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THE DIAMETER ESTIMATE AND ITS APPLICATION TO CR OBATA'S THEOREM ON CLOSED PSEUDOHERMITIAN $(2n+1)$-MANIFOLDS

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Abstract. In this paper, we obtain a sharp lower bound estimate for diameters with respect to an adapted metric in closed pseudohermitian $(2n+1)$-manifolds when a sharp lower bound estimate for the first positive eigenvalue of the sublaplacian is achieved. As a consequence, we confirm the CR Obata Conjecture on a closed pseudohermitian $(2n+1)$-manifold with an extra condition on covariant derivatives of torsion.

1. Introduction

Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold (see the next section for basic notions in pseudohermitian geometry). A. Greenleaf ([Gr]) proved the pseudohermitian analogue of Lichnerowicz's Theorem ([L]) for the lower bound of the first positive eigenvalue λ_1 of the sublaplacian for a pseudohermitian manifold M^{2n+1} with $n \geq 3$. In [LL], S.-Y. Li and H.-S. Luk proved the same result for the cases $n = 2$ and $n = 1$ with an extra condition on a covariant derivative of the pseudohermitian torsion.

Recently, it was proved by H.-L. Chiu and the first author ([Chi], [CC1]) that

$$\lambda_1 \geq \frac{nk}{n+1}$$

if

$$Ric_m(Z,Z) - \frac{n+1}{2} Tor_m(Z,Z) \geq k \langle Z, Z \rangle_{L_\theta}, \tag{1.1}$$

for all $m \in M$, $Z \in T_{1,0}$, and for some positive constant k, on a closed pseudohermitian $(2n+1)$-manifold with the nonnegative CR Paneitz operator P_0 if $n = 1$.

It is well known that

$$\lambda_1 = \frac{nk}{n+1}$$

on the standard unit sphere $(S^{2n+1}, \widehat{J}, \widehat{\theta})$ in $\mathbb{C}^{n+1}$ with

$$Ric_m(Z,Z) = k \langle Z, Z \rangle_{L_\theta}.$$

Then it is natural to conjecture the CR analogue of M. Obata's Theorem ([O]) on a closed pseudohermitian $(2n+1)$-manifold (M, J, θ).

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Conjecture 1.1. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. In addition we assume the Paneitz operator P_0 is nonnegative if $n = 1$. Suppose that*

$$Ric_m(Z, Z) - \frac{n+1}{2} Tor_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\theta},$$

for all $m \in M$, $Z \in T_{1,0}$, and for some positive constant k. Assume that

$$\lambda_1 = \frac{nk}{n+1}.$$

Then (M, J, θ) is the standard pseudohermitian $(2n+1)$-sphere $(S^{2n+1}, \widehat{J}, \widehat{\theta})$ with $Ric_m(Z, Z) = k \langle Z, Z \rangle_{L_\theta}$.

We first recall some basic definitions.

Definition 1.1. A piecewise smooth curve $\gamma : [0, 1] \to M$ is said to be horizontal if $\gamma\,'(t) \in \xi$ whenever $\gamma\,'(t)$ exists. The length of γ is then defined by

$$l(\gamma) = \int_0^1 \langle \gamma\,'(t), \gamma\,'(t) \rangle_{L_\theta}^{\frac{1}{2}} \, dt.$$

The Carnot-Carathéodory distance between two points $p,\ q \in M$ is

$$d_c(p, q) = \inf \{ l(\gamma) |\ \gamma \in C_{p,q} \},$$

where $C_{p,q}$ is the set of all horizontal curves joining p and q. By the Chow connectivity theorem [Cho], there always exists a horizontal curve joining p and q, so the distance is finite. The diameter d_c is defined by

$$d_c(M) = \sup \{ d_c(p, q) |\ p, q \in M \}.$$

Note that there is a minimizing geodesic joining p and q so that its length is equal to the distance $d_c(p, q)$.

Definition 1.2. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Define

$$P\varphi = \textstyle\sum_{\alpha=1}^n (\varphi_{\overline{\alpha}}{}^{\overline{\alpha}}{}_\beta + inA_{\beta\alpha}\varphi^\alpha)\theta^\beta = (P_\beta\varphi)\theta^\beta, \quad \beta = 1, 2, \cdots, n,$$

which is an operator that characterizes CR-pluriharmonic functions ([Le] for $n = 1$ and [GL] for $n \geq 2$). Here

$$P_\beta\varphi = \textstyle\sum_{\alpha=1}^n (\varphi_{\overline{\alpha}}{}^{\overline{\alpha}}{}_\beta + inA_{\beta\alpha}\varphi^\alpha)$$

and $\overline{P}\varphi = (\overline{P}_\beta\varphi)\theta^{\overline{\beta}}$, the conjugate of P. Moreover, we define

$$P_0\varphi = 4 \left(\delta_b(P\varphi) + \overline{\delta}_b(\overline{P}\varphi) \right),$$

which is the so-called CR Paneitz operator P_0. Here δ_b is the divergence operator that takes $(1, 0)$-forms to functions by $\delta_b(\sigma_\alpha\theta^\alpha) = \sigma_{\alpha,}{}^\alpha$ and $\overline{\delta}_b(\sigma_{\overline{\alpha}}\theta^{\overline{\alpha}}) = \sigma_{\overline{\alpha}}{}_,{}^{\overline{\alpha}}$. If we define $\partial_b\varphi = \varphi_\alpha\theta^\alpha$ and $\overline{\partial}_b\varphi = \varphi_{\overline{\alpha}}\theta^{\overline{\alpha}}$, then the formal adjoint of ∂_b on functions (with respect to the Levi form and the volume form $d\mu$) is $\partial_b^* = -\delta_b$. We observe that

$$\int \langle P\varphi + \overline{P}\varphi, d_b\varphi \rangle_{L_\theta^*} d\mu = -\frac{1}{4} \int P_0\varphi \cdot \varphi d\mu.$$

For the details about these operators, the reader can make reference to [GL] and [Hi].

Definition 1.3. On a closed pseudohermitian $(2n+1)$-manifold (M, J, θ), we call the Paneitz operator P_0 with respect to (J, θ) nonnegative if

$$\int_M P_0\varphi \cdot \varphi d\mu \geq 0$$

for all real C^∞ smooth functions φ.

Remark 1.1. (i) Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$. Then a smooth real-valued function φ satisfies $P_0\varphi = 0$ on M if and only if $P_\beta\varphi = 0$ on M for all β. It also holds for a closed pseudohermitian 3-manifold with vanishing torsion ([Hi], [GL]).

(ii) Let (M, J, θ) be a closed pseudohermitian 3-manifold with vanishing torsion. Then the corresponding CR Paneitz operator is nonnegative. Unlike $n = 1$, let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$. The corresponding CR Paneitz operator is always nonnegative ([CCC], [CC2]).

Definition 1.4 ([GL])**.** Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. We define the purely holomorphic second-order operator Q by

$$Q\varphi = 2i(A^{\alpha\beta}\varphi_\alpha)_{,\beta}\,.$$

Note that $[\Delta_b, T] = 2\,\mathrm{Im}\,Q$ and $P_0 = 2\left(\Delta_b^2 + n^2T^2 - 2n\,\mathrm{Re}\,Q\right)$.

Definition 1.5. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. We can define a family of Webster (adapted) Riemannian metrics g_ε of (M, J, θ) by

$$g_\varepsilon = d\theta + \varepsilon^{-2}\theta^2,\ \varepsilon > 0.$$

We first obtain the following lower bound estimate of diameter $d_\varepsilon(M)$ with respect to an adapted metric g_ε.

Proposition 1.2. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. In addition we assume that the Paneitz operator P_0 is nonnegative if $n = 1$. Suppose that*

$$Ric_m(Z, Z) - \frac{n+1}{2}Tor_m(Z, Z) \geq k\,\langle Z, Z\rangle_{L_\theta}$$

and

$$\left[Tor_m + \frac{4(n+1)}{k}A^2\right](Z, Z) \geq 0 \tag{1.2}$$

for all $m \in M$, $Z \in T_{1,0}$, some positive constant k and

$$A_{\alpha\gamma,\overline{\gamma}} = 0 \quad \textit{for all } \alpha.$$

Assume that

$$\lambda_1 = \frac{nk}{n+1}.$$

Then

$$d_\varepsilon(M) \geq \sqrt{\frac{n+1}{k}}\pi \quad \textit{with} \quad \frac{n+1}{k} = \varepsilon^2.$$

Remark 1.2. (i) In a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$, the corresponding condition

$$P_\beta\varphi = 0$$

is always satisfied under $Ric_m(Z, Z) - \frac{n+1}{2}Tor_m(Z, Z) \geq k\,\langle Z, Z\rangle_{L_\theta}$ and $\lambda_1 = \frac{nk}{n+1}$. It is not true in the case of $n = 1$.

(ii) Note that for $n = 1$,

$$\left[Tor + \frac{8}{k}A^2\right]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) = 2\,\mathrm{Im}(A_{11}\varphi_{\overline{1}}\varphi_{\overline{1}}) + \frac{4}{k}|A_{11}|^2\,|\nabla_b\varphi|^2\,.$$

Then from Lemma 3.4, condition (1.2) will be

$$\left[Tor + \frac{8}{k}A^2\right]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) = 0$$

if $P_1\varphi = 0$ for the corresponding eigenfunction φ.

(iii) For $n = 1$, if $Ric_m(Z, Z) \geq k\,\langle Z, Z\rangle_{L_\theta}$ and $\lambda_1 = \frac{k}{2}$ with $A_{11} = 0$, then from Lemma 3.2

$$P_0\varphi = 0$$

and from Remark 1.1

$$P_1\varphi = 0.$$

In the present paper, via the diameter estimate $d_\varepsilon(M)$ with respect to the Webster metric g_ε, we confirm Conjecture 1.1 in a closed pseudohermitian $(2n+1)$-manifold with an extra condition on covariant derivatives of torsion.

Theorem 1.3. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$. Suppose that*

$$Ric_m(Z, Z) - \frac{n+1}{2}Tor_m(Z, Z) \geq k\,\langle Z, Z\rangle_{L_\theta}\,,$$

for all $m \in M$, $Z \in T_{1,0}$, some positive constant k and

$$A_{\alpha\gamma,\overline{\gamma}} = 0; \quad A_{\alpha\beta,\gamma\overline{\gamma}} = 0 \quad \text{for all } \alpha, \beta. \tag{1.3}$$

Assume that

$$\lambda_1 = \frac{nk}{n+1}.$$

Then

$$d_\varepsilon(M) = \sqrt{\frac{n+1}{k}}\pi \quad \text{with} \quad \frac{n+1}{k} = \varepsilon^2. \tag{1.4}$$

As a consequence, (M, J, θ) is the standard pseudohermitian $(2n+1)$-sphere $(S^{2n+1}, \widehat{J}, \widehat{\theta})$.

However, regarding (iii) of Remark 1.2, we need an extra condition (1.6) to confirm Conjecture 1.1 in a closed pseudohermitian 3-manifold with nonvanishing torsion.

Theorem 1.4. *Let (M, J, θ) be a closed pseudohermitian 3-manifold with the nonnegative Paneitz operator P_0. Suppose that*

$$Ric_m(Z, Z) - Tor_m(Z, Z) \geq k\,\langle Z, Z\rangle_{L_\theta}\,,$$

for all $m \in M$, $Z \in T_{1,0}$, some positive constant k and

$$A_{11,\overline{1}} = 0. \tag{1.5}$$

Assume that

$$\lambda_1 = \frac{k}{2}$$

and there exists one corresponding eigenfunction φ satisfying

$$P_1\varphi = 0. \tag{1.6}$$

Then

$$d_\varepsilon(M) = \sqrt{\frac{2}{k}}\pi \quad \textit{with} \quad \frac{2}{k} = \varepsilon^2. \tag{1.7}$$

As a consequence, (M, J, θ) *is the standard pseudohermitian sphere* $(S^3, \widehat{J}, \widehat{\theta})$.

In particular, it follows from Remark 1.1 and (3.4) that conditions (1.5), (1.3) and (1.6) are satisfied in a closed pseudohermitian $(2n+1)$-manifold with vanishing torsion. Then we recaptured our previous result for the CR Obata's Theorem ([CC2, CC3]) as follows:

Corollary 1.5. *Let* (M, J, θ) *be a closed pseudohermitian* $(2n+1)$*-manifold with vanishing torsion. Suppose that*

$$Ric_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\theta},$$

for all $m \in M$, $Z \in T_{1,0}$ *and some positive constant* k. *Assume that*

$$\lambda_1 = \frac{nk}{n+1}.$$

Then (M, J, θ) *is the standard pseudohermitian* $(2n+1)$*-sphere* $(S^{2n+1}, \widehat{J}, \widehat{\theta})$.

2. Preliminary

Let us give a brief introduction of pseudohermitian geometry (see [Le] for more details). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ, $\dim_R \xi = 2n$. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so is $[JX, Y] + [X, JY]$, and $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$. A CR structure J can extend to $\mathbb{C} \otimes \xi$ and decomposes $\mathbb{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$, which are eigenspaces of J with respect to i and $-i$, respectively. A manifold M with a CR structure is called a CR manifold. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ. Such a choice determines a unique real vector field T transverse to ξ, which is called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathcal{L}_T \theta = 0$ or $d\theta(T, \cdot) = 0$. Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$ and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies

$$d\theta = i h_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}, \tag{2.1}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar{\beta}})$.

The Levi form $h = \langle\ ,\ \rangle_{L_\theta}$ is the hermitian form on $T_{1,0}$ defined by

$$\langle Z, W \rangle_{L_\theta} = -i \left\langle d\theta, Z \wedge \overline{W} \right\rangle.$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle \overline{Z}, \overline{W} \right\rangle_{L_\theta} = \overline{\langle Z, W \rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}$. The Levi form naturally induces a hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles.

Moreover, we can define a family of Webster (adapted) Riemannian metrics g_ε of (M, J, θ) by

$$g_\varepsilon = (1/2) d\theta + \varepsilon^{-2} \theta^2, \ \varepsilon > 0.$$

The pseudohermitian connection of (J,θ) is the connection ∇ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha\in T_{1,0}$ by

$$\nabla Z_\alpha=\theta_\alpha{}^\beta\otimes Z_\beta,\quad \nabla Z_{\bar\alpha}=\theta_{\bar\alpha}{}^{\bar\beta}\otimes Z_{\bar\beta},\quad \nabla T=0,$$

where $\theta_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta&=\theta^\alpha\wedge\theta_\alpha{}^\beta+\theta\wedge\tau^\beta,\\ 0&=\tau_\alpha\wedge\theta^\alpha,\\ 0&=\theta_\alpha{}^\beta+\theta_{\bar\beta}{}^{\bar\alpha}. \end{aligned}\tag{2.2}$$

We can write (by the Cartan lemma) $\tau_\alpha=A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma}=A_{\gamma\alpha}$. The curvature of the Webster-Stanton connection, expressed in terms of the coframe $\{\theta=\theta^0,\theta^\alpha,\theta^{\bar\alpha}\}$, is

$$\begin{aligned} \Pi_\beta{}^\alpha&=\overline{\Pi_{\bar\beta}{}^{\bar\alpha}}=d\omega_\beta{}^\alpha-\omega_\beta{}^\gamma\wedge\omega_\gamma{}^\alpha,\\ \Pi_0{}^\alpha&=\Pi_\alpha{}^0=\Pi_0{}^{\bar\beta}=\Pi_{\bar\beta}{}^0=\Pi_0{}^0=0. \end{aligned}$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written as

$$\Pi_\beta{}^\alpha=R_\beta{}^\alpha{}_{\rho\bar\sigma}\theta^\rho\wedge\theta^{\bar\sigma}+W_\beta{}^\alpha{}_\rho\theta^\rho\wedge\theta-W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho}\wedge\theta+i\theta_\beta\wedge\tau^\alpha-i\tau_\beta\wedge\theta^\alpha,\tag{2.3}$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma}=\overline{R_{\alpha\bar\beta\sigma\bar\rho}}=R_{\bar\alpha\beta\bar\sigma\rho}=R_{\rho\bar\alpha\beta\bar\sigma},\quad W_{\beta\bar\alpha\gamma}=W_{\gamma\bar\alpha\beta}.$$

We will denote components of covariant derivatives with indices preceded by a comma; thus we write $A_{\alpha\beta,\gamma}$. The indices $\{0,\alpha,\bar\alpha\}$ indicate derivatives with respect to $\{T,Z_\alpha,Z_{\bar\alpha}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $f_\alpha=Z_\alpha f,\ f_{\alpha\bar\beta}=Z_{\bar\beta}Z_\alpha f-\theta_\alpha{}^\gamma(Z_{\bar\beta})Z_\gamma f,\ f_0=Tf$ for a (smooth) function.

For a real function f, the subgradient ∇_b is defined by $\nabla_b f\in\xi$ and $\langle Z,\nabla_b f\rangle_{L_\theta}=df(Z)$ for all vector fields Z tangent to the contact plane. Locally

$$\nabla_b f=\sum_\alpha f_{\bar\alpha}Z_\alpha+f_\alpha Z_{\bar\alpha}.$$

We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2 f:T_{1,0}\oplus T_{0,1}\to T_{1,0}\oplus T_{0,1}$$

by

$$(\nabla^H)^2 f(Z)=\nabla_Z\nabla_b f.$$

Also,

$$\Delta_b f=Tr\left((\nabla^H)^2 f\right)=\textstyle\sum_\alpha(f_{\alpha\bar\alpha}+f_{\bar\alpha\alpha}).$$

The Webster-Ricci tensor and the torsion tensor on $T_{1,0}$ are defined by

$$Ric(X,Y)=R_{\alpha\bar\beta}X^\alpha Y^{\bar\beta}$$

and

$$Tor(X,Y)=i\textstyle\sum_{\alpha,\beta}(A_{\bar\alpha\bar\beta}X^{\bar\alpha}Y^{\bar\beta}-A_{\alpha\beta}X^\alpha Y^\beta).$$

We also define the following tensor A^2 as

$$A^2(X,Y)=\textstyle\sum_\gamma(A_{\alpha\gamma}A_{\overline{\beta}\overline{\gamma}}X^\alpha Y^{\overline{\beta}}),$$

where $X=X^\alpha Z_\alpha,\ Y=Y^\beta Z_\beta,\ R_{\alpha\bar\beta}=R_\gamma{}^\gamma{}_{\alpha\bar\beta}$. The Webster scalar curvature is $R=R_\alpha{}^\alpha=h^{\alpha\bar\beta}R_{\alpha\bar\beta}$.

3. The diameter estimates

Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. In this section, we will obtain a lower bound estimate for the diameter of M when a sharp lower estimate for the first positive eigenvalue of the sublaplacian on a pseudohermitian $(2n+1)$-manifold M is achieved.

First we recall the following CR Bochner formula which involves the CR Paneitz operator.

Lemma 3.1 ([CC2])**.** *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. For a (smooth) real function φ on M, we have*

$$\begin{aligned} \frac{1}{2}\Delta_b|\nabla_b\varphi|^2 &= |(\nabla^H)^2\varphi|^2 + \left(1+\frac{2}{n}\right)\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta} \\ &\quad +[2Ric-(n+2)Tor]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \\ &\quad -\frac{4}{n}\langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L^*_\theta}, \end{aligned} \tag{3.1}$$

where $(\nabla_b\varphi)_{\mathbb{C}} = \sum_\alpha \varphi_{\overline{\alpha}} Z_\alpha$ is the corresponding complex $(1,0)$-vector field of $\nabla_b\varphi$ and $d_b\varphi = \varphi_\alpha\theta^\alpha + \varphi_{\overline{\alpha}}\theta^{\overline{\alpha}}$.

Recall that formula (3.1) follows from the two identities:

$$\begin{aligned} \frac{1}{2}\Delta_b|\nabla_b\varphi|^2 &= |(\nabla^H)^2\varphi|^2 + \langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta} \\ &\quad +[2Ric-(n-2)Tor]\left((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\right) \\ &\quad +2\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} n\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta} &= \langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta} - 2nTor((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \\ &\quad -2\langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L^*_\theta}. \end{aligned} \tag{3.3}$$

We also need the following lemma which follows from an observation as in the proof of the sharp lower bound estimate for the first positive eigenvalue λ_1 of the sublaplacian Δ_b on a closed pseudohermitian 3-manifold in [Chi].

Let φ be the corresponding eigenfunction of the sublaplacian Δ_b with respect to λ_1. That is, $\Delta_b\varphi = -\lambda_1\varphi$.

Lemma 3.2. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$Ric_m(Z,Z) - \frac{n+1}{2}Tor_m(Z,Z) \geq k\,\langle Z, Z\rangle_{L_\theta},$$

for all $m \in M$, $Z \in T_{1,0}$, some positive constant k and

$$\lambda_1 = \frac{nk}{n+1}.$$

In addition, we assume that P_0 is nonnegative if $n = 1$. Then the corresponding eigenfunction φ will satisfy

$$P_0\varphi = 0 \tag{3.4}$$

and

$$\varphi_{\alpha\beta} = 0, \quad \varphi_{\alpha\overline{\beta}} = 0 \text{ with } \alpha \neq \beta, \text{ and } \varphi_{\alpha\overline{\alpha}} = \varphi_{\beta\overline{\beta}}. \tag{3.5}$$

Moreover, we have

$$[Ric_m - \frac{n+1}{2}Tor_m]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) = k\left\langle(\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\right\rangle_{L_\theta}. \tag{3.6}$$

Proof. By integrating the two identities (3.2) and (3.3), we obtain

$$\begin{aligned}\int |(\nabla^H)^2\varphi|^2 d\mu &= \int (\Delta_b\varphi)^2 d\mu + 2\int {\varphi_0}^2 d\mu \\ &\quad - \int [2Ric - (n-2)Tor]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}})d\mu\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\int {\varphi_0}^2 d\mu &= \frac{1}{n^2}\int (\Delta_b\varphi)^2 d\mu + \frac{2}{n}\int Tor((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}})d\mu \\ &\quad -\frac{1}{2n^2}\int P_0\varphi\cdot\varphi d\mu,\end{aligned} \tag{3.8}$$

since $|(\nabla^H)^2\varphi|^2 = 2\sum_{\alpha,\beta}|\varphi_{\alpha\beta}|^2 + 2\sum_{\alpha,\beta}|\varphi_{\alpha\overline{\beta}}|^2$. Substituting this on the left-hand side of (3.7) and combining with (3.8), we get

$$\begin{aligned}-2\int \sum_{\alpha,\beta}|\varphi_{\alpha\beta}|^2 d\mu &\geq -\frac{n+1}{n}\int (\Delta_b\varphi)^2 d\mu + \frac{3}{4n}\int P_0\varphi\cdot\varphi d\mu \\ &\quad + \int [2Ric - (n+1)Tor]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}})d\mu \\ &\geq \left(k - \frac{n+1}{n}\lambda_1\right)\int |\nabla_b\varphi|^2 d\mu,\end{aligned}$$

where we have used the Cauchy-Schwarz inequalities

$$\sum_{\alpha,\beta}|\varphi_{\alpha\overline{\beta}}|^2 \geq \sum_{\alpha}|\varphi_{\alpha\overline{\alpha}}|^2 \geq \frac{1}{n}\left|\sum_{\alpha}\varphi_{\alpha\overline{\alpha}}\right|^2 = \frac{1}{4n}(\Delta_b\varphi)^2 + \frac{n}{4}{\varphi_0}^2,$$

$$[Ric - \frac{n+1}{2}Tor]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \geq k\left\langle(\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\right\rangle_{L_\theta},$$

and

$$\int P_0\varphi\cdot\varphi d\mu \geq 0.$$

It follows that $k - \frac{n+1}{n}\lambda_1 \leq 0$, i.e. $\lambda_1 \geq \frac{nk}{n+1}$. In addition, if $\lambda_1 = \frac{nk}{n+1}$, then all of the above inequalities will become equalities and the results will easily follow. □

Lemma 3.3. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with vanishing torsion. Suppose that*

$$Ric_m(Z, Z) \geq k\left\langle Z, Z\right\rangle_{L_\theta}$$

for all $m \in M$, $Z \in T_{1,0}$ and some positive constant k. Then

$$d_\varepsilon(M) \leq \sqrt{\frac{n+1}{k}}\pi \quad \textit{with} \quad \frac{n+1}{k} = \varepsilon^2. \tag{3.9}$$

Proof. Since $Ric_m(Z, Z) \geq k\left\langle Z, Z\right\rangle_{L_\theta}$, we have ([CC2])

$$(R^\varepsilon_{ab}) = \begin{bmatrix} A & B & 0 \\ B^t & A & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -2\varepsilon^{-2}I_{2n\times 2n} & 0 \\ 0 & 2n\varepsilon^{-2} \end{bmatrix}. \tag{3.10}$$

Here $A = (2\operatorname{Re} R_{\alpha\bar\beta})$, $B = (2\operatorname{Im} R_{\alpha\bar\beta})$, $A + iB = [2R_{\alpha\bar\beta}]$ and $g_\varepsilon = d\theta + \varepsilon^{-2}\theta^2$ with $\frac{n+1}{k} = \varepsilon^2$. It follows that

$$Ric_{g_\varepsilon}(X,X) \geq [(2n+1)-1]\varepsilon^{-2}\left|X\right|^2. \tag{3.11}$$

It follows from Myers' theorem that

$$d_\varepsilon(M) \leq \sqrt{\frac{n+1}{k}}\pi. \tag{3.12}$$

□

Proof of Proposition 1.2. Let φ be a nonconstant eigenfunction of the sublaplacian Δ_b satisfying

$$\Delta_b\varphi = -\lambda_1\varphi.$$

Consider the smooth function

$$F = \frac{1}{2}|\nabla_b\varphi|^2 + \frac{\lambda_1}{4n}\varphi^2 + \frac{n}{4\lambda_1}{\varphi_0}^2$$

defined on M. Computing its sublaplacian

$$\begin{aligned}\Delta_b F &= |(\nabla^H)^2\varphi|^2 + \left(1+\frac{2}{n}\right)\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle \\ &\quad +[2Ric-(n+2)Tor]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \\ &\quad +\frac{\lambda_1}{2n}|\nabla_b\varphi|^2 + \frac{\lambda_1}{2n}\varphi\Delta_b\varphi + \frac{n}{4\lambda_1}\Delta_b{\varphi_0}^2,\end{aligned} \tag{3.13}$$

where we have used the CR Bochner formula (3.1) with $P_\beta\varphi = 0$ (by the assumption for $n=1$) from Lemma 3.2 and Remark 1.1, since $\lambda_1 = \frac{nk}{n+1}$. On the other hand, due to (3.5) we have ([CTW])

$$|(\nabla^H)^2\varphi|^2 = \frac{1}{2n}(\Delta_b\varphi)^2 + \frac{n}{2}{\varphi_0}^2, \tag{3.14}$$

and since $[\Delta_b, T] = 2\operatorname{Im} Q$ with the purely holomorphic second-order operator $Q\varphi = 2i(A^{\alpha\beta}\varphi_\alpha)_{,\beta}$ ([GL])

$$\begin{aligned}\Delta_b{\varphi_0}^2 &= 2\varphi_0\Delta_b\varphi_0 + 2|\nabla_b\varphi_0|^2 \\ &= -2\lambda_1{\varphi_0}^2 + 2|\nabla_b\varphi_0|^2 + 4\operatorname{Im}(Q\varphi)\varphi_0.\end{aligned} \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), one obtains

$$\begin{aligned}\Delta_b F &= \left(k - \frac{2n+3}{2n}\lambda_1\right)|\nabla_b\varphi|^2 - Tor((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \\ &\quad +\frac{n}{2\lambda_1}|\nabla_b\varphi_0|^2 + \frac{n}{\lambda_1}\operatorname{Im}(Q\varphi)\varphi_0.\end{aligned}$$

On the other hand, from Lemma 3.2, we have

$$Q\varphi = 0$$

if $A^{\alpha\beta}{}_{,\beta} = 0$. Hence

$$\Delta_b F = \left(k - \frac{2n+3}{2n}\lambda_1\right)|\nabla_b\varphi|^2 - Tor((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) + \frac{n}{2\lambda_1}|\nabla_b\varphi_0|^2. \tag{3.16}$$

Next we compute the term $|\nabla_b\varphi_0|^2$. Since $\lambda_1 = \frac{nk}{n+1}$, we have

$$P_\beta\varphi = \varphi_{\overline{\alpha}}{}^{\overline{\alpha}}{}_\beta + inA_{\beta\alpha}\varphi^\alpha = 0$$

for all β and

$$-2\varphi_{\overline{\alpha}}{}^{\overline{\alpha}} = \Box_b\varphi = -\Delta_b\varphi + in\varphi_0 = \lambda_1\varphi + in\varphi_0, \tag{3.17}$$

thus

$$inA_{\beta\alpha}\varphi^\alpha = -\varphi_{\overline{\alpha}}{}^{\overline{\alpha}}{}_\beta = \frac{1}{2}(\lambda_1\varphi + in\varphi_0)_\beta.$$

It follows that

$$\varphi_{0\beta} = 2A_{\beta\alpha}\varphi^\alpha + i\frac{\lambda_1}{n}\varphi_\beta$$

for all β. Thus

$$\begin{aligned} |\nabla_b\varphi_0|^2 &= 2\sum_\beta \varphi_{0\beta}\varphi_{0\overline{\beta}} \\ &= 2\sum_\beta \left(2A_{\beta\alpha}\varphi^\alpha + i\frac{\lambda_1}{n}\varphi_\beta\right)\left(2A_{\overline{\beta}\overline{\gamma}}\varphi^{\overline{\gamma}} - i\frac{\lambda_1}{n}\varphi_{\overline{\beta}}\right) \qquad (3.18) \\ &= \frac{\lambda_1^2}{n^2}|\nabla_b\varphi|^2 + \left(\frac{4\lambda_1}{n}Tor + 8A^2\right)((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}). \end{aligned}$$

Substituting (3.18) into (3.16), we conclude that

$$\begin{aligned} \Delta_b F &= \left(k - \frac{n+1}{n}\lambda_1\right)|\nabla_b\varphi|^2 + Tor((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \\ &\quad + \frac{4n}{\lambda_1}A^2((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \\ &= \left[Tor + \frac{4n}{\lambda_1}A^2\right]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \\ &= \left[Tor + \frac{4(n+1)}{k}A^2\right]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \\ &\geq 0. \end{aligned}$$

Since M is closed, F must be identically constant. Moreover,

$$\frac{1}{2}|\nabla_b\varphi|^2 + \frac{\lambda_1}{4n}\varphi^2 + \frac{n}{4\lambda_1}{\varphi_0}^2 = \frac{\lambda_1}{4n}|\varphi|_\infty^2,$$

where $|\varphi|_\infty = \sup_M |\varphi|$. If we normalize φ such that $|\varphi|_\infty = 1$ and observe that at the maximum and minimum points of φ its subgradient and derivative with respect to T must vanish, then we obtain that $\max\varphi = 1 = -\min\varphi$ and so

$$|\nabla_b\varphi|^2 + \frac{\lambda_1}{2n}\varphi^2 + \frac{n}{2\lambda_1}{\varphi_0}^2 = \frac{\lambda_1}{2n}.$$

Hence

$$\frac{k}{2(n+1)} = \frac{|\nabla_b\varphi|^2 + \frac{n+1}{2k}{\varphi_0}^2}{1-\varphi^2}.$$

Now let ∇^ε be the gradient with respect to g_ε. We have

$$|\nabla^\varepsilon\varphi|^2 = 2|\nabla_b\varphi|^2 + \varepsilon^2{\varphi_0}^2 = 2|\nabla_b\varphi|^2 + \frac{n+1}{k}{\varphi_0}^2$$

with $\varepsilon^2 = \frac{n+1}{k}$. Thus

$$|\nabla_b\varphi|^2 + \frac{n+1}{2k}{\varphi_0}^2 = \frac{1}{2}|\nabla^\varepsilon\varphi|^2,$$

and so

$$\frac{k}{n+1} = \frac{|\nabla^\varepsilon \varphi|^2}{1-\varphi^2}.$$

Integrating this along a minimal geodesic γ with respect to g_ε joining the points where $\varphi = 1$ and $\varphi = -1$, we have

$$\begin{aligned} d_\varepsilon \sqrt{\tfrac{k}{n+1}} &\geq \textstyle\int_\gamma \sqrt{\tfrac{k}{n+1}} ds \\ &= \textstyle\int_\gamma \frac{|\nabla^\varepsilon \varphi|}{\sqrt{1-\varphi^2}} ds \\ &\geq \textstyle\int_{-1}^1 \frac{d\varphi}{\sqrt{1-\varphi^2}} \\ &= \pi, \end{aligned}$$

which implies that

$$d_\varepsilon \geq \sqrt{\tfrac{n+1}{k}}\pi = \varepsilon\pi. \qquad \square$$

Lemma 3.4. *Let (M, J, θ) be a closed pseudohermitian 3-manifold with nonnegative CR Paneitz operator P_0. Suppose that the curvature condition (1.1) holds, $\lambda_1 = \frac{k}{2}$ and $A_{\overline{11},1} = 0$. Let φ be the corresponding eigenfunction of the sublaplacian Δ_b with respect to λ_1 and $P_1\varphi = 0$ on M. Then*

$$|A_{11}|^2 |\nabla_b \varphi|^2 = 0 \tag{3.19}$$

on M. In particular, the torsion must be vanishing

$$A_{11} = 0$$

on (M, J, θ).

Remark 3.1. For generic metrics in a closed Riemannian manifold, all of the eigenfunctions are Morse functions, and consequently their critical point sets are discrete ([U]).

Proof. From Lemma 3.2, we have $P_0\varphi = 8(P_1\varphi)_{,\overline{1}} = 0$, $\varphi_{11} = 0$, and thus

$$\begin{aligned} \int (P_1\varphi)(\overline{P_1}\varphi) d\mu &= \int (\varphi_{\overline{1}11} + iA_{11}\varphi_{\overline{1}})(\varphi_{1\overline{11}} - iA_{\overline{11}}\varphi_1) d\mu \\ &= \int [-(P_1\varphi)_{,\overline{1}}\, \varphi_{1\overline{1}} + (iA_{\overline{11}}\varphi_1)_{,1}\, \varphi_{\overline{1}1} + \frac{1}{2} |A_{11}|^2 |\nabla_b \varphi|^2] d\mu \\ &= \frac{1}{2} \int |A_{11}|^2 |\nabla_b \varphi|^2 d\mu, \end{aligned}$$

which immediately implies the first part of the lemma.

On the other hand, it follows from (3.19) that $A_{11} = 0$ on the complement of the critical point set Σ of φ. Then the torsion is vanishing on (M, J, θ) by continuity property if Σ is at most codimension one.

More precisely, $\Sigma_0 = \Sigma \cap \{p : \varphi(p) = 0\}$ is at most of codimension 2 ([HHL]) and $\Sigma \backslash \Sigma_0$ is at most of codimension one. In fact, since $\Delta_b \varphi = -\lambda_1 \varphi$, then for $p \in \Sigma \backslash \Sigma_0$

$$\varphi_{1\overline{1}}(p) \neq 0 \quad \text{and} \quad \varphi_{\overline{1}1}(p) \neq 0.$$

Define

$$f = \varphi_1 + \varphi_{\overline{1}}.$$

We have

$$f(p) = 0.$$

Hence

$$\Sigma\backslash\Sigma_0 \subset \{f = 0\}.$$

But from (3.5), for $p \in \Sigma\backslash\Sigma_0$,

$$f_1(p) = \varphi_{11}(p) + \varphi_{\overline{1}1}(p) = \varphi_{\overline{1}1}(p) \neq 0,$$

and then

$$\nabla_b f \neq 0$$

on $\Sigma\backslash\Sigma_0$.

It follows that $\Sigma\backslash\Sigma_0$ is at most of codimension one. All these imply Σ is at most of codimension one. □

Lemma 3.5. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$. Suppose that*

$$Ric_m(Z, Z) - \frac{n+1}{2} Tor_m(Z, Z) \geq k \langle Z, Z \rangle_{L_\theta},$$

for all $m \in M$, $Z \in T_{1,0}$ and

$$A_{\alpha\gamma,\overline{\gamma}} = 0; \quad A_{\alpha\beta,\gamma\overline{\gamma}} = 0 \quad \text{for all } \alpha,\ \beta.$$

Assume that

$$\lambda_1 = \frac{nk}{n+1}.$$

Then

$$A_{\alpha\beta} = 0$$

on (M, J, θ).

Proof. From Lemma 3.2, we have $P_\beta\varphi = 0$, $P_0\varphi = 8(P_\beta\varphi)_{,\overline{\beta}} = 0$, $\varphi_{\alpha\beta} = 0$, and the similar argument

$$\begin{aligned} 0 &= \int (P_\beta\varphi)(\overline{P_\beta}\varphi) d\mu = \int (\varphi_{\overline{\alpha}\alpha\beta} + inA_{\beta\alpha}\varphi_{\overline{\alpha}})(\varphi_{\gamma\overline{\gamma}\overline{\beta}} - inA_{\overline{\beta}\overline{\gamma}}\varphi_\gamma) d\mu \\ &= \int [-(P_\beta\varphi)_{,\overline{\beta}}\, \varphi_{\gamma\overline{\gamma}} + (inA_{\overline{\beta}\overline{\gamma}}\varphi_\gamma)_{,\beta}\, \varphi_{\overline{\alpha}\alpha} + n^2 A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\alpha}}\varphi_\gamma] d\mu \\ &= n^2 \int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\alpha}}\varphi_\gamma d\mu = n^2 \int A^2((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) d\mu. \end{aligned}$$

Hence from $\varphi_{\alpha\overline{\beta}} = 0$ with $\alpha \neq \beta$, $\varphi_{\alpha\overline{\alpha}} = \varphi_{\beta\overline{\beta}}$ and $A_{\beta\alpha,\overline{\alpha}} = 0$, we have

$$\begin{aligned} 0 &= \int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\alpha}}\varphi_\gamma d\mu \\ &= -\int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma},\overline{\alpha}}\varphi_\gamma\varphi d\mu - \int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma}}\varphi_{\gamma\overline{\alpha}}\varphi d\mu \\ &= -\int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma},\overline{\alpha}}\varphi_\gamma\varphi d\mu - \int A_{\beta\alpha}A_{\overline{\beta}\overline{\alpha}}\varphi_{\alpha\overline{\alpha}}\varphi d\mu \\ &= -\int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma},\overline{\alpha}}\varphi_\gamma\varphi d\mu - \int |A_{\alpha\beta}|^2 \varphi_{1\overline{1}}\varphi d\mu. \end{aligned}$$

On the other hand, from (3.17) we have

$$-\varphi_{1\overline{1}} = \frac{1}{2n}\lambda_1\varphi - \frac{1}{2}i\varphi_0.$$

Then

$$0 = -2\int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma},\overline{\alpha}}\varphi_\gamma\varphi d\mu + \frac{1}{n}\lambda_1\int |A_{\alpha\beta}|^2\varphi^2 d\mu - i\int |A_{\alpha\beta}|^2\varphi_0\varphi d\mu. \tag{3.20}$$

We compute, by the commutation relation $A_{\overline{\beta}\overline{\gamma},\overline{\alpha}} = A_{\overline{\beta}\overline{\alpha},\overline{\gamma}}$,

$$\begin{aligned}
&-2\int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma},\overline{\alpha}}\varphi_\gamma\varphi d\mu = -\int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma},\overline{\alpha}}(\varphi^2)_\gamma d\mu\\
&= \int A_{\beta\alpha,\gamma}A_{\overline{\beta}\overline{\gamma},\overline{\alpha}}\varphi^2 d\mu + \int A_{\beta\alpha}A_{\overline{\beta}\overline{\gamma},\overline{\alpha}\gamma}\varphi^2 d\mu\\
&= \int |A_{\beta\alpha,\gamma}|^2\varphi^2 d\mu + \int A_{\beta\alpha}A_{\overline{\beta}\overline{\alpha},\overline{\gamma}\gamma}\varphi^2 d\mu,
\end{aligned}$$

and substitute it into (3.20):

$$\begin{aligned}
0 \;=\; & \tfrac{1}{n}\lambda_1\int |A_{\alpha\beta}|^2\varphi^2 d\mu + \int |A_{\beta\alpha,\gamma}|^2\varphi^2 d\mu\\
& + \int A_{\beta\alpha}A_{\overline{\beta}\overline{\alpha},\overline{\gamma}\gamma}\varphi^2 d\mu - i\int |A_{\alpha\beta}|^2\varphi_0\varphi d\mu.
\end{aligned} \tag{3.21}$$

But

$$A_{\overline{\beta}\overline{\alpha},\overline{\gamma}\gamma} = 0 \quad \text{and} \quad \varphi_0 \text{ is real.}$$

Thus

$$\frac{1}{n}\lambda_1\int |A_{\alpha\beta}|^2\varphi^2 d\mu + \int |A_{\beta\alpha,\gamma}|^2\varphi^2 d\mu = 0.$$

Hence

$$|A_{\alpha\beta}|^2\varphi^2 = 0.$$

Now apply the same method as in the proof of Lemma 3.4. We have

$$|A_{\alpha\beta}| = 0$$

on (M, J, θ). □

Proof of Theorem 1.4. It follows from Lemma 3.4, (3.10) and Proposition 1.2 that

$$d_\varepsilon(M) \geq \sqrt{\frac{2}{k}}\pi$$

with

$$Ric_{g_\varepsilon}(X, X) \geq 2\varepsilon^{-2}|X|^2 = 2\times\frac{k}{2}|X|^2$$

on (M^3, g_ε). Here $g_\varepsilon = d\theta + \varepsilon^{-2}\theta^2$ with $\frac{2}{k} = \varepsilon^2$. Hence from Lemma 3.4 and Lemma 3.3

$$d_\varepsilon(M) \leq \sqrt{\frac{2}{k}}\pi.$$

All these imply

$$d_\varepsilon(M) = \sqrt{\frac{2}{k}}\pi.$$

By applying S.-Y. Cheng's theorem ([Ch]), it follows that (M, J, θ) is the standard pseudohermitian sphere $(S^3, \widehat{J}, \widehat{\theta})$. □

Proof of Theorem 1.3. It follows from Proposition 1.2, Lemma 3.5 and Lemma 3.3.

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