z = z(x) = \pi_N(\varphi(x)) \), then

(3.4) \ \grad^{N \times \mathbb{R}^\ell} g(\varphi(x)) = \grad f(x) + (\grad^{N \times \mathbb{R}^\ell} g(\varphi(x)))^\perp,

where \( \grad^{N \times \mathbb{R}^\ell} g(z, y) = \phi_b'(\varrho_N(z))\grad^N \varrho_N(z) \).

It then follows from (3.3) that

(3.5) \ \text{Hess} f(x)(X, X) = \text{Hess}^N (\phi_b \circ \varrho_N)(z) (\pi_{TN} X, \pi_{TN} X) + \langle \text{grad}^N (\phi_b \circ \varrho_N)(z), \alpha_x(X, X) \rangle

for all vector fields \( X \in T_x M \), where \( \pi_{TN} \) denotes the orthogonal projection of \( TM \) onto \( TN \). Observe also that

(3.6) \ \text{Hess}^N (\phi_b \circ \varrho_N)(z) (\pi_{TN} X, \pi_{TN} X) = \phi_b''(s) \left( \frac{\partial \varrho_N}{\partial X} \right)^2 + \phi_b'(s)\text{Hess}^N \varrho_N(z)(\pi_{TN} X, \pi_{TN} X),

where \( s = s(x) = \varrho_N(z) \) and

\[ \frac{\partial \varrho_N}{\partial X} = (\text{grad}^N \varrho_N, X). \]

Taking into account that the function \( \phi_b(t) \) satisfies the differential equation

\[ \phi_b''(t) - \phi_b'(t)C_b(t) = 0, \]

it follows from (3.5) and (3.6) that

(3.7) \ \text{Hess} f(x)(X, X) = \phi_b'(s) \left( C_b(s) \left( \frac{\partial \varrho_N}{\partial X} \right)^2 + \langle \text{grad}^N \varrho_N(z), \alpha_x(X, X) \rangle \right)

+ \phi_b'(s)\text{Hess}^N \varrho_N(z)(\pi_{TN} X, \pi_{TN} X).

Since \( m \geq \ell + 2 \), we have for each \( x \in M \) a subspace \( V_x \subset T_x M \subset T_{\varphi(x)}(N \times \mathbb{R}^\ell) \) such that \( V_x \perp \mathbb{R}^\ell \) and \( \dim V_x \geq m - \ell \geq 2 \). Choose \( \{ \partial/\partial \varrho_N, \partial/\partial \theta_2, \ldots, \partial/\partial \theta_{n-\ell} \} \).
orthonormal polar coordinates for $TN$. Then, for every $X \in V_x$ we have $\pi_{TR^\ell} X = 0$ and

$$X = a_1^X \frac{\partial}{\partial \varrho_N} + \sum_{j=2}^{n-\ell} a_j^X \frac{\partial}{\partial \theta_j},$$

where $a_1^X = \varrho_N / \partial X$. Therefore, using Theorem 13 we have that for every $X \in V_x$

$$\text{Hess}^N \varrho_N(z)(\pi_{TN} X, \pi_{TN} X) = \sum_{j=2}^{n-\ell} (a_j^X)^2 \text{Hess}^N \varrho_N(z) \left( \frac{\partial}{\partial \theta_j,} \frac{\partial}{\partial \theta_j} \right) \geq C_b(s) \left( (,) - d\varrho_N \otimes d\varrho_N \right)(X, X)$$

$$= C_b(s) |X|^2 - (a_1^X)^2,$$

since $\pi_{TN} X = X$, so that by (3.7) we have

$$1/|X|^2 \geq \text{Hess}(x_k)(X, X) \geq \phi_k'(s_k) \left( C_b(s_k) |X|^2 - |\alpha_{x_k}(X, X)| \right)$$

for every $X \in V_x$. From here and (3.3) we obtain that

$$\frac{1}{k} |X|^2 \geq \text{Hess}(x_k)(X, X) \geq \phi_k'(s_k) \left( C_b(s_k) |X|^2 - |\alpha_{x_k}(X, X)| \right)$$

for every $x_k$ and every $X \in V_{x_k}$, where $z_k = \pi_N(\varphi(x_k))$ and $s_k = s(x_k) = \varrho_N(z_k)$. Hence

$$|\alpha_{x_k}(X, X)| \geq \left( C_b(s_k) - \frac{1}{k\phi_k'(s_k)} \right) |X|^2$$

with

$$C_b(s_k) - \frac{1}{k\phi_k'(s_k)} > 0$$

for $k$ sufficiently large.

Now consider $\beta_{x_k} : V_{x_k} \times V_{x_k} \to \mathbb{R}^{n-m}$ as the restriction of the second fundamental form $\alpha_{x_k}$ to $V_{x_k}$. We have that

$$n - m \leq m - \ell - 1 \leq \dim V_{x_k} - 1$$

since $2m - 1 \geq n + \ell$, and therefore we may apply Lemma 12 to $\beta_{x_k}$. The conclusion is that there are linearly independent vectors $X_k, Y_k \in V_{x_k}$ such that

$$\alpha(X_k, X_k) = \alpha(Y_k, Y_k) \quad \text{and} \quad \alpha(X_k, Y_k) = 0$$

and $|X_k| \geq |Y_k| \geq 1$. We will now compare the sectional curvature $K_M(X_k, Y_k)$ in $M$ of the plane spanned by $X_k$ and $Y_k$ with the sectional curvature $K_{N \times \mathbb{R}^\ell}(X_k, Y_k)$ in $N \times \mathbb{R}^\ell$ of the same plane. Since $X_k, Y_k \in V_{x_k} \perp T\mathbb{R}^\ell$, then

$$K_{N \times \mathbb{R}^\ell}(X_k, Y_k) = K_N(X_k, Y_k).$$
Then, by the Gauss equation, we have that
\[ K_M(X_k, Y_k) - K_N(X_k, Y_k) = K_M(X_k, Y_k) - K_{N \times R}(X_k, Y_k) \]
\[ = \frac{|\alpha(X_k, X_k), \alpha(Y_k, Y_k)| - |\alpha(X_k, Y_k)|^2}{|X_k|^2|Y_k|^2 - (X_k, Y_k)^2} \]
\[ \geq \frac{|\alpha(X_k, X_k)|^2}{|X_k|^2|Y_k|^2} \geq \left( \frac{|\alpha(X_k, X_k)|}{|X_k|^2} \right)^2 \]
\[ \geq \left( C_b(s_k) - \frac{1}{k \phi_b'(s_k)} \right)^2. \]

Thus
\[ \sup_M K_M - \inf_{B_N(r)} K_N \geq \left( C_b(s_k) - \frac{1}{k \phi_b'(s_k)} \right)^2. \]

Observe that \( f^* = \phi_b(s^*), \) where \( s^* = \sup_M s \) and \( s_k \to s^* \leq r. \) Therefore, letting \( k \to \infty \) we have that
\[ \sup_M K_M - \inf_{B_N(r)} K_N \geq C_b^2(s^*) \geq C_b^2(r). \]

This finishes the proof of item (i) in Theorem 4.

3.2. Proof of Theorem 4 item (ii). In this case, we cannot directly apply Theorem 4 but we may apply parts of its proof. Again consider \( f : M \to \mathbb{R} \) as the function given by \( f(x) = \phi_b\pi_N(z(x)), \) with \( z(x) = \pi_N(\varphi(x)). \) Since \( \pi_N(\varphi(M)) \subset B_N(r), \) we have that \( f^* = \sup_M f \leq \phi_b(r). \) Let \( \psi : M \to [0, +\infty) \) be given by
\[ \psi(x) = \exp \left( \int_0^{\max(0, |\varphi|)} \frac{ds}{\sigma(s)} \right), \]
where \( y(x) = \pi_{N \times R}(\varphi(x)). \) Since \( \varphi \) is proper and \( \pi_N(\varphi(M)) \subset B_N(r), \) then the function \( |\psi(x)| \) satisfies \( |\psi(x)| \to +\infty \) as \( x \to \infty. \) By hypothesis we have that \( \int_0^{+\infty} 1/\sigma(s)ds = +\infty \) so that \( \psi(x) \to +\infty \) as \( x \to \infty. \)

Following the ideas of Pigola, Rigoli and Setti in the proof of [21 Theorem 1.9], we let \( x_0 \in M \) with \( \pi_N(\varphi(x_0)) \neq p \) and set
\[ f_k(x) = \frac{f(x) - f(x_0) + 1}{\psi(x)^{1/k}}. \]
Thus \( f_k(x_0) > 0, \) and since \( f^* \leq \phi_b(r) < +\infty \) and \( \psi(x) \to +\infty \) as \( x \to \infty, \) we have that \( \limsup_{x \to \infty} f_k(x) \leq 0. \) Hence \( f_k \) attains a positive absolute maximum at a point \( x_k \in M. \) This procedure yields a sequence \( \{x_k\} \) such that (passing to a subsequence if necessary) \( f(x_k) \) converges to \( f^* \) (see page 8 of [21]).

First suppose that \( x_k \to \infty \) as \( k \to +\infty. \) Since \( f_k \) attains a positive maximum at \( x_k, \) we have \( \nabla f_k(x_k) = 0 \) and \( \text{Hess} f_k(x_k)(X, X) \leq 0 \) for every \( X \in T_{x_k}M. \) This yields
\[ \nabla f(x_k) = \frac{f(x_k) - f(x_0) + 1}{k\psi(x_k)} \nabla \psi(x_k). \]
and
\[
\text{Hess } f(x_k) \leq \frac{f(x_k) - f(x_0) + 1}{k \psi(x_k)} \left( \text{Hess } \psi(x_k) + \left( \frac{1}{k} - 1 \right) \frac{1}{\psi(x_k)} \right) \]
\leq \frac{f(x_k) - f(x_0) + 1}{k \psi(x_k)} \text{Hess } \psi(x_k).
\tag{3.11}
\]

Since \( \psi(x) = \zeta(y) \), where \( y = y(x) \) and \( \zeta(y) = \exp(\int_0^{|y|} ds/\sigma(s)) \), \( y \in \mathbb{R}^\ell \), from (3.11) we have that
\[
\text{Hess } \psi(x)(X, X) = \text{Hess} \, \zeta(y)(\pi_{\mathbb{T}^\ell} X, \pi_{\mathbb{T}^\ell} X) + \langle \text{grad} \, \zeta(y), \alpha_x(X, X) \rangle
\]
for all vectors \( X \in T_x M \), where \( \pi_{\mathbb{T}^\ell} \) denotes the orthogonal projection of \( TM \) onto \( \mathbb{T}^\ell \). Also observe that
\[
\text{grad} \, \zeta(y) = \frac{\zeta(y)}{\sigma(|y|)} \text{grad} \, \zeta(|y|),
\]
and then
\[
\text{grad } \psi(x) = \frac{\psi(x)}{\sigma(|y|)} \text{grad} \, \zeta(|y|).
\tag{3.13}
\]
Thus, for every \( X \in T_x M \) such that \( \pi_{\mathbb{T}^\ell} X = 0 \), it follows from (3.12) that
\[
\text{Hess } \psi(x)(X, X) = \frac{\psi(x)}{\sigma(|y(x)|)} \langle \text{grad} \, \zeta(|y|), \alpha_x(X, X) \rangle \leq \frac{\psi(x)}{\sigma(|y(x)|)} |\alpha_x(X, X)|.
\]
Therefore, by (1.4) we obtain that
\[
\frac{1}{\psi(x)} \text{Hess } \psi(x)(X, X) \leq \frac{|\alpha_x(X, X)|}{\sigma(|y(x)|)} \leq |X|^2
\tag{3.14}
\]
for every \( X \in T_x M \) with \( \pi_{\mathbb{T}^\ell} X = 0 \).

As in the proof of item (i), since \( m \geq \ell + 2 \), we may choose for each \( x_k \in M \) a subspace \( V_{x_k} \subset T_x M \) with \( \dim V_{x_k} \geq m - \ell \geq 2 \) and such that \( V_{x_k} \perp \mathbb{T}^\ell \). Then, \( \pi_{\mathbb{T}^\ell} X = 0 \) for every \( X \in V_{x_k} \), and from (3.11) and (3.14) we get that
\[
\text{Hess } f(x_k)(X, X) \leq \frac{f(x_k) - f(x_0) + 1}{k \psi(x_k)} \text{Hess } \psi(x_k)(X, X) \leq \frac{\phi_b(r) + 1}{k} |X|^2,
\tag{3.15}
\]
for every \( X \in V_x \). Moreover, using Theorem 133 we also have here that
\[
\text{Hess } f(x)(X, X) \geq \phi_b(s) \left( C_b(s) |X|^2 - |\alpha_x(X, X)| \right)
\tag{3.16}
\]
for every \( X \in V_x \), since \( \pi_{\mathbb{T}^N} X = X \). Therefore, we obtain that
\[
\frac{\phi_b(r) + 1}{k} |X|^2 \geq \text{Hess } f(x_k)(X, X) \geq \phi_b(s_k) \left( C_b(s_k) |X|^2 - |\alpha_x(X, X)| \right)
\]
for every \( x_k \) and every \( X \in V_{x_k} \), where \( z_k = \pi_N(\varphi(x_k)) \) and \( s_k = s(x_k) = \varphi_N(z_k) \).

Hence
\[
|\alpha_{x_k}(X, X)| \geq \left( C_b(s_k) - \frac{\phi_b(r) + 1}{k \phi_b'(s_k)} \right) |X|^2
\]
with
\[
C_b(s_k) - \frac{\phi_b(r) + 1}{k \phi_b'(s_k)} > 0
\]
for $k$ sufficiently large. Reasoning now as in the last part of the proof of item (i), there exist linearly independent vectors $X_k, Y_k \in V_{x_k}$ such that, by Gauss’ equation,

$$K_M(X_k, Y_k) - K_N(X_k, Y_k) = \left( \frac{|\alpha(X_k, X_k)|}{|X_k|^2} \right)^2 \geq \left( C_b(s_k) - \frac{\phi_b(r) + 1}{k\phi_b'(s_k)} \right)^2.$$

We obtain from here that

$$sup \ K_M - inf \ B_N(r) K_N \geq \left( C_b(s_k) - \frac{\phi_b(r) + 1}{k\phi_b'(s_k)} \right)^2,$$

and letting $k \to \infty$ we conclude that

$$sup \ K_M - inf \ B_N(r) K_N \geq C^2_b(s^*) \geq C^2_b(r),$$

where $s^* = sup_M s, f^* = \phi_b(s^*)$ and $s_k \to s^* \leq r$.

To finish the proof of item (ii), we need to consider the case where the sequence \( \{x_k\} \subset M \) remains in a compact set. In that case, passing to a subsequence if necessary, we may assume that $x_k \to x_{\infty} \in M$ and $f$ attains its absolute maximum at $x_{\infty}$. Thus $Hess f(x_{\infty})(X, X) \leq 0$ for all $X \in T_{x_{\infty}} M$. In particular, it follows from \(3.16\) that for every $X \in V_{x_{\infty}}$

$$0 \geq Hess f(x_{\infty})(X, X) \geq \phi_b'(s_{\infty}) (C_b(s_{\infty})|X|^2 - |\alpha_{x_{\infty}}(X, X)|),$$

where $s_{\infty} = \phi_N(\pi_N(\varphi_{x_{\infty}})))$. Therefore

$$|\alpha_{x_{\infty}}(X, X)| \geq C_b(s_{\infty})|X|^2.$$

By applying Lemma \(12\) to $\beta_{x_{\infty}} : V_{x_{\infty}} \times V_{x_{\infty}} \to \mathbb{R}^{n-m}$, the restriction of the second fundamental form $\alpha_{x_{\infty}}$ to $V_{x_{\infty}}$, and reasoning again as in the last part of the proof of item (i), we have that there exist linearly independent vectors $X_{\infty}, Y_{\infty} \in V_{x_{\infty}}$ such that, by Gauss’ equation,

$$K_M(X_{\infty}, Y_{\infty}) - K_N(X_{\infty}, Y_{\infty}) = \left( \frac{|\alpha(X_{\infty}, X_{\infty})|}{|X_{\infty}|^2} \right)^2 \geq C^2_b(s_{\infty}).$$

Thus, we conclude from here that

$$sup \ K_M - inf \ B_N(r) K_N \geq C^2_b(s_{\infty}) \geq C^2_b(r).$$

This finishes the proof of Theorem \(4\).

4. Proof of Theorem \(7\)

We proceed as in the proof of Theorem \(5\) item (ii), to obtain a sequence \( \{x_k\} \) such that $f(x_k)$ converges to $f^*$ and satisfying

$$\text{grad} \ f(x_k) = \frac{f(x_k) - f(x_0) + 1}{k\psi(x_k)} \text{grad} \ \psi(x_k)$$

and

$$\text{Hess} \ f(x_k) \leq \frac{f(x_k) - f(x_0) + 1}{k\psi(x_k)} \text{Hess} \ \psi(x_k).$$

Recall that (see \(3.13\))

$$\text{grad} \ \psi(x) = \frac{\psi(x)}{\sigma(|y|)} \text{grad} \ b^t |y|.$$
Let us first consider the case where \( x_k \to \infty \) as \( k \to +\infty \). From (4.1) and (4.3) we know that
\[
|\langle \nabla f(x_k) \rangle| \leq \frac{(f^* + 1)}{k} \frac{1}{\sigma(|y_k|)} \leq \frac{(\phi_b(r) + 1)}{k} \frac{1}{\sigma(|y_k|)},
\]
where \( y_k = y(x_k) \). Since \( \varphi \) is proper and \( \pi_N(\varphi(M)) \subset B_N(r) \), then \( |y_k| \to +\infty \) as \( k \to +\infty \). Therefore, taking into account that \( \limsup_{t \to +\infty} 1/\sigma(t) < +\infty \) we obtain from here that
\[
\lim_{k \to +\infty} |\langle \nabla f(x_k) \rangle| = 0.
\]
Observe that
\[
\langle \text{grad}^N g(\varphi(x)) \rangle = \phi'_b(\theta_N(z)) \text{grad}^N \theta_N(z) = \text{grad} f(x) + \langle \text{grad}^N g(\varphi(x)) \rangle^\perp,
\]
where \( z = z(x) = \pi_N(\varphi(x)) \). Therefore,
\[
\phi'_b(s_k)^2 = |\langle \text{grad} f(x_k) \rangle|^2 + |\langle \text{grad}^N g(\varphi(x_k)) \rangle^\perp|^2,
\]
with \( s_k = \theta_N(z(x_k)) \), and making \( k \to \infty \) here we obtain that
\[
\lim_{k \to +\infty} |\langle \text{grad}^N g(\varphi(x_k)) \rangle^\perp| = \phi'_b(s^*) > 0,
\]
which implies that
\[
\langle \text{grad}^N g(\varphi(x_k)) \rangle^\perp \neq 0
\]
for \( k \) sufficiently large.

As in the proof of Theorem 11 since \( m = n - 1 \geq \ell + 2 \), we may choose for each \( x_k \in M \) a subspace \( V_{x_k} \subset T_{x_k}M \) with \( \dim V_{x_k} \geq n - 1 - \ell \geq 2 \) and such that \( V_{x_k} \perp TM \). Then, using Theorem 11, we also have that
\[
\text{Hess} f(x_k)(X,X) \geq \phi'_b(s_k) \left( c_b(s_k)[X]^2 - |\alpha_{x_k}(X,X)| \right)
\]
for every \( X \in V_{x_k} \leq T_{x_k}M \), since \( \pi_{TN}X = X \). On the other hand, we also know from (4.3) that
\[
\text{Hess} f(x_k)(X,X) \leq \frac{(\phi_b(r) + 1)}{k} \frac{\text{Hess} \psi(x_k)(X,X)}{\psi(x_k)}
\]
\[
= \frac{(\phi_b(r) + 1)}{k} \frac{1}{\sigma(|y_k|)} \langle \text{grad}^\ell |y|, \alpha_{x_k}(X,X) \rangle
\]
for every \( X \in T_{x_k}M \). Since \( m = n - 1 \) and \( \langle \text{grad}^N g(\varphi(x_k)) \rangle^\perp \neq 0 \) (for \( k \) large enough), then
\[
\alpha_{x_k}(X,X) = \lambda_{x_k}(X,X) \langle \text{grad}^N g(\varphi(x_k)) \rangle^\perp
\]
for a real function \( \lambda \).

Now observe that
\[
\langle \text{grad}^\ell |y|, \alpha_{x_k}(X,X) \rangle = \lambda_{x_k}(X,X) \langle \text{grad}^\ell |y|, \langle \text{grad}^N g(\varphi(x_k)) \rangle^\perp \rangle
\]
\[
= \lambda_{x_k}(X,X) \langle \text{grad}^\ell |y|, \text{grad} f(x_k) \rangle
\]
because of \( \langle \text{grad}^\ell |y|, \text{grad}^N \theta_N \rangle = 0 \). Therefore,
\[
\langle \text{grad}^\ell |y|, \alpha_{x_k}(X,X) \rangle \leq |\lambda_{x_k}(X,X)||\text{grad} f(x_k)|
\]
\[
\leq |\lambda_{x_k}(X,X)| \frac{(\phi_b(r) + 1)}{k} \frac{1}{\sigma(|y_k|)}
\]
On the other hand, from our hypothesis (1.9) we know that

\[ |\alpha_x(X, X)| \leq \sigma^2(|y(x)|)|X|^2, \]

and from (1.5) and (1.8) we have that

\[ |\alpha_{x_k}(X, X)| = |\lambda_{x_k}(X, X)| \sqrt{\phi'(s_k)^2 - |\text{grad} f(x_k)|^2} \leq \sigma^2(|y_k|)|X|^2. \]

That is,

\[ \frac{|\lambda_{x_k}(X, X)|}{\sigma(|y_k|)} \leq \frac{\sigma(|y_k|)|X|^2}{\sqrt{\phi'(s_k)^2 - |\text{grad} f(x_k)|^2}}. \]

It follows from here that

\[ \langle \text{grad}^R y, \alpha_{x_k}(X, X) \rangle \leq \frac{(\phi_b(r) + 1)^2}{k^2} \frac{\sigma(|y_k|)|X|^2}{\sqrt{\phi'(s_k)^2 - |\text{grad} f(x_k)|^2}} \]

for every \( X \in T_{x_k}M \), so that by (4.7) we get

\[ \text{Hess} f(x_k)(X, X) \leq \frac{(\phi_b(r) + 1)^2}{k^2} \frac{|X|^2}{\sqrt{\phi'(s_k)^2 - |\text{grad} f(x_k)|^2}}. \]

Therefore, from (4.6) and (4.9) we have that

\[ \phi'(s_k) (C_b(s_k)|X|^2 - |\alpha_{x_k}(X, X)|) \leq \frac{(\phi_b(r) + 1)^2}{k^2} \frac{|X|^2}{\sqrt{\phi'(s_k)^2 - |\text{grad} f(x_k)|^2}} \]

for every \( X \in V_{x_k} \). Hence

\[ |\alpha_{x_k}(X, X)| \geq \left( C_b(s_k) - \frac{(\phi_b(r) + 1)^2}{k^2 \phi'(s_k) \sqrt{\phi'(s_k)^2 - |\text{grad} f(x_k)|^2}} \right) |X|^2, \]

with

\[ \lim_{k \to +\infty} \left( C_b(s_k) - \frac{(\phi_b(r) + 1)^2}{k^2 \phi'(s_k) \sqrt{\phi'(s_k)^2 - |\text{grad} f(x_k)|^2}} \right) = C_b(s^*) \geq C_b(r) > 0, \]

where \( s^* = \sup_M s, f^* = \phi_b(s^*) \) and \( s_k \to s^* \leq r \). Reasoning now as in the last part of the proof of item (i), there exist linearly independent vectors \( X_k, Y_k \in V_{x_k} \) such that, by Gauss’ equation,

\[ K_M(X_k, Y_k) - K_N(X_k, Y_k) = \left( \frac{|\alpha(X_k, X_k)|}{|X_k|^2} \right)^2 \]

\[ \geq \left( C_b(s_k) - \frac{(\phi_b(r) + 1)^2}{k^2 \phi'(s_k) \sqrt{\phi'(s_k)^2 - |\text{grad} f(x_k)|^2}} \right)^2. \]

We obtain from here that

\[ \sup_M K_M - \inf_{B_N(r)} K_N \geq \left( C_b(s_k) - \frac{(\phi_b(r) + 1)^2}{k^2 \phi'(s_k) \sqrt{\phi'(s_k)^2 - |\text{grad} f(x_k)|^2}} \right)^2, \]

and letting \( k \to \infty \) we conclude that

\[ \sup_M K_M - \inf_{B_N(r)} K_N \geq C_b^2(s^*) \geq C_b^2(r). \]

Finally, in the case where the sequence \( \{x_k\} \subset M \) remains in a compact subset of \( M \), and passing to a subsequence if necessary, we may assume that \( x_k \to x_\infty \in M \).
and $f$ attains its absolute maximum at $x_\infty$. Thus, \( \text{Hess} f(x_\infty)(X,X) \leq 0 \) for all \( X \in T_{x_\infty} M \). Therefore, it follows again from Theorem 13 that for every \( X \in V_{x_\infty} \),

\[
0 \geq \text{Hess} f(x_\infty)(X,X) \geq \phi_N(s_\infty) \left( C_b(s_\infty)||X||^2 - |\alpha_{x_\infty}(X,X)| \right),
\]

where \( s_\infty = \phi_N(\pi_N(\varphi(x_\infty))) \) and \( V_{x_\infty} \subset T_{x_\infty} M \) is a subspace with \( \dim V_{x_\infty} \geq n - 1 - \ell \geq 2 \) and such that \( V_{x_\infty} \perp T_{x_\infty} \). The proof now finishes as at the end of item (ii) in Theorem 4.

References


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