REGULARITY OF MORREY COMMUTATORS

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Abstract. This paper is devoted to presenting a new proof of boundedness of the commutator $bI_\alpha - I_\alpha b$ (in which $I_\alpha$ and $b$ are regarded as the Riesz and multiplication operators) acting on the Morrey space $L^{p,\lambda}$ under $b \in \text{BMO}$, and naturally, developing a regularity theory of commutators for Morrey-Sobolev spaces $I_\alpha(L^{p,\lambda})$ via a completely original iteration of $I_\alpha$. Even in the special case of $I_\alpha(L^p)$, this is a new theory.

1. Introduction

This beginning section outlines our current work and collects the fundamental notation that will be used throughout the paper.

1.1. Overview. This article is two-fold: first to give an alternative argument (based on the new dual principle for Morrey spaces in [10]) for the following implication (cf. [19], [13], [14], [22]):

$$b \in \text{BMO} \implies [b, I_\alpha] : L^{p,\lambda} \to L^{p,\lambda}$$

being bounded under $\lambda > \alpha p$, thereby establishing the corresponding property for some constant $c > 0$:

$$b \in \text{BMO} \implies \sup_{\|f\|_{L^{p,\lambda}} \leq 1} \int_{B(0,1)} \exp(c\|b, I_\alpha\|f(x)\|) \, dx < \infty \quad \text{under } \lambda = \alpha p;$$

second, to develop, with much more effort on relating the traces of Morrey potentials $I_\alpha(L^{p,\lambda})$ in [11] to an entirely new way of constructing both the non-negative function $h$ enjoying

$$[b, I_\alpha]f \lesssim I_\alpha h, \quad \|h\|_{L^{p,\lambda}} \lesssim \|f\|_{L^{p,\lambda}}, \quad |\nabla b| \in L^N$$

and its iterated form, a regularity (or trace, or integrability) theory of the commutators associated with Riesz potential integral operators $I_\alpha$ acting on Morrey spaces $L^{p,\lambda}$ covering Lebesgue spaces $L^p$, namely, to explore the precise values of $q > 0$ and the essential properties of both non-negative Borel measures $\mu$ on $\mathbb{R}^N$ and BMO symbols $b$ such that

$$\sup_{\|f\|_{L^{p,\lambda}} \leq 1} \int_{B(0,1)} \|b, I_\alpha\|f|^q \, d\mu < \infty \quad \text{holds when } \lambda > \alpha p$$

and for some constant $c > 0$,

$$\sup_{\|f\|_{L^{p,\lambda}} \leq 1} \int_{B(0,1)} \exp(c[b, I_\alpha]f^q) \, d\mu < \infty \quad \text{holds when } \lambda = \alpha p.
1.2. Conventions. Before going into the next two sections detailing the above overview, let us agree to some conventions. The \( N \)-dimensional Euclidean space \( \mathbb{R}^N \), the set of positive integers \( \mathbb{Z}^+ \), and the Euclidean open ball \( B(x, r) \) with center \( x \) and radius \( r \) are our working base. On \( \mathbb{R}^N \) the Morrey functions and John-Nirenberg functions — \( f \in L^{p,\lambda} \) (where \( 1 \leq p < \infty \) and \( 0 < \lambda \leq N \) and \( b \in \text{BMO} \) (cf. [18]) mean respectively:

\[
\|f\|_{L^{p,\lambda}} = \left[ \sup_{x \in \mathbb{R}^N \text{ and } r > 0} r^{N-\lambda} \int_{B(x, r)} |f(y)|^p \, dy \right]^{\frac{1}{p}} < \infty
\]

and

\[
\|b\|_{\text{BMO}} = \sup_{x \in \mathbb{R}^N \text{ and } r > 0} \int_{B(x, r)} \left| b - \int_{B(x, r)} b \right| \, dy < \infty,
\]

where \( \int_{B(x, r)} b \) denotes the Lebesgue integral mean of \( b \) over \( B(x, r) \).

Given \( \alpha \in (0, N) \), denote the \( \alpha \)-th order Riesz commutator with symbol \( b \) by

\[
[b, I_{\alpha}]f(x) = b(x)I_{\alpha}f(x) - [I_{\alpha}(bf)](x) = \int_{\mathbb{R}^N} \frac{[b(x) - b(y)]f(y)}{|x-y|^{N+\alpha}} \, dy,
\]

where

\[
I_{\alpha}f(x) = \int_{\mathbb{R}^N} f(y)|x-y|^{\alpha-N} \, dy
\]

represents the \( \alpha \)-th order Riesz operator acting on a given function \( f \) in \( \mathbb{R}^N \).

The notation \( (\partial/\partial x)^sU \) represents the distributional derivative of \( U \) of order \( s = (s_1, \ldots, s_N) \) with \( s_j \in \mathbb{Z}^+ \), i.e.,

\[
\left( \frac{\partial}{\partial x} \right)^s = \left( \frac{\partial}{\partial x_1} \right)^{s_1} \cdots \left( \frac{\partial}{\partial x_N} \right)^{s_N},
\]

and given \( m \in \mathbb{Z}^+ \), \( |D^mU| \) is used as the Euclidean norm of the \( m \)-th order gradient

\[
D^mU = \left\{ \left( \frac{\partial}{\partial x} \right)^sU \right\}_{|s|=m}, \text{ where } |s| = \sum_{j=1}^N s_j.
\]

In the above and below, \( X \lesssim Y \), \( X \gtrsim Y \), and \( X \approx Y \) are employed to denote that there exists a constant \( c > 0 \) such that \( X \leq cy \), \( X \geq cy \), and \( c^{-1}Y \leq X \leq cY \), separately. Moreover, \( f \cdots \) is used whenever an integral domain is clear and the integral is with respect to the Lebesgue measure.

2. Morrey commutators

By a Morrey commutator we mean \([b, I_{\alpha}]f\), where \( f \in L^{p,\lambda} \). In this section, we revisit continuity of \([b, I_{\alpha}]\) with \( b \in \text{BMO} \) under the standard setting \( \lambda > p\alpha \) via giving a new proof, and find out some properties of the commutators under the endpoint setting \( \lambda = p\alpha \). To do so, we need the following Adams’ extension of the classical Sobolev imbedding to Morrey spaces (cf. [3] Theorem 3.1 or [5] Theorem 4.1 & Remark 4.1):

**Theorem 2.1** (Adams). Let \( \alpha \in (0, N) \) and \( \lambda \in (0, N] \).

(i) If \( 1 < p < \frac{\lambda}{\alpha} \), then

\[
\|I_{\alpha}f\|_{L^{\frac{\lambda}{\alpha p},\lambda}} \lesssim \|f\|_{L^{p,\lambda}}.
\]
(ii) If $1 < p = \frac{\lambda}{\alpha}$, then
\[ \|I_\alpha f\|_{\text{BMO}} \lesssim \|f\|_{L^{p,\lambda}}. \]

2.1. All exponents $p \in (1, \frac{\lambda}{\alpha})$. Following [10], which gives a new formulation of the predual to a Morrey space with $\lambda \in (0, N)$, we write
\[ H^{p,\lambda} = \left\{ f \in L^p_{\text{loc}} : \|f\|_{H^{p,\lambda}} = \inf_w \left( \int_{\mathbb{R}^N} |f(y)|^p w(y)^{1-p} \, dy \right)^{\frac{1}{p}} < \infty \right\}, \]
where $L^p_{\text{loc}}$ is the class of all $p$-locally integrable functions on $\mathbb{R}^N$ and the infimum ranges over the class $A_1^{(N-\lambda)}$ of all non-negative weights $w$ belonging to the weight class $A_1$ (whose definition and related $A_p$-weights may be read in e.g. [16], [25], and [26]) and satisfying
\[ \int_{\mathbb{R}^N} w \, d\Lambda_{N,\lambda}^{(\infty)} = \int_0^{\infty} \Lambda_{N,\lambda}^{(\infty)}(\{ x \in \mathbb{R}^N : w(x) > t \}) \, dt \leq 1. \]

In the above and below, for $0 < \lambda < N$ the symbol $\Lambda_{N,\lambda}^{(\infty)}$ is the Hausdorff capacity of order $N - \lambda$, as a set function on $\mathbb{R}^N$.

It is known that if $\alpha \in (0, N)$, $p \in (1, \infty)$, $0 < \alpha p < \lambda$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\lambda}$, then
\[ [b, I_\alpha] : L^{p,\lambda} \to L^{q,\lambda} \iff b \in \text{BMO}. \]

See also [19], [13], [14], [22] and the references therein.

Our contribution to this direction is to present an alternative proof to the above known result.

**Theorem 2.2.** Let
\[ 0 < \lambda, \alpha < N, \ 1 < p < \frac{\lambda}{\alpha}, \ p' = \frac{p}{p-1}, \ \bar{p} = \frac{\lambda p}{\lambda - \alpha p}, \ \text{and} \ \bar{p}' = \frac{\bar{p}}{\bar{p}-1}. \]

Then the following three statements are equivalent:
(i) $b \in \text{BMO};$
(ii) $[b, I_\alpha] : H^{p,\lambda} \to H^{p,\lambda}$ is bounded;
(iii) $[b, I_\alpha] : L^{\bar{p}',\lambda} \to L^{p',\lambda}$ is bounded.

**Proof.** Because (ii)$\Rightarrow$(iii)$\Rightarrow$(i) follows from the dual pairing
\[ \int ([b, I_\alpha]f)g = - \int ([b, I_\alpha]g)f \ \forall \ f \in H^{p,\lambda} \ \& \ g \in L^{\bar{p}',\lambda} \]
and the known implication in [14]-[13], it remains to prove (i)$\Rightarrow$(ii). Suppose $b \in \text{BMO}$. We rely on the following basic estimate, which appeared in [14 Lemma 2], [13 Lemma 2] and [22 Lemma 1]:
\[ ([b, I_\alpha]f)^{\#} \lesssim \|b\|_{\text{BMO}} \left( [M(I_\alpha f)^r]^\frac{1}{r} + [M_{\alpha t} f]^\frac{1}{t} \right) \]
for
\[ 1 < r, t < p < \frac{N}{\alpha} \ \& \ f \geq 0. \]
Here and henceforth,
\[
f^\#(x) = \sup_{x \in B} \left| \frac{1}{\int_B |f|} \int_B f \right|
\]
is the sharp function of \( f \) at \( x \), where the supremum is taken over all Euclidean balls \( B \) containing \( x \), and
\[
M_\beta f(x) = \sup_{r > 0} r^\beta \left| \frac{1}{\int_{B(x,r)} |f|} \int_{B(x,r)} f \right|,
\]
is the fractional maximal function of \( f \); in particular, \( M_0 \) is the standard Hardy-Littlewood maximal operator \( M \).

Instead of solving the original problem, we work on the predual space \( H^{p,\lambda} \). To see this, without loss of generality, we may assume \( 0 \leq f \in H^{p,\lambda} \) and \( 0 \leq g \in L^{q',\lambda} \), where \( q' \) is the conjugate of \( q = \bar{p} \). Due to the following duality inequality (see [25, p. 146] but also the proof of [10, Theorem 4.5]),
\[
(2.2) \quad \left| \int ([b,I_\alpha f]) g \right| \lesssim \int ([b,I_\alpha f])^\# M g,
\]
from (2.1) we need to first handle the following product:
\[
P_1 := \left( \inf_{w \in A_1} \left[ \int \left( I_\alpha f \right)^q w^{1-q} \right] \right)^{\frac{1}{q}} \| M g \|_{L^{q',\lambda}}.
\]
But \( w \in A_1 \), so \( w^{1-q} \in A_q \). However, via the reverse Hölder inequality, we have \( w^{1-q} \in A_{q/r} \) for some \( r > 1 \). So, via \( A_p \) weight theory, we use [10, Theorem 4.7 (ii)] to derive
\[
P_1 \lesssim \left( \inf_{w \in A_1} \left[ \int \left( I_\alpha f \right)^q w^{1-q} \right] \right)^{\frac{1}{q}} \| g \|_{L^{q',\lambda}}.
\]
Next, we look at
\[
P_2 := \left( \inf_{w \in A_1} \left[ \int \left( M_{\alpha t} (f^t) \right)^q w^{1-q} \right] \right)^{\frac{1}{q}} \| M g \|_{L^{q',\lambda}}.
\]
For this, we need the following.

**Lemma 2.3.** If \( w \in A_1 \), then there exists a \( \delta > 0 \) such that \( w^t \in A_1 \) whenever \( 0 < t < 1 + \delta \). In particular, this implication holds for all \( t \) close to 1. Moreover, if \( c_1 \) denotes the “\( A_1 \) constant for \( w \)”, i.e.,
\[
\int_Q w \leq c_1 \inf_Q w \quad \forall \text{ coordinate cubes } Q \subseteq \mathbb{R}^N,
\]
then, with \( c_t \) the “\( A_1 \) constant for \( w^t \)”, we have
\[
c_t \leq \begin{cases} 
  c_1, & 0 < t < 1, 
  (c_1 c_{RH})^t, & 1 < t < 1 + \delta,
\end{cases}
\]
where \( c_{RH} \) stands for the reverse Hölder constant.
In fact, when \(0 < t < 1\), we have that for any coordinate cube \(Q\),
\[
\int_Q w^t \leq \left(\int_Q w^t\right)^\frac{1}{t} \leq c_1^t \inf_Q w^t.
\]
While there is a \(\delta > 0\) such that the reverse Hölder inequality holds with constant \(c_{RN} > 0\) (cf. \[26, p. 230\]), consequently, under \(1 < t < 1 + \delta\),
\[
\left(\int_Q w^t\right)^\frac{1}{t} \leq c_{RH} \int_Q w \leq c_{RH} c_1 \inf_Q w.
\]

Now, we use this lemma as follows: Write
\[
w^{1-q} = \Theta^{1-\frac{1}{q}}, \quad \text{where} \quad \Theta = \frac{w^{\frac{q-1}{p-1}}}{w^{\frac{q}{p}} - 1} \in A_1
\]
since the map \(w \to w^t\) is bijective on \(A_1\). Thus, we have
\[
P_2 \lesssim \|M_{\alpha t}(f^t)\|_{L^{\frac{1}{p}, x}} \|g\|_{L^{q', \lambda}} \lesssim \|f^t\|_{L^{\frac{1}{p}, x}} \|g\|_{L^{q', \lambda}}
\]
thanks to
\[
M_{\alpha t}(f^t) \lesssim I_{\alpha t}(f^t)
\]
and \([10, Theorem 4.7]\). But again using Lemma 2.3, we see that
\[
P_3 : = \left(\inf_{w \in A_1^{(N-\lambda)}} \int (f^t)^{\frac{1}{p}} w^{1-\frac{1}{q}} \right)^\frac{1}{\frac{1}{p}} \|g\|_{L^{q', \lambda}}
\]
\[
\lesssim \left(\inf_{\Theta \in A_1^{(N-\lambda)}} \int f^t \Theta^{1-p} \right)^\frac{1}{\frac{1}{p}} \|g\|_{L^{q', \lambda}}
\]
\[
\lesssim \|f\|_{H^{p, \lambda}} \|g\|_{L^{q', \lambda}}
\]
thereby reaching
\[
P_2 \lesssim P_3 \lesssim \|f\|_{H^{p, \lambda}} \|g\|_{L^{q', \lambda}}.
\]

The previous estimates for \(P_1\) and \(P_2\), along with duality, yield the desired result. \(\square\)

2.2. The borderline \(p = \frac{\lambda}{N} > 1\). The foregoing argument does not apply to the endpoint case \(p = \frac{\lambda}{N}\). But we can settle this via a direct application of (2.1).

**Theorem 2.4.** Let \(1 < p = \frac{\lambda}{N}\) and \(0 < \lambda, \alpha < N\). If \(0 < \|b\|_{\text{BMO}} < \infty\), then there exists a constant \(c > 0\) such that
\[
\sup_{\|f\|_{L^{p, \lambda}} \leq 1} \int_{B(0,1)} \exp \left[\frac{c[b, I_{\alpha}]f(x)}{\|b\|_{\text{BMO}}}\right] dx < \infty.
\]

**Proof.** Without loss of generality, we may assume \(f \geq 0\). We first prove that for each \(\gamma > 0\) there is a constant \(c > 0\) such that
\[
\int_{B(0,1)} \exp[\gamma g(x)] dx \lesssim \int_{B(0,2)} \exp[c \gamma g^\#(x)] dx
\]
holds for \(g > 0\). In fact, following the arguments for \([10]\, \text{Remark 4.6}, \ [16]\, \text{Theorem 3.8} \) and \([23]\, \text{Lemmas 3.5-3.6 \& Theorem 3.7}\), we can use the Calderón-Zygmund decomposition for a cube \(2Q\) (cf. \([17]\, \text{p. 127}\)) with \(Q \subseteq B(0,1) \subseteq 2Q \subseteq B(0,2)\) to
obtain two constants $c_1, c_2 > 0$ independent of both $0 \leq g \in L^{q_0}(B(0,2))$ for some $q_0 \in [1, \infty)$ and $q \geq q_0$ such that

$$\int_{B(0,1)} [g(x)]^q \, dx \leq c_1 c_2^q \int_{B(0,2)} [g^q(x)]^q \, dx.$$  

This, along with

$$\exp t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \quad \forall \ t > 0,$$

yields (2.3).

Next, (2.1) suggests to us to handle two terms. The first one is

$$\int_{B(0,2)} [M(I_\alpha f)^r]^{\frac{k}{r}} \, dx \leq c(k) \int_{B(0,2)} [I_\alpha f(x)]^k \, dx \quad \text{for integer } k > r,$$

where (cf. [26, p. 12])

$$c(k) = \frac{(k/r)2^{k/r}5^N}{(k/r) - 1}.$$  

Now since $0 \leq f \in L^{p,\lambda}$ with $\lambda = \alpha p$, $I_\alpha f \in \text{BMO}$ follows from Theorem 2.1 (ii) with $\alpha = \frac{\lambda}{p}$. By the well-known John-Nirenberg theorem (cf. [18]), there is a constant $c_3 > 0$ such that

$$\sum_{k=0}^{\infty} \frac{c_3^k}{k!} \int_{B(0,2)} [I_\alpha f(x)]^k \, dx < \infty.$$  

This, in turn, induces a constant $c_{3,\ast} > 0$ such that

$$\sum_{k=0}^{\infty} \frac{c_{3,\ast}^k}{k!} \int_{B(0,2)} [M(I_\alpha f)^r(x)]^{k/r} \, dx < \infty.$$

The second one is

$$\int_{B(0,2)} [M_{\alpha t} f^t(x)]^{\frac{k}{t}} \, dx \leq \int_{B(0,2)} [I_{\alpha t} f^t(x)]^{k/t} \, dx \quad \text{for } k = 0, 1, 2, \ldots.$$  

This follows from $M_{\alpha t} f^t \lesssim I_{\alpha t} f^t$. Note also that

$$f^t \in L^{p/t,\lambda} \quad \text{and} \quad (\alpha t)(p/t) = \lambda.$$  

So $I_{\alpha t} f^t \in \text{BMO}$ follows by Theorem 2.1 (ii). Consequently, there is a constant $c_4 > 0$ such that

$$\sum_{k=0}^{\infty} \frac{c_4^k}{k!} \int_{B(0,2)} [M_{\alpha t} f^t(x)]^{k/t} \, dx < \infty.$$
A combination of (2.1) and (2.3) yields that for $\gamma > 0$ and $\|f\|_{L^{p,\lambda}} \leq 1$,

\[
\int_{B(0,1)} \exp \left[ \frac{\gamma \| [b, I_\alpha] f(x) \|}{\| b \|_{\text{BMO}}} \right] dx \\
\lesssim \int_{B(0,2)} \exp \left[ \frac{c\gamma \| [b, I_\alpha] f \|_{\text{BMO}}}{\| b \|_{\text{BMO}}} \right] dx \\
\lesssim \int_{B(0,2)} \exp \left[ cc' \gamma \left( [M(I_\alpha f)^{r}(x)]^{\frac{1}{r}} + [M_{a_1 f}^{s}(x)]^{\frac{1}{s}} \right) \right] dx \\
\lesssim \left( \int_{B(0,2)} \exp \left[ 2cc' \gamma [M(I_\alpha f)^{r}(x)]^{\frac{1}{r}} \right] dx \right)^{\frac{1}{r}} \\
\times \left( \int_{B(0,2)} \exp \left[ 2cc' \gamma [M_{a_1 f}^{s}(x)]^{\frac{1}{s}} \right] dx \right)^{\frac{1}{s}},
\]

where $c' > 0$ is the constant in front of $\| b \|_{\text{BMO}}$ of (2.1).

The last estimate, together with (2.4) and (2.5), derives the desired inequality, via choosing a small $\gamma > 0$.

\[\square\]

**Remark 2.5.** Let $p \geq \frac{\lambda}{\alpha} > 1$, $0 < \lambda \leq N$, $f \in L^{p,\lambda}$ be supported in $B(0,1)$ with $\|f\|_{L^{p,\lambda}} \leq 1$, and $b \in L^\infty$ with

\[\|b\|_{os} = \sup_{(x,r) \in \mathbb{R}^N \times (0,\infty)} \sup_{y \in B(x,r)} |b(x) - b(y)| > 0.\]

(i) If $p = \frac{\lambda}{\alpha}$, then

\[
\int_{B(0,1)} \exp \left[ c \frac{\| [b, I_\alpha] f(x) \|}{\| b \|_{os}} \right] dx < \infty
\]

is valid for any non-negative constant $c$ less than

\[c_{\alpha,\lambda} = \frac{\lambda}{N - \alpha} \left( \frac{\omega_{N-1}}{N} \right)^{\frac{1}{\alpha} - 1},\]

where $\omega_{N-1}$ is the surface area of $B(0,1)$.

(ii) If $p > \frac{\lambda}{\alpha}$, then

\[
\sup_{x \in B(0,1)} \frac{\| [b, I_\alpha] f(x) \|}{\| b \|_{os}} \leq \left( \frac{Np - \lambda}{\alpha p - \lambda} \right)^{\frac{1}{\alpha} - \frac{1}{p}}.
\]

Moreover,

\[ [b, I_\alpha] : L^{p,\lambda} \rightarrow C^{\alpha - \frac{1}{p}} \]

under $\|b\|_{os} < \infty$ and $\lambda/p < \alpha \leq N/p$. This corresponds nicely to the Morrey type embedding below (cf. [10, Remark 3.4]),

\[ I_\alpha : L^{p,\lambda} \rightarrow C^{\alpha - \frac{1}{p}} \quad \text{under } \frac{\lambda}{p} < \alpha \leq \frac{N}{p}, \]

extending the famous Morrey’s lemma (cf. [20] p. 30, 1.53 Theorem): \[ |\nabla u| \in L^{p,\lambda} \& \lambda < p \implies u \in C^{1 - \frac{1}{\lambda}}. \]
3. Regularity properties

Concerning the continuity of an operator, $I_\alpha$ and $[b, I_\alpha]$ (under $b \in \text{BMO}$) behave very similarly. Thus, it is reasonable to expect that the regularity estimates for the Morrey commutators are similar to those for the Morrey potentials under some further assumptions on $b$ since BMO functions are only defined with respect to the Lebesgue $N$-measure. Below is a summary of the regularity estimates for the Morrey potentials established in [11] (see also [1] regarding an early result of the case $\lambda = N$).

**Theorem 3.1** (Adams-Xiao). Let $\mu$ be a non-negative Borel measure on $\mathbb{R}^N$ such that

$$\sup_{(r,x) \in (0, \infty) \times \mathbb{R}^N} \frac{\mu(B(x,r))}{r^d} < \infty \text{ for } N \geq d > \lambda - \alpha p > 0.$$  

(i) If $1 < p < \frac{\lambda}{\alpha}$ and $0 < \lambda < N$, then

$$\sup_{\|f\|_{L^p,\lambda} \leq 1} \int_{B(0,1)} |I_\alpha f|^q \, d\mu < \infty \text{ for } q < \frac{dp}{\lambda - \alpha p},$$

and

$$\sup_{\|f\|_{L^p,\lambda} \leq 1} \int_{B(0,1)} \frac{|I_\alpha f|^p}{\ln(1 + |I_\alpha f|)} \, d\mu < \infty \text{ for } \frac{d}{\lambda - \alpha p} > 2.$$  

(ii) If $1 < p = \frac{\lambda}{\alpha}$ and $0 < \lambda \leq N$, then there exists a constant $c > 0$ such that

$$\sup_{\|f\|_{L^p,\lambda} \leq 1} \int_{B(0,1)} \exp \left( c |I_\alpha f|^q \right) \, d\mu < \infty \text{ holds for either } (\lambda, q) \in (0, N) \times (0, 1] \text{ or } (\lambda, q) = \left( N, \frac{N}{N-1} \right).$$

3.1. **All exponents** $p \in (1, \frac{\lambda}{\alpha})$. Based on Theorem 3.1(i), we discover the following result.

**Theorem 3.2.** Let $|\nabla b| \in L^N$, $\alpha \in (0, N)$, $p > 1$, and $\mu$ be a non-negative Borel measure on $\mathbb{R}^N$ such that

$$\sup_{(r,x) \in (0, \infty) \times \mathbb{R}^N} \frac{\mu(B(x,r))}{r^d} < \infty \text{ holds for } N \geq d > \lambda - \alpha p > 0.$$  

Then

$$\sup_{\|f\|_{L^p,\lambda} \leq 1} \int_{B(0,1)} ||b, I_\alpha f||_q \, d\mu < \infty$$

holds for

either $(\lambda, q) \in (0, N) \times \left( 0, \frac{dp}{\lambda - \alpha p} \right)$ or $(\lambda, q) = \left( N, \frac{dp}{\lambda - \alpha p} \right)$.

**Proof.** First of all, let us control the case

$$(\lambda, q) = \left( N, \frac{dp}{\lambda - \alpha p} \right).$$

Under this, we consider two subcases as follows.
Subcase 1: $\alpha = m \in \mathbb{Z}^+$. Set

$$U = [b, I_\alpha]f \quad \text{and} \quad K_\alpha(x, y) = |x - y|^\alpha - N.$$ 

If $|s| = m$, then according to [3, p. 769], we have

$$|U(x)| \lesssim I_m \left( \sum_{|s| = m} \left| \left( \frac{\partial}{\partial x} \right)^s U(x) \right| \right)$$

holds almost everywhere with respect to the $N$-Lebesgue measure. Note that if $m = 1$, then

$$\left( \frac{\partial}{\partial x} \right)^s [b, I_\alpha](x) = [b, \left( \frac{\partial}{\partial x} \right)^s K_\alpha]f(x) + \left( \frac{\partial}{\partial x} \right)^s b(x) (I_\alpha f)(x).$$

Thus, if $f \in L^{q,\lambda}$ (in which case $\tilde{q} > 1$ will be selected later on, but $\lambda = N$ is always assumed), then Hölder’s inequality and Theorem 2.1 yield

$$\left\| \left( \frac{\partial}{\partial x} \right)^s [b, I_\alpha]f \right\|_{L^{q,\lambda}} \lesssim \left\| b \right\|_{BMO} \left\| f \right\|_{L^{q,\lambda}} + \left\| D^1 b \right\|_{L^{q,\lambda}} \left\| f \right\|_{L^{q,\lambda}}$$

This last estimate and (3.1) give

$$\left| [b, I_\alpha]f \right| \lesssim I_\alpha h,$$

where

$$h = \sum_{|s| = 1} \left| \left( \frac{\partial}{\partial x} \right)^s [b, I_\alpha]f \right|$$

and

$$\left\| h \right\|_{L^{q,\lambda}} \lesssim \left\| \nabla b \right\|_{L^q} \left\| f \right\|_{L^{q,\lambda}}.$$

Next, we take $\theta > 0$ and write the $I_\theta$ potential of the commutator as

$$I_\theta(bI_1f - I_1(bf)) = I_\theta(bI_1f) - bI_{\theta+1}f + bI_{\theta+1}f - I_{\theta+1}(bf) = [b, I_{\theta+1}]f - [b, I_\theta](I_1 f),$$

or

$$[b, I_{\theta+1}]f = [b, I_\theta](I_1 f) + I_\theta([b, I_1]f).$$

Below, we see that identity (3.3) is the key. In fact, from the previous case $m = 1$, we have upon setting $\theta = 1$ in (3.3),

$$\left| [b, I_2]f \right| \lesssim I_1 g + I_2 h,$$

where

$$0 \leq g \in L^{p,\lambda}, \quad \tilde{p} = \frac{\lambda p}{\lambda - p}, \quad \text{and} \quad 0 \leq h \in L^{p,\lambda};$$

i.e., we apply (3.2) first with $\tilde{q} = \tilde{p}$ and then with $\tilde{q} = p$. So, the trace of the commutator can be determined by (3.4). Notice that the remark (ii) of [4, Theorem 4] ensures that

$$\int_{B(0,1)} \left| I_2 h \right|^q d\mu < \infty \quad \text{for} \quad q = \frac{dp}{\lambda - 2p}.$$
and
\[ \int_{B(0,1)} |I_1 g|^r \, d\mu < \infty \quad \text{for} \quad r = \frac{d p}{\lambda - \rho} = \frac{d p}{\lambda - 2 p}. \]

As a result, we obtain that under $|\nabla b| \in L^N$,
\[ \int_{B(0,1)} |[b, I_m] f|^\alpha \frac{d\mu}{s^{\alpha}} < \infty \quad \text{for} \quad m = 1, 2. \]

Now, assume that the assertion is valid for $\alpha = m$. When $\alpha = m + 1$, we apply (3.3) with $\theta = m$ to derive
\[ \| [b, I_{m+1}] f \| \lesssim \| [b, I_m] (I_1 f) \| + I_m \left( [b, I_1] f \right) \lesssim \sum_{j=0}^{m} I_{m+1-j} f_j, \]
where
\[ 0 \leq f_j \in L^{\frac{\lambda p}{\lambda p - \rho}} \quad \text{for} \quad j = 0, 1, \ldots, m \]
and
\[ \int_{B(0,1)} |I_{m+1-j} f_j|^q \, d\mu < \infty \quad \text{for} \quad q = \frac{d p}{\lambda - (m + 1) p}. \]

Clearly,
\[ \int_{B(0,1)} |[b, I_{m+1}] f|^q \, d\mu < \infty \quad \text{for} \quad q = \frac{d p}{\lambda - (m + 1) p}. \]

So, the general case follows by induction.

Subcase 2: $m - 1 < \alpha < m \in \mathbb{Z}^+$. If $m = 1$, i.e., $0 < \alpha < 1$, then according to (3.2) we have
\[ \| [b, I_\alpha] f \| \lesssim \sum_{|s| = 1} \left| \left( \frac{\partial}{\partial x} \right)^s I_{1-\alpha} ([b, I_\alpha] f) \right|, \]
Hence, it is enough to verify that under $|\nabla b| \in L^N$ and $|s| = 1$ one has
\[ f \in L^{p,\lambda} \implies h = \left( \frac{\partial}{\partial x} \right)^s I_{1-\alpha} ([b, I_\alpha] f) \in L^{p,\lambda}. \]

Let us form the operator
\[ T_\alpha (b, f)(x) = \left( \frac{\partial}{\partial x} \right)^s \left( I_{1-\alpha} [b, I_\alpha] f \right), \]
where $f$ may be assumed to be smooth whenever needed. After extending $T_\alpha (b, f)$ to $T_\xi (b, f)$, where $\xi = \zeta + i \eta$ and $0 < \zeta < 1$, we see that $T_\xi (b, \cdot)$ is an analytic family of operators. It remains to check the boundedness of the extended operator on $L^p$ in the cases $\zeta = 0, 1$.

For $\zeta = 0$, we have
\[ T_{i\eta} (b, f) = \left( \frac{\partial}{\partial x} \right)^s \left( I_{1-i\eta} [b, I_{i\eta}] f \right) = \left( \frac{\partial}{\partial x} \right)^s I_{1-i\eta} [b, I_{i\eta}] f. \]

Note that $\left( \frac{\partial}{\partial x} \right)^s I_{1-i\eta}$ exists as a bounded linear operator on $L^p$. Thus
\[ \| T_{i\eta} (b, f) \|_{L^{p,\lambda}} \lesssim \| [b, I_{i\eta}] f \|_{L^{p,\lambda}} \lesssim \| [b]_{BMO} \| f \|_{L^{p,\lambda}} \lesssim \| \nabla b \|_{L^N} \| f \|_{L^{p,\lambda}}. \]
For \( \zeta = 1 \), we get
\[
T_{1+i\eta}(b,f) = \left( \frac{\partial}{\partial x} \right)^*(I_{-i\eta}(bI_{1+i\eta}f - I_{1+i\eta}(bf))) = I_{-i\eta} \left( \left( \frac{\partial}{\partial x} \right)^*b \right) I_{1+i\eta}f + I_{-i\eta}([b,\tilde{K}]f),
\]
where
\[ \tilde{K}(x,\cdot) = \left( \frac{\partial}{\partial x} \right)^*|x|^{1+i\eta-N} \]
is a Calderón-Zygmund integral kernel. As a result, we have
\[
\| [b,\tilde{K}] \|_{L^p,\lambda} \lesssim \| b \|_{BMO} \| f \|_{L^p,\lambda} \lesssim \| \nabla b \|_{L^\infty} \| f \|_{L^p,\lambda}.
\]
Also note that
\[ I_{-i\eta} \]
generates a Calderón-Zygmund integral kernel. So
\[
\| T_{1+i\eta}(b,f) \|_{L^p,\lambda} \lesssim \| \nabla b \|_{L^\infty} \| f \|_{L^p,\lambda}.
\]
Using (3.8) and (3.9) as well as Stein's interpolation theorem (cf. [12, p. 209, Theorem 3.3]), we obtain that under
\[
\| [b,I_\alpha] \|_{L^p,\lambda} \lesssim \| \nabla b \|_{L^\infty} \| f \|_{L^p,\lambda}, \quad 0 < \Re \xi = \zeta < 1.
\]
Thus, \( T_\xi \) is a bilinear analytic family of operators. Consequently, (3.10) is true for \( \xi = \alpha \in (0,1) \). Therefore, (3.10), (3.7) and [8, p. 193, Theorem 7.2.2] yield the required result under \( 0 < \alpha < 1 \).

If \( m > 1 \), then an application of (3.3) and the above treatment of \( m = 1 \) gives the assertion. As a matter of fact, since \( 1 \leq m - 1 < \alpha < m \), there is a \( \theta \in (0,1) \) such that \( \alpha = m - 1 + \theta \). When \( m = 2 \), from (3.3) and the handling of \( m = 1 \) there are two non-negative functions
\[ f_1 \in L^{\frac{m}{\lambda-p},\lambda} \quad \& \quad f_0 \in L^{p,\lambda} \]
such that
\[
[b,I_\alpha]f \lesssim I_\theta f_1 + I_{1+\theta}f_0 \approx I_\theta f_1 + I_\alpha f_0
\]
with
\[
\int_{B(0,1)} |I_\theta f_1| \frac{dp}{\lambda - \alpha p} \, d\mu < \infty, \quad \int_{B(0,1)} |I_\alpha f_0| \frac{dp}{\lambda - \alpha p} \, d\mu < \infty, \quad \& \quad \lambda = N.
\]
Of course, this last argument verifies the result in the case \( m = 2 \). A similar argument works for other values of \( m \) through induction.

Next, let us turn to the case
\[
(\lambda, q) \in (0, N) \times \left( 0, \frac{dp}{\lambda - \alpha p} \right).
\]
We settle this according to two subcases.

Subcase 1: \( \alpha = m \in \mathbb{Z}^+ \). This follows from Theorem (3.4(i)) and the argument for the previously discussed Subcase 1 with
\[ q < \frac{dp}{\lambda - 2p} \quad \& \quad r < \frac{d\tilde{p}}{\lambda - \tilde{p}} \]
as well as
\[ q < \frac{dp}{\lambda - (m + 1)p} \]
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in place of
\[ q = \frac{dp}{\lambda - 2p} \quad \& \quad r = \frac{dp}{\lambda - p} \]
as well as
\[ q = \frac{dp}{\lambda - (m + 1)p}. \]

**Subcase 2:** \( m - 1 < \alpha < m \in \mathbb{Z}^+ \). If \( m = 1 \), i.e., \( 0 < \alpha < 1 \), then since it has just been proved that \( |\nabla b| \in L^N \) ensures that
\[ T_\alpha(b, \cdot) : L^p \to L^p \quad \text{for} \quad 1 < p < \frac{N}{\alpha}, \]
we now use the Garcia-Cuerva and Rubio de Francia extrapolation theorem (cf. [16, pp. 411-412]) to get boundedness of \( T_\alpha(b, \cdot) \) on the weighted Lebesgue space \( L^p(w) \) with \( w \) being an \( A_p \) weight:
\[ T_\alpha(b, \cdot) : L^p(w) \to L^p(w) \quad \text{for} \quad 1 < p < \frac{N}{\alpha}. \]

Consequently, \( T_\alpha^*(b, \cdot) \) (the dual of \( T_\alpha(b, \cdot) \)) and \( p' = \frac{p}{p-1} \) satisfy
\[ T_\alpha^*(b, \cdot) : L^{p'}(w) \to L^{p'}(w). \]

Furthermore, choosing a weight \( w \in A_p \) so that \( w^{-p} \in A_1^{(N-\lambda)} \), we get
\[ T_\alpha^*(b, \cdot) : H^{p', \lambda} \to H^{p', \lambda}. \]

Taking the dual of \( T_\alpha^*(b, \cdot) \), we achieve
\[ T_\alpha(b, \cdot) : L^{p, \lambda} \to L^{p, \lambda}, \quad \text{where} \quad 1 < p < \frac{\lambda}{\alpha} < \frac{N}{\alpha}. \]

Thus, this last boundedness plus (3.3) yields a non-negative function \( h \) such that
\[ |[b, I_\alpha f]| \lesssim I_\alpha h \quad \& \quad \|h\|_{L^{p, \lambda}} \lesssim \|f\|_{L^{p, \lambda}}, \]
thereby implying the desired result via Theorem 3.1(i).

If \( m > 1 \), then there is a \( \beta \in (0, 1) \) and \( m \) non-negative functions \( \{f_j\}_{j=0}^{m-1} \) (by (3.3)) such that \( \alpha = m - 1 + \beta \) and (see also (3.3))
\[ |[b, I_\alpha f]| \lesssim \sum_{j=0}^{m-1} I_{m-1-j + \beta} f_j \]
with
\[ f_j \in L^{\frac{\lambda}{\lambda - \alpha p}, \lambda} \quad \text{for} \quad j = 0, 1, ..., m - 1 \]
and
\[ \int_{B(0,1)} |I_{m-1-j + \beta} f_j|^q \, d\mu < \infty \quad \forall \ q < \frac{dp}{\lambda - \alpha p}. \]
As a result, we get by induction (but also by the above-treated case \( m = 1 \), i.e., \( 0 < \alpha < 1 \))
\[ \int_{B(0,1)} \|I_\alpha f\|^q \, d\mu < \infty \quad \forall \ q < \frac{dp}{\lambda - \alpha p}, \]
as desired. \( \square \)
Remark 3.3. Two comments are in order:
(i) Under $|\nabla b| \in L^N$, $\lambda \in (0, N]$ and $\alpha \in (0, N)$, we can prove a bit more; in fact, since there is a capacitary strong-type inequality (CSI) for $C_\alpha(\cdot; L^p)$ (but not for $C_\alpha(\cdot; L^{p,\lambda})$, $0 < \lambda < N$; see also [9]), one can get the following new trace inequality, actually a CSI:
\[
\int \|[b, I_\alpha]f\|^p \, dC_\alpha(\cdot; L^p) = \int_0^{\infty} C_\alpha(\{x : |[b, I_\alpha]f(x)| > t\}; L^p) \, dt^p \lesssim \|f\|_{L^p}^p.
\]
(ii) It is conjectured that Theorem [8.2] is true for $b \in I_\alpha(L^{p,\lambda}) \subseteq \text{BMO}$ with $0 < \alpha < \lambda \leq N$ (cf. [3] & [6]).

3.2. The borderline $p = \frac{\lambda}{\alpha} > 1$. Regarding the regularity estimates for the Morrey commutators at the endpoint case $\alpha p = \lambda$, we first get the following result.

Theorem 3.4. Let $|\nabla b| \in L^N$, $\alpha \in \mathbb{Z}^+ \cap (0, N)$, $p = \frac{\lambda}{\alpha} = \frac{N}{N-1} > 1$, and $\mu$ be a non-negative Borel measure on $\mathbb{R}^N$ such that
\[
\sup_{(r,x) \in (0,\infty) \times \mathbb{R}^N} \frac{\mu(B(x,r))}{r^d} < \infty \quad \text{holds for } N \geq d > 0.
\]
Then there is a constant $c > 0$ such that
\[
\sup_{\|f\|_{L^p,\lambda} \leq 1} \int_{B(0,1)} \exp\left(c\|[b, I_\alpha]f\|^q\right) \, d\mu < \infty \quad \forall \ q \in \left(0, \frac{N}{N-1}\right).
\]

Proof. First of all, we show that if
\[
E_t = \{x \in \mathbb{R}^N : |[b, I_\alpha]f(x)| > t\} \cap B(0,1) \quad \forall \ t > 0,
\]
then
\[
C_1(E_t; L^N) \lesssim t^{-N} \quad \forall \ t > 0.
\]
To get this, we use the special case $\lambda = N$ of (3.3):
\[
|[b, I_m]f| \lesssim \sum_{j=0}^{m-1} I_{m-j}f_j, \quad \text{where} \quad 0 \leq f_j \in L^{\frac{N}{N-\alpha}}
\]
and [8] Theorem 5.5.1 (d)] to obtain a constant $\kappa > 0$ such that if
\[
E_{j,t} = \left\{x \in \mathbb{R}^N : I_{m-j}f_j(x) > \frac{t}{\kappa}\right\} \cap B(0,1), \quad j = 0, 1, ..., m-1,
\]
then
\[
C_1(E_{j,t}; L^N) \lesssim \sum_{j=0}^{m-1} \left(C_{m-j}(E_{j,t}; L^{\frac{N}{N-\alpha}})^{\frac{N-1}{N-\alpha}}\right)^{\frac{N-1}{N-\alpha}}
\]
and hence, it follows from [2] (cf. [20] 1.86 Lemma) and CSI in [4] (cf. Remark 3.3 (i)) that
\[
\int_{t_0}^{\infty} C_1(E_{j,t}; L^N) \, dt^q \lesssim \sum_{j=0}^{m-1} \left(\int_{t_0}^{\infty} C_{m-j}(E_{j,t}; L^{\frac{N}{N-\alpha}}) \, dt^q\right)^{\frac{N-1}{N-\alpha}}
\]
\[
\lesssim \sum_{j=0}^{m-1} \|f\|_{L^p}^q
\]
\[
< \infty.
\]
holds for \( t_0 > 1 \) being big enough, where

\[
r_j = \left( \frac{N - j}{m - j} - 1 \right) q \text{ & } q < N.
\]

Next, using \cite{[8, Theorem 5.1.12 & Corollary 5.1.14]} it follows that for the given \( \mu \),

\[
C_1(E_t; L^N) \geq \left( \ln \frac{1}{\mu(E_t)} \right)^{1-N},
\]

where \( \mu(E_t) \) is of course assumed to be less than 1. As a result of (3.11) and (3.12), it follows that

\[
\int_{t_0}^{2t_0} \left( \ln \frac{1}{\mu(E_t)} \right)^{1-N} dt^q \lesssim 1
\]

for \( t_0 > 1 \) being big enough. Note that the constant on the right-hand side of (3.13) is independent of \( f \), \( t_0 \), and \( \| f \|_{L^p, \lambda} \leq 1 \). Thus (3.13) yields

\[
\left( \ln \frac{1}{\mu(E_{t_0})} \right)^{1-N} t_0^q \lesssim 1,
\]

whence

\[
\mu(E_t) \lesssim \exp(-c_1 t^{N-1})
\]

holds for some constant \( c_1 > 0 \) and \( t \geq t_0 \). Consequently, there is a constant \( c \in (0, c_1) \) such that

\[
\int_{B(0,1)} \exp \left( c |[b, I_\alpha]f| r \right) d\mu = \int_0^\infty \mu(E_t) d\exp(ct^r) = \left( \int_0^{t_0} + \int_{t_0}^\infty \right) \mu(E_t) d\exp(ct^r) \lesssim \mu(B(0,1)) (\exp(ct_0^r) - 1) + c \int_{t_0}^\infty \exp(-c_1 t^{N-1} + ct^r) dt^r < \infty
\]

holds for all \( r < \frac{q}{N-1} < \frac{N}{N-1} \).

**Remark 3.5.** Theorem 3.4 is optimal in the sense that if \( f \) is a nice function, say \( f \in C_c^\infty (\mathbb{R}^N) \), then \([b, I_\alpha]f\) can blow up on a zero set of \( C_1(\cdot, L^N) \), i.e.,

\[
C_1(\{x \in \mathbb{R}^N : |[b, I_\alpha]f(x)| = \infty\}; L^N) = 0.
\]

The proof uses (3.2) and \cite{[8, Theorem 5.5.1 (d)]}, yielding that if

\[
E_j = \{x \in \mathbb{R}^N : I_{m-j}f_j(x) = \infty\} \text{ & } E = \{x \in \mathbb{R}^N : |[b, I_\alpha]f(x)| = \infty\},
\]

then \( E \subseteq \bigcup_{j=0}^m E_j \) and hence

\[
C_1(E; L^N) \lesssim \sum_{j=0}^{m-1} C_1(E_j; L^N) \lesssim \sum_{j=0}^{m-1} C_{m-j}(E_j; L^N) \left( \frac{(N-1)(m-j)}{N} \right)^{(N-1)(m-j)} = 0.
\]

When a hypothesis on \( b \) stronger than \(|\nabla b| \in L^N\) is placed, the following becomes a natural thing.
**Theorem 3.6.** Let $b \in I_\alpha(L^\infty_N)$, $\alpha \in \mathbb{Z}^+ \cap (0, N)$, $p > 1$, and $\mu$ be a non-negative Borel measure on $\mathbb{R}^N$ such that

$$
\sup_{(r,x) \in (0,\infty) \times \mathbb{R}^N} \frac{\mu(B(x,r))}{r^d} < \infty \quad \text{holds for} \quad N \geq d > \lambda - \alpha p = 0.
$$

Then there is a constant $c > 0$ such that

$$
\sup_{\|f\|_{L^p,\lambda} \leq 1} \int_{B(0,1)} \exp \left( c |[b,I_\alpha]f|^q \right) d\mu < \infty
$$

holds for

either $(\lambda, q) \in (0, N) \times (0, 1]$ or $(\lambda, q) = (N, p^{\frac{p}{p-1}})$.

**Proof.** The assertion follows immediately from Theorem 3.1(ii) and the following.

**Lemma 3.7.** Let $b \in I_\alpha(L^\infty_N)$. Then for any $f \in L^p,\lambda$ there is a non-negative function $h \in L^p,\lambda$ such that

$$
|[b,I_\alpha]f| \lesssim I_\alpha h \quad \text{with} \quad \|h\|_{L^p,\lambda} \lesssim \|f\|_{L^p,\lambda}
$$

provided $\alpha = m \in \mathbb{Z}^+ \cap (0, N)$, $\lambda \in (0, N]$, and $1 < p \leq \frac{N}{\alpha}$.

So, it remains to verify this lemma. Suppose

$$
U = [b,I_\alpha]f, \quad K_\alpha(x,y) = |x-y|^{\alpha-N}, \quad f \in L^p,\lambda.
$$

By the Leibniz rule there are constants $c_{|t|,m}$ depending only on $|t|$ and $m$ such that (cf. \cite{7})

$$
\left( \frac{\partial}{\partial x} \right)^s U(x) = \left[ b, \left( \frac{\partial}{\partial x} \right)^s K_\alpha \right] f(x)
$$

$$
+ \sum_{|t|=1}^{N-1} c_{|t|,m} \int \left( \frac{\partial}{\partial x} \right)^t b(x) \left( \frac{\partial}{\partial x} \right)^{s-t} K_\alpha(x,y) f(y) dy
$$

$$
+ \left( \frac{\partial}{\partial x} \right)^s b(x) \left( I_\alpha f \right)(x)
$$

$$
=: T_1 + T_2 + T_3.
$$

Note that

$$
\left( \frac{\partial}{\partial x} \right)^s K_\alpha(x,\cdot)
$$

is a Calderón-Zygmund integral kernel, and $b \in I_\alpha(L^\infty_N) \subseteq \text{BMO}$ thanks to Theorem \cite{21}ii). So $f \in L^{p,\lambda}$ yields

(3.14)

$$
\|T_1\|_{L^{p,\lambda}} = \left\| \left[ b, \left( \frac{\partial}{\partial x} \right)^s K_\alpha \right] f \right\|_{L^{p,\lambda}} \lesssim \|b\|_{\text{BMO}} \|f\|_{L^{p,\lambda}} \lesssim \|D^m b\|_{L^{N/m}} \|f\|_{L^{p,\lambda}}
$$

by the well-known $L^{p,\lambda}$-boundedness of the commutator $[b,K]$; see e.g.\cite{15} Theorem 2.3 or \cite{21} Theorem 3.3.

Next, the conclusion that $f \in L^{p,\lambda}$ and $b \in I_\alpha L^\infty_N$ imply

(3.15)

$$
T_3 = \left( \frac{\partial}{\partial x} \right)^s b \left( I_\alpha f \right) \in L^{p,\lambda}
$$
follows from the fact that $|D^m b| \in L^{N/m}$ and the following two situations:

**Situation 1:** If $N \geq \lambda > mp$, then an application of Hölder’s inequality and Theorem 2.1(i) yields

$$
\int_{B(z,r)} \left| \frac{\partial}{\partial x} b(x) (I_\alpha f) \right|^p \leq \left( \int_{B(z,r)} \left| \frac{\partial}{\partial x} b(x) \right|^{\lambda/m} \right)^{\frac{pm}{\lambda}} \left( \int_{B(z,r)} |I_m f|^{\frac{p\lambda}{\lambda - mp}} \right)^{\frac{\lambda - mp}{\lambda}}
$$

$\lesssim r^{N-\lambda} \|D^m b\|_{L^{N/m}}^p \|f\|_{L^{p,\lambda}}^p$.

**Situation 2:** If $N \geq \lambda = mp$, then the Hölder inequality and the Sobolev inequality for higher order derivatives on balls (cf. [24 Theorem 3.1]) are applied to derive that for $N > \lambda = mp$,

$$
\int_{B(z,r)} \left| \frac{\partial}{\partial x} b(x) (I_\alpha f) \right|^p \lesssim \|D^m b\|_{L^{N/m}}^p \left( \int_{B(z,r)} |I_m f|^{\frac{pN}{N - pm}} \right)^{\frac{N - pm}{N}}
$$

$\lesssim \|D^m b\|_{L^{N/m}}^p \int_{B(z,r)} \|f\|^p$

$\lesssim r^{N-\lambda} \|D^m b\|_{L^{N/m}}^p \|f\|_{L^{p,\lambda}}^p$.

The case $N = \lambda = mp$ can be treated by using $\|I_m f\|_\infty$.

Finally, let us control the middle terms $T_2$. By Hölder’s inequality, Theorem 2.1(i), and the Sobolev embedding for higher order derivatives over balls once again, we get

$$
\int_{B(z,r)} \left| \frac{\partial}{\partial x} b(x) \left( \int \left( \frac{\partial}{\partial x} \right)^{s-t} K_\alpha(x, y) f(y) \, dy \right) \right|^p \lesssim \int_{B(z,r)} \left| \frac{\partial}{\partial x} b(x) \right|^p |I_{k|}\|I_{k|}f\|^p
$$

$\lesssim \int_{B(z,r)} |D^{k|} b|^{p|} |I_{k|}f\|^p$

$\lesssim \left( \int_{B(z,r)} |D^{k|} b|^{p|} \right)^{\frac{p|}{p}} \left( \int_{B(z,r)} |I_{k|}f\|^{\frac{p\lambda}{\lambda - p|}} \right)^{\frac{\lambda - p\lambda}{p|}}$

$\lesssim r^{N-\lambda} \|D^m b\|_{L^{N/m}}^p \|f\|_{L^{p,\lambda}}$.

whence

$$
\left( \frac{\partial}{\partial x} \right)^s b(x) \left( \int \left( \frac{\partial}{\partial x} \right)^{s-t} K_\alpha(x, y) f(y) \, dy \right) \|_{L^{p,\lambda}} \lesssim \|D^m b\|_{L^{N/m}} \|f\|_{L^{p,\lambda}}.
$$

Putting (3.14)-(3.15)-(3.16) together, we get

$$
\frac{1}{h} = \sum_{|\beta| = m} \left| \frac{\partial}{\partial x} \beta b, I_m f \right| \in L^{p,\lambda},
$$

whence completing the argument for the lemma via (3.1).

**Remark 3.8.** It is conjectured that Theorem 3.6 is valid for non-integral $\alpha \in (0, N)$. On the other hand, it follows readily from

$$
\|b, I_\alpha f\| \leq \|b\|_{L^{1}} I_\alpha |f| \& \|f\|_{L^{p,\lambda}} = \|f\|_{L^{p,\lambda}}
$$

that Theorems 3.4 and 3.2 are trivially true for $b \in L^{\infty}$. 

\[\Box\]
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