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L^2 SERRE DUALITY ON DOMAINS IN COMPLEX MANIFOLDS AND APPLICATIONS

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Abstract. An L^2 version of the Serre duality on domains in complex manifolds involving duality of Hilbert space realizations of the $\overline{\partial}$-operator is established. This duality is used to study the solution of the $\overline{\partial}$-equation with prescribed support. Applications are given to $\overline{\partial}$-closed extension of forms, as well as to Bochner-Hartogs type extension of CR functions.

1. Introduction

A fundamental result in the theory of complex manifolds is Serre's duality theorem. This establishes a duality between the cohomology of a complex manifold Ω and the cohomology of Ω with compact supports, provided the Cauchy-Riemann operator $\overline{\partial}$ has closed range in appropriate degrees.

More precisely, this can be stated as follows: let E be a holomorphic vector bundle on Ω, let $H^{p,q}(\Omega, E)$ denote the (p,q)-th Dolbeault cohomology group for E-valued forms on Ω, and let $H^{p,q}_{\text{comp}}(\Omega, E)$ denote the (p,q)-th Dolbeault cohomology group with compact support. Let E^* denote the holomorphic vector bundle on Ω dual to the bundle E, and let $n = \dim_{\mathbb{C}} \Omega$. Then (we assume that all manifolds in this paper are countable at infinity):

Serre Duality Theorem. *Suppose that each of the two operators*

$$\mathcal{C}^\infty_{p,q-1}(\Omega, E) \xrightarrow{\overline{\partial}_E} \mathcal{C}^\infty_{p,q}(\Omega, E) \xrightarrow{\overline{\partial}_E} \mathcal{C}^\infty_{p,q+1}(\Omega, E) \tag{1}$$

has closed range with respect to the natural Fréchet topology. Then the dual of the topological vector space $H^{p,q}(\Omega, E)$ (with the quotient Fréchet topology) can be canonically identified with the space $H^{n-p,n-q}_{\text{comp}}(\Omega, E^*)$ with the quotient topology, where we endow spaces of compactly supported forms with the natural inductive limit topology.

In fact, the condition that the two maps in (1) have closed range is also necessary for the duality theorem to hold (see [9]; see also [26, 27, 28] for further results of this type).

Serre's original proof [35] is based on sheaf theory and the theory of topological vector spaces. A different approach to this result, in the case when Ω is a compact complex manifold, was given by Kodaira using Hodge theory (see [23] or [7]). In this note we extend Kodaira's method to non-compact Hermitian manifolds to obtain an L^2 analog of the Serre duality. Special cases of Serre-duality using L^2

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methods have appeared before in many contexts (see [25], or [11, Theorem 5.1.7] and [19, 20], for example). The L^2-Serre duality between the maximal and minimal realizations of the $\overline{\partial}$-operator is also used in the study of the $\overline{\partial}$-operator on compact complex spaces (see e.g. [31, Proposition 1.3]) and more general duality results (of the type discussed in §3.6 below) are used as well in these investigations (see [33, Chapter 5]). Our treatment aims to streamline and systematize these results, with emphasis on non-compact manifolds, and to point out its close relation with the choice of L^2-realizations of the Cauchy-Riemann operator $\overline{\partial}$, or alternatively, the choice of boundary conditions for the L^2-realizations of the formal complex Laplacian $\overline{\partial}_E\vartheta_E + \vartheta_E\overline{\partial}_E$.

The L^2-duality can be interpreted in many ways. At one level, it is a duality between the standard $\Box$-Laplacian with $\overline{\partial}$-Neumann boundary conditions and the $\Box_c$-Laplacian with dual ("$\overline{\partial}$-Dirichlet") boundary conditions. Using another approach, results regarding the solution of the $\overline{\partial}$-equation in L^2 can be converted to statements regarding the solution of the $\overline{\partial}_c$-equation. This leads to a solution of the $\overline{\partial}$-Cauchy problem, i.e., a solution of the $\overline{\partial}$-equation with prescribed support. At the heart of the matter lies the existence of a duality between Hilbert space realizations of the $\overline{\partial}$-operator. This is explained in §3.6. However, for clarity of exposition, we concentrate on the classical duality between the well-known maximal and minimal realizations of $\overline{\partial}$ in the rest of the paper.

As an application of the duality principle, we consider the problem of $\overline{\partial}$-closed extension of forms. It is well known that solving the $\overline{\partial}$-equation with a weight can be interpreted as solving $\overline{\partial}$ with bundle-valued forms (see [8]). The weight function ϕ corresponds to the metric for the trivial line bundle with a metric under which the length of the vector 1 at the point z is $e^{-\phi(z)}$. It was used by Hörmander to study the weighted $\overline{\partial}$-Neumann operator by using weight functions which are strictly plurisubharmonic in a neighborhood of a pseudoconvex domain. When the boundary is smooth, one can also use the smooth weight functions to study the boundary regularity for pseudoconvex domains (see [24]) or pseudoconcave domains (see [36, 37]) in a Stein manifold. In this paper we will use the Serre duality to study the $\overline{\partial}$ problems with singular weight functions. The use of singular weight functions allows us to obtain the existence and regularity problem on pseudoconcave domains with Lipschitz boundary in Stein manifolds. The use of singular weights has the advantage that it only requires the boundary to be Lipschitz. Even when the boundary is smooth, the use of singular weight functions gives the regularity results much more directly (cf. the proof in [37] or [2, Chapter 9]). This method is also useful when the manifold is not Stein, as in the case of the complex projective space $\mathbb{CP}^n$. In this case, any pseudoconconvex domain in $\mathbb{CP}^n$ is Stein, but $\mathbb{CP}^n$ is not Stein. In recent years these problems have been studied by many people (see [15, 4, 3]), and are all variants of the Serre duality results.

The plan of this paper is as follows. In §2, we recall basic definitions from complex differential geometry and functional analysis. This material can be found in standard texts, e.g. [12, 43, 14]. Next, in §3 we discuss several avatars of the L^2-duality theorem: at the level of Laplacians, at the level of cohomology and for the $\overline{\partial}$ and $\overline{\partial}_c$ problems. We discuss a general form of the duality theorem using the notion of dual realizations of the $\overline{\partial}$ operator on vector bundles. In §4, we apply the results of §3 to trivial line bundles with singular metrics on pseudoconvex domains. This leads to results on the $\overline{\partial}$-closed extension of forms from pseudoconcave domains. In

the last section, we use the L^2-duality results to discuss the holomorphic extension of CR forms from the boundary of a Lipschitz domain in a complex manifold. We obtain a proof of the Bochner-Hartogs extension theorem using duality.

2. Notation and preliminaries

Throughout this article, Ω will denote a Hermitian manifold, and E a holomorphic vector bundle on Ω.

2.1. **Differential operators on Hilbert spaces.** The metrics on Ω and E induce an inner product $(,)$ on the space $\mathcal{D}(\Omega, E)$ of smooth compactly supported sections of E over Ω. The inner product is given by

$$(f, g) = \int_\Omega \langle f, g\rangle dV, \tag{2}$$

where $\langle,\rangle$ is the inner product in the metric of the bundle E, and dV denotes the volume form induced by the metric of Ω. This allows us to define the Hilbert space $L^2(\Omega, E)$ of square integrable sections of E over Ω in the usual way, as the completion of the space of smooth compactly supported sections of E over Ω under the inner product (2).

Let A be a differential operator acting on sections of E, i.e. $A : \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\Omega, E)$, and let A' be the formal adjoint of A with respect to the inner product (2). Recall that this means that for smooth sections f, g of E over Ω, at least one of which is compactly supported, we have

$$(Af, g) = (f, A'g). \tag{3}$$

The well-known facts that A' exists, that it is also a differential operator acting on sections of E, and that A' has the same order as A follow from a direct computation in local coordinates using integration by parts. It is clear that $(A')' = A$, i.e. the formal adjoint of A' is A.

By an *operator* T from a Hilbert space H_1 to another Hilbert space H_2 we mean a $\mathbb{C}$-linear map from a linear subspace $\mathrm{Dom}(T)$ of H_1 into H_2. We use the notation $T : \mathsf{H}_1 \dashrightarrow \mathsf{H}_2$ to denote the fact that T is defined on a subspace of H_1 (rather than on all of H_1 when we write $T : \mathsf{H}_1 \to \mathsf{H}_2$). Recall that such an operator is said to be *closed* if its graph is closed as a subspace of the product Hilbert space $\mathsf{H}_1 \times \mathsf{H}_2$.

The differential operator A gives rise to several closed operators on the Hilbert space $L^2(\Omega, E)$.

1. The *weak maximal realization* $A_{\max}$: we say for $f, g \in L^2(\Omega, E)$ that $Af = g$ in the *weak sense* if for all test sections $\phi \in \mathcal{D}(\Omega, E)$ we have that

$$(f, A'\phi) = (g, \phi). \tag{4}$$

(This can be rephrased in terms of the action of A on distributional sections of E, but we will not need this.) The weak maximal realization $A_{\max}$ is the densely-defined closed (cf. Lemma 1) linear operator on $L^2(\Omega, E)$ with domain $\mathrm{Dom}(A_{\max})$ consisting of all $f \in L^2(\Omega, E)$ such that $Af \in L^2(\Omega, E)$, where Af is taken in the weak sense. On $\mathrm{Dom}(A_{\max})$, we define $A_{\max} f = Af$ in the weak sense.

2. The *strong minimal realization* $A_{\min}$ is the closure of the densely defined operator $A_{\mathcal{D}}$ on $L^2(\Omega, E)$, where $A_{\mathcal{D}}$ denotes the restriction of A to the space of compactly supported sections $\mathcal{D}(\Omega, E)$. More precisely, $\mathrm{Dom}(A_{\min})$ consists of those $f \in L^2(\Omega, E)$ for which there is a $g \in L^2(\Omega, E)$ and a sequence $\{f_\nu\} \subset \mathcal{D}(\Omega, E)$

such that $f_\nu \to f$ and $Af_\nu \to g$ in $L^2(\Omega, E)$. We set $A_{\min} f = g$. The fact that $A_{\mathcal{D}}$ is closeable is a standard result in functional analysis (see [14]).

More generally, a *closed realization* of the differential operator A is a closed operator $\tilde{A} : L^2(\Omega, E) \dashrightarrow L^2(\Omega, E)$ which extends the operator $A_{\min}$. Such an operator satisfies

$$A_{\min} \subseteq \tilde{A} \subseteq A_{\max}.$$

Note that if Ω is complete in its Hermitian metric (in particular if Ω is compact), then the space $\mathcal{D}(\Omega, E)$ of compactly supported smooth sections of E is dense in $\mathrm{Dom}(A_{\max})$ in the graph norm, and it follows that $A_{\max} = A_{\min}$ and that there is a unique closed realization of A as a Hilbert-space operator. We are more interested in the case when Ω is not complete, e.g., when Ω is a relatively compact domain in a larger Hermitian manifold.

We now recall the following well-known fact, which follows immediately from (4) (see [14, Lemma 4.3]):

Lemma 1. *As operators on $L^2(\Omega, E)$, the weak maximal realization $A_{\max}$ of the differential operator A and the strong minimal realization $A'_{\min}$ of its formal adjoint A' are Hilbert space adjoints, i.e. we have $A_{\max} = (A'_{\min})^*$ (note that this implies that $A_{\max}$ is closed) and also $A'_{\min} = (A_{\max})^*$.*

Proof. Let $A'_{\mathcal{D}}$ denote the restriction of A' to the compactly supported smooth sections $\mathcal{D}(\Omega, E)$. Then $A'_{\mathcal{D}}$ is a densely defined linear operator on $L^2(\Omega, E)$ and its closure is $\overline{A'_{\mathcal{D}}} = A'_{\min}$. For a fixed $f \in L^2(\Omega, E)$, consider the linear map on $\mathrm{Dom}(A_{\mathcal{D}}) = \mathcal{D}(\Omega, E)$ given by $\phi \mapsto (f, A'\phi)$. The definition of $\mathrm{Dom}(A_{\max})$ shows that this map is bounded on $\mathrm{Dom}(A'_{\mathcal{D}})$ if and only if $f \in \mathrm{Dom}(A_{\max})$. It now follows that $(A'_{\mathcal{D}})^* = A_{\max}$. By taking the closure, we conclude that $(A'_{\min})^* = A_{\max}$. Since $T^{**} = \overline{T}$ it follows that $A'_{\min} = (A_{\max})^*$. □

We note parenthetically that all the definitions and results of this section also hold in the simpler situation when Ω is a Riemannian manifold, and E is a complex vector bundle, and are independent of the holomorphic structure of Ω and E.

2.2. **Bundle-valued forms.** We recall the standard construction of forms on Ω with values in E . Recall that an E-valued (p, q)-form on Ω is a section of the bundle $\Lambda^{p,q}T^*(\Omega) \otimes E$, where $\Lambda^{p,q}T^*(\Omega)$ is the bundle of $\mathbb{C}$-valued forms of bidegree (p, q) (see [43] for details). We denote by $\mathcal{C}^\infty_{p,q}(\Omega, E)$ the space of E-valued (p, q)-forms of class $\mathcal{C}^\infty$, so that if $\{e_\alpha\}_{\alpha=1}^k$ is a local frame of E, then locally any element ϕ of $\mathcal{C}^\infty_{p,q}(\Omega)$ has a representation

$$\phi = \sum_\alpha \phi^\alpha \otimes e_\alpha, \tag{5}$$

where the ϕ^α are ($\mathbb{C}$-valued) (p, q)-forms with smooth coefficients.

It is well known that the operator $\overline{\partial}$ gives rise to an operator $\overline{\partial} \otimes \mathbb{I}_E = \overline{\partial}_E : \mathcal{C}^\infty_{p,q}(\Omega, E) \to \mathcal{C}^\infty_{p,q+1}(\Omega, E)$, via the prescription

$$\overline{\partial}_E \phi = \sum_\alpha (\overline{\partial}\phi^\alpha) \otimes e_\alpha. \tag{6}$$

See [12] for details of this construction. For each p with $0 \le p \le n$, this gives rise to a complex $(\mathcal{C}^\infty_{p,*}(\Omega, E), \overline{\partial}_E)$ of E-valued forms on Ω.

With the holomorphic vector bundle $E \to \Omega$ we can associate the *dual bundle* $E^* \to \Omega$, which is a holomorphic vector bundle over Ω, such that over a point

$x \in \Omega$, the fiber $(E^*)_x$ of E^* coincides with the dual vector space $(E_x)^*$ of the fiber E_x of E. One then has a natural isomorphism of bundles $E \cong (E^*)^*$, and we will always make this identification. If E is endowed with a Hermitian bundle metric, this induces a Hermitian bundle metric on E^* in a natural way, via the identification of E and E^* given by the Hermitian product on each fiber.

We can also define a wedge product

$$\wedge : \mathcal{C}^\infty_{p,q}(\Omega, E) \otimes \mathcal{C}^\infty_{p',q'}(\Omega, E^*) \to \mathcal{C}^\infty_{p+p',q+q'}(\Omega)$$

of an E-valued (p,q)-form and an E^*-valued (p',q')-form with value an ordinary (i.e. $\mathbb{C}$-valued) $(p+p', q+q')$-form in the following way. Suppose that $\{e_\alpha\}_{\alpha=1}^k$ is a local frame for the bundle E over some open set in Ω, and let $\{f_\alpha\}_{\alpha=1}^k$ be a frame of E^*. Given $\phi \in \mathcal{C}^\infty_{p,q}(\Omega, E)$ and an $\psi \in \mathcal{C}^\infty_{p',q'}(\Omega, E^*)$, we locally write $\phi = \sum_\alpha \phi^\alpha \otimes e_\alpha$ and $\psi = \sum_\beta \psi^\beta \otimes f_\beta$, and define pointwise

$$\phi \wedge \psi = \sum_{\alpha,\beta} f_\beta(e_\alpha)\, \phi^\alpha \wedge \psi^\beta. \tag{7}$$

This extends by bilinearity to a wedge product on $\mathcal{C}^\infty_{*,*}(\Omega, E) \otimes \mathcal{C}^\infty_{*,*}(\Omega, E^*)$.

If E is a holomorphic vector bundle on Ω, define a linear operator σ_E on $\mathcal{C}^\infty_{*,*}(\Omega, E)$ as follows: let ϕ be a form of bidegree (p,q). Then we set

$$\sigma_E \phi = (-1)^{p+q} \phi \tag{8}$$

and extend linearly to $\mathcal{C}^\infty_{*,*}(\Omega, E)$. Clearly $(\sigma_E)^2$ is the identity map on $\mathcal{C}^\infty_{*,*}(\Omega, E)$. Further, if T is any $\mathbb{R}$-linear operator from $\mathcal{C}^\infty_{*,*}(\Omega, E)$ to $\mathcal{C}^\infty_{*,*}(\Omega, F)$ (where F is another holomorphic vector bundle on Ω) of degree d, i.e., if for a homogeneous form f we have $\deg(Tf) - \deg(f) = d$, then we have the relation

$$\sigma_F\, T = (-1)^d\, T\, \sigma_E.$$

It is easy to see that the wedge product defined in (7) satisfies the Leibniz formula

$$\overline{\partial}(\phi \wedge \psi) = \overline{\partial}_E \phi \wedge \psi + \sigma_E \phi \wedge \overline{\partial}_{E^*} \psi. \tag{9}$$

We note here that the Hermitian metric on Ω and the bundle metric on E have not played any role in this section.

2.3. **The space $L^2_*(\Omega, E)$.** We now use the fact that the manifold Ω has been endowed with a Hermitian metric which we denote by g, i.e., each tangent space $T_x\Omega$ has been endowed a Hermitian inner product $g_x(\cdot,\cdot)$, which depends smoothly on the base point x, and also the fact the holomorphic vector bundle E has been endowed with a Hermitian metric h, i.e. for each $x \in \Omega$, h_x is a Hermitian product on the fiber E_x of E over x. The dual bundle E^* can be endowed with a Hermitian metric in the natural way.

The bundle $\Lambda^{p,q}T^*\Omega \otimes E$ has a natural Hermitian inner product (cf. (10) below), so we can construct the space $L^2_{p,q}(\Omega, E) = L^2(\Omega, \Lambda^{p,q}T^*\Omega \otimes E)$ of square integrable E-valued forms using the method of §2.1. We let $L^2_*(\Omega, E)$ be the orthogonal direct sum of the Hilbert spaces $L^2_{p,q}(\Omega, E)$ for $0 \le p, q \le n$.

We write the pointwise inner product on the space of E-valued forms. Let ϕ be as in (5), and let ψ be another (p,q)-form with local representation

$$\psi = \sum_\beta \psi^\beta \otimes e_\beta,$$

with respect to the same local frame. The pointwise inner product of the E-valued (p,q) forms ϕ and ψ is given by

$$\langle \phi, \psi \rangle_x = \sum_{\alpha,\beta} \langle \phi^\alpha, \psi^\beta \rangle_x \, h_x(e_\alpha, e_\beta) \tag{10}$$

at each point x in the open set where the frame $\{e_\alpha\}$ is defined, where by $\langle , \rangle$ on the right-hand side the standard pointwise inner product on $\mathbb{C}$-valued (p,q)-forms is meant (see [2]). It is not difficult to see that this definition is independent of the choice of the local frame. We extend (10) to a pointwise inner product on $\mathcal{C}^\infty_{*,*}(\Omega, E)$ by declaring that forms of different bidegree are pointwise orthogonal.

2.4. **The Hodge star.** The pointwise inner product (10) and the wedge product (7) can be related by the *Hodge-star operator*, the map $\star_E : \mathcal{C}^\infty_{p,q}(\Omega, E) \to \mathcal{C}^\infty_{n-p,n-q}(\Omega, E^*)$, defined by

$$\langle \phi, \psi \rangle dV = \phi \wedge \star_E \psi, \tag{11}$$

where dV is the volume form on Ω induced by the Hermitian metric g. It is easy to check that (11) defines $\star_E$ as an $\mathbb{R}$-linear and $\mathbb{C}$-*antilinear map*, i.e., for a $\mathbb{C}$-valued function f and an E-valued form ϕ, we have $\star_E(f\phi) = \overline{f} \star_E \phi$. We note that

$$\star_{E^*} \star_E = \sigma_E \tag{12}$$

and that

$$\sigma_{E^*} \star_E = \star_E \sigma_E, \tag{13}$$

where σ_E, σ_{E^*} are as in (8).

Let $\vartheta_E : \mathcal{C}^\infty_{*,*}(\Omega, E) \to \mathcal{C}^\infty_{*,*}(\Omega, E)$ denote the formal adjoint of $\overline{\partial}_E$. We recall the well-known formula for ϑ_E, and take this opportunity to point out that the formula for ϑ_E given in the popular reference [12, p. 152] has a typographical error.

Lemma 2. *The following formula holds:*

$$\vartheta_E = - \star_{E^*} \overline{\partial}_{E^*} \star_E . \tag{14}$$

Proof. It is sufficient to consider the case when the smooth forms ϕ and ψ are of bidegree $(p, q-1)$ and (p,q) respectively and at least one of them has compact support and compute

$$\begin{aligned}
(\overline{\partial}_E \phi, \psi)_\Omega &= \int_\Omega \overline{\partial}_E \phi \wedge \star_E \psi \\
&= \int_\Omega \big(\overline{\partial}(\phi \wedge \star_E \psi) - \sigma_E \phi \wedge \overline{\partial}_{E^*} \star_E \psi \big) && \text{(using (9))} \\
&= -(-1)^{p+q-1} \int_\Omega \phi \wedge \overline{\partial}_{E^*} \star_E \psi && \text{(using Stokes' formula)} \\
&= - \int_\Omega \phi \wedge (-1)^{(n-p)+(n-q+1)} \overline{\partial}_{E^*} \star_E \psi \\
&= - \int_\Omega \phi \wedge \sigma_{E^*} \overline{\partial}_{E^*} \star_E \psi \\
&= - \int_\Omega \phi \wedge \star_E \star_{E^*} \overline{\partial}_{E^*} \star_E \psi && \text{(using (12))} \\
&= (\phi, - \star_{E^*} \overline{\partial}_{E^*} \star_E \psi)_\Omega. && \square
\end{aligned}$$

Corollary 1. *We also have the formula*

$$\overline{\partial}_E = \star_{E^*}\, \vartheta_{E^*}\, \star_E\,. \tag{15}$$

Proof. Using (14), we compute

$$\begin{aligned}\star_E \vartheta_E \star_{E^*} &= -\star_E \star_{E^*} \overline{\partial}_{E^*} \star_E \star_{E^*}\\ &= -\sigma_{E^*}\overline{\partial}_{E^*}\sigma_{E^*}\\ &= \overline{\partial}_{E^*}.\end{aligned}$$

The result follows by replacing E with E^*. □

3. Duality

3.1. **The basic observation.** According to the conventions of multidimensional complex analysis, we adopt the following notation: we write

$\overline{\partial}_E$	for $(\overline{\partial}_E)_{\max}$,	the weak maximal realization of $\overline{\partial}_E$ on $L^2_*(\Omega, E)$,
$\overline{\partial}_{c,E}$	for $(\overline{\partial}_E)_{\min}$,	the strong minimal realization of $\overline{\partial}_E$ on $L^2_*(\Omega, E)$,
ϑ_E	for $(\vartheta_E)_{\max}$,	the weak maximal realization of ϑ_E on $L^2_*(\Omega, E)$,
$\overline{\partial}^*_E$	for $(\vartheta_E)_{\min}$,	the strong minimal realization of ϑ_E on $L^2_*(\Omega, E)$.

By Lemma 1, the operators $\overline{\partial}_E$ and $\overline{\partial}^*_E$ are Hilbert space adjoints to each other, as are the operators $\overline{\partial}_{c,E}$ and ϑ_E.

The operator σ_E defined in (8) extends from the space $\mathcal{D}_*(\Omega, E)$ of compactly supported forms to give rise to a unitary operator on $L^2_*(\Omega, E)$. Similarly the Hodge-Star operator $\star_E$ defined in (11) extends from $\mathcal{D}_*(\Omega, E)$ to give rise to a conjugate-linear self-isometry of $L^2_*(\Omega, E)$. We continue to denote these Hilbert space realizations by σ_E and $\star_E$ respectively. We are now ready to prove the main observation behind the use of the Hodge-$\star$ operator in L^2 theory:

Proposition 1. *Let Ω be a Hermitian manifold, and E a holomorphic vector bundle on Ω equipped with a Hermitian metric. Let $\overline{\partial}_E, \overline{\partial}^*_E, \vartheta_{E^*}, \overline{\partial}_{c,E^*}$ be the Hilbert space realizations as defined above, and let $f \in L^2_*(\Omega, E)$:*

(1) *$f \in \mathrm{Dom}(\overline{\partial}^*_E)$ if and only if $\star_E f \in \mathrm{Dom}(\overline{\partial}_{c,E^*})$. Also on $\mathrm{Dom}(\overline{\partial}^*_E)$ we have the relation*

$$\overline{\partial}^*_E = -\star_{E^*}\, \overline{\partial}_{c,E^*}\, \star_E\,. \tag{16}$$

(2) *$f \in \mathrm{Dom}(\overline{\partial}_E)$ if and only if $\star_E f \in \mathrm{Dom}(\vartheta_{E^*})$. On $\mathrm{Dom}(\overline{\partial}_E)$ we have the relation*

$$\overline{\partial}_E = \star_{E^*}\, \vartheta_{E^*}\, \star_E\,. \tag{17}$$

Proof. The results are obtained by taking the minimal and maximal realizations of (14) and (15) respectively.

To justify (16), we note that if $f \in \mathrm{Dom}(\overline{\partial}^*_E)$, there is a sequence $\{f_\nu\}$ in $\mathcal{D}(\Omega, E)$ such that $f_\nu \to f$ in $L^2_*(\Omega, E)$ and $\vartheta_E f_\nu \to \overline{\partial}^*_E f$ also in $L^2_*(\Omega, E)$. Note that $\star_E f_\nu \in \mathcal{D}_*(\Omega, E^*)$, since f_ν is compactly supported. Further, since $\star_E$ extends to an isometry of $L^2_*(\Omega, E)$ with $L^2_*(\Omega, E^*)$, it follows that $\star_E f_\nu \to \star_E f$ in $L^2(\Omega, E^*)$. From (14) relating the formal adjoints, it also follows that $\overline{\partial}_{E^*}(\star_E f_\nu) = -(\star_{E^*})^{-1}\vartheta_E f_\nu \to -(\star_{E^*})^{-1}\overline{\partial}^*_E f$. Consequently, $\star_E f \in \mathrm{Dom}(\overline{\partial}_{c,E^*})$, and (16) holds. The converse assertion (where if $\star_E f \in \mathrm{Dom}(\overline{\partial}_{c,E^*})$, then $f \in \mathrm{Dom}(\overline{\partial}^*_E)$) is proved similarly.

For (17), suppose that $f \in \mathrm{Dom}(\overline{\partial}_E)$. This means that $f \in L^2_*(\Omega, E)$ and $\overline{\partial}_E \in L^2_*(\Omega, E)$ (where $\overline{\partial}_E$ is taken in the weak sense). Since $\star_E$ is an isometry of the Hilbert space $L^2_*(\Omega, E)$ with the Hilbert space $L^2_*(\Omega, E^*)$, it follows that $\star_E f \in L^2_*(\Omega, E^*)$. From (15) we see that in the weak sense, we have $\overline{\partial}_E f = \star_{E^*}\vartheta_{E^*}(\star_E f)$. Consequently, $\vartheta_{E^*}(\star_E f) = (\star_{E^*})^{-1}\overline{\partial}_E f \in L^2(\Omega, E^*)$. It follows that $\star_E f \in \mathrm{Dom}(\vartheta_{E^*})$ and (17) holds. The converse (if $\star_E f \in \mathrm{Dom}(\vartheta_{E^*})$, then $f \in \mathrm{Dom}(\overline{\partial}_E)$) is proved the same way. $\square$

3.2. **Duality of Laplacians.** Recall that the $\overline{\partial}$-*Laplacian* on E-valued forms on Ω is the operator $\square_E$ on $L^2_*(\Omega, E)$ defined by

$$\square_E = \overline{\partial}_E\overline{\partial}_{E^*} + \overline{\partial}_{E^*}\overline{\partial}_E,$$

with domain

$$\mathrm{Dom}(\square_E) = \Big\{ f \in L^2_*(\Omega, E) \mid f \in \mathrm{Dom}(\overline{\partial}_E) \cap \mathrm{Dom}(\overline{\partial}^*_E), \\ \overline{\partial}_E f \in \mathrm{Dom}(\overline{\partial}^*_E), \overline{\partial}^*_E f \in \mathrm{Dom}(\overline{\partial}_E) \Big\}.$$

The $\overline{\partial}_c$-*Laplacian* on E-valued forms is the operator

$$\begin{aligned} \square^c_E &= \overline{\partial}_{c,E}\overline{\partial}^*_{c,E} + \overline{\partial}^*_{c,E}\overline{\partial}_{c,E} \\ &= \overline{\partial}_{c,E}\vartheta_E + \vartheta_E\overline{\partial}_{c,E} \end{aligned}$$

on $L^2_*(\Omega, E)$ with domain

$$\mathrm{Dom}(\square^c_E) = \big\{ f \in L^2_*(\Omega, E) \mid f \in \mathrm{Dom}(\overline{\partial}_{c,E}) \cap \mathrm{Dom}(\vartheta_E), \\ \overline{\partial}_{c,E} f \in \mathrm{Dom}(\vartheta_E), \vartheta_E f \in \mathrm{Dom}(\overline{\partial}_{c,E}) \big\}.$$

Each $\square$ and $\square^c_E$ is a non-negative self-adjoint operator on $L^2_*(\Omega, E)$. Note that on the subspace $\mathcal{D}_*(\Omega, E)$ of compactly supported E-valued forms both $\square_E$ and $\square^c_E$ coincide with the "formal Laplacian" $\overline{\partial}_E\vartheta_E + \vartheta_E\overline{\partial}_E$. However, in general it is not true that $\square^c_E$ and $\square_E$ are equal. By [1, Lemma 3.1(2)], we have $\square_E = \square^c_E$ if and only if $\overline{\partial}_E = \overline{\partial}_{c,E}$. This happens if Ω is either compact or complete.

We define the spaces of E-valued $\overline{\partial}$-*Harmonic* and $\overline{\partial}_c$-*Harmonic* forms $\mathcal{H}_{p,q}(\Omega, E)$ and $\mathcal{H}^c_{p,q}(\Omega, E)$ by

$$\mathcal{H}_{p,q}(\Omega, E) = \ker(\square_E) \cap L^2_{p,q}(\Omega, E)$$

and

$$\mathcal{H}^c_{p,q}(\Omega, E) = \ker(\square^c_E) \cap L^2_{p,q}(\Omega, E).$$

It is easy to see that

$$\begin{aligned} \mathcal{H}_{p,q}(\Omega, E) &= \ker(\overline{\partial}_E) \cap \ker(\overline{\partial}^*_E) \cap L^2_{p,q}(\Omega, E) \\ &= \Big\{ f \in \mathrm{Dom}(\overline{\partial}_E) \cap \mathrm{Dom}(\overline{\partial}^*_E) \cap L^2_{p,q}(\Omega, E) \mid \overline{\partial}_E f = \overline{\partial}^*_E f = 0 \Big\} \end{aligned}$$

and similarly that

$$\begin{aligned} \mathcal{H}^c_{p,q}(\Omega, E) &= \ker(\overline{\partial}_{c,E}) \cap \ker(\vartheta_E) \cap L^2_{p,q}(\Omega, E) \\ &= \big\{ f \in \mathrm{Dom}(\overline{\partial}_{c,E}) \cap \mathrm{Dom}(\vartheta_E) \cap L^2_{p,q}(\Omega, E) \mid \overline{\partial}_{c,E} f = \vartheta_E f = 0 \big\}. \end{aligned}$$

The following is now easy to prove.

Theorem 1. *Let $f \in L^2_*(\Omega, E)$. Then, $f \in \mathrm{Dom}(\Box_E)$ if and only if $\star_E f \in \mathrm{Dom}(\Box^c_{E^*})$. Further, we have the relation*

$$\star_E \Box_E = \Box^c_{E^*} \star_E . \tag{18}$$

Also, the restriction of the map $\star_E$ to $\mathcal{H}_{p,q}(\Omega, E)$ gives rise to an isomorphism

$$\mathcal{H}_{p,q}(\Omega, E) \cong \mathcal{H}^c_{n-p,n-q}(\Omega, E^*). \tag{19}$$

Proof. On the space

$$\left\{ f \in L^2_*(\Omega, E) \mid f \in \mathrm{Dom}(\overline{\partial}_E), \overline{\partial}_E f \in \mathrm{Dom}(\overline{\partial}^*_E) \right\}$$

we have, using (16) and (17),

$$\begin{aligned} \overline{\partial}^*_E \overline{\partial}_E &= - \star_{E^*} \overline{\partial}_{c,E^*} \star_E \star_{E^*} \vartheta_{E^*} \star_E \\ &= - \star_{E^*} \overline{\partial}_{c,E^*} \sigma_{E^*} \vartheta_{E^*} \star_E \\ &= \star_{E^*} \sigma_{E^*} \overline{\partial}_{c,E^*} \vartheta_{E^*} \star_E . \end{aligned}$$

Similarly, we have on

$$\left\{ f \in L^2_*(\Omega, E) \mid f \in \mathrm{Dom}(\overline{\partial}^*_E), \overline{\partial}^*_E f \in \mathrm{Dom}(\overline{\partial}_E) \right\}$$

the relation

$$\overline{\partial}_E \overline{\partial}^*_E = \star_{E^*} \sigma_{E^*} \vartheta_{E^*} \overline{\partial}_{c,E^*} \star_E .$$

Combining, we have on $\mathrm{Dom}(\Box_E)$:

$$\Box_E = \star_{E^*} \sigma_{E^*} \Box^c_{E^*} \star_E .$$

Equation (18) follows by pre-composing with $\star_E$ and using (12). $\Box$

It follows that the self-adjoint operators $\Box_E$ and $\Box^c_{E^*}$ are *isospectral*: a number $\lambda \in \mathbb{R}$ belongs to the spectrum of $\Box_E$ if and only if λ belongs to the spectrum of $\Box^c_{E^*}$. Let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be a spectral family of orthogonal projections from $L^2_*(\Omega, E)$ to itself (cf. [32, Chapters VII, VIII]) such that we have the spectral representation

$$\Box_E = \int_{\mathbb{R}} \lambda dE_\lambda.$$

If $\{F_\lambda\}_{\lambda \in \mathbb{R}}$ is defined by

$$F_\lambda = \sigma_{E^*} \star_E E_\lambda \star_{E^*},$$

then F_λ is an orthogonal projection on $L^2_*(\Omega, E^*)$ and we have the spectral representation

$$\Box^c_{E^*} = \int_{\mathbb{R}} \lambda dF_\lambda.$$

These statements are purely formal consequences of (18).

3.3. **Closed-range property.** In order to apply L^2-theory to solve the $\overline{\partial}$-equation, we first need to show that the $\overline{\partial}$-operator has closed range. In this section we consider the consequences of this hypothesis on the $\overline{\partial}_c$-operator.

Recall that the notation $T : \mathsf{H}_1 \dashrightarrow \mathsf{H}_2$ means that T is a linear operator defined on a linear subspace $\mathrm{Dom}(T)$ of H_1 and taking values in H_2. Further, for notational simplicity, we will use $\overline{\partial}_E$ to denote the restriction $\overline{\partial}_E|_{L^2_{p,q}(\Omega)}$, when p, q are given, rather than introduce new subscripts, and we adopt the same convention for $\overline{\partial}_{c,E}, \vartheta_E$, and $\overline{\partial}^*_E$. We first note the following fact.

Lemma 3. *If any one of the operators in the following list of Hilbert space operators has closed range, it follows that all the others also have closed range:*

$$\begin{cases} \overline{\partial}_E : & L^2_{p,q}(\Omega, E) \dashrightarrow L^2_{p,q+1}(\Omega, E), \\ \overline{\partial}^*_E : & L^2_{p,q+1}(\Omega, E) \dashrightarrow L^2_{p,q}(\Omega, E), \\ \overline{\partial}_{c,E^*} : & L^2_{n-p,n-q-1}(\Omega, E^*) \dashrightarrow L^2_{n-p,n-q}(\Omega, E^*), \\ \vartheta_{E^*} : & L^2_{n-p,n-q}(\Omega, E^*) \dashrightarrow L^2_{n-p,n-q-1}(\Omega, E^*). \end{cases} \tag{20}$$

Proof. Thanks to the well-known fact that a closed densely-defined operator has closed range if and only if its adjoint has closed range (see [19, Theorem 1.1.1] or [2, Lemma 4.1.1]), it follows that $\overline{\partial}_E$ has closed range if and only if $\overline{\partial}^*_E$ has closed range and that $\overline{\partial}_{c,E^*}$ has closed range if and only if ϑ_{E^*} has closed range. To complete the proof, we need to show that $\overline{\partial}^*_E$ has closed range if and only if $\overline{\partial}_{c,E^*}$ has closed range. Now, (16) shows that for $f \in \mathrm{Dom}(\overline{\partial}^*_E)$, we have $\left\|\overline{\partial}^*_E f\right\| = \left\|\overline{\partial}_{c,E^*}(\star_E f)\right\|$, in particular, $f \in \ker(\overline{\partial}^*_E)$ if and only if $\star_E f \in \ker(\overline{\partial}_{c,E^*})$. This means that the inequality $\left\|\overline{\partial}^*_E f\right\| \geq C\,\|f\|$ holds for all $f \in \ker(\overline{\partial}^*_E)^\perp$ if and only if the inequality $\left\|\overline{\partial}_{c,E^*} g\right\| \geq C\,\|g\|$ holds for all $g \in \ker(\overline{\partial}_{c,E^*})^\perp$. Again by [2, Lemma 4.1.1] it follows that $\overline{\partial}^*_E$ has closed range if and only if $\overline{\partial}_{c,E^*}$ has closed range. □

3.4. **Duality of cohomologies.** We define the L^2-*cohomology* as the quotient vector space

$$H^{p,q}_{L^2}(\Omega, E) = \frac{\ker(\overline{\partial}_E) \cap L^2_{p,q}(\Omega, E)}{\mathrm{img}(\overline{\partial}_E) \cap L^2_{p,q}(\Omega, E)}.$$

Similarly, the L^2-*cohomology with the minimal realization* is defined to the space

$$H^{p,q}_{c,L^2}(\Omega, E) = \frac{\ker(\overline{\partial}_{c,E}) \cap L^2_{p,q}(\Omega, E)}{\mathrm{img}(\overline{\partial}_{c,E}) \cap L^2_{p,q}(\Omega, E)}.$$

If $\overline{\partial}_E$ (resp. $\overline{\partial}_{c,E}$) has closed range, $H^{p,q}_{L^2}(\Omega, E)$ (resp. $H^{p,q}_{c.L^2}(\Omega, E)$) is a Hilbert space with the quotient norm.

Let

$$[\cdot] : \ker(\overline{\partial}_E) \cap L^2_{p,q}(\Omega, E) \to H^{p,q}_{L^2}(\Omega, E)$$

and

$$[\cdot]_c : \ker(\overline{\partial}_{c.E}) \cap L^2_{p,q}(\Omega, E) \to H^{p,q}_{c.L^2}(\Omega, E)$$

denote the respective natural projections onto the quotient spaces. The following result was first observed by Kodaira:

Lemma 4. *Let η (resp. η_c) denote the restriction of $[\cdot]$ (resp. $[\cdot]_c$) to the vector space of $\overline{\partial}_E$-harmonic forms $\mathcal{H}_{p,q}(\Omega, E)$ (resp. the vector space of $\overline{\partial}_{c,E}$-harmonic forms $\mathcal{H}^c_{p,q}(\Omega, E)$). Then:*

(i) η (resp. η_c) is injective.

(ii) If η (resp. η_c) is also surjective, then $\mathrm{img}(\overline{\partial}_E : L^2_{p,q-1}(\Omega, E) \dashrightarrow L^2_{p,q}(\Omega, E))$ *(resp.* $\mathrm{img}(\overline{\partial}_{c,E} : L^2_{p,q-1}(\Omega, E) \dashrightarrow L^2_{p,q}(\Omega, E))$*) is closed.*

Proof. We write the proof only for the operator η. The proof for η_c is similar.

(i) Note that if $q = 0$ this is obvious, since $\mathrm{img}\left(\overline{\partial}_E : L^2_{p,q-1}(\Omega, E) \dashrightarrow L^2_{p,q}(\Omega, E)\right) = 0$. Assuming $q \geq 1$, we note that $\ker(\eta) = \ker(\overline{\partial}_E) \cap \ker(\overline{\partial}^*_E) \cap \mathrm{img}(\overline{\partial}_E)$, and

therefore a form in $\ker(\eta)$ can be written as $\overline{\partial}g$, with $\overline{\partial}^*(\overline{\partial}g) = 0$. Then

$$\begin{aligned} 0 &= (\overline{\partial}_E^*(\overline{\partial}_E g), g) \\ &= \left\|\overline{\partial}g\right\|^2. \end{aligned}$$

(ii) Since η is an isomorphism, we can identify the harmonic space $\mathcal{H}_{p,q}(\Omega, E)$ with the cohomology space $H^{p,q}_{L^2}(\Omega, E)$. Since $\mathcal{H}_{p,q}(\Omega, E)$ is a closed subspace of the Hilbert space $L^2_{p,q}(\Omega, E)$, the space $H^{p,q}_{L^2}(\Omega, E)$ also becomes a Hilbert space. We can think of the map $[\cdot]$ as an operator from the Hilbert space $\ker(\overline{\partial}_E) \cap L^2_{p,q}(\Omega, E)$ to the Hilbert space $H^{p,q}_{L^2}(\Omega, E)$. Since η is surjective, every element of $\ker(\overline{\partial}_E)$ can be written as $f + \overline{\partial}g$, where $f \in \mathcal{H}_{p,q}(\Omega, E)$. According to the identification of $\mathcal{H}_{p,q}(\Omega, E)$ and $H^{p,q}_{L^2}(\Omega, E)$, we have $[f + \overline{\partial}_E g] = f$. Since $\left\|f + \overline{\partial}_E g\right\|^2 = \|f\|^2 + \left\|\overline{\partial}g\right\|^2 \geq \|f\|^2$, then $\left\|[f + \overline{\partial}g]\right\| \leq \left\|f + \overline{\partial}g\right\|$ and it follows that $[\cdot]$ is in fact a bounded map. Therefore $\ker[.] = \operatorname{img}(\overline{\partial}_E) \cap L^2_{p,q}(\Omega, E)$ is closed, which was to be shown. □

Theorem 2 (L^2 Serre duality on non-compact manifolds)**.** *The following are equivalent:*

(1) The two operators

$$L^2_{p,q-1}(\Omega, E) \overset{\overline{\partial}_E}{\dashrightarrow} L^2_{p,q}(\Omega, E) \overset{\overline{\partial}_E}{\dashrightarrow} L^2_{p,q+1}(\Omega, E)$$

have closed range.

(2) The map $\star_E : L^2_{p,q}(\Omega, E) \to L^2_{n-p,n-q}(\Omega, E^*)$ induces a conjugate-linear isomorphism of Hilbert spaces

$$\tau = \eta_c \circ \star_E \circ \eta^{-1} : H^{p,q}_{L^2}(\Omega, E) \to H^{n-p,n-q}_{c,L^2}(\Omega, E^*). \tag{21}$$

So we can identify the Hilbert space dual of $H^{p,q}_{L^2}(\Omega, E)$ with $H^{n-p,n-q}_{c,L^2}(\Omega, E^*)$.

We note here that condition (1) is in fact the necessary and sufficient condition for the existence of the $\overline{\partial}$-Neumann operator $\mathsf{N}^E_{p,q}$, defined as the inverse (modulo kernel) of the $\square_E$ operator on (p, q)-forms.

Proof. In the diagram

$$\begin{array}{ccc} \mathcal{H}_{p,q}(\Omega, E) & \xrightarrow{\star_E} & \mathcal{H}^c_{n-p,n-q}(\Omega, E^*) \\ {\scriptstyle\eta}\downarrow & & {\scriptstyle\eta_c}\downarrow \\ H^{p,q}_{L^2}(\Omega, E) & \xrightarrow{\tau} & H^{n-p,n-q}_{c,L^2}(\Omega, E^*) \end{array}$$

the map $\star_E$ is known to be an isomorphism from $\mathcal{H}_{p,q}(\Omega, E)$ to $\mathcal{H}^c_{n-p,n-q}(\Omega, E)$ by Theorem 1 (see equation (19)). Therefore, the map τ will also be an isomorphism if and only if both η and η_c are isomorphisms. Thanks to Lemma 4 this is equivalent to the two maps $\overline{\partial}_E : L^2_{p,q-1}(\Omega, E) \dashrightarrow L^2_{p,q}(\Omega, E)$ and $\overline{\partial}_{c,E^*} : L^2_{n-p,n-q-1}(\Omega, E^*) \dashrightarrow L^2_{n-p,n-q}(\Omega, E^*)$ having closed range. Since by Lemma 3 the second map has closed range if and only if $\overline{\partial}_E : L^2_{p,q}(\Omega, E) \to L^2_{p,q+1}(\Omega, E)$ has closed range, the result follows. □

3.5. **Duality of the $\overline{\partial}$-problem and the $\overline{\partial}_c$-problem.** We can use the duality principle to solve the equation $\overline{\partial}_c u = f$, provided we know how to solve $\overline{\partial} u = f$:

Theorem 3. *Suppose that for some $0 \le p \le n$ and $0 \le q \le n-1$, the operator $\overline{\partial}_{E^*} : L^2_{n-p,n-q-1}(\Omega, E^*) \dashrightarrow L^2_{n-p,n-q}(\Omega, E^*)$ has closed range. Then the range $\operatorname{img}(\overline{\partial}_{c,E}) \cap L^2_{p,q+1}(\Omega, E)$ is closed. The condition that $f \in \operatorname{img}(\overline{\partial}_{c,E}) \cap L^2_{p,q+1}(\Omega, E)$ is equivalent to the following: for every $g \in \ker(\overline{\partial}_{E^*}) \cap L^2_{n-p,n-q-1}(\Omega, E^*)$, we have*

$$\int_\Omega f \wedge g = 0. \tag{22}$$

If Ω is a relatively compact pseudoconvex domain in a Stein manifold and $q \ne n-1$, it is further equivalent to the condition $\overline{\partial}_{c,E} f = 0$.

Proof. Since $\overline{\partial}_{E^*}$ has closed range on $L^2_{n-p,n-q-1}(\Omega, E^*)$, from Hilbert space theory, it follows that there is a bounded solution operator K from $L^2_{n-p,n-q}(\Omega, E^*)$ to $L^2_{n-p,n-q-1}(\Omega, E^*)$ such that $\overline{\partial}_{E^*} K = \mathbb{I}$ (the identity map) on $\operatorname{img}(\overline{\partial}_{E^*})$, and $K\overline{\partial}_{E^*} = \mathbb{I} - B$ on $\operatorname{Dom}(\overline{\partial}_{E^*})$, where

$$B : L^2_{n-p,n-q-1}(\Omega, E^*) \to \ker(\overline{\partial}_{E^*}) \cap L^2_{n-p,n-q-1}(\Omega, E^*)$$

is the generalized Bergman projection. Set

$$K_c = - \star_{E^*} K^* \star_E,$$

where K^* denotes the bounded operator from $L^2_{n-p,n-q-1}(\Omega, E^*)$ to $L^2_{n-p,n-q}(\Omega, E^*)$ which is the Hilbert space adjoint of the operator K defined above.

Now let $f \in \operatorname{img}(\overline{\partial}_{c,E}) \cap L^2_{p,q+1}(\Omega, E)$. Note that this means $\star_E f \in \operatorname{img}(\overline{\partial}^*_{E^*}) = \ker(\overline{\partial}_{E^*})^\perp$. It follows that $B(\star_E f) = 0$.

We set $u = K_c f$. This is well defined, since $\star_E f \in L^2_{n-p,n-q-1}(\Omega, E^*)$, which is the domain of K^*, and we have $\|u\| \le C\,\|f\|$. Also, from (16) we have $\overline{\partial}_{c,E} \star_{E^*} = -(\star_E)^{-1}\,\overline{\partial}^*_{E^*}$. Therefore,

$$\begin{aligned}
\overline{\partial}_{c,E} u &= -\overline{\partial}_{c,E} \star_{E^*} K^* \star_E f \\
&= (\star_E)^{-1}\,\overline{\partial}^*_{E^*} K^* \star_E f \\
&= (\star_E)^{-1}\left(K\overline{\partial}_{E^*}\right)^* \star_E f && \text{since } K \text{ is bounded} \\
&= (\star_E)^{-1}\,(\mathbb{I} - B)^* \star_E f \\
&= f - (\star_E)^{-1}\,B(\star_E f) && B \text{ is self-adjoint} \\
&= f.
\end{aligned}$$

We note that $g \in \ker(\overline{\partial}_{E^*}) \cap L^2_{n-p,n-q-1}(\Omega, E^*)$ if and only if $\star_{E^*} g \in \ker(\vartheta_E) \cap L^2_{p,q+1}(\Omega, E)$. Since $\operatorname{img}(\overline{\partial}_{c,E}) = \ker(\overline{\partial}^*_{c,E})^\perp = \ker(\vartheta^*_E)^\perp$, it follows that $f \in \operatorname{img}(\overline{\partial}_{c,E})$ if and only if for each $g \in \ker(\overline{\partial}_{E^*})$ we have $(f, \star_{E^*} g) = 0$, i.e.,

$$\begin{aligned}
0 &= \int_\Omega f \wedge \star_E \star_{E^*} g \\
&= \int_\Omega f \wedge \sigma_{E^*} g \\
&= (-1)^{2n-p-q-1} \int_\Omega f \wedge g,
\end{aligned}$$

which proves (22).

Now assume Ω is in a Stein manifold. Then, we know that $H^{p,q}_{L^2}(\Omega, E^*) = 0$, provided $q \neq 0$. By the L^2-Serre duality, $H^{p,q+1}_{c,L^2}(\Omega, E) = 0$, unless $q + 1 = n$. In other words, if $q \neq n-1$, if $f \in \mathrm{img}(\overline{\partial}_{c,E}) \cap L^2_{p,q+1}(\Omega, E)$, then

$$f \in \ker(\overline{\partial}_{c,E} : L^2_{p,q}(\Omega, E) \dashrightarrow L^2_{p,q+1}(\Omega, E)).$$

This completes the proof. □

3.6. **Duality of realizations of the $\overline{\partial}$ operator.** We now discuss an abstract version of L^2-duality which generalizes the duality of $\overline{\partial}_E$ and $\overline{\partial}_{c,E^*}$ discussed in the previous sections. The proofs of the statements made below are parallel to the proofs of corresponding statements (for $\overline{\partial}_E$ and $\overline{\partial}_{c,E^*}$) in the previous sections, and are omitted.

Let E be a vector bundle over Ω and let $D : L^2_*(\Omega, E) \dashrightarrow L^2_*(\Omega, E)$ be a realization of $\overline{\partial}_E$, acting on E-valued forms. Then D satisfies $\overline{\partial}_{c,E} \subseteq D \subseteq \overline{\partial}_E$. We define an operator $D^\vee$ on the Hilbert space $L^2_*(\Omega, E^*)$ by setting

$$D^\vee = \star_E D^* \star_{E^*},$$

where $D^* : L^2_*(\Omega, E) \dashrightarrow L^2_*(\Omega, E)$ is the Hilbert space adjoint of the operator D. Then the following is easy to prove using relations (14) and (15):

Lemma 5.

(1) *$D^\vee$ is a realization of the operator $\overline{\partial}_{E^*}$ on the Hilbert space $L^2_*(\Omega, E^*)$, and its domain is $\star_E(\mathrm{Dom}(D^*))$.*
(2) *$(\overline{\partial}_E)^\vee = \overline{\partial}_{c,E^*}$ and $(\overline{\partial}_{c,E})^\vee = \overline{\partial}_{E^*}$.*
(3) *The map $D \mapsto D^\vee$ is a one-to-one correspondence of the* closed *realizations of $\overline{\partial}_E$ with the closed realizations of $\overline{\partial}_{E^*}$.*

We can refer to $D^\vee$ as the realization of $\overline{\partial}_{E^*}$ dual to the realization D of $\overline{\partial}_E$. From now on we will assume that the realization D of the $\overline{\partial}_E$ operator is *closed.* Note that then $\ker(D)$ is a closed subspace of $L^2_*(\Omega, E)$.

We define the cohomology groups of the bundle E, with respect to the (closed) realization D, as

$$H^{p,q}_{L^2}(\Omega, E; D) = \frac{\ker(D) \cap L^2_{p,q}(\Omega, E)}{\mathrm{img}(D) \cap L^2_{p,q}(\Omega, E)}.$$

This becomes a Hilbert space if $\mathrm{img}(D)$ is closed in $L^2_{p,q}(\Omega, E)$.

Then, we can state the following generalized version of Serre duality, with exactly the same proof:

Theorem 4. *The following are equivalent for a closed realization D of $\overline{\partial}_E$:*

(1) The two operators

$$L^2_{p,q-1}(\Omega, E) \overset{D}{\dashrightarrow} L^2_{p,q}(\Omega, E) \overset{D}{\dashrightarrow} L^2_{p,q+1}(\Omega, E)$$

have closed range.

(2) The map $\star_E : L^2_{p,q}(\Omega, E) \to L^2_{n-p,n-q}(\Omega, E^*)$ induces a conjugate-linear isomorphism of the cohomology Hilbert space $H^{p,q}_{L^2}(\Omega, E; D)$ with $H^{n-p,n-q}_{L^2}(\Omega, E^*; D^\vee)$.

We give an example of a closed realization of $\overline{\partial}$ which is strictly intermediate between the maximal and minimal realizations. We consider a domain Ω in a product Hermitian manifold $\mathcal{M}_1 \times \mathcal{M}_2$ such that Ω is the product of smoothly

bounded, relatively compact domains $\Omega_1 \Subset \mathcal{M}_1$ and $\Omega_2 \Subset \mathcal{M}_2$. We endow Ω with the product Hermitian metric derived from $\mathcal{M}_1$ and $\mathcal{M}_2$.

If H_1 and H_2 are Hilbert spaces, we denote by $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$ the *Hilbert tensor product* of H_1 and H_2, i.e., the completion of the *algebraic* tensor product $\mathsf{H}_1 \otimes \mathsf{H}_2$ under the norm induced by the natural inner product defined on decomposable tensors by

$$(x \otimes y, z \otimes w) = (x, z)_{\mathsf{H}_1} (y, w)_{\mathsf{H}_2},$$

and extended linearly. For details see [42, §3.4]. An example of Hilbert tensor products is the space $L^2_*(\Omega)$ of square integrable forms on the product Hermitian manifold $\Omega = \Omega_1 \times \Omega_2$. In fact,

$$L^2_*(\Omega) = L^2_*(\Omega_1) \widehat{\otimes} L^2_*(\Omega_2)$$

if we make the natural identification $f \otimes g = \pi_1^* f \wedge \pi_2^* g$, where $\pi_j : \Omega \to \Omega_j$ is the natural projection.

If $T_1 : \mathsf{H}_1 \dashrightarrow \mathsf{H}_1'$ and $T_2 : \mathsf{H}_2 \dashrightarrow \mathsf{H}_2'$ are closed densely-defined operators, we can define an operator $T_1 \otimes T_2 : \mathrm{Dom}(T_1) \otimes \mathrm{Dom}(T_2) \dashrightarrow \mathsf{H}_1' \otimes \mathsf{H}_2'$, which on decomposable tensors takes the form $(T_1 \otimes T_2)(x \otimes y) = T_1 x \otimes T_2 y$. It is well known that provided T_1 and T_2 are closed, the operator $T_1 \otimes T_2$ is closable. The closure, denoted by $T_1 \widehat{\otimes} T_2$, is a closed densely defined operator from $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$ to $\mathsf{H}_1' \widehat{\otimes} \mathsf{H}_2'$.

We let $\overline{\partial}^j : L^2_*(\Omega_j) \dashrightarrow L^2_*(\Omega_j)$ denote the maximal realization of the $\overline{\partial}$-operator acting of $\mathbb{C}$-valued forms on Ω_j. Similarly, we let $\overline{\partial}^j_c : L^2_*(\Omega_j) \dashrightarrow L^2_*(\Omega_j)$ denote the minimal realization of the $\overline{\partial}$-operator. Consider the operator D on $L^2_*(\Omega)$ defined by

$$D = \overline{\partial}^1 \widehat{\otimes} I_2 + \sigma_1 \widehat{\otimes} \overline{\partial}^2_c,$$

where I_2 is the identity map on $L^2_*(\Omega_2)$ and σ_1 is the (bounded self-adjoint) operator on $L^2_*(\Omega_1)$ which, when restricted to $L^2_{p,q}(\Omega_1)$, is multiplication by $(-1)^{p+q}$. Using the techniques of [5, 6] the following properties of D can be established:

- D is a closed densely-defined operator on $L^2_*(\Omega)$.
- D is a realization of $\overline{\partial}$ on Ω, and it is strictly intermediate between the maximal and the minimal realization. We may think of D as being the realization which is maximal on the factor Ω_1 and minimal on the factor Ω_2.
- Suppose that the maximal realization $\overline{\partial}^j$ has closed range on $L^2_*(\Omega_j)$ for $j = 1$ and 2. By duality, $\overline{\partial}^j_c$ has closed range in $L^2_*(\Omega_j)$ as well. Using either of the methods of proof used in [5, Theorem 1.1] or [6, Theorem 1.2], we can conclude that the operator D also has closed range. Further, we have the Künneth formula

$$\begin{aligned} H^*_{L^2}(\Omega; D) &= H^*_{L^2}(\Omega_1; \overline{\partial}^1) \widehat{\otimes} H^*_{L^2}(\Omega_2; \overline{\partial}^2_c) \\ &= H^*_{L^2}(\Omega_1) \widehat{\otimes} H^*_{c,L^2}(\Omega_2). \end{aligned} \tag{23}$$

- The dual realization $D^\vee$ is the one which is minimal on Ω_1 and maximal on Ω_2; it can be represented as

$$D^\vee = \overline{\partial}^1_c \widehat{\otimes} I_2 + \sigma_1 \widehat{\otimes} \overline{\partial}^2.$$

Provided $\overline{\partial}$ has closed range in each of Ω_1 and Ω_2, the operator $D^\vee$ again has closed range, and the Künneth formula holds:

$$\begin{aligned} H^*_{L^2}(\Omega; D^\vee) &= H^*_{L^2}(\Omega_1; \overline{\partial}^1_c)\widehat{\otimes} H^*_{L^2}(\Omega_2; \overline{\partial}^2) \\ &= H^*_{c,L^2}(\Omega_1)\widehat{\otimes} H^*_{L^2}(\Omega_2). \end{aligned}$$

Suppose that $\dim_{\mathbb{C}} \Omega_j = n_j$, and set $n = n_1 + n_2 = \dim_{\mathbb{C}}(\Omega)$. We have by Serre duality that $H^{n-p,n-q}(\Omega; D^\vee) \cong H^{p,q}(\Omega; D)$ via the map $\star$. Note that this could also be deduced from the knowledge of Serre duality on the factors: indeed for each (p_1, q_1) we have

$$H^{p_1,q_1}_{L^2}(\Omega_1) \cong H^{n_1-p_1,n_2-q_1}_{c,L^2}(\Omega_1),$$

and for each (p_2, q_2) we have $H^{n_2-p_2,n_2-q_2}_{L^2}(\Omega_2) \cong H^{p_2,q_2}_{c,L^2}(\Omega_2)$. Therefore,

$$\begin{aligned} H^{n-p,n-q}(\Omega; D^\vee) &= \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} \Big(H^{n_1-p_1,n_2-q_1}_{c,L^2}(\Omega_1)\widehat{\otimes} H^{n_2-p_2,n_2-q_2}_{L^2}(\Omega_2)\Big) \\ &\cong \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H^{p_1,q_1}_{L^2}(\Omega_1)\widehat{\otimes} H^{p_2,q_2}_{c,L^2}(\Omega_2) \\ &= H^{p,q}_{L^2}(\Omega; D). \end{aligned}$$

4. $\overline{\partial}$-closed extension of forms

In this section, we assume that Ω is a relatively compact domain in a Hermitian manifold X. We assume that the holomorphic vector bundle E is defined on all of X.

Proposition 2. *Let Ω be a relatively compact domain with Lipschitz boundary in a Hermitian manifold X. Then a form $f \in \mathrm{Dom}(\overline{\partial}_{c,E})$ if and only if both f^0 and $\overline{\partial}(f^0)$ are in $L^2_*(\Omega, E)$, where f^0 denotes the form obtained by extending the form f by 0 on $X \setminus \Omega$. We in fact have $(\overline{\partial}_c f)^0 = \overline{\partial}(f^0)$ in the distribution sense.*

Proof. By definition, given $f \in \mathrm{Dom}(\overline{\partial}_{c,E})$, there is a sequence $\{f_\nu\}$ of smooth E-valued forms with compact support in Ω such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial}_c f$, both in $L^2_*(\Omega, E)$. Then clearly $(f_\nu)^0 \to f^0$ and $\overline{\partial}((f_\nu)^0) \to \overline{\partial} f$ in $L^2_*(\Omega)$. It is also easy to see that $\overline{\partial}((f_\nu)^0) \to \overline{\partial}((f)^0)$ in the distribution sense in X. To see that $\overline{\partial}((f)^0) = (\overline{\partial} f)^0$, we use integration-by-parts (since $\mathsf{b}\Omega$ is Lipschitz) to have that for any $\phi \in C^1_*(X)$,

$$\begin{aligned} ((\overline{\partial}_c f)^0, \phi)_X &= (\overline{\partial} f, \phi)_\Omega \\ &= \lim_{\nu\to\infty} (\overline{\partial} f_\nu, \phi)_\Omega \\ &= \lim_{\nu\to\infty} (f_\nu, \vartheta\phi)_\Omega \\ &= (f^0, \vartheta\phi)_X \\ &= (\overline{\partial}((f)^0), \phi)_X. \end{aligned}$$

This proves the "only if" part of the result.

Now assume that both f^0 and $\overline{\partial}(f^0)$ are in $L^2_*(\Omega, E)$. To show that $f \in \mathrm{Dom}(\overline{\partial}_{c,E})$, we need to construct a sequence $f_\nu \in \mathcal{D}(\Omega, E)$ which converges in the graph norm corresponding to $\overline{\partial}$ to f. By a partition of unity, this is a local

problem near each $z \in \mathsf{b}\Omega$, and we can assume that E is a trivial bundle near z. By the assumption on the boundary, for any point $z \in \mathsf{b}\Omega$, there is a neighborhood ω of z in X, and for $\epsilon \geq 0$, a continuous one parameter family t_ϵ of biholomorphic maps from ω into X such that $\Omega \cap \omega$ is compactly contained in Ω, and t_ϵ converges to the identity map on ω as $\epsilon \to 0^+$. In local coordinates near z, the map t_ϵ is simply the translation by an amount ϵ in the inward normal direction. Then we can approximate f^0 locally by $f^{(\epsilon)}$, where

$$f^{(\epsilon)} = (t_\epsilon^{-1})^* f^0$$

is the pullback of f^0 by the inverse t_ϵ^{-1} of t_ϵ. A partition of unity argument now gives a form $f^{(\epsilon)} \in L^2_*(X, E)$ such that $f^{(\epsilon)}$ is supported inside Ω and, as $\epsilon \to 0^+$,

$$\begin{cases} f^{(\epsilon)} \to f^0 & \text{in } L^2_*(X, E), \\ \overline{\partial} f^{(\epsilon)} \to \overline{\partial} f^0 & \text{in } L^2_*(X, E). \end{cases}$$

Since $\mathsf{b}\Omega$ is Lipschitz, we can apply Friedrichs' lemma (see [18] or Lemma 4.3.2 in [2]) to the form $f^{(\epsilon)}$ to construct the sequence $\{f_\nu\}$ in $\mathcal{D}(\Omega, E)$. $\square$

4.1. **Use of singular weights.** Let X be any Hermitian manifold, and let $\Omega \Subset X$ be a domain in X. We assume that Ω is pseudoconvex, and for $z \in \Omega$, let δ be a distance function on Ω. We will assume that δ satisfies the strong Oka's lemma:

$$i\partial\overline{\partial}(-\log \delta) \geq c\omega, \tag{24}$$

where $c > 0$ and ω is a positive (1,1)-form on X.

Such a distance function always exists on a Stein manifold. For example, if Ω is a pseudoconvex domain in $\mathbb{C}^n$, we can take $\delta(z)$ to be $\delta_0 e^{-t|z|^2}$, where δ_0 is the Euclidean distance from z to to $\mathsf{b}\Omega$ and $t > 0$. The distance function δ is comparable to δ_0. For each $t > 0$, let E_t denote the trivial line bundle $\mathbb{C} \times \Omega$ on Ω with pointwise Hermitian inner product $\langle u, v\rangle_z = (\delta(z))^t u\overline{v}$, where $u, v \in \mathbb{C}$ are supposed to be in the fiber over the point $z \in \Omega$. On a Stein manifold, we can take δ to be $\delta_0 e^{-t\phi}$ for sufficiently large t, where δ_0 is the distance function to the boundary with respect to the Hermitian metric on X and ϕ is a smooth strictly plurisubharmonic function on X. In classical terminology of Hörmander, this corresponds to the use of the weight function $\phi_t = -t \log \delta$. The dual bundle $(E_t)^*$ with dual metric can be naturally identified with E_{-t}, i.e. the weight $t \log \delta$. We will denote

$$L^2_{p,q}(\Omega, \delta^t) = L^2_{p,q}(\Omega, E_t) \tag{25}$$

in conformity with the classical notation. Note that for $t > 0$, the function δ^{-t} blows up at the boundary of Ω. If $t \geq 1$, a form in $L^2_{p,q}(\Omega, \delta^{-t})$ smooth up to the boundary *vanishes* on the boundary. We have the following:

Proposition 3. *Let Ω be a relatively compact pseudoconvex domain with Lipschitz boundary in a Hermitian Stein manifold X of dimension $n \geq 2$. Suppose that $f \in L^2_{(p,q)}(\Omega, \delta^{-t})$ for some $t \geq 0$, where $0 \leq p \leq n$ and $1 \leq q < n$. Assuming that*

(in the sense of distributions) $\overline{\partial} f = 0$ *in* X *with* $f = 0$ *outside* Ω*, then there exists* $u_t \in L^2_{(p,q-1)}(\Omega, \delta^{-t})$ *with* $u_t = 0$ *outside* Ω *satisfying* $\overline{\partial} u_t = f$ *in the distribution sense in* X*.*

For $q = n$*, if we assume that* f *satisfies*

$$\int_\Omega f \wedge g = 0 \qquad \text{for every } g \in \ker(\overline{\partial}) \cap L^2_{(n-p,0)}(\Omega, \delta^t), \tag{26}$$

the same result holds.

Proof. Using the notation E_t as in (25) it follows that for any $t > 0$, the map $\overline{\partial}_{E_t^*}$ has closed range in each degree following Hörmander's L^2 method [19] with weights since the weight function satisfies the strong Oka's lemma (see [16]). This is equivalent to the $\overline{\partial}$-problem on the pseudoconvex domain Ω in the bundle $E_t^* = E_{-t}$, i.e., with plurisubharmonic weight $-t \log \delta$. The result now follows by combining the solution of the $\overline{\partial}_c$ problem as given by Theorem 3 and the characterization of the $\overline{\partial}_c$-operator as given by Proposition 2. □

For real s, denote by $W^s(\Omega)$ the Sobolev space of functions on Ω with s derivatives in L^2. Let $W_0^s(\Omega)$ be the space of completion of $C_0^\infty(\Omega)$ functions under the $W^s(\Omega)$-norm.

Lemma 6. *Let* Ω *be a bounded domain with Lipschitz boundary in* $\mathbb{R}^n$ *and let* ρ *be a distance function. For any* $s \geq 0$*, if* $f \in W^s(\Omega)$ *and* $\rho^{-s+\alpha} D^\alpha f \in L^2(\Omega)$ *for every multi-integer* α *with* $|\alpha| \leq s$*, then* $f \in W_0^s(\Omega)$ *and* $f^0 \in W^s(\mathbb{R}^n)$*, where* f^0 *is the extension of* f *to be zero outside* Ω*.*

Proof. When the boundary is smooth and s is an integer, this is proved in [29, Chapter 1, Theorem 11.8]. We first note that when $s \leq \frac{1}{2}$, the space W^s and W_0^s are equal (see [29, Chapter 1, Theorem 11.1] or Grisvard [13]). When $s \neq k + \frac{1}{2}$, where $k = 0, 1, 2, \ldots$, the lemma follows from [29, Section 11.2 and Theorem 11.4] for smooth domains.

To see that when $s = k + \frac{1}{2}$ holds, we first prove it for $k = 0$. Let $f \in W^{\frac{1}{2}}(\Omega)$ and $\rho^{-\frac{1}{2}} f \in L^2(\Omega)$. We only need to show that f^0 is in $W^{\frac{1}{2}}(\mathbb{R}^n)$. Notice that for $0 \leq s \leq \frac{1}{2}$, the extension operator $u \in W^s(\Omega) = W_0^s(\Omega) \to u^0$ is continuous only when $s < \frac{1}{2}$, but is not continuous from $W^{\frac{1}{2}}(\Omega)$ to $W^{\frac{1}{2}}(\mathbb{R}^n)$ (see [29]). However, if f satisfies $\rho^{-\frac{1}{2}} f \in L^2(\Omega)$, then $f \in W_{00}^{\frac{1}{2}}(\Omega)$, which is a proper subset of $W^{\frac{1}{2}}(\Omega) = W_0^{\frac{1}{2}}(\Omega)$ (for the definition and properties of $W_{00}^{\frac{1}{2}}$, see Theorem 11.7, Chapter 1 in [29]). The extension operator $f \to f^0$ is continuous from $W_0^s(\Omega)$ to $W^s(\mathbb{R}^n)$ when $s = 0$ and $s = 1$. Thus from the interpolation theorem, it is continuous from $W_{00}^{\frac{1}{2}}(\Omega)$ to $W^{\frac{1}{2}}(\mathbb{R}^n)$ since $W_{00}^{\frac{1}{2}}(\Omega)$ is the interpolation space of $W^0(\Omega)$ and $W_0^1(\Omega)$. The case for $k > 0$ follows from induction.

The lemma also holds for Lipschitz domains since we can exhaust any Lipschitz domain Ω by smooth subdomains Ω_ν (see Lemma 0.3 in [38]). This is clear when the domain is star-shaped and the general case follows from using a partition of unity (see [13] for the corresponding properties for Sobolev spaces on Lipschitz domains). □

Combining Proposition 3 and Lemma 6, we have the following regularity results on solving $\overline{\partial}$ with prescribed support.

Proposition 4. *Let $\Omega \Subset X$ be a pseudoconvex domain with Lipschitz boundary in a Stein manifold of dimension $n \geq 3$ with a Hermitian metric. Suppose that $0 \leq p \leq n$ and $1 \leq q \leq n$ and that f is a (p,q)-form with $W_0^s(\Omega) \cap L^2(\Omega, \delta^{-2s})$ coefficients, where $s \geq 0$. We assume that*

(1) *for $1 \leq q < n$, f satisfies $f \in Dom(\overline{\partial}_c)$ and $\overline{\partial}_c f = 0$,*
(2) *for $q = n$, f satisfies*

$$\int_\Omega f \wedge g = 0 \qquad \text{for every } g \in \ker(\overline{\partial}) \cap L^2_{n-p,0}(\Omega, \delta^{2s}). \tag{27}$$

Then there exists a $(p,q-1)$-form $u \in L^2_{p,0}(\Omega, \delta^{-2s}) \cap Dom(\overline{\partial}_c)$ with $W_0^s(\Omega)$ coefficients satisfying $\overline{\partial}_c u = f$ in X.

We remark that when $s - \frac{1}{2}$ is not a non-negative integer, the assumption $f \in W_0^s(\Omega)$ implies that $f \in L^2(\Omega, \delta^{-2s})$ (see [29]). The pairing in (27) is well defined between the two spaces $L^2(\Omega, \delta^{2s})$ and $L^2(\Omega, \delta^{-2s})$.

Theorem 5. *Let X be a Stein manifold and let $\Omega \Subset X$ be a relatively compact pseudoconvex domain with Lipschitz boundary. Let $\Omega^+ = X \setminus \Omega$.*

Then for any $f \in W^s_{p,q}(\Omega^+)$, where $q \leq n-2$, with $s \geq 1$ such that $\overline{\partial} f = 0$ in Ω^+, there exists $F \in W^{s-1}_{p,q}(X)$ with $F|_{\Omega^+} = f$ and $\overline{\partial} f = 0$ on X.

For $q = n-1$, we assume that

$$\int_{b\Omega} f \wedge g = 0 \qquad \text{for every } g \in \ker(\overline{\partial}) \cap L^2_{n-p,0}(\Omega, \delta^{2(s-1)}), \tag{28}$$

and the same conclusion holds.

Proof. Since Ω has Lipschitz boundary, there is a bounded extension operator from $W^s(\Omega^+)$ to $W^s(X)$ for all $s \geq 0$ (see e.g. [13]). Let $\tilde{f} \in W^s_{p,q}(X)$ be the extension of f so that $\tilde{f}|_{\Omega^+} = f$ with $\|\tilde{f}\|_{W^s(X)} \leq C\|f\|_{W^s(\Omega^+)}$. We have $\overline{\partial}\tilde{f} \in W_0^{s-1}(\Omega) \cap L^2(\Omega, \delta^{-2(s-1)})$ (see Theorem 11.5 in [29]).

Obviously we have that $\overline{\partial}\tilde{f} \in W_0^{s-1}(\Omega)$ is $\overline{\partial}$-closed in Ω. When $q = n-1$, $\overline{\partial}\tilde{f} \in W^{s-1}_{p,n}(\Omega) \cap L^2_{p,n}(\Omega, \delta^{-2(s-1)})$ and satisfies

$$\int_\Omega \overline{\partial}\tilde{f} \wedge g = \int_{b\Omega} f \wedge g = 0 \qquad \text{for every } g \in \ker(\overline{\partial}) \cap L^2_{n-p,0}(\Omega, \delta^{2(s-1)}). \tag{29}$$

Notice that both integrals in (29) are well defined by an approximation argument using Friedrichs' lemma (see [18] or Lemma 4.3.2 in [2]).

Let $t = s - 1 \geq 0$. We define $T\tilde{f}$ by $T\tilde{f} = -\star_{(2t)}\bar{\partial}N_{2t}(\star_{(-2t)}\overline{\partial}\tilde{f})$ in Ω, where $\star_t = \star_{E_t}$. From Proposition 3 and Proposition 4, we have that there exists $u = T\tilde{f} \in L^2(\Omega, \delta^{-2t}) \cap W_0^t(\Omega)$ satisfying $\overline{\partial}(T\tilde{f})^0 = \overline{\partial}\tilde{f}$ in X.

Define

$$F = \tilde{f} - (T\tilde{f})^0 = \begin{cases} f, & x \in \overline{\Omega}^+, \\ \tilde{f} - T\tilde{f}, & x \in \Omega. \end{cases}$$

Then from Lemma 6, $F \in W^{s-1}_{p,q}(X)$ and F is a $\overline{\partial}$-closed extension of f. $\square$

Corollary 2. *Let Ω_1 and Ω be two pseudoconvex domains in a Stein manifold χ with $\Omega \Subset \Omega_1 \Subset \chi$. Let $\Omega^+ = \Omega_1 \setminus \overline{\Omega}$ be the annulus between two pseudoconvex domains Ω and Ω_1. For any $f \in W^s_{p,q}(\Omega^+)$, where $0 \le p \le n$, $1 \le q < n-1$ and $s \ge 1$, such that $\overline{\partial} f = 0$ in Ω^+, there exists $u \in W^s_{(p,q-1)}(\Omega^+)$ with $\overline{\partial} u = f$ in Ω^+. Furthermore, if $f \in C^\infty_{p,q}(\overline{\Omega}^+)$, we have $u \in C^\infty_{p,q-1}(\overline{\Omega}^+)$.*

When $q = n$, we assume that f satisfies (28) instead; then the same result holds.

We remark that Corollary 2 allows us to solve $\overline{\partial}$ smoothly up to the boundary on pseudoconcave domains with only Lipschitz boundary provided the compatibility conditions are satisfied. Results of this kind were obtained in [36] for pseudoconcave domains with smooth boundary. For Lipschitz boundary, see [30] or [15] using integral kernel methods. This is in sharp contrast with pseudoconvex domains, where solving $\overline{\partial}$ smoothly up to the boundary is known only for pseudoconvex domains with smooth boundary (see [24]) or domains with a Stein neighborhood basis (see [10]). If the boundary $\mathsf{b}\Omega$ is smooth, Theorem 5 and Corollary 2 also hold for $s = 0$ (see [37, 38]).

5. Holomorphic extension of CR forms from the boundary of a complex manifold

In this section we study the holomorphic extension of CR forms from the boundary of a domain in a complex manifold X using our L^2-duality. The use of duality in the study of the holomorphic extension of CR functions with smooth or continuous data is classical (see [34]) and has been studied by many authors (see [35, 25, 17]).

In what follows, X is a complex manifold and Ω is a relatively compact domain in X with Lipschitz boundary (see [38] for a general discussion of partial differential equations on Lipschitz domains and see [39] for a discussion of the tangential Cauchy-Riemann equations). We will assume that X has been endowed with a Hermitian metric and that the spaces $L^2_{p,q}(\Omega) = L^2_{p,q}(\Omega, \mathbb{C})$ of square integrable forms are defined with respect to the metric of X restricted to Ω. Observe that the spaces $L^2_{p,q}(\Omega)$ as well as the Sobolev spaces of forms $W^k_{p,q}(\Omega)$ are defined independently of the particular choice of metric on X. Further, it is possible to define Sobolev spaces on the boundary $\mathsf{b}\Omega$ in such a way that the usual results on existence of a trace still hold, e.g. functions in Ω of class $W^1(\Omega)$ have traces on $\mathsf{b}\Omega$ of class $W^{\frac{1}{2}}(\mathsf{b}\Omega)$ (see [21, 22]).

The main observation, which follows from the duality results in §3, is the following:

Proposition 5. *For any p, with $0 \le p \le n$, the map*

$$\overline{\partial}_c : L^2_{p,0}(\Omega) \dashrightarrow L^2_{p,1}(\Omega)$$

has closed range.

Proof. Thanks to Lemma 3 this is equivalent to the map $\overline{\partial} : L^2_{n-p,n-1}(\Omega) \dashrightarrow L^2_{n-p,n}(\Omega)$ having closed range. But it is well known that $\overline{\partial}$ has closed range in this top degree on smooth domains, a fact that is equivalent to the solvability of the Dirichlet problem for the Laplace-Beltrami operator on such domains (see [11]). For a proof of the solvability of the Dirichlet problem for domains with Lipschitz boundary, see [21, 22]. □

Recall that a holomorphic p-form is a $\overline{\partial}$-closed $(p,0)$-form. We denote the space of holomorphic p-forms on Ω by $\mathcal{O}_p(\Omega)$. We deduce a necessary condition for a $(p,0)$-form on $\mathsf{b}\Omega$ to be the boundary value of a holomorphic p-form on Ω:

Theorem 6. *Let $f \in W^{\frac{1}{2}}_{p,0}(\mathsf{b}\Omega)$ be a $(p,0)$-form on $\mathsf{b}\Omega$ with coefficients in the Sobolev space $W^{\frac{1}{2}}$. Then the following are equivalent:*

(1) *There is a holomorphic p-form $F \in \mathcal{O}_p(\Omega) \cap W^1(\Omega)$ such that $f = F|_{\mathsf{b}\Omega}$.*
(2) *For all $g \in L^2_{n-p,n-1}(\Omega) \cap \ker(\overline{\partial})$, we have*

$$\int_{\mathsf{b}\Omega} f \wedge g = 0. \tag{30}$$

(Note that it is easy to show that a $\overline{\partial}$-closed form with L^2 coefficients has a trace of class $W^{-\frac{1}{2}}$, and hence the integral above is well defined.)

(3) *For any extension $\tilde{f} \in W^1_{p,0}(\Omega)$ of f to Ω as a $(p,0)$-form with coefficients in W^1, the form $\overline{\partial}\tilde{f} \in L^2_{p,1}(\Omega)$ belongs to the range of $\overline{\partial}_c$ on Ω.*

Proof. ($1 \implies 2$) Let $g \in L^2_{n-p,n-1}(\Omega) \cap \ker(\overline{\partial})$. By Stoke's Theorem,

$$\int_{\mathsf{b}\Omega} f \wedge g = \int_{\Omega} d(F \wedge g) = \int_{\Omega} \overline{\partial}(F \wedge g) = 0.$$

($2 \implies 3$) First note that such an extension $\tilde{f}$ always exists, since $\mathsf{b}\Omega$ is Lipschitz. Again let $g \in L^2_{n-p,n-1}(\Omega) \cap \ker(\overline{\partial})$. By Stoke's Theorem,

$$\int_{\Omega} \overline{\partial}\tilde{f} \wedge g = \int_{\mathsf{b}\Omega} f \wedge g = 0.$$

Assertion (3) now follows from condition (22) given in Theorem 3 for a form to be in the range of the $\overline{\partial}_c$-operator.

($3 \implies 1$) By Proposition 5, $\overline{\partial}_c$ has closed range in degree $(p,1)$, and by hypothesis $\overline{\partial}\tilde{f}$ is in the range of $\overline{\partial}_c$. By Theorem 3, we can solve the equation

$$\overline{\partial}_c u = \overline{\partial}\tilde{f}, \tag{31}$$

with L^2 estimates for a $(p,0)$-form u. Then $F = \tilde{f} - u$ is holomorphic in Ω. Also, by Proposition 2 we have that

$$\overline{\partial}(u^0) = (\overline{\partial}u)^0 = (\overline{\partial}\tilde{f})^0,$$

where g^0 denotes the extension of the form g on Ω to all of X by setting it equal to 0 on $X \setminus \Omega$. Since $(\overline{\partial}\tilde{f})^0 \in L^2_{p,1}(X)$, by elliptic regularity, $u^0 \in W^1_{p,0}(X)$. It follows that u^0 has a trace (of class $W^{\frac{1}{2}}(\mathsf{b}\Omega)$) on the Lipschitz hypersurface $\mathsf{b}\Omega$. Since u^0 vanishes identically on $X \setminus \Omega$, it follows that this trace is 0. Consequently, $F \in W^1_{p,0}(\Omega)$ and satisfies $F|_{\mathsf{b}\Omega} = f$. □

Let f be a p-form with coefficients in $L^1(\mathsf{b}\Omega)$ which is the boundary value of a holomorphic p-form $F \in \mathcal{O}_p(\Omega)$, then f must be CR, i.e, it must satisfy in the homogeneous tangential Cauchy-Riemann equations on $\mathsf{b}\Omega$ in the weak sense. That is, for each compactly supported smooth $(n-p, n-2)$-form $\phi \in \mathcal{D}_{n-p,n-2}(X)$, we have

$$\int_{\mathsf{b}\Omega} f \wedge \overline{\partial}\phi = 0. \tag{32}$$

(See [40] for details.)

It is easy to see that (30) implies (32). But in general, the two conditions are not equivalent. One condition under which they are equivalent is the following:

Corollary 3. *Let Ω be a domain with Lipschitz boundary in a complex manifold X of complex dimension $n \geq 2$. Suppose that $H^{n-p,n-1}_{L^2}(\Omega) = 0$. Then every CR form in $f \in W^{\frac{1}{2}}_{p,0}(\mathsf{b}\Omega)$ has a holomorphic extension F to Ω with $F \in \mathcal{O}_p(\Omega) \cap W^1(\Omega)$ and $F = f$ on $\mathsf{b}\Omega$.*

Proof. Let $g \in \ker(\overline{\partial}) \cap L^2_{n-p,n-1}(\Omega)$. By the hypothesis on cohomology, there is a $u \in \mathrm{Dom}(\overline{\partial}) \cap L^2_{n-p,n-2}(\Omega)$ such that $\overline{\partial} u = g$. Since Ω is Lipschitz, by Friedrich's lemma we can find a sequence $\{u_\nu\} \subset \mathcal{C}^\infty_{n-p,n-2}(\overline{\Omega})$ such that $u_\nu \to u$ in $L^2_{n-p,n-2}(\Omega)$, and $\overline{\partial} u_\nu \to g$ in $L^2_{n-p,n-1}(\Omega)$ as $\nu \to \infty$. Let $\phi_\nu \in \mathcal{D}_{n-p,n-2}(X)$ be a smooth compactly supported extension of the form u_ν to X. Then we have

$$\int_{\mathsf{b}\Omega} f \wedge g = \lim \int_{\mathsf{b}\Omega} f \wedge \overline{\partial} \phi_\nu = 0.$$

The result now follows by Theorem 6. □

The following corollary is another extension result that can be deduced from Theorem 6:

Corollary 4. *Let $\Omega \Subset X$ be a domain with connected Lipschitz boundary in a non-compact connected complex manifold X of complex dimension $n \geq 2$. Suppose that there exists a relatively compact domain Ω' with Lipschitz boundary such that $\Omega \Subset \Omega' \Subset X$ and*

$$H^{n-p,n-1}_{L^2}(\Omega') = 0. \tag{33}$$

Then every CR form of degree $(p,0)$ on $\mathsf{b}\Omega$ of Sobolev class $W^{\frac{1}{2}}(\mathsf{b}\Omega)$ has a holomorphic extension to Ω (of class $W^1(\Omega)$).

Proof. Let $\tilde{f}$ be an extension of f to Ω (of class $W^1(\Omega)$) and let

$$g = \begin{cases} \overline{\partial} \tilde{f} & \text{on } \Omega, \\ 0 & \text{on } \Omega' \setminus \Omega. \end{cases}$$

We claim that $\overline{\partial} g = 0$ on Ω'. Indeed, let $u \in \mathcal{D}_{p,1}(\Omega')$ be a smooth $(p,1)$-form of compact support in Ω'. We have

$$\begin{aligned} (\overline{\partial} g, u)_{L^2(\Omega')} &= (g, \vartheta u)_{L^2(\Omega')} \\ &= (\overline{\partial} \tilde{f}, \vartheta u)_{L^2(\Omega)} \\ &= \int_\Omega \overline{\partial} \tilde{f} \wedge \star \vartheta u \\ &= \int_\Omega \{\overline{\partial}(\tilde{f} \wedge \star \vartheta u) - (-1)^p (\tilde{f} \wedge \overline{\partial} \star \vartheta u)\}. \end{aligned}$$

Since $\overline{\partial} \star \vartheta = -\overline{\partial} \star (\star \overline{\partial} \star) = \pm \overline{\partial}\overline{\partial} \star = 0$, the second term vanishes, and by Stoke's theorem the first integral is equal to

$$\begin{aligned}\int_{\mathsf{b}\Omega} \tilde{f} \wedge \star \vartheta u &= \pm \int_{\mathsf{b}\Omega} f \wedge (\star \vartheta \star)(\star u) \\ &= \pm \int_{\mathsf{b}\Omega} f \wedge \overline{\partial}(\star u) \\ &\quad \text{(since } \overline{\partial} = \star \vartheta \star \text{ on compactly supported forms; see (15))} \\ &= 0 \\ &\quad \text{(since } f \text{ is CR; see (32)).}\end{aligned}$$

As g vanishes near $\mathsf{b}\Omega'$ and $\overline{\partial} g = 0$, it follows that $g \in \mathrm{Dom}(\overline{\partial}_c)$ on Ω' and $\overline{\partial}_c g = 0$. Since $\overline{\partial}$ has closed range in Ω for bidegrees $(n-p, n-1)$ as well as $(n-p, n)$, it follows by duality from (33) that $H^{p,1}_{c,L^2}(\Omega') = 0$. There is then a $u \in \mathrm{Dom}(\overline{\partial}_c)$ such that $\overline{\partial}_c u = g$. By Proposition 2, the extensions by 0 satisfy $\overline{\partial}(u^0) = (\overline{\partial} u)^0 = g^0$. Since g^0 is in $L^2(X)$ it follows that $u^0 \in W^1_{p,0}(X)$. Further, u^0 is holomorphic on $X \setminus \Omega$ and $u^0 \equiv 0$ on $X \setminus \Omega'$. By analytic continuation, $u^0 \equiv 0$ on $X \setminus \Omega$. Therefore, the trace of u $\mathsf{b}\Omega$ vanishes, and the form $F = \tilde{f} - u$ on Ω is holomorphic, of class W^1 and satisfies $F = f$ on $\mathsf{b}\Omega$. □

Corollary 5. *Let Ω be a domain with Lipschitz boundary in a Stein manifold X of complex dimension $n \geq 2$. Suppose that $\mathsf{b}\Omega$ is connected. Then for every CR function on $\mathsf{b}\Omega$ of class $W^{\frac{1}{2}}(\mathsf{b}\Omega)$ has a holomorphic extension to Ω.*

Proof. In the proof of Corollary 4, we let Ω' be some strongly pseudoconvex domain in X and $\Omega \Subset \Omega'$. Then $H^{n,n-1}_{L^2}(\Omega') = H^{0,1}_{c,L^2}(\Omega') = 0$. The corollary follows. □

When $X = \mathbb{C}^n$ and $p = 0$, this gives the usual Bochner-Hartogs extension theorem. In this case, the extension function can be written explicitly as

$$F(z) = \int_{\mathsf{b}\Omega} B(\zeta, z) \wedge f(\zeta), \quad z \in \Omega,$$

where B is the Bochner-Martinelli kernel. The function F has boundary value f as z approaches the boundary (see [41] for a proof when the boundary is smooth; in this case we can allow more singular boundary values than is possible in our results with Lipschitz boundaries). This is very different from the holomorphic extension of CR functions in complex manifolds which are not Stein. We will give an example to show that the extension results on Lipschitz domain is maximal in the sense that the results might not hold if the Lipschitz condition is dropped.

We will analyze the holomorphic extension of functions on a non-Lipschitz domain. Let Ω be the Hartogs triangle in $\mathbb{CP}^2$ defined by

$$\Omega = \{[z_0, z_1, z_2] \mid |z_1| < |z_2|\},$$

where $[z_0, z_1, z_2]$ denotes the homogeneous coordinates of a point in $\mathbb{CP}^2$. As usual we endow Ω with the restriction of the Fubini-Study metric of $\mathbb{CP}^2$.

Proposition 6. *Let $\Omega \subset \mathbb{CP}^2$ be the Hartogs triangle. Then we have the following:*

(1) *The Bergman space of L^2 holomorphic functions $L^2(\Omega) \cap \mathcal{O}(\Omega)$ on the domain Ω separates points in Ω.*

(2) *There exist non-constant functions in the space $W^1(\Omega) \cap \mathcal{O}(\Omega)$. However, this space does not separate points in Ω and is not dense in the Bergman space $L^2(\Omega) \cap \mathcal{O}(\Omega)$.*
(3) *Let $f \in W^2(\Omega) \cap \mathcal{O}(\Omega)$ be a holomorphic function on Ω which is in the Sobolev space $W^2(\Omega)$. Then f is a constant.*

Remark. Statements (1) and (3) above have already been proved in [15]. Regarding (2), we would like to point out a misleading statement made in that paper, where it is claimed that $W^1(\Omega) \cap \mathcal{O}(\Omega)$ consists of constants only (see item 5 in Example 12.1 in [15]).

Proof. For (1), consider the two holomorphic functions $\frac{z_1}{z_2}$ and $\frac{z_0}{z_2}$ on Ω which separate points on Ω, and the first of which is bounded (and therefore square-integrable in the Fubini-Study metric) on Ω. To see that $\frac{z_0}{z_2}$ is in $L^2(\Omega) \cap \mathcal{O}(\Omega)$, we only need to verify that it is in $L^2(\Omega)$ near the point $[1, 0, 0]$. We choose the coordinate chart $U_0 = \{z_0 \neq 0\} \cap \Omega$ for Ω with holomorphic coordinates (z, w), where $z = \frac{z_1}{z_0}$ and $w = \frac{z_2}{z_0}$. The function $\frac{z_0}{z_2} = w^{-1}$, and it suffices to show that w^{-1} is square-integrable on $\Omega \cap P$, where P is the polydisc $\{|z| < 1, |w| < 1\}$. More generally, consider the square-integrability of $w^{-\nu}$, where $\nu \geq 1$ is an integer. We have

$$\begin{aligned}\int_{\Omega \cap P} \frac{1}{|w^\nu|^2} dV &= 4\pi^2 \iint_{r_1 < r_2 < 1} \left(\frac{1}{r_2^{2\nu}}\right) r_2 dr_2 r_1 dr_1 \\ &= 4\pi^2 \int_0^1 \left(\int_{r_1}^1 r_2^{-2\nu+1} dr_2\right) r_1 dr_1.\end{aligned}$$

When $\nu = 1$ the integral becomes

$$\begin{aligned}&= 4\pi^2 \int_0^1 -r_1 \log r_1 dr_1 \\ &< \infty.\end{aligned}$$

If $\nu > 1$, the inner integral evaluates to a constant times $(1 - r_1^{-2\nu+2})$, the double integral diverges, and consequently, $w^{-\nu} \notin L^2(\Omega \cap P)$ (cf. [15, Proposition 3]).

On the subset $\Omega \cap \{z_2 \neq 0\}$, introduce the coordinates $\widetilde{z} = \frac{z_1}{z_2}$ and $\widetilde{w} = \frac{z_0}{z_2}$. In these coordinates the set $\Omega \cap \{z_2 \neq 0\}$ is represented as the bidisc with one infinite radius $\{(\widetilde{z}, \widetilde{w}) \mid |\widetilde{z}| < 1\}$, and any function $f \in \mathcal{O}(\Omega)$ has a power series expansion on this polydisc of the form

$$f(\widetilde{z}, \widetilde{w}) = \sum_{\substack{\mu \geq 0 \\ \nu \geq 0}} C_{\mu,\nu} \widetilde{z}^\mu \widetilde{w}^\nu.$$

In the coordinate patch $\Omega \cap \{z_0 \neq 0\}$, the natural coordinates are (z, w), where $z = \frac{z_1}{z_0} = \frac{\widetilde{z}}{\widetilde{w}}$ and $w = \frac{z_2}{z_0} = \frac{1}{\widetilde{w}}$. Therefore the holomorphic function f on Ω has a Laurent expansion on $\Omega \cap \{z_0 \neq 0\}$ of the form

$$f(z, w) = \sum_{\substack{\mu \geq 0 \\ \nu \geq 0}} C_{\mu,\nu} \left(\frac{z}{w}\right)^\mu w^{-\nu}.$$

By the symmetry of the Fubini-Study metric, it follows that the terms of the series are orthogonal, provided they are in $L^2(\Omega \cap P)$, and therefore, if $f \in L^2(\Omega \cap P)$,

we have

$$\|f\|^2_{L^2(\Omega\cap P)} = \sum_{\mu,\nu\geq 0} |C_{\mu,\nu}|^2 \left\| \left(\frac{z}{w}\right)^{\mu} w^{-\nu} \right\|^2_{L^2(\Omega\cap P)}.$$

Since $\frac{z}{w} = \frac{z_1}{z_2}$ is bounded the computation of $\|w^{-\nu}\|_{L^2}$ in the last paragraph shows that non-zero terms on the right-hand side are not in L^2 if $\nu \geq 2$, which means $C_{\mu\nu} = 0$ if $\nu \geq 2$. Thus each $f \in L^2(\Omega) \cap \mathcal{O}(\Omega)$ has a Laurent expansion of the form

$$f(z,w) = \sum_{\substack{\mu\geq 0\\ 0\leq\nu\leq 1}} C_{\mu,\nu} \left(\frac{z}{w}\right)^{\mu} w^{-\nu}. \tag{34}$$

Taking a derivative we see that

$$\frac{\partial f}{\partial w}(z,w) = \sum_{\substack{\mu\geq 0\\ 0\leq\nu\leq 1}} -(\mu+\nu) C_{\mu,\nu} \left(\frac{z}{w}\right)^{\mu} w^{-(\nu+1)}.$$

By orthogonality of the terms again, if this is in $L^2(\Omega \cap P)$, then the coefficients $C_{\mu,1} = 0$. It follows that any $f \in W^1(\Omega) \cap L^2(\Omega)$ is of the form

$$f(z,w) = \sum_{\nu=0}^{\infty} b_\nu \left(\frac{z}{w}\right)^{\nu}. \tag{35}$$

Further, it is easily verified that if f is of the above form, then $\frac{\partial f}{\partial z} \in L^2(\Omega)$. Therefore any holomorphic function in $W^1(\Omega)$ is a function of $\frac{z}{w}$ alone, and it follows that $W^1(\Omega) \cap \mathcal{O}(\Omega)$ does not separate points in Ω. This proves (2).

By taking two derivatives in (35), we obtain

$$\frac{\partial^2 f}{\partial w^2}(z,w) = \sum_{\nu=1}^{\infty} -\nu(\nu+1) b_\nu \left(\frac{z}{w}\right)^{\nu} \cdot \frac{1}{w^2}.$$

None of the mutually orthogonal terms is in $L^2(\Omega \cap P)$, thanks to the computation of $\|w^{-\nu}\|_{L^2}$ above. It follows that f reduces to a constant and we have (3). □

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