ON THE GLOBAL SOLVABILITY
FOR OVERDETERMINED SYSTEMS

ADALBERTO P. BERGAMASCO, ALEXANDRE KIRILOV,
WAGNER VIEIRA LEITE NUNES, AND SÉRGIO LUÍS ZANI

ABSTRACT. We consider a class of systems of two smooth vector fields on the
3-torus associated to a closed 1-form. We prove that the global solvability
is completely determined by the connectedness of the sublevel and superlevel
sets of a primitive of this 1-form in the minimal covering.

1. INTRODUCTION

Let $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ be the n-dimensional torus, and let b be a smooth, real, closed 1-form defined on $\mathbb{T}^2$, and consider the line subbundle, T', of the complexified cotangent bundle $\mathbb{C} \otimes T^*(\mathbb{T}^3)$ spanned by the 1-form $dx - ib(t) \in \bigwedge^1 C^\infty(\mathbb{T}^3)$, where $(t, x) = (t_1, t_2, x)$ are the coordinates in $\mathbb{T}^3$. The orthogonal bundle $\mathcal{V} = (T')^\perp$ is then a vector subbundle of the complexified tangent bundle $\mathbb{C} \otimes T(\mathbb{T}^3)$ whose fibers have dimension two. The subbundle $\mathcal{V}$ is a locally integrable structure of codimension one over $\mathbb{T}^3$ spanned by the vector fields

$$(1.1) \qquad L_j = \frac{\partial}{\partial t_j} + ib_j(t)\frac{\partial}{\partial x}, \quad j = 1, 2.$$

For further information on these concepts and ideas we refer the reader to the works [18] of Treves and [13] of Berhanu, Cordaro and Hounie.

Let d_t be the exterior derivative in $\mathbb{T}^2$, and consider the differential operator $\mathbb{L}^0$ which acts on functions—or distributions—on the torus by means of

$$(1.2) \qquad \mathbb{L}^0 u = d_t u + ib(t) \wedge \frac{\partial}{\partial x} u.$$

In this work we are interested in global solvability in the sense that given any smooth f satisfying natural compatibility conditions, there should exist a distribution u such that $\mathbb{L}^0 u = f$.

The gist of the present article is that the global solvability of the operator $\mathbb{L}^0$ is completely determined by the connectedness of the sublevel and superlevel sets of a primitive of b, more precisely, a primitive of the pull-back of b to a convenient covering space of $\mathbb{T}^2$, the choice of which will depend on the periods of this 1-form.

When b is an exact 1-form, there exists a global primitive of b on the manifold itself and in this case the global solvability is equivalent to the connectedness of all superlevel and sublevel sets of the primitive, as has been proved by Cardoso

and Hounie in [14]. For the proof of sufficiency the authors used an embedding of the manifold into a space of higher dimension, via Whitney's theorem, and then applied a semiglobal result from the pioneering work of Treves in [17].

When the 1-form b is closed, but nonexact, it makes sense to ask if $\mathbb{L}^0$ is globally solvable; on the other hand, a global primitive of b does not exist on the manifold, hence it is not immediately clear what kind of condition bearing on b would yield global solvability.

In order to deal with this difficulty, Bergamasco, Nunes and Zani in [8, 9], and Bergamasco and Kirilov in [6], used an approach based on the study of the properties of a primitive of the pull-back of b via the universal covering $\Pi : \mathbb{R}^2 \to \mathbb{T}^2$.

The results obtained in [6] concern the case when the periods of b are incommensurable, that is, rationally independent. In that case, the authors proved that $\mathbb{L}^0$ is globally solvable if and only if the sublevels $\Omega_r = \{t \in \mathbb{R}^2; B(t) < r\}$ and the superlevels $\Omega^r = \{t \in \mathbb{R}^2; B(t) > r\}$ are connected, for all $r \in \mathbb{R}$, where $B(t) = \int_{t_0}^t \Pi^* b$ is a primitive of the pull-back of b via the universal covering $\Pi : \mathbb{R}^2 \to \mathbb{T}^2$.

In this work we study the remaining case, namely, the case when b is closed, nonexact and has commensurable periods. Our present results provide necessary and sufficient conditions for the global solvability by means of an analysis of the connectedness of superlevel and sublevel sets of a primitive of a pull-back of b to the cylinder $\mathbb{T} \times \mathbb{R}$.

In fact, we present a characterization of global solvability by means of what we call a minimal covering of $\mathbb{T}^2$ with respect to the 1-form b, in such a way as to unify the results obtained in [6, 14] with the ones proved in the present work.

Other works dealing with global properties such as solvability and hypoellipticity of systems of vector fields are [2, 3, 12], while [1, 4, 5, 7, 10, 11, 16] treat the case of a single vector field.

This article is organized as follows. In section 2 we present more details of the problem and two equivalent formulations of our main result. In section 3 we consider a model case and present the construction of f satisfying the natural compatibility conditions such that $\mathbb{L}^0 u = f$ has no solution. In section 4 we show that the general case (with disconnected superlevels or sublevels) can be reduced to the model case by diffeomorphisms of the torus. To reach this reduction, we make use of Sard's lemma and some properties of regular level sets. In section 5 we prove that the condition of connected superlevel and sublevel sets is sufficient for global solvability.

2. Preliminaries and statement of the main theorem

In this work the property which will play a decisive role regarding the global solvability of $\mathbb{L}^0$ is the connectedness of sublevel and superlevel sets of a global primitive of a pull-back of b to a convenient covering space of $\mathbb{T}^2$. The choice of such a covering will depend on the periods

$$(2.1) \qquad b_{10} = \frac{1}{2\pi} \int_0^{2\pi} b_1(\tau_1, 0) d\tau_1 \quad \text{and} \quad b_{20} = \frac{1}{2\pi} \int_0^{2\pi} b_2(0, \tau_2) d\tau_2.$$

The group of periods of b is defined by

$$\mathrm{Per}(b) \doteq \left\{ \frac{1}{2\pi} \int_\sigma b \, ; \, \sigma \text{ is a loop in } \mathbb{T}^2 \right\};$$

equivalently, $\mathrm{Per}(b)$ is the additive subgroup of $\mathbb{R}$ consisting of all integral combinations of the numbers b_{10}, b_{20}.

The possible values for $r \doteq \mathrm{rank}(\mathrm{Per}(b))$ are $r = 0, 1$, and 2.

- *i.* If $r = 0$, the 1-form b is exact, that is, there exists $B \in C^\infty(\mathbb{T}^2; \mathbb{R})$ with $dB = b$.
- *ii.* If $r = 1$, the periods b_{10}, b_{20} are rationally dependent but are not both equal to 0; in this case we say that the periods of b are commensurable.
- *iii.* If $r = 2$, the periods b_{10}, b_{20} are rationally independent; here we say that the periods of b are incommensurable.

We now present the definition of a minimal covering.

Definition 2.1. Let b be a smooth, real, closed 1-form defined on the two-dimensional torus $\mathbb{T}^2$. A minimal covering of $\mathbb{T}^2$, with respect to the 1-form b, is a covering space $\Pi : \mathcal{T} \to \mathbb{T}^2$ such that $\Pi^* b$ is exact and furthermore such that there is no smaller covering with the same property.

Remark 2.2. We have three possibilities for minimal coverings of $\mathbb{T}^2$, with respect to the 1-form b, namely:

- *i.* if b is exact, then $\mathcal{T} = \mathbb{T}^2$;
- *ii.* if the periods of b are commensurable, then $\mathcal{T} \simeq \mathbb{T} \times \mathbb{R}$ (diffeomorphic);
- *iii.* if the periods of b are incommensurable, then $\mathcal{T} = \mathbb{R}^2$.

We now present our definition of global solvability.

Definition 2.3. Let $\mathbb{E}$ be the set of all $f_1 dt_1 + f_2 dt_2 \in C^\infty(\bigwedge^1(\mathbb{T}^3))$ satisfying $L_1 f_2 = L_2 f_1$ and $\int_{\mathbb{T}^2} f_j(t, x) dt_j dx = 0, j = 1, 2$.

The operator $\mathbb{L}^0$ is said to be globally solvable on $\mathbb{T}^3$ if for any $f \in \mathbb{E}$ there exists $u \in \mathcal{D}'(\mathbb{T}^3)$ satisfying $\mathbb{L}^0 u = f$.

It is not difficult to prove that if $f \doteq f_1 dt_1 + f_2 dt_2 \in C^\infty(\bigwedge^1(\mathbb{T}^3))$ is such that there is $u \in \mathcal{D}'(\mathbb{T}^3)$ with $\mathbb{L}^0 u = f$, then $\int_{\mathbb{T}^2} f_j(t, x) dt_j dx = 0, j = 1, 2$ and $L_1 f_2 = L_2 f_1$. This supports our definition. We are now ready to state our main result.

Theorem 2.4. *Let $b \in \bigwedge^1 C^\infty(\mathbb{T}^2)$ be a closed 1-form and set $B(t) \doteq \int_0^t \Pi^* b$, where $\Pi : \mathcal{T} \to \mathbb{T}^2$ is the minimal covering, in the sense of Definition 2.1, and $\Pi^* b$ is the pull-back of b. The operator $\mathbb{L}^0 = d_t + ib(t) \wedge \partial/\partial x$ is globally solvable if and only if the sublevels $\Omega_r = \{\tau \in \mathcal{T}; B(\tau) < r\}$ and the superlevels $\Omega^r = \{\tau \in \mathcal{T}; B(\tau) > r\}$ are connected, for every $r \in \mathbb{R}$.*

Remark 2.5. In order to prove the theorem above we will make use of diffeomorphisms of $\mathbb{T}^2$ and also of the induced diffeomorphisms of the covering space. Note that if $\mathcal{T}$ and $\mathcal{T}'$ are covering spaces and $\Phi : \mathcal{T}' \to \mathcal{T}$ is a diffeomorphism, then the connectedness of all superlevels (sublevels) of a continuous function $B : \mathcal{T} \to \mathbb{R}$ is equivalent to the connectedness of all superlevels (sublevels) of $B \circ \Phi$.

We now make some remarks and provide examples with the purpose of clarifying the assumptions of connectedness in the theorem.

If b is an exact 1-form it is essential to consider the minimal covering in the above theorem as the following example shows: if $L_1 = \partial/\partial t_1 - i \sin t_1 \partial/\partial x$, $L_2 = \partial/\partial t_2$, then $B(t_1, t_2) = \cos t_1$ has connected sublevel and superlevel sets both in $\mathbb{T}^2$ (hence $\mathbb{L}^0$ is globally solvable) and in $\mathbb{T}_{t_1} \times \mathbb{R}_{t_2}$, but neither in $\mathbb{R}^2$ nor in $\mathbb{R}_{t_1} \times \mathbb{T}_{t_2}$.

If b is a closed nonexact 1-form with commensurable periods, then up to a diffeomorphism of the torus $\mathbb{T}^2$, the minimal covering space is $\mathcal{T} = \mathbb{T} \times \mathbb{R}$. Observe that, if we set $B_1(t) \doteq \int_0^t \Pi_1^* b$ and $B_2(t) \doteq \int_0^t \Pi_2^* b$, where $\Pi_1 : \mathbb{T} \times \mathbb{R} \to \mathbb{T}^2$ is the

minimal covering and $\Pi_2 : \mathbb{R}^2 \to \mathbb{T}^2$ is the universal covering, then for each $r \in \mathbb{R}$ we have:

> i. $\Omega_{1,r}$ is connected if and only if $\Omega_{2,r}$ is connected,
> ii. Ω_1^r is connected if and only if Ω_2^r is connected,

where $\Omega_{j,r} \doteq \{\tau \in \mathbb{R}^2; B_j(\tau) < r\}$ and $\Omega_j^r \doteq \{\tau \in \mathbb{T} \times \mathbb{R}; B_j(\tau) > r\}$ for $j = 1, 2$.

On the other hand, when Ω_1^r and Ω_2^r are disconnected, the number of connected components of these two sets is, in general, different. For example, if $b(t_1, t_2) = \cos(t_1 + t_2)dt_1 + (\cos(t_1 + t_2) - 1/2\pi)dt_2$, then $B(t_1, t_2) = \sin(t_1 + t_2) - t_2/2\pi$, with $(t_1, t_2) \in \mathbb{T} \times \mathbb{R}$. Taking $r = 0$ we have that Ω_1^0 has only two connected components, while Ω_2^0 has infinitely many connected components. The same result is true for $\Omega_{1,0}$ and $\Omega_{2,0}$.

In view of these remarks, we may rewrite Theorem 2.4 as follows.

Theorem 2.6. *Let b be a smooth closed 1-form on $\mathbb{T}^2$ and $\mathbb{L}^0 = d_t + ib(t) \wedge \partial/\partial x$.*

> i. *If b is exact and $B(t) \doteq \int_0^t b$, then the operator $\mathbb{L}^0$ is globally solvable if and only if the sublevels $\{\tau \in \mathbb{T}^2; B(\tau) < r\}$ and the superlevels $\{\tau \in \mathbb{T}^2; B(\tau) > r\}$ are connected, for every $r \in \mathbb{R}$.*
> ii. *If b is nonexact and $B(t) \doteq \int_0^t \Pi^* b$, where $\Pi : \mathbb{R}^2 \to \mathbb{T}^2$ is the universal covering, then the operator $\mathbb{L}^0$ is globally solvable if and only if the sublevels $\{\tau \in \mathbb{R}^2; B(\tau) < r\}$ and the superlevels $\{\tau \in \mathbb{R}^2; B(\tau) > r\}$ are connected, for every $r \in \mathbb{R}$.*

Taking into account the results of [6] and [14], we need only analyze the commensurable case.

3. NECESSITY: THE MODEL CASE

In this section we show that, in a special situation, which we will refer to as the model case, we are able to construct an $f \in \mathbb{E}$ (see Definition 2.3) for which the equation $\mathbb{L}^0 u = f$ has no solution $u \in \mathcal{D}'(\mathbb{T}^3)$.

We will call the *model case* the case when the 1-form b has the following properties: $b_{20} < b_{10} = 0$, $(0,0) \in \Pi^{-1}(\Sigma)$, where $\Sigma \doteq \{t \in \mathbb{T}^2 : b(t) = 0\}$ is the critical set of b, $B(0,0) = 0$ and $M > M' \geqslant 0$, where

$$(3.1) \qquad M \doteq \max_{t \in [0,2\pi]^2} B(t) = B(0, t_2^*) \quad \text{and} \quad M' \doteq \max_{t_1 \in [0,2\pi]} B(t_1, 0).$$

Let us consider M_1 and $\delta > 0$ such that $M' < M_1 < M$ and

$$(3.2) \qquad \max\{B(t_1, t_2) - B(0, t_2) : 0 \leqslant t_1 \leqslant 2\pi, |t_2| \leqslant \delta\} \leqslant M_1.$$

In order to exhibit an $f = f_1 dt_1 + f_2 dt_2 \in \mathbb{E}$ such that the equation $\mathbb{L}^0 u = f$, or equivalently, $L_1 u = f_1$, $L_2 u = f_2$, has no solution $u \in \mathcal{D}'(\mathbb{T}^3)$, we need to verify that the compatibility condition $\mathbb{L}^1 f = 0$ or, equivalently, $L_1 f_2 = L_2 f_1$ is satisfied. We choose $f_1 \equiv 0$; then f_2 must verify $L_1 f_2 = 0$. We expand f_2 in its x-Fourier series

$$f_2(t, x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}_2(t, n) e^{inx}.$$

Any possible solution to the equation $L_2 u = f_2$ can be written in the form of an x-Fourier series, namely,

$$u(t, x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{u}(t, n) e^{inx}.$$

In fact, the partial Fourier coefficients of u must be solutions to

$$(3.3) \qquad (\partial/\partial t_2 - n\, b_2(t_1, t_2))\, \hat{u}(t, n) = \hat{f}_2(t, n), \quad n \in \mathbb{Z}.$$

Since $b_{20} < 0$, we have $1 - e^{2\pi n b_{20}} \neq 0$, for all $n \neq 0$. It follows that when $n \neq 0$ this equation has a unique solution given by

$$(3.4) \qquad \hat{u}(t, n) = d_{2n} \int_0^{2\pi} e^{-in[B(t_1, t_2) - B(t_1, t_2 - s_2)]} \hat{f}_2(t_1, t_2 - s_2, n)\, ds_2,$$

where $d_{2n} \doteq (1 - e^{2\pi n\, b_{20}})^{-1}$.

In fact, we will use only positive frequencies and, thus, look for f_2 of the form

$$f_2(t, x) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \hat{f}_2(t, n) e^{inx};$$

then, up to a constant, any possible solution to $L_2 u = f_2$ can be written as

$$u(t, x) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \hat{u}(t, n) e^{inx}.$$

Now we proceed to make the choice of each $\hat{f}_2(t, n)$, with $n \geqslant 1$. Let us consider an auxiliary function $\chi_\delta \in C_c^\infty(\mathbb{R})$ satisfying the conditions

$$\chi_\delta(t_2) = 1, \text{ if } |t_2| < \delta,$$
$$\chi_\delta(t_2) = 0, \text{ if } |t_2| \geqslant 2\delta,$$
$$0 \leqslant \chi_\delta(t_2) \leqslant 1, t_2 \in \mathbb{R},$$

where $\delta > 0$ is as in (3.2). If necessary, we take a smaller δ so that $0 < 2\delta < t_2^* < 2\pi - 2\delta$, where t_2^* is as in (3.1), and we consider the smooth 2π-periodic function $q : \mathbb{R}^2 \to \mathbb{R}$ such that, for $(t_1, t_2) \in Q^* \doteq [0, 2\pi] \times [-\pi, \pi]$ it is given by

$$(3.5) \qquad q(t_1, t_2) = q(t_2) \doteq (t_2^2 - 1) \cdot \chi_\delta(t_2) + 1.$$

Finally, for each $(t_1, t_2) \in \mathbb{R}^2$, we define

$$(3.6) \qquad \hat{f}_2(t_1, t_2, n) \doteq e^{n[\varepsilon + B(t_1, t_2) - B(0, t_2) - M - Kq(t_2)]}, \; n \geqslant 1,$$

where $\varepsilon > 0$ and $K > 0$ will be chosen later on.

Proposition 3.1. *There are $\varepsilon > 0$ and $K > 0$ such that if $f_1 = 0$ and f_2 is as above, then $f \doteq f_1 dt_1 + f_2 dt_2 \in E$. Furthermore, the same conclusion will hold if we replace $\varepsilon > 0$ by any $\varepsilon' > 0$ with $0 < \varepsilon' < \varepsilon$, and also if we replace $K > 0$ by any $K' > 0$ with $0 < K < K'$.*

Proof. We will prove that $f_2 \in C^\infty(\mathbb{T}^3)$ and $L_1 f_2 = 0$.

It is easy to see that, for each $n \in \mathbb{N}$, $\hat{f}_2(\cdot, \cdot, n)$ is 2π-periodic in both variables (recall that $b_{10} = 0$) and $(\partial/\partial t_1 - n\, b_1(t_1, t_2))\, \hat{f}_2(t_1, t_2, n) = 0$, for all $(t_1, t_2) \in \mathbb{R}^2$; as soon as we prove that f_2 is smooth, it will follow from this that $L_1 f_2 = 0$. Since $f_1 = 0$, we will have $L_1 f_2 = L_2 f_1$.

Recall that, by (3.2), $B(t_1, t_2) - B(0, t_2) \leqslant M_1$ for all $0 \leqslant t_2 < \delta$. Since $q(t) \geqslant 0$ for all $t \in [0, 2\pi]^2$, if we choose $0 < \varepsilon < M - M_1$, we have

$$(3.7) \qquad \varepsilon + B(t_1, t_2) - B(0, t_2) - M - Kq(t_2) \leqslant \varepsilon + M_1 - M < 0.$$

Note that, if $0 < 2\pi - t_2 < \delta$, then

$$B(t_1, t_2) - B(0, t_2) = B(t_1, t_2 - 2\pi) - B(0, t_2 - 2\pi) < M_1.$$

Thus, we also have (3.7) for $0 < 2\pi - t_2 < \delta$.

If $t \in [0, 2\pi]^2$ and $\delta \leqslant t_2 \leqslant \pi$, then $q(t_2) \geqslant (\delta^2 - 1)\chi_\delta(t_2) + 1 \geqslant \delta^2$, since $\chi_\delta(t_2) \leqslant 1$. If $t \in [0, 2\pi]^2$ and $\pi \leqslant t_2 \leqslant 2\pi - \delta$, then $q(t_2) = ((t_2 - 2\pi)^2 - 1)\chi_\delta(t_2 - 2\pi) + 1 \geqslant \delta^2$. Now choosing K such that

$$(3.8) \qquad\qquad K > \frac{\varepsilon - m}{\delta^2},$$

where $m \doteq \min\{B(0, t_2); 0 \leqslant t_2 \leqslant 2\pi\}$, we have

$$\varepsilon + B(t_1, t_2) - B(0, t_2) - M - Kq(t_2) < \varepsilon - m - K\delta^2 < 0,$$

for all $t \in [0, 2\pi]^2$ with $\delta \leqslant t_2 \leqslant 2\pi - \delta$.

Finally, if $\lambda \doteq \max\{\varepsilon + M_1 - M, \varepsilon - m - K\delta^2\}$, we have $\lambda < 0$ and

$$|\hat{f}_2(t, n)| = e^{n[\varepsilon + B(t_1, t_2) - B(0, t_2) - M - Kq(t_2)]} \leqslant e^{n\lambda},$$

for all $t \in [0, 2\pi]^2$ and $n \in \mathbb{N}$.

Now, for $\alpha, \beta \in \mathbb{Z}_+$, we may write $\partial^{(\alpha, \beta)} \hat{f}_2(t, n) = \hat{f}_2(t, n) P(t, n, \alpha, \beta)$, where $P(t, n, \alpha, \beta)$ satisfies

$$|P(t, n, \alpha, \beta)| \leqslant C_{\alpha,\beta} \, n^{\alpha + \beta}, \ \forall n \in \mathbb{N} \ \text{ and } \ \forall t \in [0, 2\pi]^2,$$

for some $C_{\alpha,\beta} > 0$.

Therefore

$$|\partial^{(\alpha, \beta)} \hat{f}_2(t, n)| = |\hat{f}_2(t, n)| \, |P(t, n, \alpha, \beta)| \leqslant C_{\alpha,\beta} \, n^{\alpha + \beta} \, e^{n\lambda},$$

for all $n \in \mathbb{N}$ and $t \in [0, 2\pi]^2$, hence $f_2 \in C^\infty(\mathbb{T}^3)$.

In view of the fact that f_2 contains only nonzero frequencies, it is easy to see that we have $\int_{\mathbb{T}^2} f_2(t, x) dt_2 dx = 0$.

Since we trivially have $\int_{\mathbb{T}^2} f_1(t, x) dt_1 dx = 0$, the proof is complete. $\qquad\square$

We now proceed to state the main result of this section.

Proposition 3.2. *There exists no periodic distribution whose sequence of x-Fourier coefficients is given by* (3.4) *and* (3.6).

Proof. We will analyze the behavior of $\hat{u}(0, t_2^*, n)$, where t_2^* is as in (3.1). From (3.4) and (3.6) we have

$$(3.9) \qquad\qquad \hat{u}(t, n) = d_{2n} \int_0^{2\pi} e^{n[\phi(t_1, t_2, s_2) + \varepsilon]} \, ds_2,$$

where

$$\phi(t_1, t_2, s_2) = [B(t_1, t_2) - B(0, t_2 - s_2) - M - Kq(t_2 - s_2)].$$

We observe that $\phi(0, t_2^*, s_2) = -B(0, t_2^* - s_2) - Kq(t_2^* - s_2)$. From (3.9) we obtain

$$(3.10) \qquad\qquad \hat{u}(0, t_2^*, n) = d_{2n}(I_n + J_n),$$

where

$$I_n \doteq \int_{|t_2^* - s_2| < \delta} e^{n[\phi(0, t_2^*, s_2) + \varepsilon]} ds_2 \ \text{ and } \ J_n \doteq \int_{|t_2^* - s_2| > \delta} e^{n[\phi(0, t_2^*, s_2) + \varepsilon]} ds_2.$$

In the region $|t_2^* - s_2| < \delta$, we have $\chi_\delta(t_2^* - s_2) = 1$, hence

$$\phi(0, t_2^*, s_2) = -B(0, t_2^* - s_2) - K(t_2^* - s_2)^2$$

and

$$(3.11) \qquad\qquad I_n = e^{n\varepsilon} \int_{|t_2^* - s_2| < \delta} e^{-n[B(0, t_2^* - s_2) + K(t_2^* - s_2)^2]} ds_2.$$

We perform the change of variables $\sigma = t_2^* - s_2$ in (3.11) to obtain

$$I_n = e^{n\varepsilon} \int_{|\sigma| < \delta} e^{-ng(\sigma)} d\sigma,$$

where $g(\sigma) = B(0, \sigma) + K\sigma^2$.

Since $g(0) = g'(0) = 0$ and $g''(0) = \partial b_2 / \partial t_2(0, 0) + 2K$, we may choose a larger $K > 0$ and a smaller $\delta > 0$ to obtain $g''(0) > 0$ and therefore the origin will be the only critical point of the function g on $|\sigma| < \delta$, which implies that there exist constants $C_1, C_2 > 0$ such that

$$C_2 \sigma^2 \leqslant g(\sigma) \leqslant C_1 \sigma^2, \ \forall \sigma \in (-\delta, \delta).$$

It follows that

$$e^{n\varepsilon} \int_{|\sigma| < \delta} e^{-nC_1 \sigma^2} d\sigma \leqslant I_n \leqslant e^{n\varepsilon} \int_{|\sigma| < \delta} e^{-nC_2 \sigma^2} d\sigma.$$

We use new variables $\tau = \sigma \sqrt{nC_1}$ in the integral on the left and $\tau = \sigma \sqrt{nC_2}$ in the integral on the right above to obtain

$$R_1(n) e^{n\varepsilon} \frac{1}{nC_1} \leqslant I_n \leqslant R_2(n) e^{n\varepsilon} \frac{1}{nC_2},$$

where $R_j(n) = \int_{|\tau| < \delta\sqrt{nC_j}} e^{-\tau^2} d\tau$, for $j = 1, 2$.

Observe that $\{R_j(n)\}_{n \in \mathbb{N}}$ are increasing sequences, $j = 1, 2$, and

$$R_j(1) \leqslant R_j(n) \leqslant \int_{\mathbb{R}} e^{-\sigma^2} d\sigma = \sqrt{\pi}, \ j = 1, 2,$$

hence $R_1(1) e^{n\varepsilon} \frac{1}{nC_1} \leqslant I_n \leqslant \sqrt{\pi} e^{n\varepsilon} \frac{1}{nC_2}$.

Take $n_0 \in \mathbb{N}$ such that $R_1(1)(nC_1)^{-1} \exp(n\varepsilon/2) \geqslant 1$, for all $n \geqslant n_0$, which implies $I_n \geqslant e^{n\frac{\varepsilon}{2}}$, for all $n \geqslant n_0$. Since $J_n \geqslant 0$, it follows that

$$\hat{u}(0, t_2^*, n) = d_{2n}(I_n + J_n) \geqslant d_{2n} I_n \geqslant d_{2n} e^{n\frac{\varepsilon}{2}},$$

for all $n \geqslant n_0$.

We have $d_{2n} \geqslant 1/2$, thus $|\hat{u}(0, t_2^*, n)| \geqslant \frac{1}{2} e^{n\varepsilon/2}$ for all $n \geqslant n_0$. We conclude that the sequence $\{\hat{u}(0, t_2^*, n)\}$ does not correspond to any periodic distribution. $\square$

Putting together the results of this section, we reach the conclusion that, in the model case, the operator $\mathbb{L}^0$ is not globally solvable in $\mathbb{T}^3$.

4. NECESSITY: THE REDUCTION TO THE MODEL CASE

The goal of this section is, starting from the assumption of the existence of a disconnected sublevel or superlevel set of $B(t) = \int_0^t \Pi^* b$, to find a smooth diffeomorphism of $\mathbb{T}^2$ that reduces our problem to the model case.

Since b_{10} and b_{20} are commensurable, by using diffeomorphisms of the two-dimensional torus given by elements of $U(2, \mathbb{Z})$ (see [9, page 56]) we may assume that

(4.1) $$b_{20} < b_{10} = 0.$$

In fact, since b_{10} and b_{20} are rationally dependent, there exist relatively prime integers p and q such that $pb_{10} + qb_{20} = 0$. Take $r, s \in \mathbb{Z}$ such that $pr + qs = 1$ and consider the diffeomorphism $\Psi : \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$\Psi(t_1', t_2') \doteq (pt_1' - st_2', qt_1' + rt_2') \doteq (t_1, t_2).$$

Let $B(t_1, t_2)$ be a primitive of b defined on a cylinder of coordinates (t_1, t_2). Thus, $B(t_1, t_2) = P(t_1, t_2) + b_{10}t_1 + b_{20}t_2$, where P is periodic. Then $B' \doteq B \circ \Psi$ is also a primitive of b on a cylinder of coordinates (t_1', t_2'). We have $B'(t_1', t_2') = P(pt_1' - st_2', qt_1' + rt_2') + (b_{20}r - b_{10}s)t_2'$. Hence, in these new coordinates b has one null period.

Assume $b_{10} = 0$. If $b_{20} < 0$ we are done. If $b_{20} > 0$ consider the diffeomorphism $\Psi(t_1', t_2') = (t_1', -t_2') \doteq (t_1, t_2)$. Then for $B' \doteq B \circ \Psi$ we have $b_{20}' < 0$. Thus, if $B(t_1, t_2) = P(t_1, t_2) + b_{20}t_2$, then $B'(t_1', t_2') \doteq B(\Psi(t_1', t_2')) = P(t_1', -t_2') - b_{20}t_2'$. Hence, in the new coordinates, b has one null period and a negative one.

Finally, since we are assuming that B has a disconnected sublevel or superlevel set, by Remark 2.5 the same is true for B'.

In order to keep the notation light we will use the same symbols for the coefficients of the 1-form b as for the coefficients of the pull-back of $b \in \bigwedge^1 C^\infty(\mathbb{T}^2)$ via the minimal covering $\Pi : \mathbb{T} \times \mathbb{R} \to \mathbb{T}^2$; in other words we will write

$$(\Pi^* b)(t) = b_1(t)dt_1 + b_2(t)dt_2,$$

where the functions $b_1, b_2 \in C^\infty(\mathbb{T} \times \mathbb{R})$ are 2π-periodic in the second variable.

Since $\Pi^* b$ is a closed 1-form, and since $b_{10} = 0$, the function $B : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ given by

$$(4.2) \qquad B(t) = \int_0^t \Pi^* b, \quad t \in \mathbb{T} \times \mathbb{R}$$

is a well-defined primitive of $\Pi^* b$.

Remark 4.1. Since $b_{10} = 0$ we have $B(t) = P(t) + b_{20} \cdot t_2$, for all $t = (t_1, t_2) \in \mathbb{T} \times \mathbb{R}$, where $P \in C^\infty(\mathbb{T} \times \mathbb{R})$ is a 2π-periodic function in the second variable.

Let us denote by $\mathcal{F}(t)$ the level set of B which contains $t \in \mathbb{T} \times \mathbb{R}$ and by $\mathcal{F}'(t)$ the connected component of $\mathcal{F}(t)$ which contains $t \in \mathbb{T} \times \mathbb{R}$.

It follows from the above remark that each level set $\mathcal{F}(t)$ of B is contained in a cylindric strip $\mathbb{T} \times [\alpha, \beta]$. Moreover every superlevel set and every sublevel set contain a half-cylinder $\mathbb{T} \times (-\infty, \beta)$, or $\mathbb{T} \times (\alpha, \infty)$, respectively. It is clear that each superlevel and each sublevel contains a unique unbounded connected component.

4.1. Characterization of the regular level sets of B.
We say that the real number r is a regular value of the function B when there are no critical points of B at level r; that is, if $B(t) = r$, then $dB(t) \neq 0$. When r is a regular value we say that the corresponding level set is a regular level.

Definition 4.2. We say that a subset $G \subset \mathbb{T} \times \mathbb{R}$ is trivial if there exists a simply connected open subset of $\mathbb{T} \times \mathbb{R}$ containing G. Otherwise, we say that G is nontrivial.

The following result gives a characterization of the regular levels.

Proposition 4.3. *If $B(t)$ is a regular value of B and $\mathcal{F}(t)$ is the corresponding regular level set then:*

i. Each connected component of $\mathcal{F}(t)$ is diffeomorphic to $\mathbb{T}$.

ii. Each connected component, say $\mathcal{F}'$, of $\mathcal{F}(t)$ is contained in an open set, V, diffeomorphic to an annulus, which is foliated by connected components of pairwise distinct level sets, each one of these being diffeomorphic to $\mathbb{T}$. Furthermore $dB(t) \neq 0$ on this open set. Moreover, the points on one side of $\mathcal{F}'$ belong to a sublevel set of B whereas the points on the other side of $\mathcal{F}'$ belong to a superlevel set. In particular, the level $B(t)$ occurs in V only at $\mathcal{F}'$.

iii. $\mathcal{F}(t)$ has only a finite number of connected components.

iv. $\mathcal{F}(t)$ has at least one nontrivial connected component.

Proof. Since $B(t)$ is a regular value, $\mathcal{F}(t)$ is an embedded one-dimensional submanifold of $\mathbb{T} \times \mathbb{R}$. It follows that each connected component of $\mathcal{F}(t)$ is diffeomorphic either to $\mathbb{T}$ or to $\mathbb{R}$. On the other hand, $\mathcal{F}(t)$ is contained in a cylindric strip $\mathbb{T} \times [c, d]$, hence the case of a component diffeomorphic to $\mathbb{R}$ cannot occur, which proves i.

By assumption, we have $dB(t) \neq 0$ at each point of the connected component $\mathcal{F}'$, hence there is a local foliation by level sets of B. By compactness, we obtain a foliated neighborhood of $\mathcal{F}'$ by connected components of level sets of B with pairwise distinct levels. From this it is clear that all statements in ii hold true.

It follows from ii that each connected component of $\mathcal{F}(t)$ is isolated from the others. Recalling that the whole level set $\mathcal{F}(t)$ is contained in a compact strip $\mathbb{T} \times [c, d]$, we conclude that the number of components is finite, which proves iii.

To prove iv, let us suppose otherwise, that is, there exists a regular level set $\mathcal{F}(t_0)$ which has only trivial connected components.

In view of iii, the number of regular connected components having the same level is finite, hence we may write

$$(4.3) \qquad \mathbb{T} \times \mathbb{R} \setminus \mathcal{F}(t_0) = \mathcal{U}_0 \cup \bigcup_{j=1}^{m} \mathcal{U}_j,$$

where $\mathcal{U}_0$ is the unique unbounded connected component and each $\mathcal{U}_j$ is a bounded open set which satisfies $\partial \mathcal{U}_j \subseteq \mathcal{F}(t_0)$, $j = 1, \ldots, m$.

The function B is continuous on the connected set $\mathcal{U}_0$ and never attains the value $B(t_0)$ there, hence we must have either $B(\tau) < B(t_0)$, for all $\tau \in \mathcal{U}_0$, or else $B(\tau) > B(t_0)$, for all $\tau \in \mathcal{U}_0$, which contradicts the fact that both the sublevel $\Omega_{B(t_0)}$ and the superlevel $\Omega^{B(t_0)}$ contain a half-cylinder. $\qquad \square$

We remark that Proposition 4.3ii is valid assuming only that $\mathcal{F}'$ is a connected component of a level set such that $dB(p) \neq 0$ for each $p \in \mathcal{F}'$.

Definition 4.4. A normal curve is a nontrivial (in the sense of Definition 4.2) connected component of a level set of B such that $dB \neq 0$ at each of its points.

In this work, we use Sard's lemma for functions defined on $\mathbb{T} \times \mathbb{R}$, the statement of which we now recall.

Lemma 4.5 (Sard)**.** *If $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a smooth and surjective function, then the set of critical values of f has Lebesgue measure zero. In particular, the set of regular values of f is dense in $\mathbb{R}$.*

4.2. **The reduction to the model case.** We now proceed to describe the announced reduction. In order to achieve this we will rely on the notation and results of the above subsection.

We make the remark that we may work either with the case when B has a disconnected sublevel, say, Ω_ρ, or the case when B has a disconnected superlevel, say, Ω^ρ. For this, if necessary, we make a change of variables in $\mathbb{T}^3$, namely, $(t,x) \mapsto (-t,-x)$. Note that the property (4.1) is preserved under this change of variables. In order to fix the ideas we will concentrate—for now—on the case of a disconnected sublevel.

Since every sublevel of B contains a unique unbounded connected component (which contains an upper half-cylinder), the sublevel Ω_ρ has at least one bounded connected component which we denote by $\Omega_\rho^\flat$.

We may assume that ρ is a regular value of B; indeed, if ρ is a critical value, then Sard's lemma implies the existence of a regular value $r \in \mathbb{R}$, with $\rho > r$, such that $B(t^*) = r$ for some $t^* \in \Omega_\rho^\flat$. Since $B(t^*) < \rho$ and $B(t) = \rho$, for all $t \in \partial\Omega_\rho^\flat$, we have $\mathcal{F}'(t^*) \subset \Omega_\rho^\flat$. Since r is a regular value it follows from Proposition 4.3iv, that $\mathcal{F}'(t^*)$ has a neighborhood foliated by level curves, all of them with distinct values of B. Therefore B has a regular value r, whose sublevel Ω_r is disconnected and has a bounded connected component, $\Omega_r^\flat$, such that $\mathcal{F}'(t^*) \subset \partial\Omega_r^\flat$, which concludes the proof of our claim.

We claim that $\partial\Omega_\rho^\flat$ is equal to a finite union of connected components of $\mathcal{F}(t)$ for some $t \in \partial\Omega_\rho^\flat$. Indeed, B is equal to ρ on $\partial\Omega_\rho^\flat$ and ρ is a regular value, hence it follows from Proposition 4.3iii that $\partial\Omega_\rho^\flat$ is contained in a finite union of connected components of $\mathcal{F}(t)$, $t \in \partial\Omega_\rho^\flat$. On the other hand, if $t \in \partial\Omega_\rho^\flat$ then, by Proposition 4.3i, we obtain $\mathcal{F}'(t) \subset \partial\Omega_\rho^\flat$, which concludes the proof of our claim.

We may write

$$\partial\Omega_\rho^\flat = \left(\bigcup_{i=1}^{N_0} \alpha_i\right) \cup \left(\bigcup_{j=1}^{N_1} \gamma_j\right), \quad N_0, N_1 \geq 0, \quad N_0 + N_1 \geq 1,$$

where each α_i is a trivial embedded copy of $\mathbb{T}$ and each γ_j is a nontrivial embedded copy of $\mathbb{T}$ in the cylinder $\mathbb{T} \times \mathbb{R}$.

We claim that we have either $N_1 = 0$ or $N_1 = 2$.

It is easy to see that if we had $N_1 \geqslant 3$, then $\Omega_\rho^\flat$ would be disconnected by one of the intermediate curves γ_j, which is a contradiction.

Assume now that $N_1 = 1$ and let γ_1 be the nontrivial curve in $\partial\Omega_\rho^\flat$; we may assume that $\gamma_1 = \{t_2 = c_1\}$. We will prove that a contradiction occurs.

First we consider the case when $\Omega_\rho^\flat$ lies below γ_1. Let $\bar{t} = (\bar{t}_1, \bar{t}_2) \in \Omega_\rho^\flat$, with $\bar{t}$ near γ_1. Let $V \doteq \{t_2 < c_1\} \setminus \bigcup_{j=1}^{N_0} \Gamma_j$, where Γ_j is the closed bounded region whose boundary is α_j. Take $s = (s_1, s_2) \in V$, $s_2 < \bar{t}_2$ such that $B(s) > \rho$. Since V is path-connected, there exists a path β entirely contained in V, connecting the points $\bar{t} \in \Omega_\rho^\flat$ and $s \in V \setminus \overline{\Omega_\rho^\flat}$, which implies that β meets $\partial\Omega_\rho^\flat$, which is a contradiction.

Now we consider the case when $\Omega_\rho^\flat$ lies above $\gamma_1 = \{t_2 = c_1\}$. The proof is similar. Take $\bar{t} \in \Omega_\rho^\flat$ near γ_1, that is, $\bar{t} = (\bar{t}_1, \bar{t}_2)$, $\bar{t}_2 > c_1$, $\bar{t}_2$ close to c_1.

Let $V \doteq \{t_2 > c_1\} \setminus \bigcup_{j=1}^{N_0} \Gamma_j$ be as before. We know that the unique unbounded connected component of the sublevel $\{B(t) < \rho\}$ contains an upper half-cylinder. Hence V contains a point $s = (s_1, s_2)$ such that $s \notin \overline{\Omega_\rho^\flat}$. Since $\bar{t}, s \in V$ and $s \neq \bar{t}$,

there exists a path β entirely contained in V connecting $\bar{t}$ and s. But then β must cross $\partial \Omega_\rho^\flat$, a contradiction. Hence $N_1 \neq 1$.

We have completed the proof of our claim, namely, that either $N_1 = 0$ or $N_1 = 2$.

In order to go on with the reduction to the model case, it is important to distinguish between two cases which we proceed to describe.

Case 1. There exists a connected component of a regular bounded sublevel (or superlevel) such that $N_1 = 2$, that is, its boundary has exactly two nontrivial connected components of the level set of B in the cylinder.

Case 2. The boundary of every connected component of a bounded regular sublevel or superlevel of B contains only trivial copies of the unit circle $\mathbb{T}$.

We begin by dealing with Case 1.

We may assume that the nontrivial connected components of the level set ρ are $\{t_2 = c_1\}$ and $\{t_2 = c_2\}$, where $c_2 > c_1$.

Let $t_0 \in \mathbb{T} \times [c_1, c_2]$ be a point of global minimum of B on $\mathbb{T} \times [c_1, c_2]$; note that t_0 is a critical point of B, since the restriction of B to $\overline{\Omega_\rho^\flat}$ attains a global minimum at a point of $\Omega_\rho^\flat$.

Let $\alpha : [0, 2\pi] \to \mathbb{T} \times \mathbb{R}$ be a smooth parametrization of $\{t_2 = c_2\}$ such that $\alpha'(s) \neq 0$, $s \in [0, 2\pi]$, and take $0 < s_1 < s_2 < 2\pi$.

Since $\overline{\Omega_\rho^\flat}$ is a path-connected subset of the cylinder, we may replace $\alpha([s_1, s_2])$ by an arc, called γ^*, which contains the points $\alpha(s_1)$, t_0 and $\alpha(s_2)$, and lies in $\overline{\Omega_\rho^\flat}$, in such a way that the curve $\sigma_1 \doteq \alpha([0, s_1]) \cup \gamma^* \cup \alpha([s_2, 2\pi])$ is a nontrivial non-self-intersecting smooth curve in the cylinder.

We observe that for $t = (t_1, t_2) \in \mathbb{T} \times \mathbb{R} \setminus \Omega_\rho^\flat$ near $\alpha([0, 2\pi])$ with $t_2 > c_2$ we have

$$(4.4) \qquad\qquad\qquad B(t) > \rho.$$

We may assume that σ_1 is $\mathbb{T} \times \{0\}$, $t_0 = (0, 0)$ and $B(0, 0) = 0$.

It follows from (4.4) that the global maximum of B on $\mathbb{T} \times [0, 2\pi]$ occurs in $t_{\max} \in \mathbb{T} \times (0, 2\pi)$ and we may consider another nontrivial smooth curve, σ_2, without self-intersection (for example, the graph of a function of t_2) which contains the points $(0, 0)$, $t_{\max}$ and $(0, 2\pi)$. We may assume that σ_2 is $\{0\} \times \mathbb{T}$, finishing the reduction to the model case in Case 1.

It remains to treat Case 2. We begin with a preliminary result.

Proposition 4.6. *Assume that*

> *(*): the boundary of every bounded regular sublevel or superlevel*
> *of B contains only trivial copies of the unit circle $\mathbb{T}$.*

Then

> i. *every regular level set contains exactly one nontrivial copy of $\mathbb{T}$;*
> ii. *the nontrivial curve γ in (i) is part of the boundary of both the unique unbounded connected component of superlevel and the unique unbounded connected component of sublevel (with the same level r); the superlevel set lies below γ and the sublevel set lies above γ;*
> iii. *the boundary of each unbounded, regular sublevel or superlevel contains exactly one nontrivial curve γ;*
> iv. *if a normal curve γ lies below another normal curve γ', then the value of B over γ is strictly greater than the value of B over γ'; in particular, distinct normal curves have distinct values of B.*

Proof. Proposition 4.3*iv* says that every regular level set contains at least one nontrivial copy of $\mathbb{T}$. Now if this copy were not unique, then the region bounded by two consecutive nontrivial copies of $\mathbb{T}$ would contain a connected component of either a sublevel or a superlevel, the boundary of which would contain both of these nontrivial copies of $\mathbb{T}$ violating our assumption *(*)*. This concludes the proof of *i*.

From Proposition 4.3 we know that γ has a foliated neighborhood and that γ is contained in the boundary of a sublevel as well as in the boundary of a superlevel. It follows from our assumption *(*)* that *ii* holds true.

Property *iii* is an immediate consequence of *i* and *ii*.

We now move on to the proof of *iv*. Let γ_r and $\gamma_{r'}$ be nontrivial connected components of level sets, with levels r and r', respectively, such that $dB \neq 0$ at each point of the curves γ_r and $\gamma_{r'}$.

Then γ_r is the nontrivial part of the boundary of the unbounded connected component of superlevel, denoted by $\Omega^{u,r}$, and γ'_r is the nontrivial part of the boundary of $\Omega^{u,r'}$. We may assume that $\gamma_r = \{t_2 = c\}$ and $\gamma'_r = \{t_2 = c'\}$.

Then $c < c'$ if and only if $\gamma_r \subset \Omega^{u,r'}$ if and only if $\Omega^{u,r} \subset \Omega^{u,r'}$ if and only if $\{B(t) > r\} \subseteq \{B(t) > r'\}$ if and only if $r > r'$. The proof is complete. $\square$

Proposition 4.7. *Assume that*

> *(*): the boundary of every bounded regular sublevel or superlevel*
> *of B contains only trivial copies of the unit circle $\mathbb{T}$.*

Assume that r is a regular value of B and that the superlevel set $\Omega^r = \{t \in \mathbb{T} \times \mathbb{R};\ B(t) > r\}$ is disconnected.

Then there is a diffeomorphism of the three-dimensional torus such that $\gamma \doteq \{t_2 = 0\}$ is a normal curve which is the nontrivial part of the boundary of the unique unbounded connected component of the superlevel Ω^r, and, furthermore, there is a point $p \in \mathbb{T} \times (0, 2\pi)$ such that $B(p) \geq r$.

Proof. We may assume that $r = 0$. We may also start from the situation where the boundary of the unbounded connected component, $\mathcal{U}$, of the superlevel $\Omega^0 \doteq \{t \in \mathbb{T} \times \mathbb{R};\ B(t) > 0\}$ is already the normal curve $\gamma \doteq \{t_2 = 0\}$. What we have to prove is that there is a point $p \in \{0 < t_2 < 2\pi\}$ such that $B(p) \geq 0$.

Take V to be a bounded connected component of the superlevel Ω^0. By assumption (*) the boundary of V consists of a finite number of trivial copies of $\mathbb{T}$. Since γ is a normal curve, it has a neighborhood, W, foliated by nontrivial copies of $\mathbb{T}$. It follows that V cannot intersect any translate of W in the t_2 direction by numbers of the form $2k\pi$, $k \in \mathbb{Z}$. We conclude that there is $k \in \mathbb{Z}$ such that $V \subset \bar{V} \subset \{2k\pi < t_2 < 2(k+1)\pi\}$.

If $k = 0$, then any point p in the boundary of V satisfies $B(p) = 0$.

If $k > 0$ and if $p = (p_1, p_2) \in \partial V$, then $p' \doteq (p_1, p_2 - 2k\pi) \in \{0 < t_2 < 2\pi\}$ and $B(p') = -2kb_{20} > 0$.

If $k < 0$ we use the change of variables $(x, t) \to (-x, -t)$ and fall back into one of the previous two cases. The proof is complete. $\square$

Proposition 4.8. *Assume that $\gamma \doteq \{t_2 = 0\}$ is a normal curve, B is equal to 0 over γ and B has a critical point in $\mathbb{T} \times (0, 2\pi)$. Consider the set W consisting of all $t \in \mathbb{T} \times (0, 2\pi)$ for which there is a normal curve α containing t and such that the region bounded by γ and α is foliated by normal curves. Then there is a point $t_0 \in \partial W$ such that $dB(t_0) = 0$ and $B(t_0) < 0$.*

Proof. Clearly, W is open, nonempty and connected. Also, $dB(t) \neq 0$, for all $t \in W$, hence W is a proper subset of $\mathbb{T} \times (0, 2\pi)$. We may write the boundary of W as a disjoint union $\partial W = \gamma \cup \mathcal{K}$; we call $\mathcal{K}$ the upper part of ∂W. Then $\mathcal{K}$ is a compact subset of a level set, with level < 0 because B is strictly decreasing as one moves transversally to the normal curves in W, away from the initial curve γ. If we had $dB(t) \neq 0$, for each $t \in \mathcal{K}$, then K would be a normal curve contained in W, a contradiction which ends the proof. $\qquad\square$

Proposition 4.9. *In the geometrical situation of the previous propositions, assume that B satisfies $B(t_0) = 0$ and $B(p) > 0$.*

Define M by $M \doteq \max_{t \in \mathbb{T} \times [0,2\pi]} B(t)$; then $M \geq B(p) > 0$. Let $0 < \delta < 1$ be such that $B(t) < M/2$, for all t in the open disc $D(t_0; \delta)$. Then there exists a nontrivial smooth simple curve $\sigma_1 : \mathbb{T} \to \mathbb{T} \times \mathbb{R}$ with $\sigma_1(0) = t_0$ satisfying

    i. $B(\sigma_1(\tau)) < M/2$, $\tau \in \mathbb{T}$;
    ii. the global maximum M of B on $\mathbb{T} \times [0, 2\pi]$ will be attained in the upper half-cylinder relatively to σ_1.

Proof. Let $\mathcal{F}'(s)$ be a normal curve such that $\mathcal{F}'(s) \cap D(t_0, \delta) \neq \varnothing$, which can be smoothly parametrized by $\gamma : [0, 2\pi] \to \mathbb{T} \times \mathbb{R}$ such that $\gamma^{(k)}(0) = \gamma^{(k)}(2\pi)$, $k = 0, 1, \ldots$.

By reparametrizing γ, if necessary, there exist $\tau_1 < \tau_2$ such that $\gamma(\tau_1)$, $\gamma(\tau_2)$ are in $\partial D(t_0, \delta)$ and $\gamma(\tau) \notin D(t_0, \delta)$ if either $\tau < \tau_1$ or $\tau > \tau_2$.

We now replace the arc $\gamma((\tau_1, \tau_2))$ by the juxtaposition of the segments $[\gamma(\tau_1), t_0]$ and $[t_0, \gamma(\tau_2)]$ obtaining a nontrivial, continuous, piecewise smooth curve $\tilde{\gamma}$ passing through t_0. Furthermore, $B(\tilde{\gamma}(\tau)) < \frac{M}{2}$, $\tau \in \mathbb{R}$.

Finally, we smooth out $\tilde{\gamma}$ to obtain a curve σ_1 as required. $\qquad\square$

We are now finally ready to finish the reduction to the model case. We observe that $\max_{t \in \mathbb{T} \times [0,2\pi]} B(t) = B(t_{\max})$, where $t_{\max} \in \mathbb{T} \times (0, 2\pi)$, in particular, the maximum does not occur in $\mathbb{T} \times \{0\}$.

We take a nontrivial smooth simple curve, σ_2 (for example, the graph of a function of t_2) passing through the points t_0, $t_{\max}$ and $t_0 + (0, 2\pi)$.

By making use of a smooth change of variables we may take σ_1 to be $t_2 = 0$ and σ_2 to be $t_1 = 0$. This finishes the reduction to the model case in the remaining case 2.

5. Sufficiency

In this section we prove that $\mathbb{L}^0$ is globally solvable on $\mathbb{T}^3$ if all sublevels Ω_r and all superlevels Ω^r of B are connected.

Using Remark 2.5 and the arguments in the beginning of the previous section, it is enough to consider the case when $b_{10} = 0$ and $b_{20} < 0$.

Suppose, for the sake of reasoning, that there is a solution $u \in \mathcal{D}'(\mathbb{T}^3)$ to $\mathbb{L}^0 u = f$ for some f in the space $\mathbb{E}$ of Definition 2.3. Then, the x-Fourier coefficients of u must satisfy the equation

$$(d_t - nb(t)\wedge)\, \hat{u}(t, n) = \hat{f}(t, n), \ (t, n) \in \mathbb{T}^2 \times \mathbb{Z}.$$

For each $n \in \mathbb{Z}$, we lift this equation to $\mathbb{T} \times \mathbb{R}$ and proceed to analyze it.

Let $t_0 \in \mathbb{T} \times [0, 2\pi]$.

For each $n \in \mathbb{Z}$ the general solution defined on $\mathbb{T} \times \mathbb{R}$ is given by

$$(5.1) \qquad \hat{u}(t, n) = \int_{t_0}^{t} \omega(t, n) + K_n e^{nB(t)},$$

where $\omega(t, n)$ is the exact 1-form

$$(5.2) \qquad \omega(t, n)(\tau) \doteq e^{-n[B(\tau) - B(t)]} \hat{f}(\tau, n), \quad \tau \in \mathbb{T} \times \mathbb{R},$$

and K_n is an arbitrary constant.

If $n = 0$, then

$$(5.3) \qquad \hat{u}(t, 0) = \int_{t_0}^{t} \hat{f}(t, 0) + K_0.$$

In fact, for $n \neq 0$, K_n must be chosen so that $\hat{u}(\cdot, n)$ is a 2π-periodic function in the variable t_2; this yields a unique solution which may be written as

$$(5.4) \qquad \hat{u}(\gamma_{\pm}(s), n) = \int_{0}^{s} g(\sigma, n) \, d\sigma + \frac{1}{e^{\mp n b_{20}} - 1} \int_{0}^{2\pi} g(\sigma, n) \, d\sigma,$$

where g is the function

$$(5.5) \qquad g(\sigma, n) = e^{-n[B(\gamma_{\pm}(\sigma)) - B(\gamma_{\pm}(s))]} \hat{f}(\gamma_{\pm}(\sigma), n) \cdot \gamma_{\pm}{}'(\sigma), \quad \sigma \in [0, 2\pi],$$

and $\gamma_{\pm} : [0, 2\pi] \to \mathbb{T} \times \mathbb{R}$ are paths starting at a point t_0 and ending at $t_0 + (0, \pm 2\pi)$, respectively, whose projections on $\mathbb{T}^2$ are nontrivial smooth closed curves.

We will show that there are integration paths by means of which adequate estimates for the behavior of these solutions can be proved, and from this we will be able to obtain a true solution to the equation $\mathbb{L}^0 u = f$.

5.1. **Integration paths.** The next results and proofs will show how the integration paths are chosen; they are based on propositions 2.3, 2.4 and 2.9 from [6].

Proposition 5.1. *In the commensurable case, assume that all sublevels and all superlevels are connected. Then there is $M > 0$ such that, for all $t_0 \in \mathbb{T} \times [0, 2\pi]$ and $n \in \mathbb{Z}$, there exist smooth paths $\gamma_{\pm} = \gamma_{\pm}(t_0, n)$ joining the points t_0 and $t_0 + (0, \pm 2\pi)$ with length $|\gamma_{\pm}|$ satisfying*

$$|\gamma_{\pm}| < M(1 + |n|), \ \forall n \in \mathbb{Z}$$

and such that

    i. $B(\tau) \leqslant B(t) + \dfrac{1}{1 + |n|}$, for all $\tau \in \gamma_{+}$;

    ii. $B(\tau) \geqslant B(t) - \dfrac{1}{1 + |n|}$, for all $\tau \in \gamma_{-}$.

Proof. We will provide details only for the construction of the path γ_{+}; similarly, one can prove the results concerning γ_{-}.

Let $\alpha = \min_{t \in \mathbb{T} \times [0, 2\pi]} B(t)$ and $\beta = 1 + \max_{t \in \mathbb{T} \times [0, 2\pi]} B(t)$ and consider the compact interval $J \doteq [\alpha, \beta]$. Then there exists $\nu > 0$ such that $\mathbb{T} \times (\nu, \infty) \subset \Omega_r$ and $\mathbb{T} \times (-\infty, -\nu) \subset \Omega^r$ for all $r \in J$.

Set $K_r \doteq \Omega_r \cap (\mathbb{T} \times [-\nu, \nu])$. It is not difficult to prove that K_r is bounded and path-connected.

For each $j \in \mathbb{N}$, consider the collection of all squares contained in $[0, 2\pi] \times \mathbb{R}$ having side length $2\pi/2^j$ and vertices in $(2\pi/2^j) \mathbb{Z}^2$.

Since $\mathbb{T} \times \mathbb{R}$ is locally isometric to $[0, 2\pi] \times \mathbb{R}$ the collection above induces a collection $\mathcal{D}_j$ of sets, which we will continue to call squares, in $\mathbb{T} \times \mathbb{R}$.

For each $t \in \mathbb{T} \times [0, 2\pi]$ and $n \in \mathbb{Z}$, we have

$$r \doteq B(t) + \frac{1}{2(1+|n|)} \in J.$$

Consider the sets $\mathcal{A}_{jr} = \{Q \in \mathcal{D}_j \, ; \, Q \cap K_r \neq \varnothing\}$ and $\mathcal{V}_{jr} = \bigcup_{Q \in \mathcal{A}_{jr}} Q$.

Observe that any point $s \in \mathcal{V}_{jr}$ is contained in a square $Q \subset \mathbb{T} \times \mathbb{R}$ with diagonal equal to $\pi\sqrt{2}/2^{j-1}$, which intersects K_r at a point s', hence

$$B(s) \leqslant |B(s) - B(s')| + B(s') \leqslant ||b||_\infty |s - s'| + r \leqslant ||b||_\infty \frac{\pi\sqrt{2}}{2^{j-1}} + r.$$

Now we take $j \in \mathbb{N}$ such that $2^{j-1} < 4\pi\sqrt{2}(1+|n|)||b||_\infty \leqslant 2^j$, and we have

$$B(s) \leqslant ||b||_\infty \frac{\pi\sqrt{2}}{2^{j-1}} + r \leqslant \frac{1}{2(1+|n|)} + r = B(t) + \frac{1}{1+|n|}, \quad \forall s \in \mathcal{V}_{jr}.$$

K_r is path-connected, hence so is $\mathcal{V}_{jr}$, thus we can join the points t and $t+(0, 2\pi)$ by means of a path γ_+ which is piecewise linear (in the sense that it is made up of sides and diagonals of the squares above), is contained in $\mathcal{V}_{jr}$, and intersects each square in $\mathcal{V}_{jr}$ at most once.

If N is the number of squares in $\mathcal{D}_0$ intersecting K_r, then the number of squares in $\mathcal{D}_j$ intersecting K_r is at most $4^j N$. Since the diagonal of each square in $\mathcal{V}_{jr}$ is $\pi\sqrt{2}/2^{j-1}$, we have

$$|\gamma_+| \leqslant \frac{\pi\sqrt{2}}{2^{j-1}} 4^j N < (4\pi\sqrt{2})^2 N ||b||_\infty (1+|n|) = M(1+|n|).$$

It is standard that γ_+ can be replaced by a smooth path satisfying similar estimates. $\qquad\square$

5.2. Construction of the solution.

For each $f \in \mathbb{E}$, we will show that the equation $\mathbb{L}^0 u = f$ has a solution. In fact, we prove that the formal partial x-Fourier series $u(t, x) = \sum \hat{u}(t, n)e^{inx}$, with coefficients given by (5.3) and (5.4), is a C^∞ solution to this equation.

Lemma 5.2. *For each pair* (α, β) *of nonnegative integers, with* $\alpha + \beta \geqslant 1$ *and for each* $n \neq 0$, *we have*

$$\partial^{(\alpha,\beta)} \left(\int_{t_0}^t \omega(t, n) \right) = n^{\alpha+\beta} \, F_{(\alpha,\beta)} \left(\int_{t_0}^t \omega(t, n) \right) + n^{\alpha+\beta-1} \, G_{(\alpha,\beta)},$$

where

$$F_{(p,q)+e_j} = \begin{cases} b_j & \text{if} \quad p+q = 0, \\ b_j F_{(p,q)} + \frac{1}{n} \, \partial_j F_{(p,q)} & \text{if} \quad p+q \geqslant 1 \end{cases}$$

and

$$G_{(p,q)+e_j} = \begin{cases} \hat{f}_j & \text{if} \quad p+q = 0, \\ \hat{f}_j \, F_{(p,q)} + \frac{1}{n} \, \partial_j G_{(p,q)} & \text{if} \quad p+q \geqslant 1. \end{cases}$$

Proof. Using the formula

$$\partial_j \left(\int_{t_0}^t \omega(t, n) \right) = n b_j(t) \left(\int_{t_0}^t \omega(t, n) \right) + \hat{f}_j(t, n)$$

and proceeding by induction on the order of differentiation, the lemma follows. $\quad\square$

Observe that $F_{(\alpha,\beta)}$ is bounded over $\mathbb{T} \times \mathbb{R}$ because $F_{(\alpha,\beta)}$ depends only on the derivatives of b_j (of order less than $\alpha+\beta$) and on nonpositive powers of n. Moreover, $G_{(\alpha,\beta)}$ is a finite sum, each summand being uniformly bounded, and depending on the derivatives of b_j and $\hat{f}_j$ (of order less than $\alpha + \beta$) and on nonpositive powers of n. Thus, given $(\alpha, \beta) \in \mathbb{Z}_+^2$ and $N > 0$, we obtain a constant $C_{(\alpha,\beta)}$ such that $|G_{(\alpha,\beta)}| \leqslant C_{(\alpha,\beta)} \left(1 + |n|\right)^{-N}$.

Lemma 5.3. *Let $t_0 \in \mathbb{T} \times [0, 2\pi]$. For every $N \in \mathbb{N}$, there exists a constant $C > 0$ such that*

$$\text{(5.6)} \qquad \left| \int_{t_0}^t \omega(t, n) \right| \leqslant \frac{C}{(1 + |n|)^N}, \ \forall n \in \mathbb{Z},$$

where $\omega(t, n)$ was defined in (5.2) and $t \in \gamma_\pm$, where $\gamma_\pm$ is one of the paths obtained in Proposition 5.1 (this choice depends only on the sign of n).

Proof. For $n < 0$, by Proposition 5.1i we have $e^{-n(B(\tau)-B(t))} \leqslant e^{-n\frac{1}{1+|n|}} \leqslant e$, hence

$$\left| \int_{t_0}^t \omega(t, n) \right| \leqslant \sup_{\tau \in \gamma_+} \left| e^{-n\frac{1}{1+|n|}} \hat{f}(\tau, n) \right| C(1 + |n|) \leqslant C(1 + |n|)^{-N}.$$

If $n > 0$, we use Proposition 5.1ii and also obtain (5.6). The case $n = 0$ is trivial. $\qquad \square$

Definition 5.4. The operator $\mathbb{L}^0$ is globally solvable in the C^∞ sense if, for each $f \in \mathbb{E}$, there exists a solution $u \in C^\infty(\mathbb{T}^3)$ to the equation $\mathbb{L}^0 u = f$.

Proposition 5.5. *Suppose that, for each $r \in \mathbb{R}$, the sublevel Ω_r and the superlevel Ω^r are connected. Then the operator $\mathbb{L}^0 = d_t + ib(t) \wedge \partial_x$ is globally solvable in the C^∞ sense.*

Proof. Since $b_{20} < 0$, there is a constant $C_0 > 0$ such that

$$C_0^{-1} \leqslant |\ e^{\mp nb_{20}} - 1| \leqslant C_0, \ \forall n \in \mathbb{Z}, \ \text{with} \ \pm n > 0.$$

Lemma 5.3 implies $|\hat{u}(\gamma_\pm(s), 0)| \leqslant C$ and

$$|\hat{u}(t, n)| \leqslant \left| \int_{t_0}^t \omega(t, n) \right| + \frac{1}{|e^{\mp nb_{20}} - 1|} \left| \int_{t_0}^{t_0 \pm (0, 2\pi)} \omega(t, n) \right|$$

$$\leqslant \frac{C'}{(1 + |n|)^N}, \ \text{for} \ \pm n > 0, \ \text{respectively}.$$

When $\alpha + \beta \geqslant 1$, Lemma 5.2 implies

$$\left| \partial^{(\alpha,\beta)} \hat{u}(t, n) \right| \leqslant |n|^{\alpha+\beta} \frac{C_{(\alpha,\beta),N}}{(1 + |n|)^{N+\alpha+\beta}} C'_{(\alpha,\beta)} + |n|^{\alpha+\beta-1} \frac{C''_{(\alpha,\beta),N}}{(1 + |n|)^{N+\alpha+\beta-1}},$$

thus, for each $N > 0$, there is a constant $C > 0$ such that,

$$|\partial^{(\alpha,\beta)} \hat{u}(t, n)| \leqslant \frac{C}{(1 + |n|)^N}, \ \forall n \neq 0, \ t \in [0, 2\pi]^2,$$

which implies that $u \in C^\infty(\mathbb{T}^3)$. $\qquad \square$

The proof of sufficiency in Theorem 2.4 is complete.

References

[1] A. Bergamasco, *Remarks about global analytic hypoellipticity*, Trans. Amer. Math. Soc. 351 (1999), 4113–4126. MR1603878 (99m:35032)

[2] A. Bergamasco, P. Cordaro, P. Malagutti, *Globally hypoelliptic systems of vector fields*, J. Funct. Anal. 114 (1993), 267–285. MR1223704 (94e:35048)

[3] A. Bergamasco, P. Cordaro G. Petronilho, *Global solvability for certain classes of undedetermined of vector fields*, Math. Z. 223 (1996), 223–274. MR1417431 (98e:58180)

[4] A. Bergamasco, P. Dattori da Silva, *Global solvability for a special class of vector fields on the torus*, in: Contemp. Math. vol. 400, Amer. Math. Soc., Providence, RI, (2006), 11–20. MR2222462 (2006m:35024)

[5] A. Bergamasco, P. Dattori da Silva, *Solvability in the large for a class of vector fields on the torus*, J. Math. Pures Appl. 86 (2006), 427–447. MR2271625 (2007i:35012)

[6] A. Bergamasco, A. Kirilov, *Global solvability for a class of overdetermined systems*, J. Funct. Anal. 252 (2007), 603–629. MR2360930 (2008k:35338)

[7] A. Bergamasco, A. Meziani, *Solvability near the characteristic set for a class of planar vector fields of infinite type*, Ann. Inst. Fourier (Grenoble) 55 (2005), 77–112. MR2141289 (2005m:35028)

[8] A. Bergamasco, W. Nunes, S. Zani, *Global analytic hypoellipticity and pseudoperiodic functions*, Math. Contemp. 18 (2000), 43–57. MR1812862 (2001m:35062)

[9] A. Bergamasco, W. Nunes, S. Zani, *Global properties of a class of overdetermined systems*, J. Funct. Anal. 200 (2003), 31–64. MR1974087 (2004c:35295)

[10] A. Bergamasco, S. Zani, *Prescribing analytic singularities for solutions of a class of vector fields on the torus*, Trans. Amer. Math. Soc. 357 (2005), 4159–4174. MR2159704 (2006c:35002)

[11] A. Bergamasco, S. Zani, *Globally analytic hypoelliptic vector fields on compact surfaces*, Proc. Amer. Math. Soc. 136 (2008), 1305–1310. MR2367104 (2009e:35031)

[12] A. Bergamasco, S. Zani, *Global analytic regularity for structures of co-rank one*, Comm. Partial Differential Equations 33 (2008), 933–941. MR2424383 (2009i:35048)

[13] S. Berhanu, P. Cordaro, J. Hounie *An Introduction to Involutive Structures*. New Mathematical Monographs, 6. Cambridge University Press, Cambridge, 2008. MR2397326 (2009b:32048)

[14] F. Cardoso, J. Hounie, *Global solvability of an abstract complex*, Proc. Amer. Math. Soc. 65 (1977), 117–124. MR0463721 (57:3663)

[15] J. Hounie, *Globally hypoelliptic and globally solvable first order evolution equations*, Trans. Amer. Math. Soc. 252 (1979), 233–248. MR534120 (80h:35035)

[16] A. Meziani, *Hypoellipticity of nonsingular closed 1-forms on compact manifolds*, Comm. Partial Differential Equations 27 (2002), 1255–1269. MR1924466 (2003f:58002)

[17] F. Treves, *Study of a model in the theory of complexes of pseudodifferential operators*, Ann. Math. (2) 104 (1976), 269–324. MR0426068 (54:14014)

[18] F. Treves, *Hypoanalitic Structures (Local Theory)*, Princeton University Press, Princeton, NJ, 1992. MR1200459 (94e:35014)

Departamento de Matemática, ICMC-USP, Caixa Postal 668, 13560-970, São Carlos, SP, Brazil
    E-mail address: apbergam@icmc.usp.br

Departamento de Matemática, UFPR, Caixa Postal 19081, Curitiba, PR, 81531-990, Brazil
    E-mail address: akirilov@ufpr.br

Departamento de Matemática, ICMC-USP, Caixa Postal 668, 13560-970, São Carlos, SP, Brazil
    E-mail address: wvlnunes@icmc.usp.br

Departamento de Matemática, ICMC-USP, Caixa Postal 668, 13560-970, São Carlos, SP, Brazil
    E-mail address: szani@icmc.usp.br