THE DIAMETER ESTIMATE AND ITS APPLICATION TO CR OBATA’S THEOREM ON CLOSED PSEUDOHERMITIAN \((2n+1)\)-MANIFOLDS

SHU-CHENG CHANG AND CHIN-TUNG WU

Abstract. In this paper, we obtain a sharp lower bound estimate for diameters with respect to an adapted metric in closed pseudohermitian \((2n+1)\)-manifolds when a sharp lower bound estimate for the first positive eigenvalue of the sublaplacian is achieved. As a consequence, we confirm the CR Obata Conjecture on a closed pseudohermitian \((2n+1)\)-manifold with an extra condition on covariant derivatives of torsion.

1. Introduction

Let \((M, J, \theta)\) be a closed pseudohermitian \((2n+1)\)-manifold (see the next section for basic notions in pseudohermitian geometry). A. Greenleaf ([Gr]) proved the pseudohermitian analogue of Lichnerowicz’s Theorem ([L]) for the lower bound of the first positive eigenvalue \(\lambda_1\) of the sublaplacian for a pseudohermitian manifold \(M^{2n+1}\) with \(n \geq 3\). In [LL], S.-Y. Li and H.-S. Luk proved the same result for the cases \(n = 2\) and \(n = 1\) with an extra condition on a covariant derivative of the pseudohermitian torsion.

Recently, it was proved by H.-L. Chiu and the first author ([Ch], [CC1]) that

\[
\lambda_1 \geq \frac{nk}{n+1}
\]

if

\[
Ric_m(Z, Z) - \frac{n+1}{2} Tor_m(Z, Z) \geq k \langle Z, Z \rangle_{L^{\theta}},
\]

for all \(m \in M\), \(Z \in T_{1,0}\), and for some positive constant \(k\), on a closed pseudohermitian \((2n+1)\)-manifold with the nonnegative CR Paneitz operator \(P_0\) if \(n = 1\).

It is well known that

\[
\lambda_1 = \frac{nk}{n+1}
\]

on the standard unit sphere \((S^{2n+1}, \hat{J}, \hat{\theta})\) in \(\mathbb{C}^{n+1}\) with

\[
Ric_m(Z, Z) = k \langle Z, Z \rangle_{L^{\theta}}.
\]

Then it is natural to conjecture the CR analogue of M. Obata’s Theorem ([O]) on a closed pseudohermitian \((2n+1)\)-manifold \((M, J, \theta)\).
Conjecture 1.1. Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. In addition we assume the Paneitz operator \(P_0\) is nonnegative if \(n = 1\). Suppose that 

\[
\operatorname{Ric}_m(Z, Z) - \frac{n + 1}{2} \operatorname{Tor}_m(Z, Z) \geq k \langle Z, Z \rangle_{L^\theta},
\]

for all \(m \in M, Z \in T_{1,0}\), and for some positive constant \(k\). Assume that 

\[
\lambda_1 = \frac{nk}{n + 1}.
\]

Then \((M, J, \theta)\) is the standard pseudohermitian \((2n + 1)\)-sphere \((S^{2n+1}, \hat{J}, \hat{\theta})\) with \(\operatorname{Ric}_m(Z, Z) = k \langle Z, Z \rangle_{L^\theta}\).

We first recall some basic definitions.

**Definition 1.1.** A piecewise smooth curve \(\gamma : [0, 1] \rightarrow M\) is said to be horizontal if \(\gamma'(t) \in \xi\) whenever \(\gamma'(t)\) exists. The length of \(\gamma\) is then defined by

\[
l(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_{L^\theta}^{1/2} dt.
\]

The Carnot-Carathéodory distance between two points \(p, q \in M\) is

\[
d_c(p, q) = \inf \{ l(\gamma) | \gamma \in C_{p,q} \},
\]

where \(C_{p,q}\) is the set of all horizontal curves joining \(p\) and \(q\). By the Chow connectivity theorem [Cho], there always exists a horizontal curve joining \(p\) and \(q\), so the distance is finite. The diameter \(d_c\) is defined by

\[
d_c(M) = \sup \{ d_c(p, q) | p, q \in M \},
\]

Note that there is a minimizing geodesic joining \(p\) and \(q\) so that its length is equal to the distance \(d_c(p, q)\).

**Definition 1.2.** Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. Define

\[
P\varphi = \sum_{\alpha=1}^n (\varphi\overline{\theta} + \text{in}A_{\beta\alpha}\varphi^\alpha)\theta^\beta = (P\varphi)\theta^\beta, \quad \beta = 1, 2, \ldots, n,
\]

which is an operator that characterizes CR-pluriharmonic functions ([Le] for \(n = 1\) and [GL] for \(n \geq 2\)). Here

\[
P\varphi = \sum_{\alpha=1}^n (\varphi\overline{\theta} + \text{in}A_{\beta\alpha}\varphi^\alpha)
\]

and \(\overline{P}\varphi = (\overline{P}\varphi)\theta^\overline{\beta}\), the conjugate of \(P\). Moreover, we define

\[
P_0\varphi = 4 (\delta_b(P\varphi) + \overline{s_b}(\overline{P}\varphi)),
\]

which is the so-called CR Paneitz operator \(P_0\). Here \(\delta_b\) is the divergence operator that takes \((1, 0)\)-forms to functions by \(\delta_b(\sigma_\alpha) = \sigma_\alpha\) and \(\delta_b(\sigma_\beta\overline{\theta}^\alpha) = \sigma_\beta\overline{\theta}^\alpha\). If we define \(\overline{\delta}_b\varphi = \varphi_\alpha\theta^\alpha\) and \(\delta_b\varphi = \varphi_\beta\overline{\theta}^\beta\), then the formal adjoint of \(\delta_b\) on functions (with respect to the Levi form and the volume form \(d\mu\)) is \(\delta_b^* = -\delta_b\). We observe that

\[
\int \langle P\varphi + \overline{P}\varphi, d\varphi \rangle_{L^\theta} d\mu = -\frac{1}{4} \int P_0\varphi \cdot \varphi d\mu.
\]

For the details about these operators, the reader can make reference to [GL] and [Hi].
Definition 1.3. On a closed pseudohermitian \((2n + 1)\)-manifold \((M, J, \theta)\), we call the Paneitz operator \(P_0\) with respect to \((J, \theta)\) nonnegative if
\[
\int_M P_0 \varphi \cdot \varphi \, d\mu \geq 0
\]
for all real \(C^\infty\) smooth functions \(\varphi\).

Remark 1.1. (i) Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold with \(n \geq 2\). Then a smooth real-valued function \(\varphi\) satisfies \(P_0 \varphi = 0\) on \(M\) if and only if \(P_\beta \varphi = 0\) on \(M\) for all \(\beta\). It also holds for a closed pseudohermitian 3-manifold with vanishing torsion (\([HI]\), \([GL]\)).

(ii) Let \((M, J, \theta)\) be a closed pseudohermitian 3-manifold with vanishing torsion. Then the corresponding CR Paneitz operator is nonnegative. Unlike \(n = 1\), let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold with \(n \geq 2\). The corresponding CR Paneitz operator is always nonnegative (\([CC]\), \([CC2]\)).

Definition 1.4 (\([GL]\)). Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. We define the purely holomorphic second-order operator \(Q\) by
\[
Q \varphi = 2i(A^{\alpha\beta} \varphi_\alpha)_\beta.
\]
Note that \([\Delta_\theta, T] = 2 \text{Im} \, Q\) and \(P_0 = 2(\Delta_\theta^2 + n^2 T^2 - 2n \text{Re} \, Q)\).

Definition 1.5. Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. We can define a family of Webster (adapted) Riemannian metrics \(g_\varepsilon\) of \((M, J, \theta)\) by
\[
g_\varepsilon = d\theta + \varepsilon^{-2} \theta^2, \quad \varepsilon > 0.
\]
We first obtain the following lower bound estimate of diameter \(d_\varepsilon(M)\) with respect to an adapted metric \(g_\varepsilon\).

Proposition 1.2. Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. In addition we assume that the Paneitz operator \(P_0\) is nonnegative if \(n = 1\). Suppose that
\[
\text{Ric}_m(Z, Z) - \frac{n + 1}{2} \text{Tor}_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\theta}
\]
and
\[
[\text{Tor}_m + \frac{4(n + 1)}{k} A^2](Z, Z) \geq 0
\]
for all \(m \in M, \ Z \in T_{1,0}\), some positive constant \(k\) and
\[
A_{\alpha\gamma\tau} = 0 \quad \text{for all } \alpha.
\]
Assume that
\[
\lambda_1 = \frac{nk}{n + 1}.
\]
Then
\[
d_\varepsilon(M) \geq \sqrt{\frac{n + 1}{k}} \pi \quad \text{with} \quad \frac{n + 1}{k} = \varepsilon^2.
\]
Remark 1.2. (i) In a closed pseudohermitian \((2n + 1)\)-manifold with \(n \geq 2\), the corresponding condition
\[
P_\beta \varphi = 0
\]
is always satisfied under \(\text{Ric}_m(Z, Z) - \frac{n + 1}{2} \text{Tor}_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\theta}\) and \(\lambda_1 = \frac{nk}{n + 1}\). It is not true in the case of \(n = 1\).
(ii) Note that for $n = 1$,
\[
    \left[ Tor + \frac{8}{k} A^2 \right] ((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) = 2 \text{Im}(A_{11} \varphi \bar{\varphi}^\gamma) + \frac{4}{k} |A_{11}|^2 |\nabla_b \varphi|^2.
\]
Then from Lemma 3.4, condition (1.2) will be
\[
    \left[ Tor + \frac{8}{k} A^2 \right] ((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) = 0
\]
if $P_1 \varphi = 0$ for the corresponding eigenfunction $\varphi$.

(iii) For $n = 1$, if $\text{Ric}(Z, Z) \geq k \langle Z, Z \rangle_{L_0}$ and $\lambda_1 = \frac{k}{2}$ with $A_{11} = 0$, then from Lemma 3.2
\[
P_0 \varphi = 0
\]
and from Remark 1.1
\[
P_1 \varphi = 0.
\]

In the present paper, via the diameter estimate $d_\varepsilon(M)$ with respect to the Webster metric $g_\varepsilon$, we confirm Conjecture 1.1 in a closed pseudohermitian $(2n + 1)$-manifold with an extra condition on covariant derivatives of torsion.

**Theorem 1.3.** Let $(M, J, \theta)$ be a closed pseudohermitian $(2n + 1)$-manifold with $n \geq 2$. Suppose that
\[
    \text{Ric}(Z, Z) - \frac{n+1}{2} \text{Tor}(Z, Z) \geq k \langle Z, Z \rangle_{L_0},
\]
for all $m \in M, Z \in T_{1,0}$, some positive constant $k$ and
(1.3) $A_{\alpha\gamma,\bar{\gamma}} = 0; \ A_{\alpha\beta, \gamma} = 0$ for all $\alpha, \beta$.
Assume that
\[
    \lambda_1 = \frac{nk}{n+1}.
\]
Then
(1.4) $d_\varepsilon(M) = \sqrt{\frac{n+1}{k}} \pi$ with $\frac{n+1}{k} = \varepsilon^2$.
As a consequence, $(M, J, \theta)$ is the standard pseudohermitian $(2n + 1)$-sphere $(S^{2n+1}, \tilde{J}, \tilde{\theta})$.

However, regarding (iii) of Remark 1.2, we need an extra condition (1.6) to confirm Conjecture 1.1 in a closed pseudohermitian 3-manifold with nonvanishing torsion.

**Theorem 1.4.** Let $(M, J, \theta)$ be a closed pseudohermitian 3-manifold with the nonnegative Paneitz operator $P_0$. Suppose that
\[
    \text{Ric}(Z, Z) - \text{Tor}(Z, Z) \geq k \langle Z, Z \rangle_{L_0},
\]
for all $m \in M, Z \in T_{1,0}$, some positive constant $k$ and
(1.5) $A_{11, \bar{\tau}} = 0$.
Assume that
\[
    \lambda_1 = \frac{k}{2}
\]
and there exists one corresponding eigenfunction $\varphi$ satisfying
(1.6) $P_1 \varphi = 0$.  

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Then
\begin{equation}
(1.7) \quad d_\epsilon(M) = \sqrt{\frac{g}{k}} \pi \quad \text{with} \quad \frac{2}{k} = \epsilon^2.
\end{equation}
As a consequence, \((M, J, \theta)\) is the standard pseudohermitian sphere \((S^3, \hat{J}, \hat{\theta})\).

In particular, it follows from Remark 1.1 and (5.4) that conditions (1.5), (1.3) and (1.6) are satisfied in a closed pseudohermitian \((2n+1)\)-manifold with vanishing torsion. Then we recaptured our previous result for the CR Obata’s Theorem \((\text{CC2}, \text{CC3})\) as follows:

**Corollary 1.5.** Let \((M, J, \theta)\) be a closed pseudohermitian \((2n+1)\)-manifold with vanishing torsion. Suppose that
\[
\text{Ric}_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\alpha},
\]
for all \(m \in M\), \(Z \in T_{1,0}\) and some positive constant \(k\). Assume that
\[
\lambda_1 = \frac{nk}{n+1}.
\]
Then \((M, J, \theta)\) is the standard pseudohermitian \((2n+1)\)-sphere \((S^{2n+1}, \hat{J}, \hat{\theta})\).

2. **Preliminary**

Let us give a brief introduction of pseudohermitian geometry (see [Le] for more details). Let \((M, \xi)\) be a \((2n+1)\)-dimensional, orientable, contact manifold with contact structure \(\xi\), \(\dim_R \xi = 2n\). A CR structure compatible with \(\xi\) is an endomorphism \(J : \xi \to \xi\) such that \(J^2 = -1\). We also assume that \(J\) satisfies the following integrability condition: If \(X\) and \(Y\) are in \(\xi\), then so is \([JX, Y] + [X, JY]\), and \(J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]\). A CR structure \(J\) can extend to \(\mathbb{C} \otimes \xi\) and decomposes \(\mathbb{C} \otimes \xi\) into the direct sum of \(T_{1,0}\) and \(T_{0,1}\), which are eigenspaces of \(J\) with respect to \(i\) and \(-i\), respectively. A manifold \(M\) with a CR structure is called a CR manifold. A pseudohermitian structure compatible with \(\xi\) is a CR structure \(J\) compatible with \(\xi\) together with a choice of contact form \(\theta\). Such a choice determines a unique real vector field \(T\) transverse to \(\xi\), which is called the characteristic vector field of \(\theta\), such that \(\theta(T) = 1\) and \(L_T \theta = 0\) or \(d\theta(T, \cdot) = 0\). Let \(\{T, Z_\alpha, Z_{\bar{\alpha}}\}\) be a frame of \(TM \otimes \mathbb{C}\), where \(Z_\alpha\) is any local frame of \(T_{1,0}\), \(Z_{\bar{\alpha}} = Z_{\bar{\alpha}} \in T_{0,1}\) and \(T\) is the characteristic vector field. Then \(\{\theta, 2\alpha, \theta^3\}\), which is the coframe dual to \(\{T, Z_\alpha, Z_{\bar{\alpha}}\}\), satisfies
\begin{equation}
(2.1) \quad d\theta = ih_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}},
\end{equation}
for some positive definite hermitian matrix of functions \((h_{\alpha\bar{\beta}})\).

The Levi form \(h = \langle \cdot, \cdot \rangle_{L_\alpha}\) is the hermitian form on \(T_{1,0}\) defined by
\[
\langle Z, W \rangle_{L_\alpha} = -i \langle d\theta, Z \wedge W \rangle.
\]
We can extend \(\langle \cdot, \cdot \rangle_{L_\alpha}\) to \(T_{0,1}\) by defining \(\langle Z, W \rangle_{L_{\bar{\alpha}}} = \langle Z, W \rangle_{L_\alpha}\) for all \(Z, W \in T_{1,0}\). The Levi form naturally induces a hermitian form on the dual bundle of \(T_{1,0}\), denoted by \(\langle \cdot, \cdot \rangle_{L_{\bar{\alpha}}}\), and hence on all the induced tensor bundles.

Moreover, we can define a family of Webster (adapted) Riemannian metrics \(g_\epsilon\) of \((M, J, \theta)\) by
\[
g_\epsilon = (1/2)d\theta + \epsilon^{-2}\theta^2, \quad \epsilon > 0.
\]
The pseudohermitian connection of \((J, \theta)\) is the connection \(\nabla\) on \(TM \otimes \mathbb{C}\) (and extended to tensors) given in terms of a local frame \(Z_\alpha \in T_{1,0}\) by

\[
\nabla Z_\alpha = \theta^\beta_\alpha \otimes Z_\beta, \quad \nabla Z_\bar{\alpha} = \theta^\bar{\beta}_{\bar{\alpha}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,
\]

where \(\theta_{\alpha}^{\beta}\) are the 1-forms uniquely determined by the following equations:

\[
d\theta^\beta = \theta^\alpha_\beta + \theta \wedge \tau^\beta,  \\
0 = \tau_\alpha \wedge \theta^\alpha,  \\
0 = \theta^\beta_\alpha + \theta^{\bar{\beta}}_{\bar{\alpha}}.
\]  (2.2)

We can write (by the Cartan lemma) \(\tau_\alpha = A_{\alpha\gamma} \theta^\gamma\) with \(A_{\alpha\gamma} = A_{\gamma\alpha}\). The curvature of the Webster-Stanton connection, expressed in terms of the coframe \(\{\theta = \theta^0, \theta^\alpha, \theta^{\bar{\alpha}}\}\), is

\[
\Pi^\beta_\alpha = \Pi^\bar{\beta}_{\bar{\alpha}} = d\omega^\alpha_\beta - \omega^\beta_\gamma \wedge \omega^\gamma_\alpha,  \\
\Pi^0_\alpha = \Pi^0_{\bar{\alpha}} = \Pi^\bar{\beta}_{\bar{\alpha}} = 0.
\]

Webster showed that \(\Pi^\alpha_\alpha\) can be written as

\[
\Pi^\alpha_\alpha = R_{\beta\rho\bar{\sigma}}^\alpha \rho^\beta \wedge \theta^\rho + W_{\beta\rho\bar{\sigma}}^\alpha \rho^\beta \wedge \theta + W_{\bar{\alpha}\bar{\beta}\bar{\sigma}} \beta_\sigma \wedge \theta + i\theta_{\beta\sigma} \wedge \tau^\alpha - i\tau_{\beta\sigma} \wedge \theta^\alpha,
\]  (2.3)

where the coefficients satisfy

\[
R_{\beta\rho\bar{\sigma}}^\alpha = \overline{R_{\alpha\beta\rho\bar{\sigma}}}, \quad R_{\alpha\beta\rho\bar{\sigma}} = R_{\rho\bar{\sigma}\beta\alpha}, \quad W_{\beta\rho\bar{\sigma}}^\alpha = W_{\bar{\beta}\bar{\rho}\bar{\sigma}}^\alpha.
\]

We will denote components of covariant derivatives with indices preceded by a comma; thus we write \(A_{\alpha\beta\gamma}\). The indices \(\{0, \alpha, \bar{\alpha}\}\) indicate derivatives with respect to \(\{T, Z_\alpha, Z_{\bar{\alpha}}\}\). For derivatives of a scalar function, we will often omit the comma, for instance, \(f_\alpha = Z_\alpha f, \ f_{\bar{\alpha}} = Z_{\bar{\alpha}} f - \theta_{\bar{\alpha}} (Z_\beta f)\). \(f_0 = T f\) for a (smooth) function.

For a real function \(f\), the subgradient \(\nabla f\) is defined by \(\nabla f \in \xi\) and \((Z, \nabla f)_{L_0} = df(Z)\) for all vector fields \(Z\) tangent to the contact plane. Locally

\[
\nabla f = \sum f_\alpha Z_\alpha + f_\alpha Z_{\bar{\alpha}}.
\]

We can use the connection to define the subhessian as the complex linear map

\[(\nabla H)^2 f : T_{1,0} \oplus T_{0,1} \rightarrow T_{1,0} \oplus T_{0,1}\]

by

\[(\nabla H)^2 f(Z) = \nabla Z \nabla b f.
\]

Also,

\[
\Delta b f = \text{Tr} ((\nabla H)^2 f) = \sum_a (f_{a\bar{a}} + f_{\bar{a}a}).
\]

The Webster-Ricci tensor and the torsion tensor on \(T_{1,0}\) are defined by

\[
\text{Ric}(X, Y) = R_{\alpha\beta} X^\alpha Y^\beta
\]

and

\[
\text{Torr}(X, Y) = i \sum_{\alpha\beta} (A_{\alpha\beta} X^\alpha Y^\beta - A_{\beta\alpha} X^\beta Y^\alpha).
\]

We also define the following tensor \(A^2\) as

\[
A^2(X, Y) = \sum_{\gamma} (A_{\alpha\gamma} A_{\beta\bar{\gamma}} X^\alpha Y^\beta),
\]

where \(X = X^\alpha Z_\alpha, \ Y = Y^\beta Z_{\bar{\beta}}, \ R_{\alpha\beta} = R_{\gamma\beta} \gamma_{\alpha\bar{\beta}}\). The Webster scalar curvature is \(R = R_{\alpha}^\alpha = h^{\alpha\beta} R_{\alpha\beta}\).
3. THE DIAMETER ESTIMATES

Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. In this section, we will obtain a lower bound estimate for the diameter of \(M\) when a sharp lower estimate for the first positive eigenvalue of the sublaplacian on a pseudohermitian \((2n + 1)\)-manifold \(M\) is achieved.

First we recall the following CR Bochner formula which involves the CR Paneitz operator.

**Lemma 3.1** (**CC2**). Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. For a (smooth) real function \(\varphi\) on \(M\), we have

\[
\frac{1}{2} \Delta_b |\nabla_b \varphi|^2 = |(\nabla^H \varphi)^2| + \left(1 + \frac{2}{n}\right) \langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle_{L_\theta} 
+ 2\text{Ric} - (n + 2)\text{Tor}\langle (\nabla_b \varphi)_C, (\nabla_b \varphi)_C \rangle 
- \frac{4}{n} \langle P\varphi + \nabla \varphi, d_b \varphi \rangle_{L^*_\theta},
\]

where \((\nabla_b \varphi)_C = \sum_\alpha \varphi_\alpha \sigma_\alpha\) is the corresponding complex \((1, 0)\)-vector field of \(\nabla_b \varphi\) and \(d_b \varphi = \varphi_\alpha \sigma_\alpha + \varphi_\beta \sigma_\beta\).

Recall that formula (3.1) follows from the two identities:

\[
\frac{1}{2} \Delta_b |\nabla_b \varphi|^2 = |(\nabla^H \varphi)^2| + \langle (\nabla_b \varphi)_C, (\nabla_b \varphi)_C \rangle 
+ 2\langle J\nabla_b \varphi, \nabla_b \varphi_0 \rangle_{L_\theta} 
- 2\langle P\varphi + \nabla \varphi, d_b \varphi \rangle_{L^*_\theta}.
\]

We also need the following lemma which follows from an observation as in the proof of the sharp lower bound estimate for the first positive eigenvalue \(\lambda_1\) of the sublaplacian \(\Delta_b\) on a closed pseudohermitian 3-manifold in [Chi].

Let \(\varphi\) be the corresponding eigenfunction of the sublaplacian \(\Delta_b\) with respect to \(\lambda_1\). That is, \(\Delta_b \varphi = -\lambda_1 \varphi\).

**Lemma 3.2.** Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. Suppose that

\[
\text{Ric}_m(Z, Z) - \frac{n + 1}{2} \text{Tor}_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\theta},
\]

for all \(m \in M\), \(Z \in T_{1,0}\), some positive constant \(k\) and

\[
\lambda_1 = \frac{nk}{n + 1}.
\]

In addition, we assume that \(P_0\) is nonnegative if \(n = 1\). Then the corresponding eigenfunction \(\varphi\) will satisfy

\[
P_0 \varphi = 0
\]

and

\[
\varphi_{\alpha\beta} = 0, \quad \varphi_{\alpha\beta} = 0 \text{ with } \alpha \neq \beta, \quad \varphi_{\alpha\beta} = \varphi_{\beta\alpha}.
\]
Moreover, we have

\begin{equation}
[Ric_m - \frac{n+1}{2} Tor_m]((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) = k \langle (\nabla_b \varphi)_C, (\nabla_b \varphi)_C \rangle_{L_\alpha}.
\end{equation}

**Proof.** By integrating the two identities (3.2) and (3.3), we obtain

\begin{equation}
\int |(\nabla^H)^2 \varphi|^2 d\mu = \int (\Delta_b \varphi)^2 d\mu + 2 \int \varphi_0^2 d\mu
\end{equation}

and

\begin{equation}
\int \varphi_0^2 d\mu = \frac{1}{n} \int (\Delta_b \varphi)^2 d\mu + \frac{2}{n} \int Tor((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) d\mu
\end{equation}

since \( |(\nabla^H)^2 \varphi|^2 = 2 \sum_{\alpha, \beta} |\varphi_{\alpha \beta}|^2 + 2 \sum_{\alpha, \beta} |\varphi_{\alpha \pi}|^2 \). Substituting this on the left-hand side of (3.7) and combining with (3.8), we get

\[-2 \int \sum_{\alpha, \beta} |\varphi_{\alpha \beta}|^2 d\mu \geq -\frac{n+1}{n} \int (\Delta_b \varphi)^2 d\mu + \frac{3}{4n} \int P_0 \varphi \cdot \varphi d\mu + \int [2 Ric - (n+1) Tor]((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) d\mu \geq \left( k - \frac{n+1}{n} \lambda_1 \right) \int |\nabla_b \varphi|^2 d\mu,\]

where we have used the Cauchy-Schwarz inequalities

\[\sum_{\alpha, \beta} |\varphi_{\alpha \beta}|^2 \geq \sum_{\alpha} |\varphi_{\alpha \pi}|^2 \geq \frac{1}{n} \left| \sum_{\alpha} \varphi_{\alpha \pi} \right|^2 = \frac{1}{4n} (\Delta_b \varphi)^2 + \frac{n}{4} \varphi_0^2,\]


\[Re \left( \frac{n+1}{2} Tor((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) \right) \geq k \langle (\nabla_b \varphi)_C, (\nabla_b \varphi)_C \rangle_{L_\alpha},\]

and

\[\int P_0 \varphi \cdot \varphi d\mu \geq 0.\]

It follows that \( k - \frac{n+1}{n} \lambda_1 \leq 0 \), i.e. \( \lambda_1 \geq \frac{n k}{n+1} \). In addition, if \( \lambda_1 = \frac{n k}{n+1} \), then all of the above inequalities will become equalities and the results will easily follow. \( \square \)

**Lemma 3.3.** Let \((M, J, \theta)\) be a closed pseudohermitian \((2n+1)\)-manifold with vanishing torsion. Suppose that

\[Ric_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\alpha}\]

for all \( m \in M \), \( Z \in T_{1, 0} \) and some positive constant \( k \). Then

\begin{equation}
d_\varepsilon(M) \leq \sqrt{\frac{n+1}{k}} \pi \quad \text{with} \quad \frac{n+1}{k} = \varepsilon^2.
\end{equation}

**Proof.** Since \( Ric_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\alpha} \), we have (CC2)

\begin{equation}
(R^c_{ab}) = \begin{bmatrix} A & B & 0 \\ B^t & A & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -2\varepsilon^{-2}I_{2n \times 2n} & 0 \\ 0 & 2\varepsilon^{-2} \end{bmatrix}.
\end{equation}
Here $A = (2 \Re R_{\alpha \beta})$, $B = (2 \Im R_{\alpha \beta})$, $A + iB = [2R_{\alpha \beta}]$ and $g_\epsilon = d\theta + \epsilon^{-2}\theta^2$ with $\frac{n+1}{k} = \epsilon^2$. It follows that

$$ (3.11) \quad \text{Ric}_g(X, X) \geq [(2n + 1) - 1]\epsilon^{-2} |X|^2. $$

It follows from Myers’ theorem that

$$ (3.12) \quad d_\epsilon(M) \leq \sqrt{\frac{n+1}{k}} \pi. \quad \Box $$

Proof of Proposition 1.2. Let $\varphi$ be a nonconstant eigenfunction of the sublaplacian $\Delta_b$ satisfying

$$ \Delta_b \varphi = -\lambda_1 \varphi. $$

Consider the smooth function

$$ F = \frac{1}{2} |\nabla_b \varphi|^2 + \frac{\lambda_1}{4n} \varphi^2 + \frac{n}{4\lambda_1} \varphi_0^2 $$

defined on $M$. Computing its sublaplacian

$$ (3.13) \quad \Delta_b F = |(\nabla^H \varphi)|^2 + 2\text{Ric} - (n + 2)\text{Tor}[(\nabla_b \varphi)_C, (\nabla_b \varphi)_C] $$

$$ + \frac{\lambda_1}{2n} |\nabla_b \varphi|^2 + \frac{\lambda_1}{2n} \varphi \Delta_b \varphi + \frac{n}{4\lambda_1} \Delta_b \varphi_0^2, $$

where we have used the CR Bochner formula (3.1) with $P_{\beta} \varphi = 0$ (by the assumption for $n = 1$) from Lemma 3.2 and Remark 1.1, since $\lambda_1 = \frac{n}{n+1}$. On the other hand, due to (3.5) we have (\[CTW\])

$$ (3.14) \quad |(\nabla^H \varphi)|^2 = \frac{1}{2n} (\Delta_b \varphi)^2 + \frac{n}{2} \varphi_0^2, $$

and since $[\Delta_b, T] = 2\text{Im}Q$ with the purely holomorphic second-order operator $Q \varphi = 2i(A^{\alpha \beta} \varphi_\alpha)_\beta$ (\[GL2\])

$$ (3.15) \quad \Delta_b \varphi_0^2 = 2\varphi_0 \Delta_b \varphi_0 + 2|\nabla_b \varphi_0|^2 $$

$$ = -2\lambda_1 \varphi_0^2 + 2|\nabla_b \varphi_0|^2 + 4\text{Im}(Q \varphi) \varphi_0. $$

Substituting (3.14) and (3.15) into (3.13), one obtains

$$ \Delta_b F = \left( k - \frac{2n + 3}{2n} \lambda_1 \right) |\nabla_b \varphi|^2 - \text{Tor}((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) $$

$$ + \frac{n}{2\lambda_1} |\nabla_b \varphi_0|^2 + \frac{n}{\lambda_1} \text{Im}(Q \varphi) \varphi_0. $$

On the other hand, from Lemma 3.2 we have

$$ Q \varphi = 0 $$

if $A^{\alpha \beta, \beta} = 0$. Hence

$$ (3.16) \quad \Delta_b F = \left( k - \frac{2n + 3}{2n} \lambda_1 \right) |\nabla_b \varphi|^2 - \text{Tor}((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) + \frac{n}{2\lambda_1} |\nabla_b \varphi_0|^2. $$

Next we compute the term $|\nabla_b \varphi_0|^2$. Since $\lambda_1 = \frac{n}{n+1}$, we have

$$ P_{\bar{\beta}} \varphi = \varphi \bar{\beta} + inA_{\beta \alpha} \varphi = 0 $$

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for all $\beta$ and

\[ -2\varphi_0 = \Box_b \varphi = -\Delta_b \varphi + in\varphi_0 = \lambda_1 \varphi + in\varphi_0, \]

thus

\[ inA_{\beta\alpha} \varphi^\alpha = -\varphi_0 = \frac{1}{2} (\lambda_1 \varphi + in\varphi_0)_{\beta}. \]

It follows that

\[ \varphi_0 = 2A_{\beta\alpha} \varphi^\alpha + i\frac{\lambda_1}{n} \varphi_0. \]

Substituting (3.18) into (3.16), we conclude that

\[ \Delta_b F = \left( k - \frac{n+1}{n} \lambda_1 \right) |\nabla_b \varphi|^2 + Tor((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) + \frac{4n}{\lambda_1} A^2 ((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) \]

\[ = \left[ Tor + \frac{4n}{\lambda_1} A^2 \right] ((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) \]

\[ = \left[ Tor + \frac{4(n+1)}{k} A^2 \right] ((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) \]

\[ \geq 0. \]

Since $M$ is closed, $F$ must be identically constant. Moreover,

\[ \frac{1}{2} |\nabla_b \varphi|^2 + \frac{\lambda_1}{4n} \varphi^2 + \frac{n}{4\lambda_1} \varphi_0^2 = \frac{\lambda_1}{4n} |\varphi|_\infty^2, \]

where $|\varphi|_\infty = \sup_M |\varphi|$. If we normalize $\varphi$ such that $|\varphi|_\infty = 1$ and observe that at the maximum and minimum points of $\varphi$ its subgradient and derivative with respect to $T$ must vanish, then we obtain that $\max \varphi = 1 = -\min \varphi$ and so

\[ |\nabla_b \varphi|^2 + \frac{\lambda_1}{2n} \varphi^2 + \frac{n}{2\lambda_1} \varphi_0^2 = \frac{\lambda_1}{2n}. \]

Hence

\[ \frac{k}{2(n+1)} = \frac{|\nabla_b \varphi|^2 + \frac{n+1}{2k} \varphi_0^2}{1 - \varphi^2}. \]

Now let $\nabla^\varepsilon$ be the gradient with respect to $g^\varepsilon$. We have

\[ |\nabla^\varepsilon \varphi|^2 = 2|\nabla_b \varphi|^2 + \varepsilon^2 \varphi_0^2 = 2|\nabla_b \varphi|^2 + \frac{n+1}{k} \varphi_0^2 \]

with $\varepsilon^2 = \frac{n+1}{k}$. Thus

\[ |\nabla_b \varphi|^2 + \frac{n+1}{2k} \varphi_0^2 = \frac{1}{2} |\nabla^\varepsilon \varphi|^2, \]
and so
\[ \frac{k}{n+1} = \frac{|\nabla^c \varphi|^2}{1 - \varphi^2}. \]
Integrating this along a minimal geodesic \( \gamma \) with respect to \( g_\varepsilon \) joining the points where \( \varphi = 1 \) and \( \varphi = -1 \), we have
\[ d_\varepsilon \sqrt{k/n+1} \geq \int_\gamma \sqrt{\frac{k}{n+1}} \sqrt{1 - \varphi^2} ds = \int^{1}_{-1} \frac{d\varphi}{\sqrt{1 - \varphi^2}} = \pi, \]
which implies that
\[ d_\varepsilon \geq \sqrt{\frac{n+1}{k}} \pi = \varepsilon \pi. \]

\[ \square \]

**Lemma 3.4.** Let \((M, J, \theta)\) be a closed pseudohermitian 3-manifold with nonnegative CR Paneitz operator \( P_0 \). Suppose that the curvature condition \((1.1)\) holds, \( \lambda_1 = \frac{k}{n} \) and \( A_{111} = 0 \). Let \( \varphi \) be the corresponding eigenfunction of the sublaplacian \( \Delta_b \) with respect to \( \lambda_1 \) and \( P_1 \varphi = 0 \) on \( M \). Then
\[ |A_{111}|^2 |\nabla_b \varphi|^2 = 0 \]
on \( M \). In particular, the torsion must be vanishing
\[ A_{111} = 0 \]
on \((M, J, \theta)\).

**Remark 3.1.** For generic metrics in a closed Riemannian manifold, all of the eigenfunctions are Morse functions, and consequently their critical point sets are discrete (\([\mathbb{U}]\)).

**Proof.** From Lemma 3.2, we have \( P_0 \varphi = 8(P_1 \varphi)_T = 0 \), \( \varphi_{11} = 0 \), and thus
\[ \int (P_1 \varphi)(P_1 \varphi) d\mu = \int (\varphi_{111} + iA_{111} \varphi_T)(\varphi_{11} - iA_{11T} \varphi_1) d\mu = \int (\varphi_{111} + iA_{111} \varphi_T)(\varphi_{11} + iA_{11T} \varphi_1) + \frac{1}{2} |A_{111}|^2 |\nabla_b \varphi|^2 d\mu = \frac{1}{2} \int |A_{111}|^2 |\nabla_b \varphi|^2 d\mu, \]
which immediately implies the first part of the lemma.

On the other hand, it follows from \((3.19)\) that \( A_{111} = 0 \) on the complement of the critical point set \( \Sigma \) of \( \varphi \). Then the torsion is vanishing on \((M, J, \theta)\) by continuity property if \( \Sigma \) is at most co-dimension one.

More precisely, \( \Sigma_0 = \Sigma \cap \{ p : \varphi(p) = 0 \} \) is at most of codimension 2 \((\mathbb{HHL})\) and \( \Sigma \setminus \Sigma_0 \) is at most of codimension one. In fact, since \( \Delta_b \varphi = -\lambda_1 \varphi \), then for \( p \in \Sigma \setminus \Sigma_0 \)
\[ \varphi_{1T}(p) \neq 0 \quad \text{and} \quad \varphi_{T1}(p) \neq 0. \]
Define
\[ f = \varphi_1 + \varphi_T. \]
We have
\[ f(p) = 0. \]
Hence
\[ \Sigma \setminus \Sigma_0 \subset \{ f = 0 \}. \]

But from (3.3), for \( p \in \Sigma \setminus \Sigma_0 \),
\[ f_1(p) = \varphi_{11}(p) + \varphi_{T_1}(p) = \varphi_{T_1}(p) \neq 0, \]
and then
\[ \nabla_b f \neq 0 \]
on \( \Sigma \setminus \Sigma_0 \).

It follows that \( \Sigma \setminus \Sigma_0 \) is at most of codimension one. All these imply \( \Sigma \) is at most of codimension one. \( \square \)

Lemma 3.5. Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold with \( n \geq 2 \). Suppose that
\[ \text{Ric}_m(Z, Z) - \frac{n+1}{2} \text{Tor}_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\theta}, \]
for all \( m \in M, Z \in T_{1,0} \) and
\[ A_{\alpha \gamma \tau} = 0; \quad A_{\alpha \beta \gamma \tau} = 0 \quad \text{for all } \alpha, \beta. \]

Assume that
\[ \lambda_1 = -\frac{nk}{n+1}. \]

Then
\[ A_{\alpha \beta} = 0 \]
on \((M, J, \theta)\).

Proof. From Lemma 3.2, we have \( \beta_i = 0, P_0 \varphi = 8(P_\beta \varphi)_\tau = 0, \varphi_{\alpha \beta} = 0 \), and the similar argument
\[ 0 = \int (P_\beta \varphi)(\overline{P_\beta} \varphi) d\mu = \int (\varphi_{\alpha \beta} + in A_{\beta \alpha} \varphi_{\tau} - i A_{\beta \alpha} \varphi_{\tau}) d\mu \]
\[ = \int [-P_\beta \varphi]_\tau \varphi_{\gamma \tau} + (in A_{\beta \alpha} \varphi_{\tau})_\beta \varphi_{\alpha} + n^2 A_{\beta \alpha} \overline{A_{\beta \alpha}} \varphi_{\gamma \tau} d\mu \]
\[ = n^2 \int A_{\beta \alpha} A_{\beta \alpha} \varphi_{\gamma \tau} \varphi_{\gamma} d\mu = n^2 \int A^2((\nabla_b \varphi)_\xi, (\overline{\nabla_b} \varphi)_\xi) d\mu. \]

Hence from \( \varphi_{\alpha \beta} = 0 \) with \( \alpha \neq \beta, \varphi_{\alpha \pi} = \varphi_{\beta \pi} \) and \( A_{\beta \alpha, \pi} = 0 \), we have
\[ 0 = \int A_{\beta \alpha} A_{\beta \alpha} \varphi_{\gamma \tau} \varphi_{\gamma} d\mu \]
\[ = -\int A_{\beta \alpha} A_{\beta \alpha} \varphi_{\gamma \tau} \varphi_{\gamma} d\mu - \int A_{\beta \alpha} A_{\beta \alpha} \varphi_{\gamma \tau} \varphi_{\gamma} d\mu \]
\[ = -\int A_{\beta \alpha} A_{\beta \alpha} \varphi_{\gamma \tau} \varphi_{\gamma} d\mu - \int A_{\beta \alpha} A_{\beta \alpha} \varphi_{\gamma \tau} \varphi_{\gamma} d\mu \]
\[ = -\int A_{\beta \alpha} A_{\beta \alpha} \varphi_{\gamma \tau} \varphi_{\gamma} d\mu - \int |A_{\beta \alpha}|^2 \varphi_{1T} \varphi d\mu. \]

On the other hand, from (3.34) we have
\[ -\varphi_{1T} = \frac{1}{2n} \lambda_1 \varphi - \frac{1}{2} i \varphi_0. \]
Then
\[(3.20) \quad 0 = -2 \int A_{\alpha \beta} A_{\gamma} \beta \gamma d\mu + \frac{1}{n} \lambda_1 \int |A_{\alpha \beta}|^2 \varphi^2 d\mu - i \int |A_{\alpha \beta}|^2 \varphi_0 \varphi d\mu.\]

We compute, by the commutation relation $A_{\beta \gamma}, \alpha = A_{\beta \alpha}, \gamma$,
\[ -2 \int A_{\beta \alpha} A_{\gamma} \beta \gamma \varphi d\mu = - \int A_{\beta \alpha} A_{\gamma} \beta \gamma (\varphi^2) \varphi d\mu = \int A_{\beta \alpha} A_{\gamma} \beta \gamma \varphi^2 d\mu + \int A_{\beta \alpha} A_{\gamma} \beta \gamma \varphi^2 d\mu = \int |A_{\beta \alpha} \gamma|^2 \varphi^2 d\mu + \int A_{\beta \alpha} A_{\gamma} \beta \gamma \varphi^2 d\mu, \]
and substitute it into (3.20):
\[(3.21) \quad 0 = \frac{1}{n} \lambda_1 \int |A_{\alpha \beta}|^2 \varphi^2 d\mu + \int |A_{\beta \alpha} \gamma| \varphi^2 d\mu - i \int |A_{\alpha \beta}|^2 \varphi_0 \varphi d\mu. \]

But $A_{\beta \alpha} \gamma = 0$ and $\varphi_0$ is real.
Thus
\[ \frac{1}{n} \lambda_1 \int |A_{\alpha \beta}|^2 \varphi^2 d\mu + \int |A_{\beta \alpha} \gamma| \varphi^2 d\mu = 0. \]
Hence
\[ |A_{\alpha \beta}|^2 \varphi^2 = 0. \]

Now apply the same method as in the proof of Lemma 3.4. We have
\[ |A_{\alpha \beta}| = 0 \]
on $(M, J, \theta)$.

Proof of Theorem 1.4. It follows from Lemma 3.3, 3.4 and Proposition 1.2 that
\[ d_\varepsilon (M) \geq \sqrt{\frac{2}{k}} \pi \]
with
\[ \text{Ric}_{g_\varepsilon} (X, X) \geq 2 \varepsilon^{-2} |X|^2 = 2 \times \frac{k}{2} |X|^2 \]
on $(M^3, g_\varepsilon)$. Here $g_\varepsilon = d\theta + \varepsilon^{-2} \theta^2$ with $\frac{2}{k} = \varepsilon^2$. Hence from Lemma 3.4 and Lemma 3.3
\[ d_\varepsilon (M) \leq \sqrt{\frac{2}{k}} \pi. \]
All these imply
\[ d_\varepsilon (M) = \sqrt{\frac{2}{k}} \pi. \]
By applying S.-Y. Cheng’s theorem ([Ch]), it follows that $(M, J, \theta)$ is the standard pseudohermitian sphere $(S^3, \hat{J}, \hat{\theta})$. □

Proof of Theorem 1.3. It follows from Proposition 1.2, Lemma 3.5 and Lemma 3.3.
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References


[Li] P. Li, Lecture notes on Geometric Analysis, 1996.


Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan, Republic of China – and – Taida Institute for Mathematical Sciences (TIMS), National Taiwan University, Taipei 10617, Taiwan, Republic of China

E-mail address: scchang@math.ntu.edu.tw

Department of Applied Mathematics, National PingTung University of Education, PingTung, Taiwan 90003, Republic of China

E-mail address: ctwu@mail.npue.edu.tw