TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 364, Number 10, October 2012, Pages 5571–5587
S 0002-9947(2012)05568-1
Article electronically published on May 21, 2012

COMMUTATOR ESTIMATES IN W^*-FACTORS

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ABSTRACT. Let $\mathcal{M}$ be a W^*-factor and let $S(\mathcal{M})$ be the space of all measurable operators affiliated with $\mathcal{M}$. It is shown that for any self-adjoint element $a \in S(\mathcal{M})$ there exists a scalar $\lambda_0 \in \mathbb{R}$, such that for all $\varepsilon > 0$, there exists a unitary element u_ε from $\mathcal{M}$, satisfying $|[a, u_\varepsilon]| \geq (1-\varepsilon)|a - \lambda_0 \mathbf{1}|$. A corollary of this result is that for any derivation δ on $\mathcal{M}$ with the range in an ideal $I \subseteq \mathcal{M}$, the derivation δ is inner, that is, $\delta(\cdot) = \delta_a(\cdot) = [a, \cdot]$, and $a \in I$. Similar results are also obtained for inner derivations on $S(\mathcal{M})$.

1. Introduction

Let $\mathcal{M}$ be a W^*-algebra and $\mathcal{N}$ its W^*-subalgebra, let I be an ideal in $\mathcal{M}$ and let δ be a derivation on $\mathcal{N}$ with the range in an ideal I. The problem studied in [8, 11, 13] can be stated as follows: *What are the conditions on $\mathcal{M}$, $\mathcal{N}$ and I which guarantee that $\delta(\cdot) = \delta_a(\cdot) := [a, \cdot]$, where $a \in I$?* In the present article, we show that the answer is affirmative when $\mathcal{N} = \mathcal{M}$ is an arbitrary W^*-factor and I is an arbitrary ideal in $\mathcal{M}$ (see Corollaries 1 and 2). Our methods are completely different from the methods employed in [8, 11, 13] and are strong enough to enable us (see Corollaries 3 and 4) to also treat an analogous question in a much more general setting of the theory of non-commutative integration on von Neumann algebras, initiated by I.E. Segal [15] (for an alternative approach to this theory, see E. Nelson's paper [12]). All necessary definitions will be given in the next section.

Recall that the classical algebras of measurable operators associated with a von Neumann algebra $\mathcal{M}$ and/or with a pair $(\mathcal{M}, \tau)$ consisting of a semi-finite von Neumann algebra $\mathcal{M}$ and a faithful normal semi-finite trace τ are the following:

(i) the space of all measurable operators $S(\mathcal{M})$ [15];
(ii) the space $S(\mathcal{M}, \tau)$ of all τ-measurable operators [12].

It should be noted that we always have $S(\mathcal{M}, \tau) \subseteq S(\mathcal{M})$, but in the important case when $\mathcal{M}$ is a semi-finite factor (respectively, of type I or III), we have $S(\mathcal{M}) = S(\mathcal{M}, \tau)$ (respectively, $S(\mathcal{M}, \tau) = \mathcal{M}$).

Our main result in this paper is the following theorem.

Theorem 1. *Let $\mathcal{M}$ be a W^*-factor and let $a = a^* \in S(\mathcal{M})$.*

(i) *If $\mathcal{M}$ is a finite factor or else a purely infinite σ-finite factor, then there exists $\lambda_0 \in \mathbb{R}$ and $u_0 = u_0^* \in U(\mathcal{M})$, such that*

$$|[a, u_0]| = u_0^* |a - \lambda_0 \mathbf{1}| u_0 + |a - \lambda_0 \mathbf{1}|, \tag{1}$$

where $U(\mathcal{M})$ is a group of all unitary operators in $\mathcal{M}$;

Received by the editors November 18, 2010 and, in revised form, February 15, 2011.
2010 *Mathematics Subject Classification.* Primary 46L57, 46L51, 46L52.
Key words and phrases. Derivations in von Neumann algebras, measurable operators, ideals of compact operators.

(ii) *there exists* $\lambda_0 \in \mathbb{R}$, *so that for any* $\varepsilon > 0$ *there exists* $u_\varepsilon = u_\varepsilon^* \in U(\mathcal{M})$ *such that*

$$|[a, u_\varepsilon]| \geq (1-\varepsilon)|a - \lambda_0 \mathbf{1}|. \tag{2}$$

If $\mathcal{M}$ *is an infinite semi-finite* σ*-finite factor, then the result stated in* (ii) *above is sharp. More precisely, in this case there exists* $0 \leq a \in S(\mathcal{M})$ *such that for all* $\lambda \in \mathbb{C}$ *and all* $u \in U(\mathcal{M})$ *the inequality* $|[a, u]| \geq |a - \lambda \mathbf{1}|$ *fails.*

The authors thank the referee for a very careful reading of the manuscript and useful suggestions.

2. Preliminaries

For details on von Neumann algebra theory, the reader is referred to e.g. [4], [9], [14] or [17]. General facts concerning measurable operators may be found in [12], [15] (see also [18, Chapter IX]). For the convenience of the reader, some of the basic definitions are recalled.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H equipped with a semi-finite normal faithful trace τ. The set of all self-adjoint projections (respectively, all unitary elements) in $\mathcal{M}$ is denoted by $P(\mathcal{M})$ (respectively, $U(\mathcal{M})$). The algebra $B(H)$ of all bounded linear operators on H is equipped with its standard trace Tr. The commutant of a set $D \subset B(H)$ is denoted by D'. We use the notation $s(x), l(x), r(x)$ to denote the support, left support, right support, respectively, of an element $x \in \mathcal{M}$.

Let $p, q \in P(\mathcal{M})$. The projections p and q are said to be equivalent if there exists a partial isometry $v \in \mathcal{M}$ such that $v^*v = p,\ vv^* = q$. In this case, we write $p \sim q$. The fact that the projections p and q are not equivalent is recorded as $p \nsim q$. If there exists a projection $q_1 \in P(\mathcal{M})$ such that $q_1 \leq p,\ q_1 \sim q$, then we write $q \preceq p$. If $q \preceq p$ and $p \nsim q$, then we employ the notation $q \prec p$.

A linear operator $x : \mathfrak{D}(x) \to H$, where the domain $\mathfrak{D}(x)$ of x is a linear subspace of H, is said to be *affiliated* with $\mathcal{M}$ if $yx \subseteq xy$ for all $y \in \mathcal{M}'$ (which is denoted by $x\eta\mathcal{M}$). A linear operator $x : \mathfrak{D}(x) \to H$ is termed *measurable* with respect to $\mathcal{M}$ if x is closed, densely defined, affiliated with $\mathcal{M}$ and there exists a sequence $\{p_n\}_{n=1}^\infty$ in $P(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subseteq \mathfrak{D}(x)$ and $p_n^\perp$ is a finite projection (with respect to $\mathcal{M}$) for all n. It should be noted that the condition $p_n(H) \subseteq \mathfrak{D}(x)$ implies that $xp_n \in \mathcal{M}$. The collection of all measurable operators with respect to $\mathcal{M}$ is denoted by $S(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products (denoted simply by $x + y$ and xy for all $x, y \in S(\mathcal{M})$).

Let a be a self-adjoint operator affiliated with $\mathcal{M}$. We denote its spectral measure by $\{e^a\}$. It is known that if x is a closed operator in H with the polar decomposition $x = u|x|$ and $x\eta\mathcal{M}$, then $u \in \mathcal{M}$ and $e \in \mathcal{M}$ for all projections $e \in \{e^{|x|}\}$. Moreover, $x \in S(\mathcal{M})$ if and only if x is closed, densely defined, affiliated with $\mathcal{M}$ and $e^{|x|}(\lambda, \infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that in the case when $\mathcal{M}$ is a von Neumann algebra of type III or a type I factor, we have $S(\mathcal{M}) = \mathcal{M}$. For type II von Neumann algebras, this is no longer true.

An operator $x \in S(\mathcal{M})$ is called τ*-measurable* if there exists a sequence $\{p_n\}_{n=1}^\infty$ in $P(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subseteq \mathfrak{D}(x)$ and $\tau\left(p_n^\perp\right) < \infty$ for all n. The collection $S(\tau)$ of all τ-measurable operators is a unital $*$-subalgebra of $S(\mathcal{M})$ denoted by $S(\mathcal{M}, \tau)$. It is well known that a linear operator x belongs to $S(\mathcal{M}, \tau)$ if and only if $x \in S(\mathcal{M})$ and there exists $\lambda > 0$ such that $\tau(e^{|x|}(\lambda, \infty)) < \infty$.

In this paper, we shall frequently assume that $\mathcal{M}$ is a factor. If $\mathcal{M}$ is a semi-finite factor with the trace τ, then the notions of τ-finite and (algebraically) finite projections coincide. An immediate corollary of this observation is that the algebras $S(\mathcal{M})$ and $S(\mathcal{M},\tau)$ coincide in this setting.

3. The proof of Theorem 1

For better readability, we break the theorem's proof into the following series of lemmas.

Lemma 1. *Let $p,q,r \in P(\mathcal{M})$, $p < q$, $p \prec r \prec q$. Then there exists $r_1 \in P(\mathcal{M})$, such that $r_1 \sim r$ and $p < r_1 < q$.*

Proof. There exists $p_1 \in P(\mathcal{M})$, such that $p \sim p_1 < r$. Assume that $r - p_1 \succeq q - p$. Then $r = (r-p_1)+p_1 \succeq (q-p)+p = q$, which contradicts our assumption. Therefore $r - p_1 \prec q - p$. Hence, there exists $p_2 \in P(\mathcal{M})$ such that $r - p_1 \sim p_2 < q - p$. Then $p < p + p_2 < q$ and $p + p_2 \sim p_1 + (r - p_1) = r$. Setting $r_1 = p + p_2$ completes the proof. □

Lemma 2. *Let p be an infinite projection in $\mathcal{M}$. Then:*

(i) *If $P(\mathcal{M}) \ni q_1, ..., q_n, ... \preceq p$, $q_n q_m = 0$ for all $n \neq m$, then $\bigvee_{n=1}^{\infty} q_n \preceq p$.*
(ii) *If $P(\mathcal{M}) \ni q_1, ..., q_n \prec p$, $q_i q_j = 0$ for all $i \neq j$, then $\bigvee_{i=1}^{n} q_i \prec p$.*
(iii) *If $p \succeq \mathbf{1} - p$, then $p \sim \mathbf{1}$.*
(iv) *If $P(\mathcal{M}) \ni q \prec p$, $qp = pq$, then $p(\mathbf{1} - q) \sim p$.*

Proof. (i) Since p is an infinite projection, there exist pairwise disjoint projections $p_1, ..., p_n, ... \in P(\mathcal{M})$ such that $p = \bigvee_{n=1}^{\infty} p_n$, $p_n \sim p$ for all $n \in \mathbb{N}$. Then $p_n \succeq q_n$ for all $n \in \mathbb{N}$. Hence, $p = \bigvee_{n=1}^{\infty} p_n \succeq \bigvee_{n=1}^{\infty} q_n$.

(ii) Since $\mathcal{M}$ is a factor, every projection is comparable to every other projection. Thus reordering if necessary, we may assume that $q_1 \preceq q_2 \preceq ... \preceq q_n$. If q_n is a finite projection, then $\bigvee_{i=1}^{n} q_i$ is a finite projection and $\bigvee_{i=1}^{n} q_i \prec p$. If q_n is infinite, then by (i), $\bigvee_{i=1}^{n} q_i \preceq q_n \prec p$.

(iii) Since $\mathbf{1} = p + (\mathbf{1} - p)$, it follows from (i) that $\mathbf{1} \preceq p$.

(iv) We have $p = qp + (\mathbf{1} - q)p$, $qp \leq q \prec p$. If it were true that $(\mathbf{1} - q)p \prec p$, then by (ii) we would have $p \prec p$, which is false. Thus $(1 - q)p \succeq p$ and certainly $(1 - q)p \preceq p$. The result follows immediately. □

In the special case when $\mathcal{M}$ is semi-finite and a is positive, it may be of interest to compare the result given below with [6, Theorem 3.5] and [2, Lemma 4.1].

Lemma 3. *Let $a \in S_h(\mathcal{M})$ and $p, q \in P(\mathcal{M})$, $p \succeq q$. Suppose that one of the following conditions holds:*

(i) *q is finite and there exists a sequence of finite projections $\{p_n\}$ in $\mathcal{M}$ such that $p_n \uparrow p$ and $ap_n = p_n a$ for all $n \in \mathbb{N}$;*
(ii) *q is an infinite projection and $ap = pa \in \mathcal{M}$.*

Then there exists a projection $q_1 \in P(\mathcal{M})$ such that $q_1 \sim q$, $aq_1 = q_1 a$ and such that $q_1 \leq p$.

Proof. Assume (i) holds. By the assumption $\mathcal{M}$ contains finite projections and therefore $\mathcal{M}$ is a factor of type I or else of type II. Therefore $\mathcal{M}$ admits a faithful normal semifinite trace τ. Let D be a commuting family given by the spectral measure $\{e^a\}$ and let $\mathcal{A}_1 := D' \cap \mathcal{M}$. Since $ap_n = p_n a$ for all $n \in \mathbb{N}$ and $p_n \uparrow p$, we also have $ap = pa$. Therefore $p \in \mathcal{A}_1$. Then $\mathcal{A} := p\mathcal{A}_1 p = \mathcal{A}_1 p$ is a W^*-subalgebra in $\mathcal{M}$ with the unit p. Let e be an atom in $\mathcal{A}$ and let $f \in P(\mathcal{M})$ be such that $f < e$. Then for every $t \in \{e^a\}$ we have $tp = pt \in P(\mathcal{A})$ and so $tf = t(p(ef)) = ((tp)e)f \in \{0, e\}f = \{0, ef\} = \{0, f\}$, that is, $tf = ft$. Therefore $f \in P(\mathcal{A})$ and since e is an atom in $\mathcal{A}$ we conclude that $f = 0$. Therefore e is also an atom in $\mathcal{M}$.

In the set $P(\mathcal{A})$ we select the subset $M(q) = \{r \in P(\mathcal{A}) : \ \tau(r) \geq \tau(q)\}$. If $\tau(q) = \tau(p)$, then $q_1 := p \sim q$ and the proof is finished. Therefore, we assume below that $\tau(q) < \tau(p)$. Observing that $p_n \in \mathcal{A}_1$, $n \geq 1$ and that $\tau(q) < \infty$, $\tau(p) > \tau(q)$, $\tau(p_n) \uparrow \tau(p)$, we see that there exists $n \geq 1$, such that $\tau(p_n) > \tau(q)$. This shows, in particular, that $M(q)$ is not empty. Let $\mathfrak{C}$ be a linearly ordered family in $M(q)$. Then the mapping $\tau|_{\mathfrak{C}}$ into the interval $[\tau(q), \tau(p)]$ is injective and order-preserving. Since the trace τ is normal, we have $\tau(\bigwedge \mathfrak{C}) = \bigwedge_{r \in \mathfrak{C}} \tau(r) \geq \tau(q)$. Therefore $\bigwedge \mathfrak{C} \in M(q)$. This shows that the set $M(q)$ satisfies Zorn's lemma assumption and therefore it has a minimal element. Let r_0 be a minimal element in $M(q)$. If $\tau(r_0) = \tau(q)$, then we set $q_1 := r_0 \sim q$ and the proof is finished. Suppose that $\tau(r_0) > \tau(q)$. Moreover, consider the set $N(q) = \{r \in P(\mathcal{A}) : \ \tau(r) \leq \tau(q), \ r \leq r_0\}$. This set is not empty, in particular $0 \in N(q)$. Arguing as above, we see that $N(q)$ has a maximal element r_1. We claim that $\tau(r_1) < \tau(q)$. Indeed, if it is not so, then $\tau(r_1) = \tau(q)$ and $r_0 > r_1 \in M(q)$, which contradicts the assumption that r_0 is minimal. Thus, $\tau(r_1) < \tau(r_0)$, that is, $r_0 - r_1 > 0$. Observe that $r_0 - r_1$ is an atom in $\mathcal{A}$. Indeed, if there exists $0 < f < r_0 - r_1, \ f \in P(\mathcal{A})$, then either $\tau(r_1) < \tau(r_1 + f) \leq \tau(q)$ (which contradicts the assumption that r_1 is maximal), or else $\tau(r_0) > \tau(r_1 + f) \geq \tau(q)$, which contradicts the assumption that r_0 is minimal. Thus, $r_0 - r_1$ is an atom in $\mathcal{A}$ and hence, as we already observed above, it is also an atom in $\mathcal{M}$. On the other hand, it follows from Lemma 1 that there exists $r_2 \in P(\mathcal{M})$, such that $r_2 \sim q, \ r_1 < r_2 < r_0$. In particular, $0 < r_2 - r_1 < r_0 - r_1$, that is, $r_0 - r_1$ is not an atom. We have arrived at the contradiction. Therefore, $q_1 = r_0 \sim q$.

Assume (ii) holds. By the assumption there exists a projection $q_1^0 \in \mathcal{M}$ such that $q_1^0 \leq p$ and $q_1^0 \sim q$. We set $q_1^n := l(a^n q_1^0)$ for all $n > 0$, $q_1 := \bigvee_{k=0}^{\infty} q_1^k$. We claim that $q_1 \sim q$. Indeed, since $q_1 \geq q_1^0 \sim q$, we have $q_1 \succeq q$. On the other hand, we have $q_1^n \sim r(a^n q_1^0) \leq q_1^0 \sim q$, which implies $q_1^n \preceq q$ for all $n \geq 0$. Now, we shall show that in fact $q_1 \preceq q$. Note that although q is an infinite projection we cannot simply refer to Lemma 2(i) since the sequence $\{q_1^k\}_{k \geq 0}$ does not necessarily consist of pairwise orthogonal elements. However, representing the projection q_1 as

$$q_1 = \bigvee_{k=0}^{\infty} q_1^k = \sum_{m=1}^{\infty} (\bigvee_{k=0}^{m} q_1^k - \bigvee_{k=0}^{m-1} q_1^k) + q_1^0 = \sum_{m=1}^{\infty} (q_1^m \vee \bigvee_{k=0}^{m-1} q_1^k - \bigvee_{k=0}^{m-1} q_1^k) + q_1^0,$$

and noting that $q_1^0 \sim q$ and

$$q_1^m \vee \bigvee_{k=0}^{m-1} q_1^k - \bigvee_{k=0}^{m-1} q_1^k \sim q_1^m - (q_1^m \wedge \bigvee_{k=0}^{m-1} q_1^k) \leq q_1^m \preceq q$$

we infer via Lemma 2(i) that $q_1 \preceq q$. This completes the proof of the claim.

Since $ap = pa$ and $q_1^0 \leq p$, we have $pa^n q_1^0 = a^n p q_1^0 = a^n q_1^0$, and so $q_1^n \leq p$ for all $n > 0$. Hence, $q_1 \leq p$. It remains to show that $aq_1 = q_1 a$. The subspace $q_1(H)$ coincides with the closure of the linear span of the set $Q := \{a^n q_1^0(H) : \ n > 0\}$. By the assumption the operator ap is bounded, and since $q_1 \leq p$, the operator aq_1

is also bounded. Thus, for every vector $\xi \in Q$, the vector $a\xi = aq_1\xi$ again belongs to Q. Again appealing to the fact that aq_1 is bounded, we infer $q_1aq_1 = aq_1$. From this we conclude that $aq_1 = q_1a$. □

Lemma 4. *Let $\mathcal{M}$ be an infinite factor and let $e^a(-\infty,0] \prec e^a(0,+\infty)$, $e^a(0,\lambda] \succ e^a(-\infty,0]$ and $e^a(0,\lambda] \succ e^a(\lambda,+\infty)$ for all $\lambda > 0$. Then for all $\varepsilon > 0$ there exists $u_\varepsilon = u_\varepsilon^* \in U(\mathcal{M})$ such that $||[a,u_\varepsilon]| \geq (1-\varepsilon)|a|$.*

Proof. Certainly the result is trivial for $\varepsilon \geq 1$ and so we restrict ourselves to the case of $\varepsilon < 1$. Our aim is to build a decreasing sequence of positive scalars $\{\lambda_n\}_{n=0}^\infty$ converging to zero and two sequences $\{p_n\}_{n=0}^\infty$, $\{q_n\}_{n=0}^\infty$ of pairwise orthogonal projections from $\mathcal{M}$, which satisfy:

(i) $p_nq_m = 0,\ ap_n = p_na,\ aq_n = q_na,\ p_n \sim q_n$ for all $n, m \geq 0$.
(ii) $p_n \leq e^a(\lambda_n,+\infty),\ q_n \leq e^a(-\infty,\varepsilon\lambda_n]$ for all $n \geq 0$ and $q_0 \geq e^a(-\infty,0]$.
(iii) $\bigvee_{n=0}^\infty p_n \vee \bigvee_{n=0}^\infty q_n = \mathbf{1}$.

Consider the three cases:

(a) Suppose that the projection $e^a(-\infty,0]$ and all the projections $e^a(\lambda,+\infty)$ for all $\lambda > 0$ are finite. Then $e^a(0,\lambda]$ is the supremum of an increasing sequence of finite projections $\{e^a(\lambda/n,\lambda]\}_{n\geq 1}$ for all $\lambda > 0$. We claim that there exists $\lambda_0 > 0$ such that $e^a(\lambda_0,+\infty) \succ e^a(-\infty,0]$. Indeed, $e^a(0,+\infty) = \bigvee_{n=1}^\infty e^a(1/n,+\infty)$, where, by the assumption every projection $e^a(1/n,+\infty)$, $n \geq 1$ is finite. Therefore, if it were $e^a(1/n,+\infty) \preceq e^a(-\infty,0]$ for all $n \in \mathbb{N}$, we would then have $e^a(0,+\infty) \preceq e^a(-\infty,0]$ (see e.g. [17, Chapter V, Lemma 2.2]), which is not the case. Thus, our claim holds and there exists $r \in \mathcal{M}$ such that $e^a(-\infty,0] \sim r < e^a(\lambda_0,+\infty)$. Now, we claim the existence of a converging to zero sequence $\{\lambda_n\}_{n=0}^\infty$ of positive numbers such that $e^a(\lambda_{n+1},+\infty) \succ e^a(\lambda_n,+\infty)$ for all $n \geq 0$. This claim is justified by the same argument as above: since $e^a(0,+\infty)$ is an infinite projection and since $e^a(1/n,+\infty) \uparrow e^a(0,+\infty)$ for $n \to \infty$, we see that for any finite projection $q \in \mathcal{M}$ there exists $n \geq 1$, such that $e^a(1/n,+\infty) \succ q$ (again by [17, Chapter V, Lemma 2.2]). Indeed, if it were true that $e^a(1/n,+\infty) \preceq q$, for all $n \geq 1$, we would then have $e^a(0,+\infty) \preceq q$, which is false, since the projection $e^a(0,+\infty)$ is infinite, whereas the projection q is finite.

(b) Suppose that the projection $e^a(-\infty,0]$ is finite and there exists a number $\lambda_0 > 0$, such that $e^a(\lambda_0,+\infty)$ is an infinite projection. Then there exists a projection $r \in \mathcal{M}$ such that $e^a(-\infty,0] \sim r < e^a(\lambda_0,+\infty)$. In addition, $e^a(\lambda_0,+\infty) - r$ is an infinite projection.

(c) Suppose that the projection $e^a(-\infty,0]$ is infinite. Then there exists a scalar $\lambda_0 > 0$ such that $e^a(-\infty,0] \prec e^a(\lambda_0,+\infty)$. Indeed if the opposite inequality were to hold for every $\lambda > 0$, then Lemma 2(i) would yield the estimate

$$e^a(0,+\infty) = \sum_{k=1}^\infty e^a(1/(k+1),1/k] + e^a(1,+\infty) \preceq e^a(-\infty,0],$$

which contradicts the assumption $e^a(-\infty,0] \prec e^a(0,\lambda] \leq e^a(0,+\infty)$ for any $\lambda > 0$. Therefore there exists a projection $r \in \mathcal{M}$ such that $e^a(-\infty,0] \sim r < e^a(\lambda_0,+\infty)$ and $e^a(\lambda_0,+\infty) - r$ is an infinite projection. The latter follows from the equivalence $e^a(\lambda_0,+\infty) - r \sim e^a(\lambda_0,+\infty)$, which, in turn, follows from Lemma 2(iv) with $p = e^a(\lambda_0,+\infty)$, $q = r$.

In all these cases, let us set $p_0 := e^a(\lambda_0, +\infty)$. Since, by the assumption, $e^a(0, \varepsilon\lambda_0] \succ e^a(\varepsilon\lambda_0, +\infty) \geq e^a(\lambda_0, +\infty)$, we have $e^a(0, \varepsilon\lambda_0] \succ p_0$, it follows from Lemma 3(i) in case (a) and from Lemma 3(ii) in cases (b) and (c) that there exists a projection $q_0^1 \in \mathcal{M}$, for which $p_0 - r \sim q_0^1 < e^a(0, \varepsilon\lambda_0]$ and $aq_0^1 = q_0^1 a$. Let us set $q_0 := e^a(-\infty, 0] + q_0^1$. Since $r \sim e^a(-\infty, \lambda)$ in all three cases, we have $q_0 \sim p_0$.

Now, similar to case (a), we shall show that in the cases of (b) and (c) there also exists a decreasing sequence of positive real numbers $\{\lambda_n\}_{n=0}^{\infty}$, which converges to 0 and such that $e^a(\lambda_{n+1}, +\infty) \succ e^a(\lambda_n, +\infty)$ for all $n \geq 0$. To this end, it is sufficient to show that for every $\lambda > 0$ the inequality $e^a(t, +\infty) \sim e^a(\lambda, +\infty)$ for all $t \in (0, \lambda)$ does not hold. Suppose the opposite, and let a scalar λ be such that for all $t \in (0, \lambda)$, we have $e^a(t, +\infty) \sim e^a(\lambda, +\infty)$. Then we have $e^a(\lambda/(k+1), \lambda/k] \leq e^a(\lambda/(k+1), +\infty) \sim e^a(\lambda, +\infty)$ for every $k \geq 1$, that is, $e^a(\lambda/(k+1), \lambda/k] \preceq e^a(\lambda, +\infty)$, and so by Lemma 2(i), it follows

$$e^a(0, \lambda] = \sum_{k=1}^{\infty} e^a(\lambda/(k+1), \lambda/k] \preceq e^a(\lambda, +\infty).$$

However, this contradicts our initial assumption that $e^a(0, \lambda] \succ e^a(\lambda, +\infty)$.

Now, we are well equipped to proceed with the construction of the sequences $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$.

Suppose the projections $p_0, ..., p_n$, $q_0, ..., q_n$ have already been constructed. Set

$$p_{n+1} = e^a(\lambda_{n+1}, +\infty) \prod_{k=0}^{n} (\mathbf{1} - p_k) \prod_{k=0}^{n} (\mathbf{1} - q_k).$$

In case (a), all the projections p_k, q_k for $k \leq n$ are finite and $e^a(0, \varepsilon\lambda_{n+1}]$ is an infinite projection. Hence, $e^a(0, \varepsilon\lambda_{n+1}] \prod_{k=0}^{n} (\mathbf{1} - p_k) \prod_{k=0}^{n} (\mathbf{1} - q_k)$ is an infinite projection for all $n \geq 1$. We shall now explain to the reader that we are now in a position to apply Lemma 3(i) and infer that there exists a projection $q_{n+1} \in \mathcal{M}$ such that

$$p_{n+1} \sim q_{n+1} < e^a(0, \varepsilon\lambda_{n+1}] \prod_{k=0}^{n} (\mathbf{1} - p_k) \prod_{k=0}^{n} (\mathbf{1} - q_k)$$

and $aq_{n+1} = q_{n+1} a$. To see that Lemma 3(i) is indeed applicable, set $p := e^a(0, \varepsilon\lambda_{n+1}] \prod_{k=0}^{n} (\mathbf{1} - p_k) \prod_{k=0}^{n} (\mathbf{1} - q_k)$ and $q := p_{n+1}$. Observe that here p is infinite and q is finite, in particular $p \succ q$. The role of finite projections p_m from that lemma is then played by the sequence $\{e^a(1/m, \varepsilon\lambda_{n+1}) \prod_{k=0}^{n} (\mathbf{1} - p_k) \prod_{k=0}^{n} (\mathbf{1} - q_k)\}_{m \geq 1}$ Observe that $e^a(1/m, \varepsilon\lambda_{n+1}) \uparrow_m e^a(0, \varepsilon\lambda_{n+1})$ and this sequence obviously commutes with the operator a. This completes the construction in case (a).

Now let us consider cases (b) and (c). Since $p_k \leq e^a(\lambda_k, +\infty)$, we have

$$\sum_{k=0}^{n} p_k = \bigvee_{k=0}^{n} p_k \leq \bigvee_{k=0}^{n} e^a(\lambda_k, +\infty) = e^a(\lambda_n, +\infty)$$

and so

$$\sum_{k=0}^{n} p_k \leq e^a(\lambda_n, +\infty) \prec e^a(\lambda_{n+1}, +\infty),\ n \geq 1.$$

Since $q_k \sim p_k \prec e^a(\lambda_{n+1}, +\infty)$ for all $k = 0, 1, 2, \ldots, n$, we obtain, via an application of Lemma 2(ii), that

$$\sum_{k=0}^{n} p_k + \sum_{k=0}^{n} q_k \prec e^a(\lambda_{n+1}, +\infty).$$

We shall now explain that it easily follows from the preceding estimate that the projection

$$p_{n+1} = e^a(\lambda_{n+1}, +\infty) \prod_{k=0}^{n} (\mathbf{1} - p_k) \prod_{k=0}^{n} (\mathbf{1} - q_k)$$

is infinite. Indeed, assume for a moment that the projection $e^a(\lambda_{n+1}, +\infty)[\mathbf{1} - \bigvee_{k=0}^n p_k \vee \bigvee_{k=0}^n q_k]$ is finite. In this case, $p_{n+1} \prec e^a(\lambda_{n+1}, +\infty)$ and so $e^a(\lambda_{n+1}, +\infty) = e^a(\lambda_{n+1}, +\infty)[\bigvee_{k=0}^n p_k \vee \bigvee_{k=0}^n q_k] + p_{n+1} \prec e^a(\lambda_{n+1}, +\infty)$ (Lemma 2(ii)). This contradiction shows that the assumption just made is false.

Now, by Lemma 2(iv), we first deduce that

$$p_{n+1} \sim e^a(\lambda_{n+1}, +\infty),$$

next, by the assumption of Lemma 4, we have

$$e^a(\lambda_{n+1}, +\infty) \leq e^a(\varepsilon\lambda_{n+1}, +\infty) \prec e^a(0, \varepsilon\lambda_{n+1}],$$

and finally, again by Lemma 2(iv),

$$e^a(0, \varepsilon\lambda_{n+1}] \sim e^a(0, \varepsilon\lambda_{n+1}] \prod_{k=0}^{n} (\mathbf{1} - p_k) \prod_{k=0}^{n} (\mathbf{1} - q_k).$$

Thus,

$$p_{n+1} \prec e^a(0, \varepsilon\lambda_{n+1}] \prod_{k=0}^{n} (\mathbf{1} - p_k) \prod_{k=0}^{n} (\mathbf{1} - q_k),$$

and, therefore, it follows from Lemma 3(ii) that there exists

$$P(\mathcal{M}) \ni q_{n+1} < e^a(0, \varepsilon\lambda_{n+1}] \prod_{k=0}^{n} (\mathbf{1} - p_k) \prod_{k=0}^{n} (\mathbf{1} - q_k)$$

such that $q_{n+1} \sim p_{n+1}$ and $aq_{n+1} = q_{n+1}a$.

Thus the projections p_{n+1} and q_{n+1} are defined and so the construction of the sequences $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ is also completed for cases (b) and (c).

It is clear from the construction that for all these sequences conditions (i) and (ii) hold. To see that condition (iii) also holds, we first make the claim that

$$e^a(-\infty, 0] + e^a(\lambda_n, +\infty) \leq \bigvee_{k=0}^{n} p_k \vee \bigvee_{k=0}^{n} q_k, \ n \geq 1.$$

To see that the estimate above indeed holds, observe that by the construction, we have $e^a(-\infty, 0] \leq q_0$ and that by the definition $p_{n+1} := e^a(\lambda_{n+1}, +\infty)[\mathbf{1} - \bigvee_{k=0}^n p_k \vee \bigvee_{k=0}^n q_k]$. Therefore $\bigvee_{k=0}^n p_k \vee \bigvee_{k=0}^n q_k \vee p_{n+1} \geq e^a(\lambda_{n+1}, +\infty)$ for all $n \geq 1$, which completes the justification of the claim above. Now, running $n \to \infty$ we arrive at condition (iii).

Now, we can proceed with the construction of the unitary operator $u_\varepsilon \in \mathcal{M}$ from the assertion.

Let $v_n \in \mathcal{M}$ be a partial isometry such that $v_n^* v_n = p_n,\ v_n v_n^* = q_n,\ n = 0, 1, \ldots.$ We set

$$u_\varepsilon = \sum_{n=0}^{\infty} v_n + \sum_{n=0}^{\infty} v_n^*$$

(here, the sums are taken in the strong operator topology). Then we have

$$u_\varepsilon^* u_\varepsilon = \sum_{n=0}^{\infty} p_n + \sum_{n=0}^{\infty} q_n = \mathbf{1},\ u_\varepsilon u_\varepsilon^* = \sum_{n=0}^{\infty} q_n + \sum_{n=0}^{\infty} p_n = \mathbf{1}.$$

Observe that

$$u_\varepsilon p_n = q_n u_\varepsilon,\ u_\varepsilon q_n = p_n u_\varepsilon,\ a p_n = p_n a,\ q_n a = a q_n,\ n \geq 0,$$

and so the element $u_\varepsilon^* a u_\varepsilon$ commutes with all the projections p_n and q_n, $n \geq 0$. Moreover, since for all $n \geq 0$,

$$a p_n = a e^a(\lambda_n, +\infty) p_n \geq \lambda_n e^a(\lambda_n, +\infty) p_n = \lambda_n p_n,$$
$$a q_n = a e^a(-\infty, \varepsilon\lambda_n) q_n \leq \varepsilon\lambda_n e^a(-\infty, \varepsilon\lambda_n) q_n = \varepsilon\lambda_n q_n$$

hold, we obtain immediately for all such n's that

$$u_\varepsilon^* a u_\varepsilon p_n = u_\varepsilon^* a q_n u_\varepsilon \leq \varepsilon\lambda_n u_\varepsilon^* q_n u_\varepsilon = \varepsilon\lambda_n p_n,$$
$$u_\varepsilon^* a u_\varepsilon q_n = u_\varepsilon^* a p_n u_\varepsilon \geq \lambda_n u_\varepsilon^* p_n u_\varepsilon = \lambda_n q_n.$$

In particular, $(u_\varepsilon^* a u_\varepsilon - a) p_n \leq \varepsilon\lambda_n p_n - \lambda_n p_n = -\lambda_n(1-\varepsilon) p_n \leq 0$. Taking into account that $a p_n \geq \lambda_n p_n$, we now obtain

$$\begin{aligned} |u_\varepsilon^* a u_\varepsilon - a| p_n &= (a - u_\varepsilon^* a u_\varepsilon) p_n \geq a p_n - \varepsilon\lambda_n p_n \\ &\geq a p_n - \varepsilon a p_n = (1-\varepsilon) a p_n \\ &= (1-\varepsilon)|a| p_n. \end{aligned}$$

Analogously, for every $n \geq 0$, we have $(u_\varepsilon^* a u_\varepsilon - a) q_n \geq \lambda_n q_n - \varepsilon\lambda_n q_n = (1-\varepsilon)\lambda_n q_n \geq 0$. Therefore,

$$\begin{aligned} |u_\varepsilon^* a u_\varepsilon - a| q_n = (u_\varepsilon^* a u_\varepsilon - a) q_n &\geq (1-\varepsilon)\lambda_n q_n \\ &\geq (1-\varepsilon) a q_n. \end{aligned}$$

Observe that the inequalities above hold for all $n \geq 0$. If $n > 0$, then $q_n < e^a(0, \varepsilon\lambda_n]$, $q_n a = a q_n$ by the construction and so $a q_n = |a| q_n$; that is, we have

$$|u_\varepsilon^* a u_\varepsilon - a| q_n \geq (1-\varepsilon)|a| q_n.$$

A little bit more care is required when $n = 0$. In this case, recall that $q_0 = e^a(-\infty, 0] + q_0^1$, where $q_0^1 < e^a(0, \varepsilon\lambda_0]$. Obviously, $a e^a(-\infty, 0] \leq 0$, and so $a e^a(-\infty, 0] = -|a| e^a(-\infty, 0]$. Therefore since (see above) $u_\varepsilon^* a u_\varepsilon q_0 \geq \lambda_0 q_0$ and $a q_0 = a e^a(-\infty, 0] + a q_0^1 = -|a| e^a(-\infty, 0] + a q_0^1$, we have

$$\begin{aligned} |u_\varepsilon^* a u_\varepsilon - a| q_0 &\geq (u_\varepsilon^* a u_\varepsilon - a) q_0 \geq \lambda_0 q_0 - a q_0^1 + |a| e^a(-\infty, 0] \\ &\geq \lambda_0 q_0^1 - \varepsilon\lambda_0 q_0^1 + |a| e^a(-\infty, 0] = (1-\varepsilon)\lambda_0 q_0^1 + |a| e^a(-\infty, 0] \\ &\geq (1-\varepsilon) a q_0^1 + |a| e^a(-\infty, 0] = (1-\varepsilon)|a| q_0^1 + |a| e^a(-\infty, 0] \\ &\geq (1-\varepsilon)(|a| q_0^1 + |a| e^a(-\infty, 0]) = (1-\varepsilon)|a| q_0. \end{aligned}$$

Collecting all preceding inequalities, we see that for every $k \geq 0$ we have

$$|u_\varepsilon^* a u_\varepsilon - a| \sum_{n=0}^{k} (p_n + q_n) \geq (1-\varepsilon)|a| \sum_{n=0}^{k} (p_n + q_n)$$

and since $\sum_{n=0}^{\infty} (p_n + q_n) = \mathbf{1}$, we conclude

$$|u_\varepsilon^* a u_\varepsilon - a| \geq (1-\varepsilon)|a|.$$

The assertion of the lemma now follows by observing that $|u_\varepsilon^* a u_\varepsilon - a| = |[a, u_\varepsilon]|$. □

The following lemma is somewhat similar to [1, Proposition 5.6] proved there for II_1-factors. However, we need its modification (and strengthening) for general W^*-factors.

Lemma 5. *Suppose that there exists $\lambda \in \mathbb{R}$ and projections $p, q \in P(\mathcal{M})$ such that $p, q \leq e^a\{\lambda\}$, $pq = 0$ and $e^a(-\infty, \lambda) + p \sim e^a(\lambda, +\infty) + q$. Then there exists an element $u = u^* \in U(\mathcal{M})$, satisfying* (1).

Proof. Set $r := \mathbf{1} - (e^a(-\infty, \lambda) + p + e^a(\lambda, +\infty) + q)$. Then $p, q, r \leq e^a\{\lambda\}$ and so $ap = \lambda p$, $aq = \lambda q$, $ar = \lambda r$. We claim that there exists a self-adjoint unitary element u such that $u(e^a(-\infty, \lambda) + p) = (e^a(\lambda, +\infty) + q)u$, $ur = r$. Indeed, since $e^a(-\infty, \lambda) + p \sim e^a(\lambda, +\infty) + q$, there exists a partial isometry v such that $v^*v = e^a(-\infty, \lambda) + p$, $vv^* = e^a(\lambda, +\infty) + q$. Set $u := v + v^* + r$. We have $u^*u = e^a(-\infty, \lambda) + p + e^a(\lambda, +\infty) + q + r = \mathbf{1}$, $uu^* = e^a(\lambda, +\infty) + q + e^a(-\infty, \lambda) + p + r = \mathbf{1}$, $u^* = v^* + v + r = u$. This establishes the claim. It now remains to verify that (1) holds.

To this end, first of all observe that the operators a and u^*au commute with the projections $e^a(-\infty, \lambda) + p$, $e^a(\lambda, +\infty) + q$ and r. This observation guarantees that

$$(u^*au - a)(e^a(-\infty, \lambda) + p) = |u^*au - a|(e^a(-\infty, \lambda) + p),$$
$$(u^*au - a)(e^a(\lambda, +\infty) + q) = |u^*au - a|(e^a(\lambda, +\infty) + q)$$

and so

$$\begin{aligned} |u^*au - a|(e^a(-\infty, \lambda) + p) &= u^*a(e^a(\lambda, +\infty) + q)u - a(e^a(-\infty, \lambda) + p) \\ &= u^*a(e^a(\lambda, +\infty) + q)u - \lambda u^*(e^a(\lambda, +\infty) + q)u \\ &\quad + \lambda(e^a(-\infty, \lambda) + p) - a(e^a(-\infty, \lambda) + p) \\ &= u^*|a(e^a(\lambda, +\infty) + q) - \lambda(e^a(\lambda, +\infty) + q)|u \\ &\quad + |\lambda(e^a(-\infty, \lambda) + p) - a(e^a(-\infty, \lambda) + p)| \\ &= u^*|a - \lambda\mathbf{1}|u(e^a(-\infty, \lambda) + p) \\ &\quad + |a - \lambda\mathbf{1}|(e^a(-\infty, \lambda) + p), \end{aligned}$$

and similarly

$$\begin{aligned} |u^*au - a|(e^a(\lambda, +\infty) + q) &= u^*a(e^a(-\infty, \lambda) + p)u - a(e^a(\lambda, +\infty) + q) \\ &= u^*a(e^a(-\infty, \lambda) + p)u - \lambda u^*(e^a(-\infty, \lambda) + p)u \\ &\quad + \lambda(e^a(\lambda, +\infty) + q) - a(e^a(\lambda, +\infty) + q) \\ &= -u^*|a - \lambda\mathbf{1}|u(e^a(\lambda, +\infty) + q) \\ &\quad - |a - \lambda\mathbf{1}|(e^a(\lambda, +\infty) + q). \end{aligned}$$

Finally, $(u^*au - a)r = \lambda r - \lambda r = 0$, that is, $|u^*au - a|r = 0$. We now obtain (1) as follows:

$$\begin{aligned}|u^*au - a| &= |u^*au - a|[(e^a(-\infty,\lambda) + p) + (e^a(\lambda,+\infty) + q) + r]\\ &= |u^*au - a|(e^a(-\infty,\lambda)+p) + |u^*au - a|(e^a(\lambda,+\infty) + q) + |u^*au - a|r\\ &= (u^*|a - \lambda\mathbf{1}|u + |a - \lambda\mathbf{1}|)[(e^a(-\infty,\lambda) + p) + (e^a(\lambda,+\infty) + q) + r]\\ &= u^*|a - \lambda\mathbf{1}|u + |a - \lambda\mathbf{1}|.\end{aligned}$$

□

The following lemma is well known. We include a short proof for convenience.

Lemma 6. *Let I be an arbitrary ideal in an arbitrary W^*-algebra $\mathcal{A}$. Then $x \in I \Leftrightarrow |x| \in I \Leftrightarrow x^* \in I$. Furthermore, if $0 \le x \le y \in I$, then $x \in I$.*

Proof. If $x \in I$ and $x = v|x|$ is polar decomposition of x, then $|x| = v^*x \in I$ and $x^* = |x|v^* \in I$.

Let $x, y \in \mathcal{A}$, $0 \le x \le y \in I$. In this case there exists an element $z \in \mathcal{A}$ such that $x^{1/2} = zy^{1/2}$ [4, Ch.11, Lemma 2]. Then $x^{1/2} = (x^{1/2})^* = y^{1/2}z^*$ and $x = x^{1/2}x^{1/2} = zy^{1/2}y^{1/2}z^* = zyz^* \in I$. □

We are now fully equipped to prove Theorem 1.

Proof of Theorem 1. We concentrate first on proving assertions (i) and (ii) of Theorem 1. Let us consider the splitting of the set $\mathbb{R}$ of all real numbers into the following pairwise disjoint subsets:

$$\begin{aligned}\Lambda_- &:= \{\lambda \in \mathbb{R} : \ e^a(-\infty,\lambda) \prec e^a(\lambda,+\infty)\},\\ \Lambda_0 &:= \{\lambda \in \mathbb{R} : \ e^a(-\infty,\lambda) \sim e^a(\lambda,+\infty)\},\\ \Lambda_+ &:= \{\lambda \in \mathbb{R} : \ e^a(-\infty,\lambda) \succ e^a(\lambda,+\infty)\}.\end{aligned}$$

If $\Lambda_0 \ne \emptyset$, then the assumptions of Lemma 5 hold for all $\lambda \in \Lambda_0$. Thus in this case for a, assertion (i) of Theorem 1 follows immediately from that lemma and hence assertion (ii) of that theorem trivially holds as well.

In the rest of the proof, we shall assume that $\Lambda_0 = \emptyset$.

Note that if $\lambda \in \Lambda_-$ and $\mu < \lambda$, then

$$e^a(-\infty,\mu) \le e^a(-\infty,\lambda) \prec e^a(\lambda,+\infty) \le e^a(\mu,+\infty),$$

that is, $\mu \in \Lambda_-$. The analogous assertion for Λ_+ is proved similarly. These observations immediately imply that Λ_- and Λ_+ are connected subsets in $\mathbb{R}$, and so for all $\lambda_- \in \Lambda_-$ and $\lambda_+ \in \Lambda_+$, we have $\lambda_- < \lambda_+$. We shall now show that both sets Λ_- and Λ_+ are non-empty. Suppose for a moment that $\Lambda_- = \emptyset$. Since $a \in S_h(\mathcal{M})$ there exists some $\lambda_1 > 0$, such that all projections $e^a(-\infty,\mu)$ for $\mu < -\lambda_1$ and $e^a(\mu,+\infty)$ for $\mu > \lambda_1$ are finite, and $e^a(-\infty,\mu) \to 0$ as $\mu \to -\infty$ and $e^a(\mu,\infty) \to 0$ as $\mu \to \infty$. Let $\lambda_n \downarrow -\infty$, $\lambda_1 = -\mu$. By the assumption, $\lambda_n \notin \Lambda_-$ for all $n \ge 1$. Fixing n and tending k to infinity we have $e^a(-\infty,\lambda_{n+k}) \succeq e^a(\lambda_{n+k},+\infty) \ge e^a(\lambda_n,+\infty)$. However, all projections $e^a(-\infty,\lambda_{n+k})$ are finite and $e^a(-\infty,\lambda_{n+k}) \downarrow 0$; therefore $e^a(\lambda_n,+\infty) = 0$ for any $n \in \mathbb{N}$ (see [1, Lemma 6.11]). On the other hand, $e^a(\lambda_n,+\infty) \uparrow \mathbf{1}$. This contradiction shows that $\Lambda_- \ne \emptyset$. The assertion $\Lambda_+ \ne \emptyset$ is established with a similar argument.

Therefore, there exists such a unique $\lambda_0 \in \mathbb{R}$, satisfying $(-\infty,\lambda_0) \subset \Lambda_-$ and $(\lambda_0,+\infty) \subset \Lambda_+$.

Consider the case when both projections $e^a(-\infty,\lambda_0)$ and $e^a(\lambda_0,+\infty)$ are finite. Since $\Lambda_0=\emptyset$, we have that these two projections are not pairwise equivalent. For definiteness, let us assume $e^a(-\infty,\lambda_0)\prec e^a(\lambda_0,+\infty)$ (the case when $e^a(\lambda_0,+\infty)\prec e^a(-\infty,\lambda_0)$ is treated similarly). Then there exists $r\in P(\mathcal{M})$ such that $e^a(-\infty,\lambda_0)\sim r<e^a(\lambda_0,+\infty)$. If $\mathcal{M}$ is an infinite factor, then $e^a\{\lambda_0\}$ is an infinite projection. Therefore, there exists $p\in P(\mathcal{M})$ such that $e^a(\lambda_0,+\infty)-r\sim p<e^a\{\lambda_0\}$. Then the pair (a,λ_0) satisfies the assumption of Lemma 5. Indeed, setting $q=0$, we have $e^a(-\infty,\lambda_0)+p\sim r+(e^a(\lambda_0,+\infty)-r)=e^a(\lambda_0,+\infty)$, $q=0$. As above this yields assertions (i) and (ii) of Theorem 1.

If $\mathcal{M}$ is a finite factor, then there exists a faithful normal trace τ on $\mathcal{M}$ such that $\tau(\mathbf{1})=1$ (that is, τ is normalized) [9, §8.5]. Certainly, we have $\tau(e^a(-\infty,\lambda))\le 1/2$ for all $\lambda\in\Lambda_-$ and $\tau(e^a(\lambda,+\infty))\le 1/2$ for all $\lambda\in\Lambda_+$. Therefore, by the normality of τ it follows that $\tau(e^a(-\infty,\lambda_0))\le 1/2$ and $\tau(e^a(\lambda_0,+\infty))\le 1/2$. Thus if we have $e^a(-\infty,\lambda_0)\preceq e^a(\lambda_0,+\infty)$, then there exists a projection $p\le e^a\{\lambda_0\}$ such that $e^a(-\infty,\lambda_0)+p\sim e^a(\lambda_0,+\infty)$. Hence, in this case, both projections $e^a(-\infty,\lambda_0)$ and $e^a(\lambda_0,+\infty)$ are finite, and (setting $q=0$ as above) we see that the assumption of Lemma 5 holds. We note, in passing, that a similar argument also occurred in [7, Corollary 2.7]. So, in this case, again assertions (i) and (ii) of Theorem 1 hold.

Note that by now we have completed the proof of Theorem 1(i), (ii) for the case when $\mathcal{M}$ is a finite factor. Moreover, we have also finished the proof for the case when $\mathcal{M}$ is an infinite factor and both projections $e^a(-\infty,\lambda_0)$ and $e^a(\lambda_0,+\infty)$ are finite.

Let us now consider the case when $\mathcal{M}$ is a purely infinite σ-finite factor. In such a factor, all non-zero projections are infinite and are equivalent to each other [17, Proposition V.1.39]. Therefore, in this case, we may assume that both projections $e^a(-\infty,\lambda_0)$ and $e^a(\lambda_0,+\infty)$ are infinite or otherwise one of these projections must be 0. Our strategy is to show that in this case $\Lambda_0\ne\emptyset$. This would yield a contradiction with the assumption $\Lambda_0=\emptyset$ made earlier and would complete the proof.

Suppose that $e^a(-\infty,\lambda_0)\prec e^a(\lambda_0,+\infty)$; that is, assume that $\lambda_0\in\Lambda_-$. Then $e^a(-\infty,\lambda_0)=0$ since all non-zero projections in $\mathcal{M}$ are equivalent. Furthermore, since for all $\lambda>\lambda_0$ we have $\lambda\in\Lambda_+$, a similar argument yields $e^a(\lambda,+\infty)=0$ for all such λ's. Thus

$$e^a(\lambda_0,+\infty)=\bigvee_{\lambda>\lambda_0}e^a(\lambda,+\infty)=0,$$

and we obtain $0=e^a(-\infty,\lambda_0)\sim e^a(\lambda,+\infty)=0$, that is, $\lambda_0\in\Lambda_0$. However, this contradicts our assumption that $\Lambda_0=\emptyset$. The case when $e^a(-\infty,\lambda_0)\succ e^a(\lambda_0,+\infty)$ is considered analogously. This completes the proof of assertions (i) and (ii) of Theorem 1 for the case of a purely infinite σ-finite factor $\mathcal{M}$.

To finish the proof of assertion (ii), it remains to consider the case of an infinite factor $\mathcal{M}$, that is, when $\mathcal{M}$ is of type II_∞ or else when $\mathcal{M}$ is of type I_∞, or else when $\mathcal{M}$ is a non-σ-finite factor of type III and when at least one of the projections $e^a(-\infty,\lambda_0)$ and $e^a(\lambda_0,+\infty)$ is properly infinite.

In fact, it is sufficient to consider only the case when

$$e^a(-\infty,\lambda_0)\prec e^a(\lambda_0,+\infty), \tag{3}$$

that is, $\lambda_0 \in \Lambda_-$. Indeed, if assertion (ii) of Theorem 1 holds under assumption (3), then the remaining case (when $e^a(-\infty, \lambda_0) \succ e^a(\lambda_0, +\infty)$, that is, $\lambda_0 \in \Lambda_+$) is reduced to (3) by substituting a for $-a$ and λ_0 for $-\lambda_0$.

Assume now that (3) holds (in this case, the projection $e^a(\lambda_0, +\infty)$ is necessarily infinite).

We may also further assume that the assumptions of Lemma 5 do not hold (otherwise, there is nothing to prove).

We shall now show that

$$e^a(-\infty, \lambda_0] \prec e^a(\lambda_0, +\infty). \tag{4}$$

Suppose the contrary, that is, that either

$$e^a(-\infty, \lambda_0] = e^a(-\infty, \lambda_0) + e^a\{\lambda_0\} \succ e^a(\lambda_0, +\infty) \tag{5}$$

or else that

$$e^a(-\infty, \lambda_0] \sim e^a(\lambda_0, +\infty). \tag{6}$$

If (6) holds, then setting $p = e^a\{\lambda_0\},\ q = 0$, we have $e^a(-\infty, \lambda_0] = e^a(-\infty, \lambda_0) + p$, so we arrive at the setting when the assumptions of Lemma 5 hold and we are done. Suppose now that (5) holds. Then by Lemma 1, it follows from (3) that there exists a projection $p \in P(\mathcal{M})$, for which $e^a(\lambda_0, +\infty) \sim e^a(-\infty, \lambda_0) + p$ and $p < e^a\{\lambda_0\}$. However, this again means that the assumptions of Lemma 5 hold (with $q = 0$). This completes the proof of (4).

Our next claim is that

$$e^a(\lambda_0, \lambda] \succ e^a(-\infty, \lambda_0] + e^a(\lambda, +\infty) \tag{7}$$

for all $\lambda > \lambda_0$. Suppose the contrary:

$$e^a(\lambda_0, \lambda] \preceq e^a(-\infty, \lambda_0] + e^a(\lambda, +\infty) \tag{8}$$

for some $\lambda > \lambda_0$. Setting for a moment $p := e^a(-\infty, \lambda_0] + e^a(\lambda, +\infty),\ \mathbf{1} - p = e^a(\lambda_0, \lambda]$, we rewrite (8) as $p \succeq \mathbf{1} - p$, and, thanks to Lemma 2(iii), conclude that

$$e^a(-\infty, \lambda_0] + e^a(\lambda, +\infty) \sim \mathbf{1}.$$

However, (4) implies

$$e^a(-\infty, \lambda_0] \prec \mathbf{1},$$

and, by Lemma 2(ii), we obtain $e^a(\lambda, +\infty) \sim \mathbf{1}$. However, for $\lambda > \lambda_0$ we have $\lambda \in \Lambda_+$, and so $e^a(-\infty, \lambda) \succ e^a(\lambda, +\infty) \sim \mathbf{1}$, that is, $e^a(-\infty, \lambda) \succ \mathbf{1}$, which is obviously impossible. Hence, (8) fails and (7) holds.

Now, observe that condition (7) means that the element $a - \lambda_0\mathbf{1}$ satisfies the assumptions of Lemma 4. Assertion (ii) of Theorem 1 now follows from that lemma. This completes the proof of assertions (i) and (ii).

Let us prove the final assertion of Theorem 1. To this end, let $\mathcal{M}$ be an infinite semi-finite σ-finite factor. Fix a sequence $\{p_n\}_{n\geq 1}$ of pairwise disjoint finite projections $p_1, p_2, ..., p_n, ...$ such that $\bigvee_{n=1}^{\infty} p_n = \mathbf{1}$ (any maximal family of pairwise disjoint finite projections in $\mathcal{M}$ is countable) and set

$$a := \sum_{n=1}^{\infty} n^{-1} p_n.$$

We have $a = a^* \in \mathcal{M} \cap \mathcal{F}$, where $\mathcal{F}$ is the norm-closed ideal, generated by the elements $x \in \mathcal{F}$ such that $r(x)$ (and hence $l(x)$) is a finite projection in $\mathcal{M}$. Moreover, the support of a, $s(a)$, is equal to $\mathbf{1}$. Suppose that

$$|[a,u]| \geq |a - \lambda \mathbf{1}| \tag{9}$$

for some $\lambda \in \mathbb{C}$ and some $u \in U(\mathcal{M})$. Since $[a,u] \in \mathcal{F}$, we have by Lemma 6 that also $a - \lambda\mathbf{1} \in \mathcal{F}$. However, the set $\{a - \lambda\mathbf{1} : \ \lambda \in \mathbb{C}\}$ may contain at most one element and can belong to $\mathcal{F}$, since $\mathcal{F}$ is a proper ideal in $\mathcal{M}$ (see [9, Theorem 6.8.7]). This guarantees that $\lambda = 0$, that is, $|u^*au - a| = |[a,u]| \geq |a| = a$. Let $e_+ := s((u^*au - a)_+)$, $e_- := \mathbf{1} - e_+$. We have $e_-(a - u^*au)e_- \geq e_-ae_-$, or equivalently, $e_-u^*aue_- \leq 0$. Since $u^*au \geq 0$, we conclude $e_-u^*aue_- = 0$, or equivalently, $a^{1/2}ue_- = 0$, which in turn implies $aue_-u^* = 0$. Thus,

$$a = au(e_+ + e_-)u^* = aue_+u^*;$$

in particular, $ue_+u^* \geq s(a) = \mathbf{1}$ and therefore $e_+ = \mathbf{1}$. On the other hand, due to the definition of e_+ and the inequality $|u^*au - a| \geq a$, we have $e_+(u^*au - a)e_+ \geq e_+ae_+$, or equivalently, $u^*au \geq 2a$. However, the preceding inequality implies that $1 = \|a\| \geq 2\|a\| = 2$ and therefore is false. This contradiction shows that λ and u satisfying (9) do not exist. □

4. Applications of Theorem 1 to derivations

Recall that a *derivation* on a complex algebra A is a linear map $\delta : A \to A$ such that

$$\delta(xy) = \delta(x)y + x\delta(y), \quad x, y \in A.$$

If $a \in A$, then the map $\delta_a : A \to A$, given by $\delta_a(x) = [a,x]$, $x \in A$, is a derivation. A derivation of this form is called *inner*.

Our first result here is somewhat similar (at least in spirit) to some results in [8, 11, 13].

Corollary 1. *Let $\mathcal{M}$ be a W^*-factor and I be an ideal in $\mathcal{M}$ and let $\delta : \mathcal{M} \to I$ be a derivation. Then there exists an element $a \in I$, such that $\delta = \delta_a = [a,.]$.*

Proof. Since δ is a derivation on a W^*-algebra, it is necessarily inner [14, Theorem 4.1.6]. Thus there exists an element $d \in \mathcal{M}$ such that $\delta(\cdot) = \delta_d(\cdot) = [d,.]$. It follows from our hypothesis that $[d, \mathcal{M}] \subseteq I$.

Using Lemma 6, we obtain $[d^*, \mathcal{M}] = -[d, \mathcal{M}]^* \subseteq I^* = I$ and $[d_k, \mathcal{M}] \subseteq I$, $k = 1,2$, where $d = d_1 + id_2$, $d_k = d_k^* \in \mathcal{M}$, for $k = 1,2$. It follows now from Theorem 1, that there exist scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in U(\mathcal{M})$ such that $|[d_k, u_k]| \geq \frac{1}{2}|d_k - \lambda_k\mathbf{1}|$ for $k = 1,2$. Again applying Lemma 6, we obtain $d_k - \lambda_k\mathbf{1} \in I$, for $k = 1,2$. Setting $a := (d_1 - \lambda_1\mathbf{1}) + i(d_2 - \lambda_2\mathbf{1})$, we deduce that $a \in I$ and $\delta = [a,.]$. □

Classical examples of ideals I satisfying the assumptions of Corollary 1 above are given by symmetric operator ideals.

Definition 1. If $\mathcal{I}$ is a $*$-ideal in a von Neumann algebra $\mathcal{N}$ which is complete in a norm $\|\cdot\|_{\mathcal{I}}$, then we will call $\mathcal{I}$ a symmetric operator ideal if
(1) $\|S\|_{\mathcal{I}} \geq \|S\|$ for all $S \in \mathcal{I}$,
(2) $\|S^*\|_{\mathcal{I}} = \|S\|_{\mathcal{I}}$ for all $S \in \mathcal{I}$,
(3) $\|ASB\|_{\mathcal{I}} \leq \|A\| \, \|S\|_{\mathcal{I}}\|B\|$ for all $S \in \mathcal{I}$, $A, B \in \mathcal{N}$.

Since $\mathcal{I}$ is an ideal in a von Neumann algebra, it follows from I.1.6, Proposition 10 of [4] that if $0 \leq S \leq T$ and $T \in \mathcal{I}$, then $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$.

Corollary 2. *Let $\mathcal{M}$ be a W^*-factor and I be a symmetric operator ideal in $\mathcal{M}$ and let $\delta : \mathcal{M} \to I$ be a self-adjoint derivation. Then there exists an element $a \in I$ such that $\delta = \delta_a = [a, .]$ and that $\|a\|_I \leq \|\delta\|_{\mathcal{M}\to I}$.*

Proof. First, we observe that $\|\delta\|_{\mathcal{M}\to I} < \infty$. Indeed, we have $\delta = \delta_a$, $a \in I$ and therefore $\|\delta(x)\|_I = \|ax - xa\|_I \leq \|ax\|_I + \|xa\|_I \leq 2\|a\|_I\|x\|_{\mathcal{M}}$, that is, $\|\delta\|_{\mathcal{M}\to I} \leq 2\|a\|_I < \infty$.

Now let δ be a self-adjoint derivation on $\mathcal{M}$, that is, $\delta(\cdot) = \delta_d(\cdot) = [d, \cdot]$ for some $d \in \mathcal{M}$, such that $[d, x]^* = [d, x^*]$ for all $x \in \mathcal{M}$. We have $x^*d^* - d^*x^* = dx^* - x^*d$, that is, $x^*(d^* + d) = (d^* + d)x^*$ for all $x \in \mathcal{M}$. This immediately implies $\mathfrak{Re}(d) \in Z(\mathcal{M})$, and so we can safely assume that $\delta = \delta_{id} = [id, .]$, where d is a self-adjoint operator from $\mathcal{M}$. Fix $\varepsilon > 0$ and let $\lambda_0 \in \mathbb{R}$, $u_\varepsilon \in U(\mathcal{M})$ be such that

$$|[d, u_\varepsilon]| \geq (1 - \varepsilon)|d - \lambda_0\mathbf{1}|.$$

The assumption on $(I, \|\cdot\|)$ guarantees that $(1-\varepsilon)\|d-\lambda_0\mathbf{1}\|_I \leq \|\delta(u_\varepsilon)\|_I \leq \|\delta\|_{\mathcal{M}\to I}$. Since ε was chosen arbitrarily, we conclude that $\|d - \lambda_0\mathbf{1}\|_I \leq \|\delta\|_{\mathcal{M}\to I}$. Setting $a = i(d - \lambda_0\mathbf{1})$ completes the proof. □

If the von Neumann algebra $\mathcal{N}$ is equipped with a faithful normal semi-finite trace τ, then the set

$$\mathcal{L}_p(\mathcal{N}) = \{S \in \mathcal{N} : \ \tau(|S|^p) < \infty\}$$

equipped with a standard norm

$$\|S\|_{\mathcal{L}_p(\mathcal{N})} = \max\{\|S\|_{B(H)}, \ \tau(|S|^p)^{1/p}\}$$

is said to be of Schatten-von Neumann p-class. In the Type I setting these are the usual Schatten-von Neumann ideals. The result of Corollary 1 complements results given in [11, Section 6].

A closely linked example is the following. Consider the ideal $\mathcal{K}_{\mathcal{N}}$ of τ-compact operators in $\mathcal{N}$ (that is, the norm closed ideal generated by the projections $E \in P(\mathcal{N})$ with $\tau(E) < \infty$). In this special case, the result of Corollary 2 is analogous to the classical result that any derivation on $B(H)$ taking values in the ideal of compact operators on H can be represented as δ_a with a being a compact operator (see e.g. [8, Lemma 3.2]).

We now consider analogues of Corollary 1 for ideals of (unbounded) τ-measurable operators.

Corollary 3. *Let $\mathcal{M}$ be a W^*-factor and let $\mathcal{A}$ be a linear subspace in $S(\mathcal{M})$ such that $\mathcal{A}^* = \mathcal{A}$, $x \in \mathcal{A} \Leftrightarrow |x| \in \mathcal{A}$, $0 < x < y \in \mathcal{A} \Rightarrow x \in \mathcal{A}$. Fix $a \in S(\mathcal{M})$ and consider an inner derivation $\delta = \delta_a$ on the algebra $S(\mathcal{M})$ given by $\delta(x) = [a, x]$, $x \in S(\mathcal{M})$. If $\delta(\mathcal{M}) \subseteq \mathcal{A}$, then there exists $d \in \mathcal{A}$ such that $\delta(x) = [d, x]$.*

Proof. Let $a = a_1 + ia_2$, where $a_1 = \mathfrak{Re}(a)$ and $a_2 = \mathfrak{Im}(a)$. We have $2[a_1, x] = [a + a^*, x] = [a, x] - [a, x^*]^* = \mathcal{A} - \mathcal{A}^* \subseteq \mathcal{A}$ for any $x \in \mathcal{M}$. Analogously, $[a_2, x] \in \mathcal{A}$ for any $x \in \mathcal{M}$. By Theorem 1, there is a scalar $\lambda_k \in \mathbb{R}$ and a unitary element $u_k \in U(\mathcal{M})$, such that $|[a_k, u_k]| \geq \frac{1}{2}|a_k - \lambda_k\mathbf{1}|$ for $k = 1, 2$. The assumption on $\mathcal{A}$ guarantees that $a_k - \lambda_k\mathbf{1} \in \mathcal{A}$, for $k = 1, 2$. Setting $d = (a_1 - \lambda_1\mathbf{1}) + i(a_2 - \lambda_2\mathbf{1})$, we deduce that $d \in \mathcal{A}$ and $\delta = [d, \cdot]$. □

Numerous examples of absolutely solid subspaces $\mathcal{A}$ in $S(\mathcal{M},\tau)$ satisfying the assumptions of the preceding corollary are given by $\mathcal{M}$-bimodules of $S(\mathcal{M},\tau)$.

Definition 2. A linear subspace E of $S(\mathcal{M},\tau)$, is called an $\mathcal{M}$-bimodule of τ-measurable operators if $uxv \in E$ whenever $x \in E$ and $u, v \in \mathcal{M}$. If an $\mathcal{M}$-bimodule E is equipped with a (semi-)norm $\|\cdot\|_E$, satisfying

$$\|uxv\|_E \leq \|u\|_{B(H)} \|v\|_{B(H)} \|x\|_E, \quad x \in E,\ u, v \in \mathcal{M}, \tag{10}$$

then E is called a (semi-)normed $\mathcal{M}$-bimodule of τ-measurable operators.

We omit a straightforward verification of the fact that every $\mathcal{M}$-bimodule of τ-measurable operators satisfies the assumption of Corollary 3.

The best known examples of normed $\mathcal{M}$-bimodules of $S(\mathcal{M},\tau)$ are given by the so-called symmetric operator spaces (see e.g. [5, 16, 10]). We briefly recall relevant definitions.

Let L_0 be a space of Lebesgue measurable functions either on $(0,1)$ or on $(0,\infty)$, or on $\mathbb{N}$ finite almost everywhere (with identification $m-$a.e.). Here m is Lebesgue measure or else counting measure on $\mathbb{N}$. Define S_0 as the subset of L_0 which consists of all functions x such that $m(\{|x| > s\})$ is finite for some s.

Let E be a Banach space of real-valued Lebesgue measurable functions either on $(0,1)$ or $(0,\infty)$ (with identification $m-$a.e.). E is said to be an *ideal lattice* if $x \in E$ and $|y| \leq |x|$ implies that $y \in E$ and $||y||_E \leq ||x||_E$.

The ideal lattice $E \subseteq S_0$ is said to be a *symmetric space* if for every $x \in E$ and every y the assumption $y^* = x^*$ implies that $y \in E$ and $||y||_E = ||x||_E$.

Here, x^* denotes the non-increasing right-continuous rearrangement of x given by

$$x^*(t) = \inf\{s \geq 0 :\ m(\{|x| \geq s\}) \leq t\}.$$

If $E = E(0,1)$ is a symmetric space on $(0,1)$, then

$$L_\infty \subseteq E \subseteq L_1.$$

If $E = E(0,\infty)$ is a symmetric space on $(0,\infty)$, then

$$L_1 \cap L_\infty \subseteq E \subseteq L_1 + L_\infty.$$

Let a semi-finite von Neumann algebra $\mathcal{N}$ be equipped with a faithful normal semi-finite trace τ. Let $x \in S(\mathcal{N},\tau)$. The generalized singular value function of x is $\mu(x) : t \to \mu_t(x)$, where, for $0 \leq t < \tau(\mathbf{1})$,

$$\mu_t(x) = \inf\{s \geq 0 \mid \tau(e^{|x|}(s,\infty) \leq t\}.$$

Let $\mathcal{E}$ be a linear subset in $S(\mathcal{N},\tau)$ equipped with a complete norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a *symmetric operator space* (on $\mathcal{N}$) if $x \in E$ and for every $y \in S(\mathcal{N},\tau)$, the assumption $\mu(y) \leq \mu(x)$ implies that $y \in E$ and $\|y\|_{\mathcal{E}} \leq \|x\|_{\mathcal{E}}$. The fact that every symmetric operator space $\mathcal{E}$ is (an absolutely solid) $\mathcal{M}$-bimodule of $S(\mathcal{M},\tau)$ is well known (see e.g. [16] and the references therein).

There exists a strong connection between symmetric functions and operator spaces.

Let E be a symmetric function space on the interval $(0,1)$ (respectively, on the semi-axis or on $\mathbb{N}$) and let $\mathcal{N}$ be a type II_1 (respectively, II_∞ or type I) von Neumann algebra. Define

$$E(\mathcal{N},\tau) := \{S \in S(\mathcal{N},\tau) :\ \mu_t(S) \in E\},\ \|S\|_{E(\mathcal{N},\tau)} := \|\mu_t(S)\|_E.$$

The main results of [10] assert that $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ is a symmetric operator space. If $E=L_p$, $1\le p<\infty$, then $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ coincides with the classical non-commutative L_p-space associated with the algebra $(\mathcal{N},\tau)$. If $\mathcal{N}$ is a semi-finite factor, then the converse result is trivially true. That is, assume for definiteness that $\mathcal{N}$ is II_∞-factor and that $\mathcal{E}$ is a symmetric operator space on $\mathcal{N}$. Then,

$$E(0,\infty):=\{f\in S_0((0,\infty)):\ f^*=\mu(x)\text{ for some }x\in\mathcal{E}\},\ \|f\|_E:=\|x\|_{\mathcal{E}}$$

is a symmetric function space on $(0,\infty)$. It is obvious that $\mathcal{E}=E(\mathcal{N},\tau)$.

We are now fully equipped to provide a full analogue of Corollaries 1 and 2.

Corollary 4. *Let $\mathcal{M}$ be a semi-finite W^*-factor and let $\mathcal{E}$ be a symmetric operator space. Fix $a=a^*\in S(\mathcal{M})$ and consider an inner derivation $\delta=\delta_a$ on the algebra $S(\mathcal{M})$ given by $\delta(x)=[a,x]$, $x\in S(\mathcal{M})$. If $\delta(\mathcal{M})\subseteq\mathcal{E}$, then there exists $d\in\mathcal{E}$ such that $\delta(x)=[d,x]$. Furthermore, $\|\delta\|_{\mathcal{M}\to\mathcal{E}}<\infty$ and the element $d\in\mathcal{E}$ can be chosen so that $\|d\|_{\mathcal{E}}\le\|\delta\|_{\mathcal{M}\to\mathcal{E}}$.*

Proof. The existence of $d\in\mathcal{E}$ such that $\delta(x)=[d,x]$ follows from Corollary 3. Now, if $u\in U(\mathcal{M})$, then $\|\delta(u)\|_{\mathcal{E}}=\|du-ud\|_{\mathcal{E}}\le\|du\|_{\mathcal{E}}+\|ud\|_{\mathcal{E}}=2\|d\|_{\mathcal{E}}$. Hence, if $x\in\mathcal{M}_1=\{x\in\mathcal{M}:\ \|x\|\le1\}$, then $x=\sum_{i=1}^4\alpha_iu_i$, where $u_i\in U(\mathcal{M})$ and $|\alpha_i|\le1$ for $i=1,2,3,4$, and so $\|\delta(x)\|_{\mathcal{E}}\le\sum_{i=1}^4\|\delta(\alpha_iu_i)\|_{\mathcal{E}}\le8\|d\|_{\mathcal{E}}$, that is, $\|\delta\|_{\mathcal{M}\to\mathcal{E}}\le8\|d\|_{\mathcal{E}}<\infty$.

The final assertion is established exactly as in the proof of Corollary 2. □

An interesting illustration of the result above can be obtained already for the situation when the space E is given by the norm closure of the subspace $L_1\cap L_\infty$ in the space L_1+L_∞. In this case, the space $\mathcal{E}=E(\mathcal{M},\tau)$ can be equivalently described as the set of all $x\in L_1+L_\infty(\mathcal{M},\tau)$ such that $\lim_{t\to\infty}\mu_t(x)=0$. This space is a natural counterpart of the ideal $\mathcal{K}_{\mathcal{M}}$ of τ-compact operators in $\mathcal{M}$.

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