TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 364, Number 11, November 2012, Pages 5765–5790
S 0002-9947(2012)05489-4
Article electronically published on June 22, 2012

COHOMOLOGY ALGEBRA OF PLANE CURVES, WEAK COMBINATORIAL TYPE, AND FORMALITY

J. I. COGOLLUDO AGUSTÍN AND D. MATEI

Abstract. We determine an explicit presentation by generators and relations of the cohomology algebra $H^*(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ of the complement to an algebraic curve $\mathcal{C}$ in the complex projective plane $\mathbb{P}^2$ via the study of log-resolution logarithmic forms on $\mathbb{P}^2$. As a first consequence, we derive that $H^*(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ depends only on the following finite pieces of data: the number of irreducible components of $\mathcal{C}$ together with their degrees and genera, the number of local branches of each component at each singular point, and the intersection numbers of every two distinct local branches at each singular point of $\mathcal{C}$. This finite set of data is referred to as the weak combinatorial type of $\mathcal{C}$. A further corollary is that the twisted cohomology jumping loci of $H^*(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ containing the trivial character also depend on the weak combinatorial type of $\mathcal{C}$. Finally, the explicit construction of the generators and relations allows us to prove that complements of plane projective curves are formal spaces in the sense of Sullivan.

1. Introduction

The combinatorial type $K_{\mathcal{C}}$ of a complex projective curve $\mathcal{C} \subset \mathbb{P}^2$ consists of the following list of data: the set of irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_r$ of $\mathcal{C}$ together with their degrees $\bar{d} := (d_1, \ldots, d_r)$, the set of singular points $\mathrm{Sing}(\mathcal{C})$ of $\mathcal{C}$ together with their topological types $\Sigma(\mathcal{C})$, and, for every $P \in \mathrm{Sing}(\mathcal{C})$, the correspondence ϕ_P that associates to each local branch at P the global irreducible component it belongs to.

The combinatorial type of $\mathcal{C}$ determines the abstract topology of $\mathcal{C}$ itself. This is not the case for the topology of the embedding $\mathcal{C} \subset \mathbb{P}^2$, as shown by Zariski's classical work where he established that the fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is not determined by $K_{\mathcal{C}}$.

In this paper, we consider a topological invariant of $\mathcal{C}$, called the weak combinatorial type $W_{\mathcal{C}}$ of $\mathcal{C}$, which is coarser than $K_{\mathcal{C}}$, and yet it contains enough information to determine the cohomology algebra $H^*(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$. Roughly speaking, $W_{\mathcal{C}}$ consists of the following pieces of data: the set of irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_r$ of $\mathcal{C}$ together with their degrees $\bar{d} := (d_1, \ldots, d_r)$ and genera, the set of singular points $\mathrm{Sing}(\mathcal{C})$ of $\mathcal{C}$, the correspondence ϕ_P as above, and the intersection numbers of every two distinct local branches at each singular point of $\mathcal{C}$. Note that

Received by the editors July 10, 2009 and, in revised form, October 14, 2010.
2010 *Mathematics Subject Classification.* Primary 14F25, 14F40, 14H50, 58A10, 58A12, 58A14, 14B05, 14E15, 32A27, 32S22, 55P62.
Key words and phrases. Plane algebraic curves, singularities.
The first author was partially supported by the Spanish Ministry of Education MTM2010-21740-C02-02. The second author has been partially supported by SB2004-0181 and grant CNCSIS PNII-IDEI 1189/2008.

the Betti numbers of $S_{\mathcal{C}}$ depend only on the number, degrees, and genera of the irreducible components of $\mathcal{C}$.

In [5], the first author proves that $H^*(S_{\mathcal{C}},\mathbb{C})$ depends only on $W_{\mathcal{C}}$, in the case where $\mathcal{C}$ is an arrangement of rational curves. Here we extend that result to arbitrary curves. The result follows from an explicit presentation of the cohomology algebra $H^*(S_{\mathcal{C}},\mathbb{C})$ which is obtained by means of the Poincaré residue operators of [12, 11]. For other interesting attempts to describe a presentation of $H^*(S_{\mathcal{C}},\mathbb{C})$ via differential forms, see [16].

An outline of the construction proceeds as follows. Fix a resolution $\pi : \bar{S} \to \mathbb{P}^2$ of the singular locus of $\mathcal{C}$ in $\mathbb{P}^2$ such that $\bar{\mathcal{C}}$, the reduced divisor associated with $\pi^*\mathcal{C}$, is a simple normal crossing divisor in $\bar{S}$. Denote by $\bar{\mathcal{C}}^{[k]}$, $k = 0, 1, 2$, the disjoint union of the codimension k intersections of the components of $\bar{\mathcal{C}}$. Let $\mathcal{W}_\ell^{[m]}$ be the weight filtration on the sheaf of logarithmic forms on $\bar{\mathcal{C}}^{[m]}$ with respect to the divisor $\bar{\mathcal{C}}^{[m+1]}$. Consider ${}^\ell R_m^{[k]}$ as the residue operator on $\mathcal{W}_\ell^{[m]}$. Note that these filtrations are compatible with the exterior differential d and that the residues ${}^\ell \tilde{R}_m^{[k]}$ defined on $\mathcal{W}_\ell^{[m]}/\mathcal{W}_{\ell-1}^{[m]}$ induce isomorphisms in d-cohomology. In particular, for $k = 1, 2$ one has the residue operators $R^{[k]} := {}^k R_0^{[k]}$ mapping the sheaf $\mathcal{W}_k := \mathcal{W}_k^{[0]}$ of logarithmic forms of weight k on $\bar{S}$ with respect to the divisor $\bar{\mathcal{C}}$ to the sheaf of differential forms $\mathcal{W}_0^{[k]}$. Considering the complexes $(\mathcal{W}_k, d)$, the exact sequence $0 \to \mathcal{W}_{i-1} \to \mathcal{W}_i \to \mathcal{W}_i/\mathcal{W}_{i-1} \to 0$, and using the resolution π and deRham isomorphisms, we can construct from the coboundary maps the following residue maps:

$$\mathrm{Res}^{[i]} : H^i(S_{\mathcal{C}},\mathbb{C}) \to H^0(\bar{\mathcal{C}}^{[i]}), \quad i = 1, 2.$$

They are key to our approach in understanding the cohomology groups $H^i(S_{\mathcal{C}},\mathbb{C})$, $i = 1, 2$.

First of all, $\mathrm{Res}^{[1]} : H^1(S_{\mathcal{C}},\mathbb{C}) \to H^0(\bar{\mathcal{C}}^{[1]})$ turns out to be an injection. Then a basis for $H^1(S_{\mathcal{C}},\mathbb{C})$ can be chosen to be the cohomology classes of the logarithmic 1-forms $\sigma_i = d(\log \frac{C_i}{C_0^{d_i}})$, $1 \le i \le r$, where $\mathcal{C}_i$ are the irreducible components of $\mathcal{C}$ and $\mathcal{C}_0$ is a transversal line at infinity. This condition is not strictly necessary, but we use it for technical reasons. For a general description see Remark 3.30.

The map $\mathrm{Res}^{[2]} : H^2(S_{\mathcal{C}},\mathbb{C}) \to H^0(\bar{\mathcal{C}}^{[2]})$ will not be an injection in general, unless all the components of $\mathcal{C}$ are rational. Nevertheless, we can find a decomposition of $H^2(S_{\mathcal{C}},\mathbb{C})$ of the form $\mathcal{V}_{\mathcal{C}}^2 \oplus \mathcal{K}_{\mathcal{C}} \oplus \overline{\mathcal{K}}_{\mathcal{C}}$, where $\ker \mathrm{Res}^{[2]} = \mathcal{K}_{\mathcal{C}} \oplus \overline{\mathcal{K}}_{\mathcal{C}}$, with $\mathcal{K}_{\mathcal{C}}$ a g-dimensional vector space of classes of holomorphic 2-forms of weight 1 such that ${}^1\tilde{R}_0^{[1]}\mathcal{K}_{\mathcal{C}}$ exhausts the holomorphic 1-forms on $\bar{\mathcal{C}}^{[1]}$ and $\overline{\mathcal{K}}_{\mathcal{C}}$ is the conjugate of $\mathcal{K}_{\mathcal{C}}$. Note that $\overline{\mathcal{K}}_{\mathcal{C}}$ will necessarily consist of classes having non-holomorphic representatives. The vector space $\mathcal{V}_{\mathcal{C}}^2$ will be generated by the classes of certain log-resolution logarithmic 2-forms which are constructed by the same method employed in [5] for the rational arrangements case. The basic ingredients are logarithmic ideals associated with the resolution trees appearing in the construction of π and ideal sheaves associated to pairs of branches at the singular points of $\mathcal{C}$. The choice of the log-resolution logarithmic 2-forms is made by imposing appropriate normalizing conditions.

An important feature of the decomposition $H^2(S_{\mathcal{C}},\mathbb{C}) = \mathcal{V}_{\mathcal{C}}^2 \oplus \mathcal{K}_{\mathcal{C}} \oplus \overline{\mathcal{K}}_{\mathcal{C}}$ is that $\mathcal{V}_{\mathcal{C}}^2 \supset H^1(S_{\mathcal{C}},\mathbb{C}) \cup H^1(S_{\mathcal{C}},\mathbb{C})$, the cup product of 1-forms. Moreover, by a residue computation we determine the map $H^1(S_{\mathcal{C}},\mathbb{C}) \times H^1(S_{\mathcal{C}},\mathbb{C}) \overset{\cup}{\to} \mathcal{V}_{\mathcal{C}}^2$ in terms of the

above constructed generators and see that it depends only on the degrees of the components of $\mathcal{C}$ and the intersection numbers of the local branches at the singular points of $\mathcal{C}$. By another residue computation, we determine the relations among the generators of $\mathcal{V}^2_{\mathcal{C}}$. Finally, adding the trivial relations $H^1(S_{\mathcal{C}},\mathbb{C}) \cup H^2(S_{\mathcal{C}},\mathbb{C}) = 0$, we obtain a presentation for the cohomology algebra $H^*(S_{\mathcal{C}},\mathbb{C})$.

After normalizing the generators in $\mathcal{V}^2_{\mathcal{C}}$ appropriately, the relations imposed by $H^1(S_{\mathcal{C}},\mathbb{C}) \cup H^1(S_{\mathcal{C}},\mathbb{C}) \subset \mathcal{V}^2_{\mathcal{C}}$ will be satisfied as differential forms. We thus derive an analogue of the Brieskorn lemma in the theory of hyperplane arrangements, thereby obtaining an embedding of the algebra of differential forms on $S_{\mathcal{C}}$ into the cohomology algebra $H^*(S_{\mathcal{C}},\mathbb{C})$. This immediately implies the formality of $S_{\mathcal{C}}$.

The implications of this description of the cohomology algebra $H^*(S_{\mathcal{C}},\mathbb{C})$ and the cohomology jumping loci of the space of local systems on $\mathbb{C}$ (along the lines of [8]) will be addressed in an upcoming paper.

2. Settings

2.1. C^∞ **log complex of quasi-projective algebraic varieties.** For the sake of completeness, in this section we will describe an appropriate setting for the study of the cohomology ring of the complement to plane algebraic curves. This includes definitions and basic properties of logarithmic sheaves and the definition of a very useful operator on these sheaves: the Poincaré residue operator. The basic ideas in this section follow from [11, Chapter 5], but we present a slight generalization of their results which will be necessary for the rest of the paper.

Let X be a smooth, quasi-projective algebraic variety of dimension n over $\mathbb{C}$ and $\overline{X}$ be a smooth compactification of X. We will assume $\overline{X}$ to be a smooth projective variety such that $X = \overline{X} \setminus \overline{\mathcal{D}}$, where $\overline{\mathcal{D}}$ is a simple normal crossing divisor, that is, a union $\overline{\mathcal{D}}_1 \cup \cdots \cup \overline{\mathcal{D}}_N$ of smooth divisors on $\overline{X}$ with normal crossings. The condition of normal crossing on $\overline{\mathcal{D}}$ means that locally at $P \in \overline{X}$, the divisor $\overline{\mathcal{D}}$ is given by

$$\{(z_1,\dots,z_n) \mid z_{i_1} \cdot \dots \cdot z_{i_m} = 0\} = \{(z_1,\dots,z_n) \mid z_{I_P} = 0\},$$

where $I_P = \{i_1,\dots,i_m\} \subset \{1,\dots,n\}$. Each coordinate of I_P must correspond locally to a different global component of $\overline{\mathcal{D}}$ (since each component $\overline{\mathcal{D}}_i$ is smooth). We will use a tilde, as in $\tilde{I}_P$, to indicate the ordered set of such subindices, that is, $\tilde{I}_P \subset \{1,\dots,N\}$.

Definition 2.1. Let $\mathcal{A}_{\overline{X}}$ be the sheaf of C^∞-forms on $\overline{X}$. Denote by $\mathcal{A}^0_{\overline{X}}$ the sheaf of C^∞ functions on $\overline{X}$. Note that $\mathcal{A}_{\overline{X}}$ is a sheaf of graded algebras over $\mathcal{A}^0_{\overline{X}}$. The *sheaf of C^∞ log forms* $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$ can be defined locally at a point $P \in \overline{X}$ as the graded algebra over $(\mathcal{A}^0_{\overline{X}})_P$ of C^∞-forms $\varphi \in (\mathcal{A}_X)_P$ such that

$$z_{I_P}\varphi \quad \text{and} \quad z_{I_P} d\varphi$$

are C^∞-forms in $(\mathcal{A}_{\overline{X}})_P$.

A form φ on $U \subset \overline{X}$ shall be called *logarithmic on U (with respect to $\overline{\mathcal{D}}$)* if $\varphi \in \mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)(U)$.

The sheaf $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$ is a locally free and finitely generated $\mathcal{A}^0_{\overline{X}}$-algebra, as follows from

Lemma 2.2 ([11, Lemma 5.7])**.** $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)(U_P) \cong \mathcal{A}_{\overline{X}}(U_P)\{\frac{dz_i}{z_i}\}_{i\in I_P}$.

By definition, $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$ is closed under the exterior derivation d. This lemma shows that it is in fact closed under the exterior product and generated by $\mathcal{A}^1_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$.

In what follows, a weight filtration is defined in this sheaf of graded algebras that is compatible with the differential d.

Definition 2.3. If $\ell \geq 0$ we shall define the *sheaf of C^∞ log forms of weight ℓ* as the subsheaf of $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$ given locally as the $(\mathcal{A}_{\overline{X}})^0_P$-submodule of $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)_P$ of those forms φ such that

$$\varphi \in \sum_{\substack{I\subset I_P,\\ |I|\leq \ell}} \mathcal{A}_{\overline{X}}\left\{\frac{dz_i}{z_i}\right\}_{i\in I}.$$

Such a sheaf will be denoted by $\mathcal{W}_\ell := \mathcal{W}_\ell\left(\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)\right)$. If $\ell < 0$, we will assume $\mathcal{W}_\ell := \{0\}$.

Remark 2.4. Note that $\mathcal{W}_\ell \subset \mathcal{W}_{\ell+1}$, $d\mathcal{W}_\ell \subset \mathcal{W}_\ell$, and $\mathcal{W}_\ell \wedge \mathcal{W}_{\ell'} \subset \mathcal{W}_{\ell+\ell'}$ are obvious consequences of Definition 2.3.

Notation 2.5.

(1) Let us denote by $\overline{\mathcal{D}}^{[k]}$ the disjoint union of the codimension k intersections of components of $\overline{\mathcal{D}}$, that is;

$$\overline{\mathcal{D}}^{[k]} := \bigsqcup_{|I|=k} \overline{\mathcal{D}}_I,$$

where $\overline{\mathcal{D}}_I = \bigcap_{i\in I} \overline{\mathcal{D}}_i$.

(2) There is a natural inclusion $\overline{D}_I \overset{i_I}{\hookrightarrow} \overline{X}$ for each $\overline{\mathcal{D}}_I \in \overline{\mathcal{D}}^{[k]}$. Denoting by i_k the corresponding map on $\overline{\mathcal{D}}^{[k]}$, one has the following sheaf on $\overline{X}$:

$$\mathcal{A}^*_{\overline{\mathcal{D}}^{[k]}} = (i_k)_* \bigoplus_{|I|=k} \mathcal{A}^*_{\overline{\mathcal{D}}_I}.$$

Definition 2.6. Under the notation above, the *Poincaré residue operator*

$$R^{[k]} : \mathcal{W}_k\left(\mathcal{A}^*_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)\right) \longrightarrow \mathcal{A}^{*-k}_{\overline{\mathcal{D}}^{[k]}}$$

can be defined locally by

$$R^{[k]}\left(\alpha_P \wedge \frac{dz_I}{z_I}\right) = (-1)^{\sigma(\tilde{I})}\alpha_P \mid_{\overline{\mathcal{D}}_I},$$

where:

i) $\frac{dz_I}{z_I}$ denotes $\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_k}}{z_{i_k}}$, and

ii) if $\tilde{I} = \{\tilde{i}_1, \ldots, \tilde{i}_k\} \subset \{1, \ldots, N\}$, then $\sigma(\tilde{I}) := \mathrm{sign}(\tilde{i}_{k+1}, \ldots, \tilde{i}_N, \tilde{i}_1, \ldots, \tilde{i}_k)$, where $\tilde{i}_{k+1} < \cdots < \tilde{i}_N$ are the ordered elements of $\{1, \ldots, N\} \setminus \tilde{I}$.

Remark 2.7. Note that for any $\overline{\mathcal{D}}_{I'}$ and $\overline{\mathcal{D}}_I$ with $|I'| = k+1$ and $|I| = k$, one can define a smooth divisor on $\overline{\mathcal{D}}_I$ as follows:

$$\overline{\mathcal{D}}_I \mid_{\overline{\mathcal{D}}_{I'}} := \begin{cases} \overline{\mathcal{D}}_{I'} & \text{if } I \subset I', \\ \emptyset & \text{otherwise.} \end{cases}$$

Moreover, the union

$$\overline{\mathcal{D}}_I \mid_{\overline{\mathcal{D}}^{[k+1]}} := \sum_{|I'|=k+1} \overline{\mathcal{D}}_I \mid_{\overline{\mathcal{D}}_{I'}}$$

provides a simple normal crossing divisor in $\overline{\mathcal{D}}_I \subset \overline{\mathcal{D}}^{[k]}$. Hence, $\overline{\mathcal{D}}^{[k]}$ can be regarded as a disjoint union of smooth compact algebraic varieties, each component containing a divisor with normal crossings. Therefore, one can consider the sheaf of C^∞ log forms on each smooth algebraic variety $\overline{\mathcal{D}}_I$ with respect to $\overline{\mathcal{D}}_I \mid_{\overline{\mathcal{D}}^{[k+1]}}$, denoted by $\mathcal{A}_{\overline{\mathcal{D}}_I}(\log\langle\overline{\mathcal{D}}^{[k+1]}\rangle)$.

Definition 2.8. By means of the inclusions $\overline{\mathcal{D}}_I \overset{i_k}{\hookrightarrow} \overline{X}$, one can also define *log sheaves on* $\overline{\mathcal{D}}^{[k]}$ *relative to* $\overline{\mathcal{D}}^{[k+1]}$ as subsheaves of the direct sum of log sheaves, for each component satisfying certain compatibility relations. That is,

$$\mathcal{A}_{\overline{\mathcal{D}}^{[k]}}(\log\langle\overline{\mathcal{D}}^{[k+1]}\rangle) \subset \bigoplus_{|I|=k} \mathcal{A}_{\overline{\mathcal{D}}_I}(\log\langle\overline{\mathcal{D}}^{[k+1]}\rangle),$$

defined by the following natural local condition: for any strings I_1, I_2, I_1', I_2' such that $|I_i| = |I_i'| = k_i$, $i = 1, 2$, and $\{I_1 + I_2\} = \{I_1' + I_2'\}$, and for any pair of forms

$$\alpha_P \frac{dz_{I_2}}{z_{I_2}} \in \left(\mathcal{A}^*_{\overline{\mathcal{D}}_{I_1}}(\log\langle\overline{\mathcal{D}}^{[k_1+1]}\rangle)\right)_P \quad \text{and} \quad \beta_P \frac{dz_{I_2'}}{z_{I_2'}} \in \left(\mathcal{A}^*_{\overline{\mathcal{D}}_{I_1'}}(\log\langle\overline{\mathcal{D}}^{[k_1+1]}\rangle)\right)_P,$$

one has

$$(-1)^{\sigma(\tilde{I}_1)}(-1)^{\sigma(\widetilde{I_2+I_1})}\alpha_P \mid_{\overline{\mathcal{D}}_{I_1}} = (-1)^{\sigma(\tilde{I}_1')}(-1)^{\sigma(\widetilde{I_2'+I_1'})}\beta_P \mid_{\overline{\mathcal{D}}_{I_1'}}, \tag{1}$$

where $\tilde{I}_i$ and $\tilde{I}_i'$ are as in Definition 2.6 and $I + I'$ denotes juxtaposition of strings. For simplicity, this sheaf will be denoted by $\mathcal{A}^*_k(\log\langle\overline{\mathcal{D}}\rangle)$ and its restriction to $\overline{\mathcal{D}}_I$ (for $|I| = k$) by $\mathcal{A}^*_{k,I}(\log\langle\overline{\mathcal{D}}\rangle)$.

There also exists an obvious weight filtration on $\mathcal{A}_k(\log\langle\overline{\mathcal{D}}\rangle)$, denoted by $\mathcal{W}_\ell^{[k]}$. Note that $\mathcal{W}_\ell^{[0]} = \mathcal{W}_\ell(\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle))$ and $\mathcal{W}_0^{[k]} = \mathcal{A}_{\overline{\mathcal{D}}^{[k]}}$. The compatibility relations described in (1) allow for a generalization of the Poincaré residue operator to all the log sheaves relative to $\overline{\mathcal{D}}$, namely

$${}^\ell R_m^{[k]} : \mathcal{W}_\ell^{[m]} \longrightarrow \mathcal{W}_{\ell-k}^{[m+k]}. \tag{2}$$

In order to give a local description of ${}^\ell R_m^{[k]}$ let us consider a point $P \in \overline{X}$ and a form $\varphi \in \left(\mathcal{A}^*_k(\log\langle\overline{\mathcal{D}}\rangle)\right)_P$. Let us denote by $\left({}^\ell R_m^{[k]}\varphi\right)_I$ the coordinate of ${}^\ell R_m^{[k]}\varphi$ on $\mathcal{A}^{*-k}_{k+m,I}(\log\langle\overline{\mathcal{D}}\rangle)_P$, where $|I| = m + k$. In order to calculate this coordinate take two disjoint strings I_1 and I_2 such that $|I_1| = m$ and $\overline{\mathcal{D}}_I = \{z_{I_1}z_{I_2} = 0\}$, and hence $|I_2| = k$. The form φ can be written as

$$\alpha \frac{dz_{I_2}}{z_{I_2}} \in \left(\mathcal{A}^*_{k,I_1}(\log\langle\overline{\mathcal{D}}\rangle)\right)_P.$$

Thus one can define

$$\left({}^\ell R_m^{[k]}\varphi\right)_I := (-1)^{\sigma(\tilde{I}_1)}(-1)^{\sigma(\widetilde{I_2+I_1})}\alpha \mid_{\overline{\mathcal{D}}_I}.$$

Again by (1) the definition of $\left({}^\ell R_m^{[k]}\varphi\right)_I$ does not depend on the choice of I_1 and I_2.

The main result about these generalized residue maps, which will be intensively used throughout the paper, is the following:

Theorem 2.9 ([11, Theorem 5.15] and [5, Theorem 1.28])**.** *Any generalized residue mapping*

$$ {}^{\ell}\tilde{R}_m^{[k]} : (W_\ell^{[m],*}/W_{\ell-1}^{[m],*}) \longrightarrow (W_{\ell-k}^{[m+k],*-k}/W_{\ell-k-1}^{[m+k],*-k}) $$

on the complex of global sections induces an isomorphism on d-cohomology. Moreover,

$$ {}^{\ell-k_1}\tilde{R}_{m+k_1}^{[k_2]} \circ {}^{\ell}\tilde{R}_m^{[k_1]} = {}^{\ell}\tilde{R}_m^{[k_1+k_2]}. $$

2.2. **The spaces $H^k(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{C})$ and the residue maps.** As a general setting, let S be a smooth compact surface, $\mathcal{C} \subset S$ a reduced divisor. Let us denote by $S_{\mathcal{C}}$ the complement of $\mathcal{C}$ in S. Consider a resolution $\pi : \overline{S}_{\mathcal{C}} \to S$ of $\mathcal{C}$ in S such that $\overline{S}_{\mathcal{C}}$ is a compactification of $S_{\mathcal{C}}$ by a simple normal crossing divisor, and let $\overline{\mathcal{C}}$ be the reduced structure of $\pi^*\mathcal{C}$. Note that $S_{\mathcal{C}}$ is isomorphic to $\overline{S}_{\mathcal{C}} \setminus \overline{\mathcal{C}}$ via π.

Definition 2.10. A *log-resolution logarithmic form* on $\mathcal{C}$ at $P \in S$ is a differential form $\varphi \in (\mathcal{A}^*_{S_{\mathcal{C}}})_P$ such that $\pi^*(\varphi)_P \in \mathcal{A}^*(\log\langle\overline{\mathcal{C}}\rangle)_P$, that is, $\varphi \in \pi_*\mathcal{A}^*_{S_{\mathcal{C}}}$. The sheaf of log-resolution logarithmic forms on $\mathcal{C}$ will be denoted by $\mathcal{A}^{\log}_S(\mathcal{C}) = \pi_*\mathcal{A}^*_{S_{\mathcal{C}}}$.

Remark 2.11. Consider $\mathcal{C} \subset S$ a simple normal crossing divisor, $P \in S$, and $\varphi \in \mathcal{A}^*(\log\langle\mathcal{C}\rangle)_P$ a differential logarithmic form. Denote by $\pi : \bar{S} \to S$ the blow-up of P in S. Note that $\pi^*\varphi$ is also a logarithmic form on $\pi^{-1}\mathcal{C}$ at any point $Q \in \pi^{-1}(P)$. This, together with the fact that any two sequences of blow-ups of S are dominated by a third one, implies that the notion of a log-resolution logarithmic form on $\mathcal{C}$ is independent of the given embedded resolution of $\mathcal{C}$.

Note that $\mathcal{A}^{\log}_S(\mathcal{C}) \subset \mathcal{A}_S(\mathcal{C})$, where $\mathcal{A}_S(\mathcal{C})$ is the classical sheaf of logarithmic differential forms on $\mathcal{C}$ locally defined as

$$ (\mathcal{A}_S(\mathcal{C}))_P := \{\varphi \in (\mathcal{A}^*_{S_{\mathcal{C}}})_P \mid C_P\varphi \in (\mathcal{A}^*_{\overline{S}_{\mathcal{C}}})_P, \quad C_P d\varphi \in (\mathcal{A}^*_{\overline{S}_{\mathcal{C}}})_P\}, $$

where C_P is a reduced equation of $\mathcal{C}$ at P.

In fact, $\mathcal{A}^{\log}_S(\mathcal{C})$ is the largest subsheaf of $\mathcal{A}_S(\mathcal{C})$ that is stable under blow-ups. Moreover, by Lemma 2.2, $\mathcal{A}^{\log}_S(\mathcal{C})$ is locally free.

Construction 2.12. Let $\mathcal{C} \subset \mathbb{P}^2$ be an algebraic curve with irreducible components $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_r$. Fix $\pi : \overline{S}_{\mathcal{C}} \longrightarrow \mathbb{P}^2$ as a resolution of the singularities of $\mathcal{C}$ so that the reduced divisor $\overline{\mathcal{C}} = (\pi^*(\mathcal{C}))_{\text{red}}$ is a simple normal crossing divisor on $\overline{S}_{\mathcal{C}}$, as described in section 2.1.

Consider the following short exact sequence of complexes $0 \to W_{i-1} \to W_i \to W_i/W_{i-1} \to 0$, where W_i denotes the complex $(W_i^{[0],*}\mathcal{A}_{\overline{S}_{\mathcal{C}}}(\log\langle\overline{\mathcal{C}}\rangle), d)$. Let us consider its corresponding long exact sequence of d-cohomology

$$ \begin{aligned} \ldots &\to H^{k-1}(W_i/W_{i-1}) \to H^k(W_{i-1}) \to H^k(W_i) \xrightarrow{\delta_i^k} H^k(W_i/W_{i-1}) \\ &\to H^{k+1}(W_{i-1}) \to \ldots. \end{aligned} \tag{3} $$

Using the de Rham Theorem and Theorem 2.9, one can define the *residue map* $\text{Res}^{[i]} : H^i(S_{\mathcal{C}}; \mathbb{C}) \to H^0(\overline{\mathcal{C}}^{[i]}; \mathbb{C})$ as the following composition:

$$ H^i(S_{\mathcal{C}}; \mathbb{C}) \overset{\pi^{-1}}{\cong} H^i(\overline{S}_{\mathcal{C}} \setminus \overline{\mathcal{C}}; \mathbb{C}) \overset{DR}{\cong} H^i(\overline{S}_{\mathcal{C}}, W_i) \xrightarrow{\delta_i^i} H^i(\overline{S}_{\mathcal{C}}, W_i/W_{i-1}) \overset{R^{[i]}}{\cong} H^0(\overline{\mathcal{C}}^{[i]}; \mathbb{C}). \tag{4} $$

Proposition 2.13 ([5, Proposition 2.2])**.** *If $\mathcal{C}$ is an algebraic plane curve with complement $S_{\mathcal{C}}$, then*

$$H^2(S_{\mathcal{C}};\mathbb{C}) \cong H_1(\mathcal{C};\mathbb{C}), \text{ and } H^1(S_{\mathcal{C}};\mathbb{C}) \cong H_2(\mathcal{C};\mathbb{C})/\mathbb{C}.$$

Notation 2.14. Let Y be a topological space. In what follows, we will denote by $h_i(Y)$ (resp. $h^i(Y)$) the dimension of the vector space $H_i(Y;\mathbb{C})$ (resp. $H^i(Y;\mathbb{C})$). Note that, by the Universal Coefficient Theorem, $h_i(Y) = h^i(Y)$.

One has the following result.

Proposition 2.15. *The first residue map $H^1(S_{\mathcal{C}}) \xrightarrow{\mathrm{Res}^{[1]}} H^0(\overline{\mathcal{C}}^{[1]})$ is injective. On the other hand,*

$$\ker\left(H^2(S_{\mathcal{C}}) \xrightarrow{\mathrm{Res}^{[2]}} H^0(\overline{\mathcal{C}}^{[2]})\right) \subset H^2(W_1)$$

has dimension $2g$, where $g = \sum_{i=1}^{r} g(\mathcal{C}_i)$ is the sum of the genera of the irreducible components of $\mathcal{C}$.

Proof. The injectivity of $\mathrm{Res}^{[1]}$ follows immediately from (3) for the case $i = k = 1$, and $H^1(W_0, d) = 0$. Let us now consider (3) for $k = 2$, $i = 1$.

(5)
$$\begin{array}{ccccccccccccc}
H^1(W_0) & \to & H^1(W_1) & \to & H^1(W_1/W_0) & \to & H^2(W_0) & \to & H^2(W_1) & \to & H^2(W_1/W_0) & \to & H^3(W_0) \\
\| & & \| & & \| & & \| & & & & \| & & \| \\
H^1(\overline{S}_{\mathcal{C}}) & & H^1(S_{\mathcal{C}}) & & H^0(\overline{\mathcal{C}}^{[1]}) & & H^2(\overline{S}_{\mathcal{C}}) & & & & H^1(\overline{\mathcal{C}}^{[1]}) & & H^3(\overline{S}_{\mathcal{C}}) \\
\| & & \| & & \| & & \| & & & & \| & & \| \\
0 & & \mathbb{C}^r & & \mathbb{C}^{r+e+1} & & \mathbb{C}^{e+1} & & & & \mathbb{C}^{2g} & & 0
\end{array}$$

where e is the number of exceptional components in the resolution of $\mathcal{C}$. The equalities in the second column are a consequence of de Rham and Proposition 2.13. The others are a consequence of $H^k(W_0) = H^k(\overline{S}_{\mathcal{C}})$, de Rham, and Theorem 2.9.

Computing the Euler characteristic of this long exact sequence, one obtains that $H^2(W_1, d) \cong \mathbb{C}^{2g}$, and therefore, using (3) for the case $i = k = 2$, one obtains

$$0 \to H^2(W_1) = \mathbb{C}^{2g} \to H^2(W_2) = H^2(S_{\mathcal{C}}) \xrightarrow{\mathrm{Res}^{[2]}} H^0(\overline{\mathcal{C}}^{[2]}) \to H^3(W_1) \to \dots,$$

which proves that $\ker\left(H^2(S_{\mathcal{C}}) \xrightarrow{\mathrm{Res}^{[2]}} H^0(\overline{\mathcal{C}}^{[2]})\right) = H^2(\overline{S}_{\mathcal{C}}; W_1) \cong \mathbb{C}^{2g}$. Finally, by the Leray spectral sequence, since all these sheaves are flasque, one has the projection formula $H^2(\overline{S}_{\mathcal{C}}; W_1) \cong H^2(\mathbb{P}^2; \pi_* W_1)$. □

2.3. **Classical combinatorics.** In this paragraph we just want to give a general outline of the classical concept of a *combinatorial type* of a curve. This concept is generally accepted and used, but is seldom explicitly defined. In [2] there is a detailed explanation of the matter. For the sake of completeness, we summarize the main ideas.

Definition 2.16. Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane projective curve. The *combinatorial type* of $\mathcal{C}$ is given by a sextuplet

$$K_{\mathcal{C}} := (\mathbf{r}, \bar{d}, S, \Sigma, \sigma, \Delta, \phi),$$

where:

- (i) The elements of $\mathbf{r}$ are in bijection with the irreducible components of $\mathcal{C}$,
- (ii) $\bar{d} : \mathbf{r} \to \mathbb{N}$ is the degree map that assigns to each irreducible component of $\mathcal{C}$ its degree,
- (iii) $S := \mathrm{Sing}(\mathcal{C})$, the set of singular points of $\mathcal{C}$,

(*iv*) Σ is the set of topological types of the points in S,
(*v*) $\sigma : S \to \Sigma$ assigns to each singular point its topological type,
(*vi*) $\Delta := \{\Delta_P\}_{P\in S}$, where Δ_P is the set of local branches of $\mathcal{C}$ at $P \in S$ (a local branch can be seen as an arrow in the dual graph of the minimal resolution of $\mathcal{C}$ at P; see [10, Chapter II.8] for details), and $\phi := \{\phi_P\}_{P\in S}$, where $\phi_P : \Delta \to \mathbf{r}$ assigns to each local branch the global irreducible component that contains it.

We say that two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have the *same combinatorial type* (or simply the *same combinatorics*) if their combinatorial data $K_{\mathcal{C}_1}$ and $K_{\mathcal{C}_2}$ are equivalent. That is, if $\Sigma_1 = \Sigma_2$, and there exist bijections:

(1) $\varphi_{\mathbf{r}} : \mathbf{r}_1 \to \mathbf{r}_2$,
(2) $\varphi_S : S_1 \to S_2$, and
(3) $\varphi_P : \Delta_{1,P} \to \Delta_{2,\varphi_S(P)}$ (the restriction of a bijection of dual graphs) for each $P \in S_1$

such that:

(1) $\bar{d}_1 = \bar{d}_2 \circ \varphi_{\mathbf{r}}$,
(2) $\sigma_1 = \sigma_2 \circ \varphi_S$, and
(3) $\varphi_{\mathbf{r}} \circ \phi_{1,P} = \phi_{2,\varphi_S(P)} \circ \varphi_P$.

In the irreducible case, two curves have the same combinatorial type if they have the same degree and the same topological types for local singularities. On the other extreme, for line arrangements combinatorial type is determined by the incidence graph. In higher dimensions, the concept of combinatorics still makes sense but becomes much harder to describe, except for the case of hyperplane (or in general linear) arrangements where the incidence relations are enough to determine the combinatorial type.

The main interest in (and motivation for) considering combinatorial types of curves is due to the following (see [2]).

Proposition 2.17. *Consider two curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$, and $T(\mathcal{C}_1)$, $T(\mathcal{C}_2)$ as their regular neighborhoods with boundary. Then the pairs $(T(\mathcal{C}_1), \mathcal{C}_1)$ and $(T(\mathcal{C}_2), \mathcal{C}_2)$ are homeomorphic if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same combinatorial type.*

Proof. In one direction, the self intersections of the components of $\mathcal{C}_i$ and the topological types of the singularities of $\mathcal{C}_i$ are well defined and preserved under homeomorphisms of pairs $(T(\mathcal{C}_i), \mathcal{C}_i)$. This determines degrees and topological types of singularities, as well as the incidence of local branches. Therefore their combinatorial types coincide. Conversely, the coincidence of the combinatorial type allows one to recover the minimal resolutions of the singularities, together with a homeomorphism between them. Since the self intersections coincide, one can extend this to a homeomorphism of the tubular neighborhoods of each component (including exceptional components) and glue them along the intersections as prescribed by the multiplicities of the components. By contracting the exceptional components one can define a homeomorphism of pairs between $(T(\mathcal{C}_1), \mathcal{C}_1)$ and $(T(\mathcal{C}_2), \mathcal{C}_2)$. □

A pair of plane curves $(\mathcal{C}_1, \mathcal{C}_2)$, such that $(T(\mathcal{C}_1), \mathcal{C}_1)$ and $(T(\mathcal{C}_2), \mathcal{C}_2)$ are homeomorphic but $(\mathbb{P}^2, \mathcal{C}_1)$ and $(\mathbb{P}^2, \mathcal{C}_2)$ are not (that is, whose embeddings in $\mathbb{P}^2$ are not homeomorphic), is called a *Zariski pair*. The existence of Zariski pairs and the search for invariants of the embedding of a curve that can tell two combinatorially-equivalent curves apart has been a very productive field of research started by

O. Zariski in [21, 22] (see [2] and the references therein for an extended survey on Zariski pairs).

Alternatively, one can also define a weaker concept of combinatorics as follows.

Definition 2.18. Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane projective curve. The *weak combinatorial type* of $\mathcal{C}$ is given by a septuplet

$$W_{\mathcal{C}} := (\mathbf{r}, S, \Delta, \phi, \mu, \bar{d}, \bar{g}),$$

where $\mathbf{r}$, $\bar{d}$, S, Δ, and ϕ are defined as above, $\bar{g} : \mathbf{r} \to \mathbb{N}$ is the list of genera, ($\mu := \{\mu_P\}_{P \in S}$), where $\mu_P : \Sigma^2_{\phi_P}\Delta_P \to \mathbb{N}$, $\Sigma^2_{\phi_P}\Delta_P := \frac{(\Delta_P \times \Delta_P)\setminus(\Delta_P \times_{\phi_P} \Delta_P)}{S_2}$ is the symmetric product of Δ_P outside the ϕ_P-diagonal (that is, the fibered product $\{(\delta_1, \delta_2) \mid \phi_P(\delta_1) = \phi_P(\delta_2)\}$), and $\mu_P(\delta_1, \delta_2)$ represents the intersection multiplicity of δ_1 and δ_2 at P.

We say that two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have the *same weak combinatorial data* (or simply the *same combinatorics*) if their weak combinatorial types $W_{\mathcal{C}_1}$ and $W_{\mathcal{C}_2}$ are equivalent, that is, if there exist bijections:

(1) $\varphi_{\mathbf{r}} : \mathbf{r}_1 \to \mathbf{r}_2$,
(2) $\varphi_S : S_1 \to S_2$, and
(3) $\varphi_P : \Delta_{1,P} \to \Delta_{2,\varphi_S(P)}$ (restriction of a bijection of dual graphs) for each $P \in S_1$

such that:

(1) $\bar{d}_1 = \bar{d}_2 \circ \varphi_{\mathbf{r}}$,
(2) $\varphi_{\mathbf{r}} \circ \phi_1 = \phi_2 \circ \varphi_P$, and
(3) $\mu_1(\delta_1, \delta_2) = \mu_2(\varphi_P(\delta_1), \varphi_P(\delta_2))$.

It is obvious that $K_{\mathcal{C}}$ determines $W_{\mathcal{C}}$ using the intersection multiplicity formula. The converse is also true for smooth arrangements (a curve whose irreducible components are smooth), but not true in general, as Example 5.1 shows.

The question immediately arises as to what extent the combinatorial type of a curve determines well-known invariants of its embedding in $\mathbb{P}^2$. We will refer to such invariants as *combinatorial*. Fundamental groups of complements of curves are known not to be combinatorial, as first shown by Zariski in [21]. The cohomology ring of the complement of a curve was only known to be combinatorial when the curve was a line arrangement [1, 3] or, more generally, a rational arrangement [5]. The purpose of the upcoming section is to prove that the cohomology ring of the complement of a curve is a combinatorial invariant.

3. Cohomology ring structure

In what follows we will describe generators for $H^*(S_{\mathcal{C}})$. For simplicity, we will assume $\mathcal{C}_0$ is a transversal line. We will consider coordinates $[X : Y : Z]$ in $\mathbb{P}^2$ such that $\mathcal{C}_0 := \{Z = 0\}$, and define $\omega := XdY \wedge dZ + YdZ \wedge dX + ZdX \wedge dY$ as the contraction of the volume form in the affine space $\mathbb{A}^3$ by the Euler vector field.

As in the classical cases, the subspace $H^1(S_{\mathcal{C}})$ is generated by the log-resolution logarithmic 1-forms $\sigma_i := d\log \frac{C_i}{C_0^{d_i}}$, $i = 1, \ldots, r$, where C_i is an equation for the component $\mathcal{C}_i$.

Theorem 3.1 ([5, Theorems 2.10 and 2.11])**.** *The* 1*-forms* σ_i*,* $i = 1, \ldots, r$*, defined above verify the following properties:*

(*i*) $\sigma_i \in W^1\mathcal{A}^{\log}_{\mathbb{P}^2}(\mathcal{C})$,

(*ii*) $\Sigma := \{\sigma_1, \dots, \sigma_r\}$ *generate* $H^1(S_{\mathcal{C}})$ *as a vector space, and*

(*iii*) $\left(\mathrm{Res}^{[1]}\sigma_i\right)_{\tilde{\mathcal{C}}_j} = \begin{cases} (-1)^{r-i} & \text{if } i = j, \\ 0 & \text{if } i \neq j \neq 0, \\ (-1)^{r+1} d_i & \text{if } j = 0. \end{cases}$

Moreover, $\langle \Sigma \rangle_{\mathbb{C}} = H^1(S_{\mathcal{C}}) = W_1^1 = W^1$.

In order to obtain generators for $H^2(S_{\mathcal{C}})$ we will define special 2-forms as global forms of ideal sheaves. Such ideal sheaves are supported on the singular points of $\mathcal{C}$. Their definition will be given in terms of the logarithmic trees, which are isomorphic (as directed trees) to multiplicity trees but whose weights are different.

3.1. **Logarithmic trees. Local setting.** Let us first recall the concept of multiplicity trees.

Let $f \in \mathbb{C}\{x, y\}$ be a germ of a holomorphic function at P whose set of zeroes is a curve germ $V_f \subset S_0$ with an isolated singularity at the point P. Consider the sequence of blow-ups

$$S_0 \xleftarrow{\varepsilon_1} S_1 \xleftarrow{\varepsilon_2} S_2 \xleftarrow{\varepsilon_3} \dots \xleftarrow{\varepsilon_m} S_m = \overline{S}$$

in the resolution of S_0 at P, and denote by π_k the composition of the first k blow-ups $\pi_k = \varepsilon_k \circ \dots \circ \varepsilon_1$. The curve germ $\tilde{V}_{f,k} = \overline{\pi_k^{-1}(V_f \setminus \{P\})}$ shall be called the *strict transform of* V_f *in* S_k and its equation denoted by $\tilde{f}_k$. The divisor $\pi_k^*(V_f)$ shall be denoted by $\overline{V}_{f,k}$ and called the *total transform of* V_f *in* S_k. For simplicity, let us write $\tilde{V}_f := \tilde{V}_{f,m}$ and $\overline{V}_f := \overline{V}_{f,m}$. The exceptional divisor in S_k resulting from the blow-up of a point in S_{k-1} shall be denoted by E_k, and the points $P_k^1, \dots, P_k^{N_k}$ in $E_k \cap \tilde{V}_{f,k}$ will be called the *infinitely near points to* P *in* E_k. For convenience, the point P is also considered to be infinitely near to itself. Finally, the multiplicity of $\tilde{V}_{f,k} \subset S_k$ at the point P_k^ℓ shall be denoted by $\nu_{P_k^\ell}(f)$, i.e.

$$\nu_{P_k^\ell}(f) := \mathrm{mult}_{P_k^\ell}(\tilde{V}_{f,k}).$$

To each resolution of singularities π one can assign the *multiplicity tree of* π *at* P – denoted by $\mathcal{T}_P(f, \pi)$ or simply by $\mathcal{T}_P(f)$ if the resolution π of S_0 is fixed. $\mathcal{T}_P(f)$ is a tree with weights at each vertex and is defined as follows:

(a) The vertices of $\mathcal{T}_P(f)$ are in bijection with the infinitely near points to P.
(b) Two vertices of $\mathcal{T}_P(f)$, say Q_1 and Q_2, are joined by an edge if and only if one of the points, say Q_2, belongs to S_k for some k, and the other point, Q_1, belongs to S_{k-1} and $Q_2 \in \varepsilon_k^{-1}(Q_1)$.
(c) For convenience, this tree is considered to simply be a vertex if P is not a singular point of f. If $f(P) \neq 0$, then $\mathcal{T}_P(f) := \emptyset$.
(d) The weight $w(\mathcal{T}_P(f), Q)$ of a vertex Q is $\nu_Q(f)$.

Example 3.2. Let $f = (x^3 - y^5)(x - y^2)(y^2 - x^3)y$. The tree given in Figure 1 is the multiplicity tree $\mathcal{T}_0(f)$ of the minimal resolution of $\{f = 0\}$ at 0.

The set of vertices $|\mathcal{T}_P(f)|$ of a multiplicity tree $\mathcal{T}_P(f)$ is endowed with a partial order as follows. Consider P as the root of the tree and direct the edges of the tree towards P. In this *directed tree*, a point Q_2 is said to be *greater* than Q_1 – denoted $Q_2 \geq Q_1$ – if there is a directed path from Q_2 to Q_1. In graph theory this situation

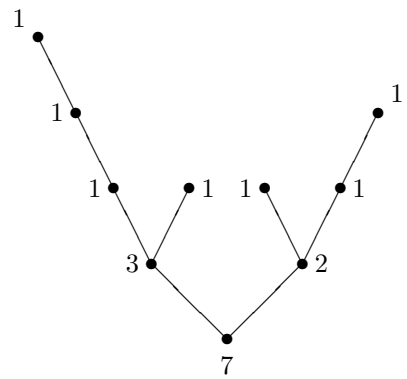

FIGURE 1. Multiplicity tree of $(f, 0)$.

is commonly described by calling Q_2 an *ancestor* of Q_1 or Q_1 a *descendant* of Q_2. Given a set of points $\{P_1, \ldots, P_n\} \subset \mathcal{T}_P(f)$ one can define

$$\mathrm{Asc}(P_1, \ldots, P_n) = \{Q \in \mathcal{T}_P(f) \mid Q \geq P_i \quad i = 1, \ldots, n\}$$

and

$$\mathrm{Desc}(P_1, \ldots, P_n) = \{Q \in \mathcal{T}_P(f) \mid Q \leq P_i \quad i = 1, \ldots, n\}.$$

Multiplicity trees are *quasi-strongly connected trees*, which means that the set of common descendants $\mathrm{Desc}(P_1, \ldots, P_n)$ is non-empty and inherits a linear order from $\mathcal{T}_P(f)$. The maximal element in $\mathrm{Desc}(P_1, \ldots, P_n)$ is called the *greatest common descendant* and is denoted by $\gcd(P_1, \ldots, P_n)$.

The *degree* of a weighted tree $\mathcal{T}$ shall be defined as

$$\deg(\mathcal{T}) := \sum_{Q \in |\mathcal{T}|} \binom{w(\mathcal{T}, Q) + 1}{2}, \tag{6}$$

where $w(\mathcal{T}, Q)$ denotes the weight of $\mathcal{T}$ at Q.

Note that if $\mathcal{T} = \mathcal{T}_P(f)$, then $\deg(\mathcal{T})$ is the δ-invariant of the singularity of f at P.

In order to simplify, we shall write $\mathcal{T} \cong \mathcal{T}'$ for two weighted trees that are isomorphic as trees, and $\mathcal{T} = \mathcal{T}'$ (resp. $\geq$, $\leq$, $<$ or $>$) if $\mathcal{T} \cong \mathcal{T}'$ and $\hat{w}(\mathcal{T}, Q) = \hat{w}(\mathcal{T}', Q)$ (resp. $\geq$, $\leq$, $<$ or $>$) for any $Q \in |\mathcal{T}| = |\mathcal{T}'|$, where $\hat{w}(\mathcal{T}, Q) := \sum_{Q' \in \mathrm{Desc}(Q)} w(\mathcal{T}, Q')$ (we are using the isomorphism of trees to identify the vertices). Note that $\hat{w}(\mathcal{T}, Q)$ is the multiplicity of the total transform of f at Q. Also, $\mathcal{T} - k$ will denote a tree $\mathcal{T}' \cong \mathcal{T}$ so that $w(\mathcal{T}', Q) = \max\{w(\mathcal{T}, Q) - k, 0\}$ for any $Q \in |\mathcal{T}|$. Particularly useful will be the tree

$$\mathcal{T}_P^{\mathfrak{n}}(f) := \mathcal{T}_P(f) - 1. \tag{7}$$

Sometimes it will be necessary to compare empty trees. In this case, the conditions $=, \leq, \geq$ are vacuous and hence always satisfied.

Let $g \in \mathbb{C}\{x, y\}$ be another germ at P. Then one can consider the *restriction of g to a weighted tree $\mathcal{T}$* (e.g. $\mathcal{T}_P(f)$) –denoted by $\mathcal{T}|_g$– as a weighted graph

isomorphic to $\mathcal{T}$ whose weight at each vertex Q is $\nu_Q(g)$. One can check that the set $I := \{g \in \mathbb{C}\{x,y\} \mid \mathcal{T}|_g \geq \mathcal{T}\}$ defines an ideal. Note that $\dim_{\mathbb{C}}\left(\mathbb{C}\{x,y\}/\mathfrak{m}^k\right) = \binom{k+1}{2}$. Hence

$$\deg(\mathcal{T}) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{I}. \tag{8}$$

Let $\mathcal{C}$ be a plane projective curve and $P \in \operatorname{Sing}\mathcal{C}$. Note that its multiplicity tree does not depend on the equation of $\mathcal{D}$, hence it can be denoted as $\mathcal{T}_P(\mathcal{C})$ or $\mathcal{T}_P(\mathcal{C},\pi)$ in case we want to specify the underlying resolution.

In case $\mathcal{C}$ is irreducible and has degree d, from (6) and (7) one can rewrite the Noether formula for the genus [4, p. 614] as follows:

$$g(\mathcal{C}) = \frac{(d-1)(d-2)}{2} - \sum_{P\in\operatorname{Sing}\mathcal{C}} \deg(\mathcal{T}_P^{\mathfrak{n}}(\mathcal{C})). \tag{9}$$

Now consider π as a resolution of singularities for the plane curve $\mathcal{C}$. We define the *basic ideal sheaf of* $\mathcal{C}$ *with respect to* π as follows:

$$(\mathcal{I}^{\mathfrak{n}}_{\mathcal{C},\pi})_P := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(\mathcal{C},\pi)|_h \ \geq \ \mathcal{T}_P^{\mathfrak{n}}(\mathcal{C},\pi)\}. \tag{10}$$

If no possible confusion results from the underlying resolution, the sheaf $\mathcal{I}^{\mathfrak{n}}_{\mathcal{C},\pi}$ will be denoted simply by $\mathcal{I}^{\mathfrak{n}}_{\mathcal{C}}$.

Remark 3.3. Since π also induces a log resolution of the ideal $C = I(\mathcal{C})$ at any point P, one can also see $\mathcal{I}^{\mathfrak{n}}_{\mathcal{C}}$ as the multiplier ideal sheaf of C, that is, $\pi_*\mathcal{O}_{\bar{S}_{\mathcal{C}}}(K_{\bar{S}_{\mathcal{C}}/\mathbb{P}^2} - F)$, where $C \cdot \mathcal{O}_{\bar{S}_{\mathcal{C}}} = \mathcal{O}_{\bar{S}_{\mathcal{C}}}(-F)$. Analogously, $\mathcal{I}^{\mathfrak{n}}_{\mathcal{C}}$ corresponds to the special ideal of quasi-adjunction $\mathcal{A}_0(C)$ as defined in [14, 15].

This leads naturally to the idea of a logarithmic ideal of a germ. Let $f \in \mathbb{C}\{x,y\}$ be a holomorphic germ at P and π a resolution of V_f. We can define an ideal $I \subset \mathbb{C}\{x,y\}$ satisfying that for any germ $h \in I$ the 2-form

$$h\ \frac{dx \wedge dy}{f} \tag{11}$$

is log-resolution logarithmic at P –with respect to V_f and the resolution π. Such an ideal will be called a *logarithmic ideal for* f *at* P.

Remark 3.4. Using Remark 2.11, it is easy to see that the set of logarithmic ideals associated with a singularity is independent of the resolution. Hence, from now on, when referring to logarithmic ideals any reference to a resolution will be omitted.

A very useful way to encode the information required to construct logarithmic ideals is given by weighted trees.

Definition 3.5. Let $(f,0)$ be a germ and $\mathcal{T}(f)$ its multiplicity tree. A weighted tree $\mathcal{T}$ is said to be a *logarithmic tree for* $(f,0)$ if it satisfies the following properties:

(*i*) $\mathcal{T} \cong \mathcal{T}(f)$ and
(*ii*) the ideal $I := \{h \in \mathcal{O}_0 \mid \mathcal{T}(f)|_h \ \geq \ \mathcal{T}\}$ is logarithmic.

In addition, if δ_1 and δ_2 are local branches of $(f,0)$, we say that $\mathcal{T}$ is a *logarithmic tree* (*for* $(f,0)$) *relative to* δ_1 *and* δ_2 if $\mathcal{T}$ satisfies (*i*), (*ii*) above and

(*iii*) if $\varphi \in M_I$, where $M_I := \{h \in \mathcal{O}_0 \mid \mathcal{T}(f)|_h = \mathcal{T}\} \subset I$, then

$$\left(\mathrm{Res}^{[2]}\varphi \, \frac{dx \wedge dy}{f}\right)_Q \neq 0$$

if and only if Q is a vertex of the unique subtree $\gamma(\delta_1, \delta_2) \subset \mathcal{T}$ joining δ_1 and δ_2.

Example 3.6. Note that $\mathcal{T}_P^{\mathfrak{n}}(f)$ is a logarithmic tree for f, but it is *not* relative to any two branches δ_1, δ_2. One can check that properties (*i*) and (*ii*) are satisfied but $\mathcal{T}_P^{\mathfrak{n}}(f)$ does not satisfy property (*iii*). Moreover, if $\varphi \in \mathbb{C}\{x, y\}$ is a germ at P such that $\psi := \varphi \frac{dx \wedge dy}{f}$ with $\mathcal{T}_P(f)|_\varphi \geq \mathcal{T}_P^{\mathfrak{n}}(f)$, then one can check that $\psi \in \pi_* W_1^2$, that is, it has weight one, and hence $\left(\mathrm{Res}^{[2]}\psi\right)_Q = 0$ for any vertex Q of $\mathcal{T}_P(f)$.

The main result of this part is the existence of logarithmic trees relative to any two local branches of any reduced germ f.

Theorem 3.7 ([5, Lemma 2.34])**.** *For any given two local branches δ_1 and δ_2 of f at P, there exists a logarithmic tree for (f, P) relative to δ_1 and δ_2.*

Theorem 3.7 is constructive. We will denote such a tree by $\mathcal{T}_P^{\delta_1, \delta_2}$, and it will be referred to as the *basic logarithmic tree relative to δ_1 and δ_2*.

3.2. **Logarithmic ideal sheaves. Global setting.** Let us return to the situation presented at the beginning of this section, where $\mathcal{C}$ is a plane projective curve and π is a resolution of singularities. The concept of logarithmic ideal translates globally as follows:

Definition 3.8. We call an ideal sheaf $\mathcal{I}$ on $\mathbb{P}^2$ a *logarithmic ideal sheaf* for $\mathcal{C}$ if its stalks $\mathcal{I}_P$ are logarithmic ideals for the germs C_P of $\mathcal{C}$ at any $P \in \mathbb{P}^2$.

Remark 3.9. By Example 3.6, the basic ideal sheaf of $\mathcal{C}$ denoted by $\mathcal{I}_{\mathcal{C}}^{\mathfrak{n}}$ defined in (10) is a logarithmic ideal sheaf for $\mathcal{C}$.

Also note that by Remark 3.4, such sheaves are independent of the given resolution.

Let $\mathcal{C}_{ij} := \mathcal{C}_i \cup \mathcal{C}_j$, $d_{ij} := \deg \mathcal{C}_{ij}$, and $g_{ij} := g(\mathcal{C}_{ij})$. We denote by C_{ij} an equation of $\mathcal{C}_{ij}$ (which is C_i if $i = j$ and $C_i C_j$ if $i \neq j$). We will first check that the basic ideal sheaf of $\mathcal{C}_{ij}$ has non-trivial sections of degree $d_{ij} - 2$ except for the obvious case of lines.

Proposition 3.10. $\dim H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{C}_{ij}}^{\mathfrak{n}}(d_{ij} - 2)) \geq d_{ij} + g_{ij} - \#\{i, j\}$.

Proof. To ease the notation let us write $\mathcal{I}$ for $\mathcal{I}_{\mathcal{C}_{ij}}^{\mathfrak{n}}$. From the exact sequence

$$0 \to \mathcal{I}(d_{ij} - 2) \to \mathcal{O}_{\mathbb{P}^2}(d_{ij} - 2) \to \mathcal{O}/\mathcal{I}(d_{ij} - 2) \to 0$$

and the fact that $H^\ell(\mathcal{O}(k)) = 0$ for any k and $\ell > 0$, one obtains that

$$h^0(\mathbb{P}^2, \mathcal{I}(d_{ij} - 2)) \geq \binom{d_{ij}}{2} - h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I}). \tag{12}$$

In what follows, we will assume $i \neq j$. The case $i = j$ is analogous. First, we will calculate $h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I})$. Note that, by (8), one has

$$h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I}) = \sum_{P \in \mathrm{Sing}\, \mathcal{C}_{ij}} \deg \mathcal{T}_P^{\mathfrak{n}}(\mathcal{C}_{ij}) = \sum_{P \in \mathrm{Sing}\, \mathcal{C}_{ij}} \sum_{Q \in |\mathcal{T}_P(\mathcal{C}_{ij})|} \binom{\nu_Q(C_{ij})}{2}.$$

Since $\nu_Q(C_{ij}) = \nu_Q(C_i) + \nu_Q(C_j)$ and $\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab$, one obtains that

$$h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I}) = \sum_{P \in \operatorname{Sing} \mathcal{C}_{ij}} \sum_{Q \in |\mathcal{T}_P(\mathcal{C}_{ij})|} \binom{\nu_Q(C_i)}{2} + \binom{\nu_Q(C_j)}{2} + \nu_Q(C_i)\nu_Q(C_j),$$

and finally, using (9), one obtains

$$h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I}) = \binom{d_i - 1}{2} + \binom{d_j - 1}{2} - g_{ij} + d_i d_j. \tag{13}$$

Therefore (12) becomes

$$\begin{aligned} h^0(\mathbb{P}^2, \mathcal{I}(d_{ij} - 2)) \geq & \left[\binom{d_{ij}}{2} - \binom{d_i - 1}{2} - \binom{d_j - 1}{2} - d_i d_j \right] + g_{ij} \\ = & \ (d_i - 1) + (d_j - 1) + g_{ij}. \end{aligned}$$

□

In order to construct global forms we will proceed as follows. First, for each irreducible component $\mathcal{C}_i$ of $\mathcal{C}$, we will order the $d_i = \deg \mathcal{C}_i$ points of $\mathcal{C}_i$ at infinity $\mathcal{C}_i \cap \mathcal{C}_0 = \{P_1^i, \ldots, P_{d_i}^i\}$.

Definition 3.11. Let $P \in \mathcal{C}_{ij}$, and let δ_1 (resp. δ_2) be a local branch of the irreducible component $\mathcal{C}_i$ (resp. $\mathcal{C}_j$) at P. The *ideal sheaf* $\mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1,\delta_2}$ *associated with* δ_1 *and* δ_2 shall be defined as

$$(\mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1,\delta_2})_Q := \left\{ h \in \mathcal{O}_Q \;\middle|\; \begin{array}{lll} \mathcal{T}_Q(\mathcal{C}_{ij})|_h & \geq & \mathcal{T}_P^{\delta_1,\delta_2}(\mathcal{C}_{ij}) \quad \text{if } Q = P, \\ \mathcal{T}_Q(\mathcal{C}_{ij})|_h & \geq & \mathcal{T}_Q(\mathcal{C}_{ij}) - 2 \quad \text{if } Q \in \{P_1^i, P_1^j\}, \\ \mathcal{T}_Q(\mathcal{C}_{ij})|_h & \geq & \mathcal{T}_Q^{\mathfrak{n}}(\mathcal{C}_{ij}) \quad \text{otherwise} \end{array} \right\}.$$

A global section s of $\mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1,\delta_2}(d)$ shall be called *essential* if $s_Q \in M_{I_Q}$ for every $Q \in \mathbb{P}^2$, where s_Q is the section s localized at Q, $I_Q = (\mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1,\delta_2})_Q$ and M_{I_Q} is as in Definition 3.5.

Analogously to [5, Lemma 3.35] one can prove the following.

Proposition 3.12. $\deg \mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1,\delta_2} = \deg \mathcal{I}_{\mathcal{C}_{ij}}^{\mathfrak{n}} + d_{ij} - \#\{i,j\} - 1$.

Therefore Propositions 3.10 and 3.12 imply the following.

Proposition 3.13. $\dim H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1,\delta_2}(d_{ij} - 2)) > g_{ij}$.

One can give a description of a section in such sheaf ideals. In order to do so, let us denote by $\gamma_P(\delta_1, \delta_2)$ the minimal subtree in $\mathcal{T}_P(\mathcal{C}_{ij})$ (see Definition 3.5) containing δ_1 and δ_2. We can consider $\gamma_P(\delta_1, \delta_2)$ as a subset of the total transform $\overline{\mathcal{C}}$ of $\mathcal{C}$ (in particular it should contain $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}_j$). We also denote by $v(\gamma_P(\delta_1, \delta_2))$ the set of vertices of $\gamma_P(\delta_1, \delta_2)$.

Proposition 3.14. *Let φ be a section in $H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1,\delta_2}(d_{ij} - 2))$. Consider the 2-form $\varphi \frac{\omega}{C_0 C_{ij}}$. One has the following basic properties:*

(1)

$$\left(\operatorname{Res}^{[2]} \varphi \frac{\omega}{C_0 C_{ij}} \right)_Q = \begin{cases} \pm\lambda & \text{if } Q \in v(\gamma_P(\delta_1, \delta_2)), \\ \pm\varepsilon_{ij}\lambda & \text{if } Q \in \{P_1^i, P_1^j\}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\lambda \neq 0$ *if and only if* $\varphi \in M_{\mathcal{I}_{C_{ij}}^{\delta_1,\delta_2}}$ *is essential (as in Definition* 3.5*).*

(2) *The signs of the residues described in* (1) *are such that if* $\mathcal{D} \subset \gamma_P(\delta_1, \delta_2) \cup \mathcal{C}_0 \subset \overline{\mathcal{C}}^{[1]}$ *is an irreducible component, then*

$$\left({}^2\tilde{R}_0^{[1]}\varphi\frac{\omega}{C_0C_{ij}}\right)_F = \begin{cases} 0 & \text{if } F \notin \mathcal{D}, \\ \lambda\omega_1 & \text{if } F \in \mathcal{D}, F \neq \tilde{\mathcal{C}}_0, \tilde{\mathcal{C}}_i, \tilde{\mathcal{C}}_j, \\ \varepsilon_{ij}\lambda\omega_1 & \text{if } F = \tilde{\mathcal{C}}_0, \\ \eta_i^{\delta_1,\delta_2} & \text{if } F = \tilde{\mathcal{C}}_i, \\ \eta_j^{\delta_1,\delta_2} & \text{if } F = \tilde{\mathcal{C}}_j, \end{cases}$$

where $\omega_1 := \frac{XdY+YdX}{XY}$ *is the Euler* 1*-form on* $\mathbb{P}^1 \cong F$ *that has poles on* $[0:1], [1:0]$ *and whose residues are* λ *and* $-\lambda$ *respectively, and where* η_i, η_j *are* 1*-forms on* $\tilde{\mathcal{C}}_i$ *(resp.* $\tilde{\mathcal{C}}_j$*) with only two poles. Moreover, the poles are at the point on* $\tilde{\mathcal{C}}_i$ *(resp.* $\tilde{\mathcal{C}}_j$*) determined by* δ_1 *(resp.* δ_2*) and at the point determined by the branch of* $\tilde{\mathcal{C}}_i$ *(resp.* $\tilde{\mathcal{C}}_j$*) at* P_1^i *(resp.* P_1^j*).*

Proof. Since the proof of Theorem 3.7 is constructive, one can use such a construction to check part (1). Part (2) is a consequence of the commutativity of the generalized residue maps (Theorem 2.9) and the fact that the residues of a meromorphic function on a compact Riemann surface add up to zero and the difference between the number of zeroes and poles is the Euler characteristic $2g-2$, where g is the genus. □

By Theorem 2.9 and the exact sequence (5) note that

$$H^2(W_1) \cong H^2(W_1/W_0) \overset{{}^1\tilde{R}_0^{[1]}}{\cong} H^1(W_0^{[1]}) = H^1(\overline{\mathcal{C}}^{[1]}). \tag{14}$$

Using the inclusion $\Omega^* \overset{i^*}{\hookrightarrow} W_0^*$, from the complex of global holomorphic forms on $\overline{\mathcal{C}}^{[1]}$ to the complex of global differential forms, one has a map $H^1(\overline{\mathcal{C}}^{[1]}, \Omega^1) \overset{i^1}{\to} H^1(\overline{\mathcal{C}}^{[1]})$. Also note that $\dim H^1(\overline{\mathcal{C}}^{[1]}, \Omega^1) = g$. In the following, we will describe generators for $H^1(\overline{\mathcal{C}}^{[1]}, \Omega^1)$.

Proposition 3.15. *Let* $K_i := \{\psi = \varphi\frac{\omega}{C_i} \mid \varphi \in H^0(\mathbb{P}^2, \mathcal{O}(d_i-3)), \mathcal{T}_P(\mathcal{C}_i)|_\varphi \geq \mathcal{T}_P^{\mathrm{n}}(\mathcal{C}_i)\}$ *and* $K_{\mathcal{C}} = \sum_{i=1}^r K_i$*. One has the following properties:*

(1) $K_i \subset W_1^2\left(\mathcal{A}_{\mathbb{P}^2}^{\log}(\mathcal{C})\right)$,

(2) $\mathcal{K}_{\mathcal{C}} \subset \ker\left(H^2(S_{\mathcal{C}}) \xrightarrow{\mathrm{Res}^{[2]}} H^0(\overline{\mathcal{C}}^{[2]})\right)$*, where* $\mathcal{K}_{\mathcal{C}}$ *is the projection of* $K_{\mathcal{C}}$ *on* $H^2(W_1) \subset H^2(S_{\mathcal{C}})$*,*

(3) $K_{\mathcal{C}} = \bigoplus_{i=1}^r K_i$ *and* $\mathcal{K}_{\mathcal{C}} = \bigoplus_{i=1}^r \mathcal{K}_i$*,*

(4) ${}^1\tilde{R}_0^{[1]}\mathcal{K}_{\mathcal{C}} = i^1H^1(\overline{\mathcal{C}}^{[1]}, \Omega^1)$.

Moreover, if $\overline{K}_{\mathcal{C}}$ *denotes the conjugate of* $K_{\mathcal{C}}$*, then*

$$\mathcal{K}_{\mathcal{C}} \oplus \overline{\mathcal{K}}_{\mathcal{C}} = \ker\left(H^2(S_{\mathcal{C}}) \xrightarrow{\mathrm{Res}^{[2]}} H^0(\overline{\mathcal{C}}^{[2]})\right).$$

Proof. Let us start with parts (1) and (2). The result is local, and according to Example 3.6, it is enough to check it at the points at infinity $\{P_1^i, \dots, P_{d_i}^i\} = \mathcal{C}_0 \cap \mathcal{C}_i$. Since such points are smooth on $\mathcal{C}_i$ by hypothesis, the condition $\mathcal{T}_{P_k^i}(\mathcal{C}_i)|_\varphi \geq \mathcal{T}^{\mathfrak{n}}_{P_k^i}(\mathcal{C}_i)$ is vacuous. Hence the local equation of ψ at P_k^i is

$$\varphi(u,v)\frac{du \wedge dv}{v},$$

up to a unit, where $\{v = 0\}$ (resp. $\{u = 0\}$) is the local equation of $\mathcal{C}_i$ (resp. of $\mathcal{C}_0$). Hence part (1) follows, as well as the first statement of part (2). On the one hand note that

$$\dim K_i \geq \binom{d_i - 1}{2} - \sum_{P \in \mathrm{Sing}(\mathcal{C}_i)} \deg \mathcal{T}^{\mathfrak{n}}_P(\mathcal{C}_i) = g_i.$$

In order to prove the first statement of part (3), note that if $\psi_1 + \cdots + \psi_r = 0$, $\psi_i \in K_i$, then multiplying by C, one has

$$\varphi_1 C_2 \cdot \ldots \cdot C_r + C_1 \varphi_2 \cdot \ldots \cdot C_r + \cdots + C_1 C_2 \cdot \ldots \cdot \varphi_r = 0. \tag{15}$$

Consider $\{Q_1, \dots, Q_{d_1-2}\} \subset \mathcal{C}_1 \setminus (\mathcal{C}_2 \cup \cdots \cup \mathcal{C}_r)$ and evaluate such points in (15). One obtains that $\varphi_1(Q_1) = \cdots = \varphi_1(Q_{d_1-2}) = 0$. Since $\mathcal{C}_1$ is irreducible and $\deg \varphi_1 = d_1 - 3$, one obtains that $\varphi_1 = 0$. Proceeding analogously for every φ_i, one obtains that $K_{\mathcal{C}} = K_1 \oplus K_2 \oplus \cdots \oplus K_r$. This same idea shows that ${}^1\tilde{R}_0^{[1]}\psi = 0$ if and only if $\psi = 0$. Therefore $\dim {}^1\tilde{R}_0^{[1]}\mathcal{K}_{\mathcal{C}} \geq g = h^1(\overline{\mathcal{C}}^{[1]}, \Omega^1)$. The inclusion ${}^1\tilde{R}_0^{[1]}\mathcal{K}_{\mathcal{C}} \subset i^1 H^1(\overline{\mathcal{C}}^{[1]}, \Omega^1)$ forces i^1 to be an injection, and thus the second statement of part (3) and part (4) follow for dimension reasons.

The *moreover* statement is a consequence of Proposition 2.15. □

Remark 3.16. Note that Proposition 3.15(4) implies in particular that cohomology classes outside $\mathcal{K}_{\mathcal{C}}$ do not have holomorphic representatives.

Notation 3.17. One can normalize any log-resolution logarithmic 2-form $\varphi \frac{\omega}{C_0 C_{ij}}$ as in Proposition 3.14 in such a way that $\left(\mathrm{Res}^{[2]} \varphi \frac{\omega}{C_0 C_{ij}}\right)_{P_1^i} = 1$.

Note that if $\psi_{ij} \in K_{ij} = K_i \oplus K_j$, then $\left(\mathrm{Res}^{[2]} \varphi \frac{\omega}{C_0 C_{ij}} + \psi_{ij}\right)_{P_1^i} = 1$. The set of classes of such normalized 2-forms will be denoted by $v_1 := \{\psi_P^{\delta_1, \delta_2} + K_{ij}\}_{P, \delta_1, \delta_2}$.

Analogously, one needs to consider certain 2-forms with residues on the line at infinity.

Definition 3.18. For any $P_k^i \in \mathcal{C}_0 \cap \mathcal{C}_i$, $k = 2, \dots, d_i$, the *ideal sheaf* $\mathcal{I}_{\mathcal{C}_i}^{P_k^i}$ *associated with* P_k^i shall be defined as

$$(\mathcal{I}_{\mathcal{C}_i}^{P_k^i})_Q := \left\{ h \in \mathcal{O}_Q \mid \begin{array}{lll} \mathcal{T}_Q(\mathcal{C}_i)|_h & \geq & \mathcal{T}_Q(\mathcal{C}_i) - 2 \quad \text{if } Q \in \{P_1^i, P_k^i\}, \\ \mathcal{T}_Q(\mathcal{C}_i)|_h & \geq & \mathcal{T}^{\mathfrak{n}}_Q(\mathcal{C}_i) \quad \text{otherwise} \end{array} \right\}.$$

As above, a global section s of $\mathcal{I}_{\mathcal{C}_i}^{P_k^i}(d)$ shall be called *essential* if $\mathcal{T}_Q(\mathcal{C}_i)|_{s_Q} = \mathcal{T}_Q(\mathcal{C}_i) - 2$ for every $Q \in \{P_1^i, P_k^i\}$, where s_Q is the section s localized at Q.

One can also describe such sections in terms of their residues as in Proposition 3.14. Its analogue reads as follows.

Proposition 3.19. *Let φ be a section in $H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{C}_i}^{P_k^i}(d_i-2))$. Consider the 2-form $\psi_\infty^{i,k} := \varphi\frac{\omega}{C_0C_i}$. One has the following basic properties:*

(1)
$$\left(\mathrm{Res}^{[2]}\psi_\infty^{i,k}\right)_Q = \begin{cases} \pm\lambda & \text{if } Q \in \{P_1^i, P_k^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *$\lambda \neq 0$ if and only if $\varphi \in M_{\mathcal{I}_{\mathcal{C}_i}^{P_k^i}}$ is essential (as defined in Definition 3.18).*

(3) *The signs of the residues described in (1) are such that if $\mathcal{D} \in \{\mathcal{C}_i, \mathcal{C}_0\} \subset \overline{\mathcal{C}}^{[1]}$, then ${}^2\tilde{R}_0^{[1]}\psi_\infty^{i,k}$ has exactly two poles along $\mathcal{D}$ whose residues are λ and $-\lambda$ so that they add up to zero.*

Proof. The proof follows immediately from Example 3.6 and the fact that the singularities at infinity are always nodes. Hence the local trees are as shown in Figure 2. Therefore $\mathcal{T}_{P_1^i}(\mathcal{C}_i) - 2$ imposes no conditions on φ; and thus

$$\left(\mathrm{Res}^{[2]}\psi_\infty^{i,k}\right)_{P_1^i} = \pm\varphi(P_1^i) = \pm\lambda.$$

The same argument works for P_k^i, and hence (1)-(2) follow. Finally, part (3) follows from the same ideas as in Proposition 3.14(2). □

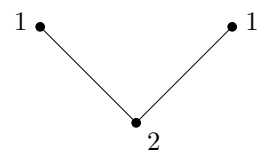

FIGURE 2. Multiplicity tree of a node.

Notation 3.20. For any $P_k^i \in \mathcal{C}_0 \cap \mathcal{C}_i$, $k = 2, \ldots, d_i$, let $\psi_\infty^{i,k}$ be a log-resolution logarithmic 2-form as constructed in Proposition 3.19, again, with the extra normalizing condition that $\left(\mathrm{Res}^{[2]}\psi_\infty^{i,k}\right)_{P_1^i} = 1$. Again, as in Notation 3.17, the set of all classes of such normalized 2-forms will be denoted by $v_\infty := \{\psi_\infty^{i,k} + K_i\}_{i,k}$.

Note that in the line arrangement case, v_∞ is trivial and each class in v_1 contains exactly one representative, which is the form $\frac{d\ell_i \wedge d\ell_j}{\ell_i\ell_j}$ described by Brieskorn [3] and Orlik-Solomon [19]. Note that their cohomology classes are not linearly independent. In particular, if ℓ_i, ℓ_j, and ℓ_k intersect at P, then the 2-form

$$\frac{d\ell_i \wedge d\ell_j}{\ell_i\ell_j} + \frac{d\ell_j \wedge d\ell_k}{\ell_j\ell_k} + \frac{d\ell_k \wedge d\ell_i}{\ell_k\ell_i} \tag{16}$$

is not only cohomologically trivial, but even 0 as a 2-form, as one can easily see as follows: by the concurrence condition we can assume $\ell_i = x$, $\ell_j = y$, and $\ell_k = \alpha x + \beta y$. Hence $d\ell_i \wedge d\ell_j = dx \wedge dy$, $d\ell_j \wedge d\ell_k = -\alpha dx \wedge dy$, and $d\ell_k \wedge d\ell_i = -\beta$. Thus, after removing denominators (16) becomes $(\ell_k - \alpha x - \beta y)dx \wedge dy$, which is zero.

The following results will be used as a tool to come up with the relations amongst cohomology classes. Additionally, we will prove that such relations also hold for the 2-forms in v_1 and not just for their cohomology classes.

Lemma 3.21. *Let* $\psi := \varphi\frac{\omega}{qC_i} \in W_1^2$. *If* $\left({}^1\tilde{R}_0^{[1]}(\psi)\right)_{\tilde{\mathcal{C}}_i} = 0$, *then* $\varphi = pC_i$ *for some function* p. *Moreover, if* φ, q, *and* C_i *are homogeneous polynomials, then* $p \in \mathbb{C}[X,Y,Z]$ *is a homogeneous polynomial.*

Proof. Note that for any $P \in \mathcal{C}_i \setminus \operatorname{Sing}\mathcal{C}$, one has that $\left({}^1R_0^{[1]}(\psi)\right)_{\tilde{\mathcal{C}}_i}|_P = \varphi_P\frac{dz_i}{q_P} = 0$ (that is, at the level of forms, and not only at the level of cohomology classes by (14)), where z_i is a local system of coordinates around P (note that there is no anti-holomorphic component). Since C_i is irreducible one has $\varphi = pC_i$. □

Analogously to the rational case, consider $P \in \operatorname{Sing}\mathcal{C}$ and three local branches δ_i, δ_j, and δ_k belonging to the global components $\mathcal{C}_i$, $\mathcal{C}_j$, and $\mathcal{C}_k$, respectively.

Proposition 3.22. *Let us assume that* $\psi := \varphi\frac{\omega}{C_0C_iC_jC_k}$ *is trivial in* $H^2(S_{\mathcal{C}})$ *for* $\varphi = \varphi_iC_i+\varphi_jC_j+\varphi_kC_k$, *where* φ *is a homogeneous polynomial of degree* $d_i+d_j+d_k-1$. *In this case* $\varphi = 0$.

Proof. Since $\operatorname{Res}^{[2]}\psi = 0$, one has that $\psi \in W_1^2$. Furthermore, ${}^1\tilde{R}_0^{[1]}(\psi) = 0$, and hence, by Lemma 3.21, $\varphi = fC_iC_jC_k$, which, by the degree condition, implies $f = 0$. □

Finally, one can prove that there is a choice of forms satisfying the desired relations.

Theorem 3.23. *There is a choice of representatives* $\{\psi_P^{\delta_1,\delta_2}\}_{P,\delta_1,\delta_2}$ *in* v_1 *and* $\{\psi_\infty^{i,k}\}_{i,k}$ *in* v_∞ *such that the following equalities of* 2*-forms hold:*

$$\psi_P^{\delta_1,\delta_2} + \psi_P^{\delta_2,\delta_3} + \psi_P^{\delta_3,\delta_1} = 0 \tag{17}$$

for any $P \in \mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k$, δ_1, δ_2, *and* δ_3 *local branches of* $\mathcal{C}_i$, $\mathcal{C}_j$, *and* $\mathcal{C}_k$, *respectively, and*

$$\sigma_i \wedge \sigma_j = \sum_{\substack{P \in \operatorname{Sing}(\mathcal{C}_{ij})\\ \delta_1 \in \Delta_P(\mathcal{C}_i)\\ \delta_2 \in \Delta_P(\mathcal{C}_j).}} \mu_P(\delta_1,\delta_2)\psi_P^{\delta_1,\delta_2} + d_j\sum_{k=2}^{d_i}\psi_\infty^{i,k} - d_i\sum_{k=2}^{d_j}\psi_\infty^{j,k}, \tag{18}$$

where δ_1 *(resp.* δ_2*) runs over the local branches of* $\mathcal{C}_i$ *(resp.* $\mathcal{C}_j$*) at* P *and* $\mu_P(\delta_1,\delta_2)$ *denotes the intersection number.*

Proof. For any $P \in \operatorname{Sing}\mathcal{C}$, one can order the set Δ_P of local branches and denote by δ_P the first local branch under such ordering. Fixing arbitrary representatives $\psi_P^{\delta_P,\delta}$ in v_1 for any $\delta \in \Delta_P$ one can complete the choices of representatives as follows: $\psi_P^{\delta_1,\delta_2} := \psi_P^{\delta_P,\delta_2} - \psi_P^{\delta_P,\delta_1}$ for any $\delta_1,\delta_2 \in \Delta_P$.

As a consequence of this, if δ_1, δ_2, and δ_3 are local branches of $\mathcal{C}_i$, $\mathcal{C}_j$, and $\mathcal{C}_k$ at P, respectively, then one has

$$\psi_P^{\delta_1,\delta_2}+\psi_P^{\delta_2,\delta_3}+\psi_P^{\delta_3,\delta_1} = (\psi_P^{\delta_P,\delta_2}-\psi_P^{\delta_P,\delta_1})+(\psi_P^{\delta_P,\delta_3}-\psi_P^{\delta_P,\delta_2})+(\psi_P^{\delta_P,\delta_1}-\psi_P^{\delta_P,\delta_3}) = 0.$$

In order to prove the second equality we only have freedom on the choice of $\psi_P^{\delta_P,\delta}$ and $\psi_\infty^{j,k}$.

First of all, note that

$$\sigma_i \wedge \sigma_j = \operatorname{Jac}(C_i,C_j,C_0)\frac{\omega}{C_0C_iC_j} = \begin{vmatrix} C_{i,X} & C_{i,Y} & C_{i,Z}\\ C_{j,X} & C_{j,Y} & C_{j,Z}\\ C_{0,X} & C_{0,Y} & C_{0,Z}\end{vmatrix}\frac{\omega}{C_0C_iC_j}.$$

Denoting by ψ_{ij} the right-hand side of (18), it is easy to check that $\mathrm{Res}^{[2]}\sigma_i\wedge\sigma_j = \mathrm{Res}^{[2]}\psi_{ij}$ (see [5, Theorem 2.47] for details). As a consequence of this, $\sigma_i\wedge\sigma_j-\psi_{ij} \in K_{ij} \subset W_1^2$.

By Lemma 3.21 and Proposition 3.22, one only needs to find representatives $\psi_P^{\delta_P,\delta}$ verifying ${}^1R_0^{[1]}(\sigma_i\wedge\sigma_j-\psi_{ij})=0$.

One can proceed as follows: choose arbitrary representatives $\dot\psi_P^{\delta_P,\delta}$ and $\dot\psi_\infty^{i,k}$. We are looking for 2-forms $\psi_P^{\delta_P,\delta} := \dot\psi_P^{\delta_P,\delta} + \varphi_P^{i,\delta}\frac{\omega}{\bar{C}_i}$ and $\psi_\infty^{i,k} := \dot\psi_\infty^{i,k} + \varphi_\infty^{i,k}\frac{\omega}{\bar{C}_i}$ for certain $\varphi_P^{i,\delta}, \varphi_P^{j,\delta}$, and $\varphi_\infty^{i,k}$ satisfying equation (18).

Since the projection of $W_0(\log\langle\bar{\mathcal{C}}\rangle)\xrightarrow{{}^1R_0^{[1]}}W_0^{[1]} \to \bigoplus \mathcal{O}_{\bar{\mathcal{C}}_i}(\bar{\mathcal{C}}_i)$ is injective (because it is constant when restricted to the exceptional divisors), one only needs to make sure that, for a certain choice of representatives, such projections are zero.

This gives rise to an affine system of equations on the vector space $\bigoplus K_i$, where the variables are $\varphi_P^{i,\delta}$ and $\varphi_\infty^{i,k}$ mentioned above. All there is left to do is to check that such a system is compatible.

Using the relations in (17) one obtains that

$$E_{ij} \equiv \sigma_i\wedge\sigma_j - \dot\psi_{ij} = \sum_{\substack{P\,\in\,\mathrm{Sing}(\mathcal{C}_{ij})\\ \delta_1\,\in\,\Delta_P(\mathcal{C}_i)\\ \delta_2\,\in\,\Delta_P(\mathcal{C}_j).}} \mu_P(\delta_1,\delta_2)\left(\varphi_P^{j,\delta_2}\frac{\omega}{C_j} - \varphi_P^{i,\delta_1}\frac{\omega}{C_i}\right) + d_j\sum_{k=2}^{d_i}\varphi_\infty^{i,k}\frac{\omega}{C_i} - d_i\sum_{k=2}^{d_j}\varphi_\infty^{j,k}\frac{\omega}{C_j}, \tag{19}$$

where

$$\dot\psi_{ij} := \sum_{\substack{P\,\in\,\mathrm{Sing}(\mathcal{C}_{ij})\\ \delta_1\,\in\,\Delta_P(\mathcal{C}_i)\\ \delta_2\,\in\,\Delta_P(\mathcal{C}_j).}} \mu_P(\delta_1,\delta_2)(\dot\psi_P^{\delta_P,\delta_2} - \dot\psi_P^{\delta_P,\delta_1}) + d_j\sum_{k=2}^{d_i}\dot\psi_\infty^{i,k} - d_i\sum_{k=2}^{d_j}\dot\psi_\infty^{j,k}. \tag{20}$$

Therefore, the given system is compatible, if for any linear combination of $\sigma_i\wedge\sigma_j - \dot\psi_{ij}$ that maps to zero, then $\sigma_i\wedge\sigma_j-\dot\psi_{ij}=0$.

After a more careful study of (19), one can prove that a combination of such equations is zero on the right-hand side if and only if it can be written as a linear combination of equations of the form $E_{\bar{i},\bar{j}} + E_{\bar{j},\bar{k}} + E_{\bar{k},\bar{i}}$, where $E_{\bar{i},\bar{j}} = \sum_{i_*\in\bar{i}}\sum_{j_*\in\bar{j}} E_{i_*,j_*}$ and where $\mathcal{C}_{\bar{i}}, \mathcal{C}_{\bar{j}}, \mathcal{C}_{\bar{k}}$ form a combinatorial pencil.

All there is left to do is to check that the left-hand side of (19) is also zero. By the definition given in (20) it is immediate that $\dot\psi_{\bar{i},\bar{j}} + \dot\psi_{\bar{j},\bar{k}} + \dot\psi_{\bar{k},\bar{i}} = 0$.

In order to check $\sigma_{\bar{i}}\wedge\sigma_{\bar{j}} + \sigma_{\bar{j}}\wedge\sigma_{\bar{k}} + \sigma_{\bar{k}}\wedge\sigma_{\bar{i}} = 0$ one can proceed as follows. By the Combinatorial Noether Theorem [6] we can assume that $\alpha\mathcal{C}_{\bar{i}} + \beta\mathcal{C}_{\bar{j}} = \mathcal{C}_{\bar{k}}$. In that case,

$$C_0C_{\bar{i}}C_{\bar{j}}C_{\bar{k}}\cdot(\sigma_{\bar{j}}\wedge\sigma_{\bar{k}}) = C_0C_{\bar{i}}\cdot\mathrm{Jac}(C_0,C_{\bar{j}},C_{\bar{k}})\omega$$

$$= C_0C_{\bar{i}}\cdot\mathrm{Jac}(C_0,C_{\bar{j}},\alpha\mathcal{C}_{\bar{i}}+\beta\mathcal{C}_{\bar{j}})\omega = -\alpha C_0C_{\bar{i}}\cdot\mathrm{Jac}(C_0,C_{\bar{i}},C_{\bar{j}})\omega.$$

Analogously,

$$C_0C_{\bar{i}}C_{\bar{j}}C_{\bar{k}}\cdot(\sigma_{\bar{k}}\wedge\sigma_{\bar{i}}) = -\beta C_0C_{\bar{j}}\cdot\mathrm{Jac}(C_0,C_{\bar{i}},C_{\bar{j}})\omega.$$

Hence

$$C_0 C_{\bar{i}} C_{\bar{j}} C_{\bar{k}} \cdot (\sigma_{\bar{i}} \wedge \sigma_{\bar{j}} + \sigma_{\bar{j}} \wedge \sigma_{\bar{k}} + \sigma_{\bar{k}} \wedge \sigma_{\bar{i}}) = 0,$$

which ends the proof. □

We will denote by V_1 (resp. V_∞) the subspace of $W^2\left(\mathcal{A}^{\log}_{\mathbb{P}^2}(\mathcal{C})\right)$ generated by the 2-forms in v_1 (resp. v_∞) described in Theorem 3.23 and by $\mathcal{V}_1$ (resp. $\mathcal{V}_\infty$) their projection on $H^2(S_\mathcal{C}) = W^2/d(W^1)$. Now we are in the position to give a decomposition of $H^2(S_\mathcal{C})$.

Corollary 3.24. *Under the above conditions*

$$H^2(S_\mathcal{C}) = \mathcal{V}^2_\mathcal{C} \oplus \mathcal{K}_\mathcal{C} \oplus \overline{\mathcal{K}}_\mathcal{C}.$$

Proof. By Proposition 3.15, it is enough to check that

$$H^2(S_\mathcal{C})/(\mathcal{K}_\mathcal{C} \oplus \overline{\mathcal{K}}_\mathcal{C}) \cong H^2(S_\mathcal{C})/\ker \mathrm{Res}^{[2]}$$

is isomorphic to $\mathcal{V}^2_\mathcal{C} := \mathcal{V}_1 \oplus \mathcal{V}_\infty$. Notice that the residue map $\mathrm{Res}^{[2]}$ is injective on the quotient. Let us consider $\psi \in V_1$. By Proposition 3.14 $\left(\mathrm{Res}^{[2]}\psi\right)_{P^i_k} = 0$ for any $i = 1, \dots, r$ and any $k = 2, \dots, d_i$. On the other hand, by Proposition 3.19 it is immediate that any 2-form $\psi \in V_\infty$ satisfies the fact that $\sum_{k=1}^{d_i} \left(\mathrm{Res}^{[2]}\psi\right)_{P^i_k} = 0$. Therefore, if $\psi \in V_1 \cap V_\infty$, $\left(\mathrm{Res}^{[2]}\psi\right)_{P^i_k} = 0$, $k = 1, \dots, d_i$, and thus $\psi = 0$. □

As a consequence of the previous results one obtains a description of the cohomology ring of the complement of a projective plane curve $H^*(S_\mathcal{C})$.

Theorem 3.25. *The cohomology ring $H^*(S_\mathcal{C})$ of $S_\mathcal{C}$ can be decomposed as follows:*

$$\mathcal{V}_\mathcal{C} \oplus \mathcal{K}_\mathcal{C} \oplus \overline{\mathcal{K}}_\mathcal{C},$$

where $H^*(S_\mathcal{C})$ is trivial in degree ≥ 3, $\mathcal{K}_\mathcal{C}$ and $\overline{\mathcal{K}}_\mathcal{C}$ are homogeneous subrings of degree 2 and dimension g each, and $\mathcal{V}_\mathcal{C}$ can be described as:

- *Generated in degrees 1 and 2 by*
  - (G1) *(Generators of $\mathcal{V}^1_\mathcal{C}$) $\sigma_1, \dots, \sigma_r$,*
  - (G2) *(Generators of $\mathcal{V}^2_\mathcal{C} = \mathcal{V}_1 \oplus \mathcal{V}_\infty$) $v_1 \cup v_\infty$ from Theorem 3.23.*
- *The relations given in (18), (17), and*

$$\psi^{\delta_1,\delta_2}_P + \psi^{\delta_2,\delta_1}_P = 0 \tag{21}$$

is a complete system of relations. □

Note that the ring structure depends on some local and global data which will be described in what follows. Because of the general condition about the transversal line we will repeat Definition 2.18 with a slight change in notation.

Let $\tilde{\mathcal{C}} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_r \subset \mathbb{P}^2$, $\mathcal{C}_0$ a transversal line, and $\mathcal{C} := \mathcal{C}_0 \cup \tilde{\mathcal{C}}$. Analogously to Definition 2.18 consider the following.

Definition 3.26. The family $W_\mathcal{C} := (\mathbf{r}, S, \Delta, \phi, \bar{d}, \bar{g})$ is called the *weak combinatorial type of $\mathcal{C}$ with respect to $\mathcal{C}_0$* or simply the *weak combinatorial type of $\mathcal{C}$* if no ambiguity seems likely to arise.

Corollary 3.27. *The cohomology ring $H^*(S_\mathcal{C})$ of $S_\mathcal{C}$ only depends on $W_\mathcal{C}$ being the weak combinatorial type of $\mathcal{C}$.* □

Remark 3.28. Corollary 3.27 is also true in the case where the curve does not contain a transversal line – as we have assumed throughout section 3. In this case one can add a transversal line and consider $\mathcal{C} = \tilde{\mathcal{C}} \cup \mathcal{C}_0$. The ring $H^*(S_{\tilde{\mathcal{C}}})$ fits in the following exact sequence:

$$0 \quad \to \quad H^*(S_{\tilde{\mathcal{C}}}) \quad \to \quad H^*(S_{\mathcal{C}}) \quad \xrightarrow{\pi_{\mathcal{C}_0} \circ \mathrm{Res}^{[1]}} \quad \mathbb{C}_{\mathcal{C}_0} \quad \to \quad 0,$$

where $\mathrm{Res}^{[1]}$ is the residue defined in Proposition 2.15 and $\pi_{\mathcal{C}_0}$ is the projection of $H^0((\tilde{\mathcal{C}}_0 \cup \overline{\mathcal{C}})^{[1]})$ on the coordinate corresponding to $\mathcal{C}_0$.

Example 3.29. Consider the two conics $\mathcal{C}_1 := \{y(y-z) + (x+y)^2 = 0\}$, $\mathcal{C}_2 := \{y(y-z) + (x-y)^2 = 0\}$ and the line $\mathcal{C}_3 := \{y = 0\}$ (see Figure 3). The weak combinatorial type of $\tilde{\mathcal{C}} := \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is

$$W_{\tilde{\mathcal{C}}} := (\{1,2,3\}, S, \{\Delta_P, \phi_P, (\bullet,\bullet)_P\}_{P \in S}, (2,2,1), (0,0,0)),$$

where $S := \{P_1, P_2, P_2', P_3\}$, $\Delta_{P_1} := \{\delta_1^1, \delta_2^1\}$, $\Delta_{P_2} := \{\delta_2^2, \delta_3^2\}$, $\Delta_{P_2'} := \{\delta_1'^2, \delta_3'^2\}$, $\Delta_{P_2} := \{\delta_1^3, \delta_2^3\}$, $\phi_{P_i}(\delta_j^i) := j$, and $(\delta_j^i, \delta_k^i)_{P_i} = i$. The ring $H^*(S_{\tilde{\mathcal{C}}})$ is generated by the 1-forms $\omega_i := 2\sigma_3 - \sigma_i$, $i = 1, 2$, and the 2-forms $\psi_1 := \psi_{P_3}^{\delta_1^3, \delta_2^3} + \psi_{P_2}^{\delta_2^2, \delta_3^2} - \psi_{P_2'}^{\delta_1'^2, \delta_3'^2}$ and $\psi_2 := \psi_{P_1}^{\delta_1^1, \delta_2^1} + \psi_{P_2}^{\delta_2^2, \delta_3^2} - \psi_{P_2'}^{\delta_1'^2, \delta_3'^2}$. The only relation is given by $\omega_1 \wedge \omega_2 = 3\psi_1 + \psi_2$. Hence

$$H^*(S_{\tilde{\mathcal{C}}}) = \langle \omega_1, \omega_2, \psi_1, \psi_2 \mid \omega_1 \wedge \omega_2 = 3\psi_1 + \psi_2 \rangle.$$

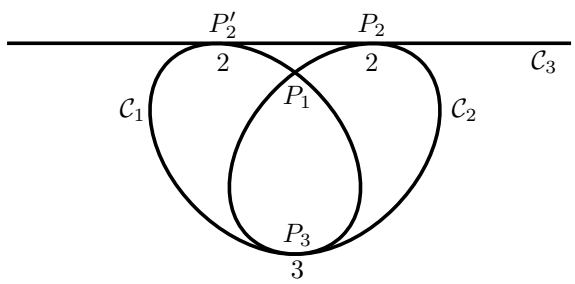

FIGURE 3. Projective realization of $\mathcal{C}$ and multiplicities of intersection.

Remark 3.30. As in Definition 3.26, the curve $\tilde{\mathcal{C}} \subset \mathbb{P}^2$ will not be assumed to have a transversal line, and usually we will denote by $\mathcal{C}$ the union of $\tilde{\mathcal{C}}$ and a transversal line. In the future, we will always consider this situation unless otherwise stated.

4. Formality of complements to projective plane curves

All the basic definitions of minimal algebras, minimal models, homotopy, etc. required in the definition of formality and in the theory of homotopy theory of algebras can be found for instance in any of the foundational papers [7, 18, 20].

Definition 4.1. Two graded differential algebras (A, d_A) and (B, d_B) are called *quasi-isomorphic* if there exists a morphism of graded algebras $f : A \to B$ such that the induced morphism $f^* : H^*(A, d_A) \to H^*(B, d_B)$ is an isomorphism. Note that "being quasi-isomorphic" is not an equivalence relation. We will refer to the *quasi-isomorphism class* of a graded differential algebra as the minimal equivalence class generated by the quasi-isomorphism relation.

A minimal differential graded algebra is called *formal* if it is quasi-isomorphic to its cohomology algebra using a zero differential. A differential graded algebra is called *formal* if its minimal model is formal. Finally, a complex space X is called *formal* if the algebra of differential forms $(\mathcal{E}(X), d)$ is formal.

The concept of formal algebra is well defined since any differential graded algebra has a unique (up to homotopy) minimal model (cf. [20, Section §5]). Also note that a minimal model for (A, d_A) consists of a minimal algebra $(\mathcal{M}(A), d_{\mathcal{M}(A)})$ plus a quasi-isomorphism $\mathcal{M}(A) \to A$. Therefore, if one finds a quasi-isomorphism between $(\mathcal{E}(X), d)$ and $(H(X), 0)$, then X is formal. Moreover, if X is a smooth complex variety and $\overline{X}$ is a completion of X by a simple normal crossing divisor, then the minimal model of $\mathcal{E}(X)$ and the minimal model of $\mathcal{A}_{\overline{X}}(\log\langle D\rangle)$ are isomorphic (cf. [18, Section §6]).

Theorem 4.2. *There is a well-defined quasi-isomorphism* $H^*(S_{\mathcal{C}}) \to \mathcal{A}^*_{\overline{S}_{\mathcal{C}}}(\log\langle\overline{\mathcal{C}}\rangle)$.

Proof. According to Theorem 3.25 $H^*(S_{\mathcal{C}})$ admits a decomposition

$$\mathbb{C} \oplus \mathcal{V}^1_{\mathcal{C}} \oplus \mathcal{V}^2_{\mathcal{C}} \oplus \mathcal{K}_{\mathcal{C}} \oplus \overline{\mathcal{K}}_{\mathcal{C}},$$

where $\mathcal{V}^1_{\mathcal{C}}$ is generated by 1-forms $V^1_{\mathcal{C}} \subset W^1$ and $\mathcal{V}^2_{\mathcal{C}}$ (resp. $\mathcal{K}_{\mathcal{C}}$, $\overline{\mathcal{K}}_{\mathcal{C}}$) is generated by 2-forms $V^2_{\mathcal{C}} \subset W^2_2$ (resp. $K_{\mathcal{C}} \subset W^2_1$, $\overline{K}_{\mathcal{C}} \subset W^2_1$). Each cohomology class $\varphi \in H^*(S_{\mathcal{C}})$ can be thus described as the cohomology class of a combination of forms as follows: $\varphi = [z + \psi^1 + \psi^2_1 + \psi^2_2]$. The map is defined by $\varphi \mapsto z + \psi^1 + \psi^2_1 + \psi^2_2$. By Theorem 3.23 this map is well defined and is obviously a quasi-isomorphism. □

As a consequence of the discussion at the beginning of this section one has the following.

Theorem 4.3. *The complement of a plane projective curve* $S_{\mathcal{C}}$ *is a formal space.*

Remark 4.4. Theorem 4.3 is the global version of the formality of algebraic links proved by Durfee-Hain in [9]. The result is a consequence of a more general fact proved paper by A. Macinic in [17]: a 2-complex X which is 1-formal is also a formal space.

The 1-minimal model $\mathcal{M}_1(A)$ of a differential graded algebra (A, d_A) is the subalgebra generated by the degree 1 part in $\mathcal{M}(A)$. Then a space X is 1-formal if $\mathcal{M}_1(\mathcal{E}(X), d)$ is quasi-isomorphic to $\mathcal{M}_1(H^*(X), 0)$. This condition can be restated in terms of the fundamental group as follows. A finitely presented group G is 1-formal if its Malcev completion is filtered isomorphic to its holonomy Lie algebra, completed with respect to bracket length. Fundamental groups of complements to algebraic plane curves are known to be 1-formal (see [13] and [18]).

5. Examples

5.1. **Weak combinatorics do not determine classical combinatorics.** Let $\ell_0 := \{x = 0\}$, $\ell_1 := \{y = 0\}$, and $\ell_2 := \{z = 0\}$ be three lines in general position and consider:

(1) $\tilde{\mathcal{C}}_1 := \{(x-y)^2 - (x+y)z = 0\}$ a conic tangent to ℓ_2 at $(1, 1, 0)$ and passing through $\ell_0 \cap \ell_1$,
(2) $\tilde{\mathcal{C}}^{(1)}_2 := \{x - y + z = 0\}$ the line passing through $\ell_0 \cap \tilde{\mathcal{C}}_1$ and $\ell_2 \cap \tilde{\mathcal{C}}_1$, and
(3) $\tilde{\mathcal{C}}^{(2)}_2 := \{3x - y + z = 0\}$ the line tangent to $\tilde{\mathcal{C}}_1$ at $\ell_0 \cap \tilde{\mathcal{C}}_1$.

The Cremona transformation based on ℓ_0, ℓ_1, and ℓ_2 transforms $\tilde{\mathcal{C}}^{(k)} := \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2^{(k)}$ into $\mathcal{C}^{(k)}$, a union of a cuspidal cubic $\mathcal{C}_1$ and a conic $\mathcal{C}_2^{(k)}$. Note that $\mathcal{C}^{(k)}$ has three singular points $\{P_1, P_2, P_3\}$, where $\Delta_{\mathcal{C}^{(k)},P_i} := \{\delta_i^1, \delta_i^{2,k}\}$, $\phi_{\mathcal{C}^{(k)},P_i}(\delta_i^1) = \mathcal{C}_1$, $\phi_{\mathcal{C}^{(k)},P_i}(\delta_i^{2,k}) = \mathcal{C}_2^{(k)}$, $(\delta_i^1, \delta_i^{2,1})_{\mathcal{C}^{(1)},P_i} = i$, and $(\delta_i^1, \delta_i^{2,2})_{\mathcal{C}^{(2)},P_i} = \sigma_{(2,3)}(i)$ (where $\sigma_{(2,3)}(i)$ represents the permutation $(2,3)$ applied to i). Figure 4 represents the singular points of $\mathcal{C}^{(k)}$, the local branches at those points (a solid line for $\mathcal{C}_1$ and a broken line for $\mathcal{C}_2^{(k)}$), and the multiplicity of intersection in brackets. Note that the bijection φ of singular points that permutes P_2 and P_3 induces an equivalence between $W_{\mathcal{C}^{(1)}}$ and $W_{\mathcal{C}^{(2)}}$, since $\varphi_{\mathbf{r}_{\mathcal{C}^{(1)}}}$ and φ_{P_i} are forced by their compatibility with the degrees and with φ. The combinatorial types $K_{\mathcal{C}^{(1)}}$ and $K_{\mathcal{C}^{(2)}}$ cannot be equivalent since the topological types of their singularities do not coincide.

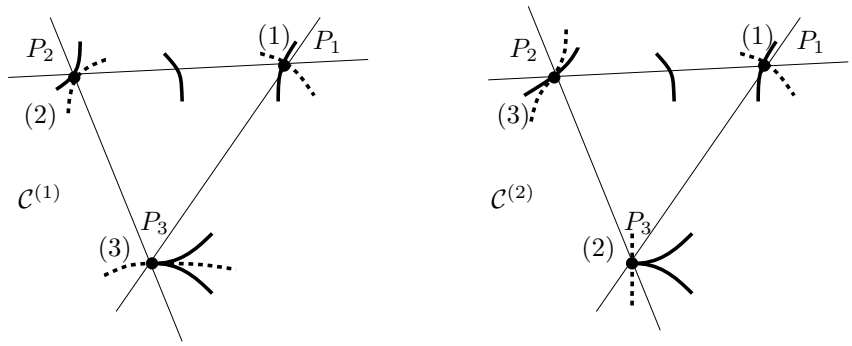

FIGURE 4. Singularities of $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$, respectively.

5.2. **An explicit computation of the cohomology ring in the non-rational case.** We will present a simple example of a non-rational arrangement of curves in order to show how to compute the forms described in section 3. Let $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, where $\mathcal{C}_0 := \{x+y+z=0\}$, $\mathcal{C}_1 := \{y-z=0\}$, $\mathcal{C}_2 := \{xy+xz+yz=0\}$, and $\mathcal{C}_3 := \{x^2(y+z)+y^2(x+z)+z^2(x+y)=0\}$. In this case, for simplicity it is more convenient to consider the line at infinity $\mathcal{C}_0$ with an equation different from $\{z=0\}$. Consider ξ a primitive third root of unity (a root of $t^2+t+1=0$) and denote $\mathcal{C}_0 \cap \mathcal{C}_1 = \{P_{01} = [-2:1:1]\}$, $\mathcal{C}_0 \cap \mathcal{C}_2 = \{R_1 = [-\bar{\xi}-1:\bar{\xi}:1], R_2 = [-1-\xi:\xi:1]\}$, $\mathcal{C}_0 \cap \mathcal{C}_3 = \{Q_1 = [0:1:-1], Q_2 = [-1:0:1], Q_3 = [-1:1:0]\}$, $\mathcal{C}_1 \cap \mathcal{C}_2 = \{P_1, P_{12} = [1:-2:-2]\}$, $\mathcal{C}_1 \cap \mathcal{C}_3 = \{P_1, P_{13} = [\xi:1:1], \bar{P}_{13} = [\bar{\xi}:1:1]\}$, and $\mathcal{C}_2 \cap \mathcal{C}_3 = \{P_1 = [1:0:0], P_2 = [0:1:0], P_3 = [0:0:1]\}$,

Since all the local branches of the irreducible components at any singular point are irreducible, we will denote by $\psi_P^{i,j}$ the 2-form associated with the singular point P and the local branches at P of $\mathcal{C}_i$ and $\mathcal{C}_j$. For example, in order to compute $\psi_{P_1}^{2,3} = \varphi_{P_1}^{2,3} \frac{\omega}{C_0 C_2 C_3}$, one needs a section of $H^0(\mathbb{P}^2, \mathcal{I}_{P_1}^{2,3}(3))$. Note that

$$\left(\mathcal{I}_{P_1}^{2,3}\right)_P := \begin{cases} \{\varphi \in \mathcal{O}_P \mid \mathcal{T}_P(\mathcal{C}_{2,3})|_\varphi \geq \mathcal{T}_{\mathbb{A}_3}\} = \mathfrak{m}_P & \text{if } P = P_1, \\ \{\varphi \in \mathcal{O}_P \mid \mathcal{T}_P(\mathcal{C}_{2,3})|_\varphi \geq \mathcal{T}_{\mathbb{A}_3}^{\mathfrak{n}}\} & \text{if } P = P_2, P_3, \\ \mathcal{O}_P & \text{otherwise,} \end{cases}$$

where Figure 5 describes the local conditions at the tacnodes P_i.

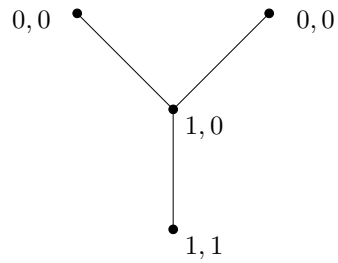

FIGURE 5. Description of $\mathcal{T}^{\mathbf{n}}_{\mathbb{A}_3}$ and $\mathcal{T}_{\mathbb{A}_3}$, respectively.

Therefore $\varphi^{2,3}_{P_1}$ is the equation of a cubic $\alpha(xz+x^2+(1-\xi)xy+yz)z+\beta C_0C_2$. In order to obtain a normal form one has to require the different residues of $\psi^{2,3}_{P_1}$ at P_1 and at an exceptional divisor E joining δ_2 and δ_3 to be equal to ± 1. It is a simple computation that

$$\mathrm{Res}^{[2]}_{P_1}\psi^{2,3}_{P_1}=\frac{\alpha}{3}$$

and that

$$\left({}^1R^{[1]}_0\psi^{2,3}_{P_1}\right)_E=\frac{1}{3}(\beta-\xi).$$

Since $\left({}^1R^{[1]}_0\frac{d\mathcal{C}_2}{\mathcal{C}_2}\wedge\frac{d\mathcal{C}_3}{\mathcal{C}_3}\right)_E=-\frac{2}{3}$ and $(\delta_2,\delta_3)_{P_1}=2$, one concludes that

$$\varphi^{2,3}_{P_1}=3(xz+x^2+(1-\xi)xy+yz)z+(\xi-1)C_0C_2.$$

Analogously one can proceed with $\psi^{1,2}_{P_1}=\varphi^{1,2}_P\frac{\omega}{C_0C_1C_2}$ and $\psi^{3,1}_{P_1}=\varphi^{3,1}_{P_1}\frac{\omega}{C_0C_3C_1}$ obtaining $\varphi^{1,2}_{P_1}:=2x-\xi y+(1+\xi)z$ and $\varphi^{1,3}_{P_1}:=2(x^2+xz+2yz-y^2)+C_0C_1$.

Note that $\varphi^{1,2}_{P_1}C_3+\varphi^{2,3}_{P_1}C_1-\varphi^{1,3}_{P_1}C_2=0$ and hence $\psi^{1,2}_{P_1}+\psi^{2,3}_{P_1}+\psi^{3,1}_{P_1}=0$ (equation (17) of Theorem 3.23).

The following list describes the polynomials $\varphi^{i,j}_P$ for the generating 2-forms $\psi^{i,j}_P:=\varphi^{i,j}_P\frac{\omega}{C_0C_iC_j}$:

$$\begin{aligned}
\varphi^{2,3}_{P_1}&:=(\xi+2)(zx^2+\xi yx^2+xz^2+\xi y^2x+z^2y+\xi zy^2),\\
\varphi^{2,3}_{P_2}&:=(\xi+2)(y^2x+xz^2+yx^2+zx^2+z^2y+(1-\xi)zxy+\xi zy^2),\\
\varphi^{2,3}_{P_3}&:=(\xi-1)(y^2x+xz^2+yx^2+zx^2+y^2z+(\xi+2)zxy-(1+\xi)z^2y),\\
\varphi^{1,2}_{P_1}&:=2x-\xi y+(1+\xi)z,\\
\varphi^{1,2}_{P_{12}}&:=(\xi-1)((\xi+1)y+z),\\
\varphi^{1,3}_{P_1}&:=2x^2+xz+xy-z^2+4yz-y^2,\\
\varphi^{1,3}_{P_{13}}&:=-(\xi+2)(xz+xy+z^2+2\xi zy+y^2),\\
\varphi^{1,3}_{\bar{P}_{13}}&:=(\xi-1)(xz+xy+z^2-2(\xi+1)yz+y^2).
\end{aligned}$$

Finally we also describe the polynomials $\varphi^{i,k}_\infty$ for the generating 2-forms $\psi^{i,k}_\infty:=\varphi^{i,k}_\infty\frac{\omega}{C_0C_i}$:

$$\begin{aligned}
\varphi^{3,Q_2}_\infty&:=-3(x+y),\\
\varphi^{3,Q_3}_\infty&:=3(x+z),\\
\varphi^{2,R_2}_\infty&:=-(2\xi+1).
\end{aligned}$$

One can then easily verify that

$$\mathrm{Jac}(C_2, C_3, C_0) = 2\varphi^{2,3}_{P_1} + 2\varphi^{2,3}_{P_2} + 2\varphi^{2,3}_{P_3} + 3\varphi^{2,R_2}_{\infty} C_3 - 2\varphi^{3,Q_2}_{\infty} C_2 - 2\varphi^{3,Q_3}_{\infty} C_2,$$

and hence

$$\sigma_2 \wedge \sigma_3 = 2\psi^{2,3}_{P_1} + 2\psi^{2,3}_{P_2} + 2\psi^{2,3}_{P_3} + 3\psi^{2,R_2}_{\infty} - 2\psi^{3,Q_2}_{\infty} - 2\psi^{3,Q_3}_{\infty},$$

which corresponds to equation (18) of Theorem 3.23.

The same can be checked for $\mathrm{Jac}(C_1, C_2, C_0)$ and $\mathrm{Jac}(C_1, C_3, C_0)$.

References

1. V.I. Arnold, *The cohomology ring of the colored braid group*, Math. Notes of the Academy of Sci. of the USSR **5** (1969), 138–140. MR0242196 (39:3529)
2. E. Artal, J.I. Cogolludo, and H.O Tokunaga, *A survey on Zariski pairs*, Algebraic geometry in East Asia—Hanoi 2005, Adv. Stud. Pure Math., vol. 50, Math. Soc. Japan, Tokyo, 2008, pp. 1–100. MR2409555 (2009g:14030)
3. E. Brieskorn, *Sur les groupes de tresses [d'après V. I. Arnol′d]*, Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, Springer, Berlin, 1973, pp. 21–44. Lecture Notes in Math., Vol. 317. MR0422674 (54:10660)
4. E. Brieskorn and H. Knörrer, *Plane algebraic curves*, Birkhäuser Verlag, Basel, 1986, Translated from the German by John Stillwell. MR886476 (88a:14001)
5. J.I. Cogolludo, *Topological invariants of the complement to arrangements of rational plane curves*, Mem. Amer. Math. Soc. **159** (2002), no. 756. MR1921584 (2003k:32037)
6. J.I. Cogolludo-Agustín and M.Á. Marco Buzunáriz, *The Max Noether Fundamental Theorem is combinatorial*, Preprint available at `arXiv:1002.2325v1 [math.AG]`, 2010.
7. P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274. MR0382702 (52:3584)
8. Alexandru Dimca, Ştefan Papadima, and Alexander I. Suciu, *Topology and geometry of cohomology jump loci*, Duke Math. J. **148** (2009), no. 3, 405–457. MR2527322
9. A.H. Durfee and R.M. Hain, *Mixed Hodge structures on the homotopy of links*, Math. Ann. **280** (1988), no. 1, 69–83. MR928298 (89c:14012)
10. D. Eisenbud and W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematics Studies, vol. 110, Princeton University Press, Princeton, NJ, 1985. MR817982 (87g:57007)
11. P. Griffiths and W. Schmid, *Recent developments in Hodge theory: A discussion of techniques and results*, Proc. Int. Colloq. on Discrete Subgroups and Lie groups (1973), 31–127, Bombay. MR0419850 (54:7868)
12. P.A. Griffiths, *Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems*, Bull. Amer. Math. Soc. **76** (1970), 228–296. MR0258824 (41:3470)
13. T. Kohno, *On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces*, Nagoya Math. J. **92** (1983), 21–37. MR726138 (85d:14032)
14. A. Libgober, *Alexander invariants of plane algebraic curves*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 135–143. MR713242 (85h:14017)
15. F. Loeser and M. Vaquié, *Le polynôme d'Alexander d'une courbe plane projective*, Topology **29** (1990), no. 2, 163–173. MR1056267 (91d:32053)
16. D. Lubicz, *Une description de la cohomologie du complément à un diviseur non réductible de* $\mathbb{P}^2$, Bull. Sci. Math. **124** (2000), no. 6, 447–458. MR1796280 (2001k:14046)
17. A.D. Măcinic, *Cohomology rings and formality properties of nilpotent groups*, J. Pure Appl. Algebra **214** (2010), no. 10, 1818–1826. MR2608110
18. J.W. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1978), no. 48, 137–204. MR516917 (80e:55020)
19. P. Orlik and L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), 167–189. MR558866 (81e:32015)
20. D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269–331 (1978). MR0646078 (58:31119)

21. O. Zariski, *On the irregularity of cyclic multiple planes*, Ann. of Math. (2) **32** (1931), no. 3, 485–511. MR1503012
22. ———, *The topological discriminant group of a Riemann surface of genus p*, Amer. J. Math **59** (1937), 335–338. MR1507244

Departamento de Matemáticas, IUMA, Universidad de Zaragoza, C/Pedro Cerbuna 12, CP50009 Zaragoza, Spain
E-mail address: jicogo@unizar.es

Departamento de Matemáticas, Universidad de Zaragoza, C/Pedro Cerbuna 12, CP50009 Zaragoza, Spain – and – Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania
E-mail address: daniel.matei@imar.ro