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WEIGHTED NORM INEQUALITIES FOR MULTILINEAR FOURIER MULTIPLIERS

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ABSTRACT. In this paper, we consider weighted norm inequalities for multilinear Fourier multipliers. Our result can be understood as a multilinear version of the result by Kurtz and Wheeden.

1. INTRODUCTION

Coifman and Meyer [4] proved the boundedness of multilinear Fourier multipliers: if $m \in C^L(\mathbb{R}^{Nn} \setminus \{0\})$ satisfies

$$|\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} m(\xi_1, \dots, \xi_N)| \le C_{\alpha_1,\dots,\alpha_N} (|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1|+\dots+|\alpha_N|)} \tag{1.1}$$

for all $|\alpha_1|+\dots+|\alpha_N| \le L$, where L is a sufficiently large natural number, then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \dots + 1/p_N = 1/p$. Here the N-linear Fourier multiplier operator T_m is defined by

$$\begin{aligned} &T_m(f_1, \dots, f_N)(x) \\ &\quad = \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\dots+\xi_N)} m(\xi_1, \dots, \xi_N) \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N)\, d\xi_1 \dots d\xi_N \end{aligned}$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$. This result was studied and applied to nonlinear partial differential equations by many mathematicians. But there seem to be few literatures investigating the used number of derivatives of m. In [17], the second author gave a Hörmander type theorem for multilinear Fourier multipliers. As a corollary of it, we see that (1.1) with $L = [Nn/2] + 1$ assures the boundedness of T_m from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \dots + 1/p_N = 1/p$, where $[Nn/2]$ is the integer part of $Nn/2$. Grafakos and Si [8] gave similar results for the case $p \le 1$ by using the L^r-based Sobolev spaces, $1 < r \le 2$.

This paper is concerned with weighted norm inequalities for multilinear operators. For example, see Cruz-Uribe, Martell and Pérez [5], Grafakos and Torres [10], Hu and Yang [11], Lerner, Ombrosi, Pérez, Torres and Trujillo-González [14], and Maldonado and Naibo [15] for this direction. The purpose of the paper is to consider a weighted version of [17]. In particular, we focus on the result of Kurtz and Wheeden [13] in linear theory and recall their result (see also [7, Chapter 4, Theorem 3.9]).

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Theorem 1.1. *Let $1 < p < \infty$ and $n/2 < \ell \le n$, where ℓ is a positive integer. Assume*

$$\text{(i) } p > n/\ell \text{ and } w \in A_{p\ell/n} \quad \text{or} \quad \text{(ii) } p < (n/\ell)' \text{ and } w^{1-p'} \in A_{p'\ell/n}.$$

If $m \in L^\infty(\mathbb{R}^n)$ satisfies

$$\sup_{R>0} \left(R^{2|\alpha|-n} \int_{R<|\xi|<2R} |\partial^\alpha m(\xi)|^2 \, d\xi \right)^{1/2} < \infty \qquad \text{for all } |\alpha| \le \ell, \tag{1.2}$$

then $m(D)$ is bounded on $L^p(w)$.

Theorem 1.1 says that if ℓ is large (that is, m is smooth), then we can weaken the assumption on w, since $A_{p_1} \subset A_{p_2}$ when $p_1 \le p_2$. Note that we cannot use the pointwise estimate for the kernel $K = \mathcal{F}^{-1}m$, because m has only the limited smoothness.

Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\operatorname{supp} \Psi \subset \{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\}, \quad \sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1 \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}, \tag{1.3}$$

and note that condition (1.2) is equivalent to

$$\sup_{k\in\mathbb{Z}} \|m(2^k\cdot)\Psi\|_{H^\ell(\mathbb{R}^n)} < \infty,$$

where $H^\ell(\mathbb{R}^n)$ is the Sobolev space and Ψ is as in (1.3) with $d = n$. We denote by N a positive integer satisfying $N \ge 2$. For $m \in L^\infty(\mathbb{R}^{Nn})$, we set

$$m_k(\xi) = m(2^k\xi_1, \dots, 2^k\xi_N)\Psi(\xi_1, \dots, \xi_N), \tag{1.4}$$

where $k \in \mathbb{Z}$, $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and Ψ is as in (1.3) with $d = Nn$. The following is our main result:

Theorem 1.2. *Let $1 < p_1, \dots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$ and $Nn/2 < s \le Nn$. Assume*

(i) $\min\{p_1, \dots, p_N\} > Nn/s$ *and* $w \in A_{\min\{p_1 s/(Nn), \dots, p_N s/(Nn)\}}$ *or*

(ii) $\min\{p_1, \dots, p_N\} < (Nn/s)'$, $1 < p < \infty$ *and* $w^{1-p'} \in A_{p's/(Nn)}$.

If $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies

$$\sup_{k\in\mathbb{Z}} \|m_k\|_{H^s(\mathbb{R}^{Nn})} < \infty,$$

then T_m is bounded from $L^{p_1}(w) \times \cdots \times L^{p_N}(w)$ to $L^p(w)$.

Theorem 1.2 under assumption (i) is a simplified version of Theorem 6.2 below, and note that we do not assume any conditions on the target space $L^p(w)$ in it. On the other hand, in order to use the duality argument, we assume $1 < p < \infty$ in Theorem 1.2 under assumption (ii).

The organization of this paper is as follows: In Sections 2 and 3, definitions and lemmas are given. Sections 4 and 5 are devoted to the proofs of Theorem 1.2 under assumptions (i) and (ii), respectively. In Section 6, by using the Sobolev space of product type and multiple weights, we give an improvement of Theorem 1.2 under assumption (i).

In the first draft of this paper, the case $p \le 1$ in Theorem 1.2 under assumption (i) was not treated. But, by using the argument of Grafakos and Si [8], we have succeeded in treating its case (see Remark 2.6).

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

To distinguish linear and multilinear multipliers, we use the following notation: For $m \in L^\infty(\mathbb{R}^n)$, the (linear) Fourier multiplier operator $m(D)$ is defined by $m(D)f = \mathcal{F}^{-1}[m\widehat{f}]$ for $f \in \mathcal{S}(\mathbb{R}^n)$.

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^s} = \|(I-\Delta)^{s/2} f\|_{L^2} < \infty,$$

where $(I-\Delta)^{s/2} f = \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\widehat{f}]$. Let $1 < p < \infty$ and $w \geq 0$. The weighted Lebesgue space $L^p(w)$ consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty.$$

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n}\int_{|x-y|<r} |f(y)|\,dy$$

for locally integrable functions f on $\mathbb{R}^n$. Let $1 < p < \infty$. We say that a weight $w \geq 0$ belongs to the Muckenhoupt class A_p if

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, $|B|$ is the Lebesgue measure of B, and p' is the conjugate exponent of p (that is, $1/p + 1/p' = 1$). It is well known that M is bounded on $L^p(w)$ if and only if $w \in A_p$ ([6, Theorem 7.3]).

In the sequel, we will use the following functions $\varphi, \psi, \widetilde{\psi}, \zeta, \widetilde{\zeta} \in \mathcal{S}(\mathbb{R}^n)$:

$$\operatorname{supp}\varphi \subset \{|\eta| \leq 16N\}, \qquad \varphi = 1 \text{ on } \{|\eta| \leq 8N\}, \tag{2.1}$$

$$\begin{cases} \operatorname{supp}\psi \subset \{1/2 \leq |\eta| \leq 2\}, & \sum_{j\in\mathbb{Z}} \psi(\eta/2^j) = 1 \quad \text{for all } \eta \neq 0, \\ \operatorname{supp}\widetilde{\psi} \subset \{1/4 \leq |\eta| \leq 4\}, & \widetilde{\psi} = 1 \text{ on } \{1/2 \leq |\eta| \leq 2\}, \end{cases} \tag{2.2}$$

$$\begin{cases} \operatorname{supp}\zeta \subset \{1/(16N) \leq |\eta| \leq 16N\}, & \zeta = 1 \text{ on } \{1/(8N) \leq |\eta| \leq 8N\}, \\ \operatorname{supp}\widetilde{\zeta} \subset \{1/(32N) \leq |\eta| \leq 32N\}, & \widetilde{\zeta} = 1 \text{ on } \{1/(16N) \leq |\eta| \leq 16N\}, \end{cases} \tag{2.3}$$

where $\eta \in \mathbb{R}^n$. The following lemmas will be used later on:

Lemma 2.1 ([6, Proposition 2.7])**.** *Let $\epsilon > 0$. Then there exists a constant $C > 0$ such that*

$$\sup_{r>0}\left(r^n \int_{\mathbb{R}^n} \frac{|f(y)|}{(1+r|x-y|)^{n+\epsilon}}\,dy\right) \leq CMf(x)$$

for all locally integrable functions f on $\mathbb{R}^n$.

Lemma 2.2 ([16, Section V])**.** *Let $1 < p < \infty$ and $w \in A_p$. Then*

(1) $w^{1-p'} \in A_{p'}$, (2) *there exists $q < p$ such that $w \in A_q$.*

Lemma 2.3 ([1]). *Let $1 < p, q < \infty$ and $w \in A_p$. Then there exists a constant $C > 0$ such that*

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} (Mf_k)^q \right\}^{1/q} \right\|_{L^p(w)} \le C \left\| \left\{ \sum_{k\in\mathbb{Z}} |f_k|^q \right\}^{1/q} \right\|_{L^p(w)}$$

for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^n$.

Lemma 2.4 ([12]). *Let $1 < p < \infty$, $w \in A_p$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp}\psi \subset \{\xi \in \mathbb{R}^n : 1/r \le |\xi| \le r\}$ for some $r > 1$. Then there exists a constant $C > 0$ such that*

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} |\psi(D/2^k)f|^2 \right\}^{1/2} \right\|_{L^p(w)} \le C\|f\|_{L^p(w)} \qquad \text{for all } f \in L^p(w),$$

where $\psi(D/2^k)f = \mathcal{F}^{-1}[\psi(\cdot/2^k)\widehat{f}]$. Moreover, if $\sum_{k\in\mathbb{Z}} \psi(\xi/2^k) = 1$ for all $\xi \ne 0$, then

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} |\psi(D/2^k)f|^2 \right\}^{1/2} \right\|_{L^p(w)} \approx \|f\|_{L^p(w)} \qquad \text{for all } f \in L^p(w).$$

In the case $s = 0$, the following lemma appears as Triebel [18, Proposition 1.3.2]:

Lemma 2.5 ([17, Lemma 3.3]). *Let $2 \le q < \infty$, $r > 0$, $s \ge 0$ and $w_s(x) = (1 + |x|^2)^{s/2}$. Then there exists a constant $C > 0$ such that*

$$\|\widehat{f}\|_{L^q(w_s)} \le C\|f\|_{H^{s/q}} \quad \text{for all } f \in H^{s/q}(\mathbb{R}^n) \text{ with } \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| \le r\}.$$

We end this section by giving the following remark which is a key to proving the case $p \le 1$ in Theorem 1.2 under assumption (i):

Remark 2.6. In the unweighted case, by using [8, Lemma 2.4], Grafakos and Si treated the case $p \le 1$, where p is the index defined by $1/p = 1/p_1 + \cdots + 1/p_N$. In the weighted case, we modify their argument as follows.

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \Phi(x)\,dx = 1$, and set $\Phi_t(x) = t^{-n}\Phi(x/t)$. Then

$$\begin{aligned} |T_m(f_1, \dots, f_N)(x)| &= \lim_{t\to 0} |\Phi_t * T_m(f_1, \dots, f_N)(x)| \\ &\le \sup_{t>0} |\Phi_t * T_m(f_1, \dots, f_N)(x)| \end{aligned}$$

for all $m \in L^\infty(\mathbb{R}^{Nn})$ and $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$. On the other hand, it is known that if $w \in A_\infty$, then the weighted Hardy space $\mathcal{H}^p(w)$ coincides with the weighted Triebel-Lizorkin space $\dot{F}^{p,2}_0(w)$ for $0 < p < \infty$ (see, for example, [2, 3]). Hence, if $w \in A_\infty$, then

$$\begin{aligned} \|T_m(f_1, \dots, f_N)\|_{L^p(w)} &\le \|\sup_{t>0} |\Phi_t * T_m(f_1, \dots, f_N)|\|_{L^p(w)} \\ &= \|T_m(f_1, \dots, f_N)\|_{\mathcal{H}^p(w)} \approx \|T_m(f_1, \dots, f_N)\|_{\dot{F}^{p,2}_0(w)} \\ &= \left\| \left(\sum_{j\in\mathbb{Z}} |\psi(D/2^j)T_m(f_1, \dots, f_N)|^2 \right)^{1/2} \right\|_{L^p(w)} \end{aligned}$$

for $0 < p < \infty$, where ψ is as in (2.2).

3. Lemmas

Let N be a natural number, and let ϕ_1 be a C^∞-function on $[0,\infty)$ satisfying

$$\phi_1(t) = 1 \quad \text{on} \quad [0, 1/(4N)], \qquad \operatorname{supp}\phi_1 \subset [0, 1/(2N)].$$

We also set $\phi_2(t) = 1 - \phi_1(t)$. For $(i_1, i_2, \dots, i_N) \in \{1,2\}^N$, we define the function $\Phi_{(i_1,i_2,\dots,i_N)}$ on $\mathbb{R}^{Nn} \setminus \{0\}$ by

$$\Phi_{(i_1,i_2,\dots,i_N)}(\xi) = \phi_{i_1}(|\xi_1|/|\xi|)\phi_{i_2}(|\xi_2|/|\xi|)\dots\phi_{i_N}(|\xi_N|/|\xi|), \tag{3.1}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n$ and $|\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2 + \dots + |\xi_N|^2}$. Note that $\Phi_{(1,1,\dots,1)}(\xi) = 0$. In fact, if $|\xi_i|/|\xi| \le 1/(2N)$ for all $1 \le i \le N$, then

$$|\xi| \le |\xi_1| + \dots + |\xi_N| \le |\xi|/(2N) + \dots + |\xi|/(2N) = |\xi|/2,$$

and this is a contradiction. The following lemma seems to be known to many people, but we shall give a proof for the reader's convenience.

Lemma 3.1. *Let $\Phi_{(i_1,\dots,i_N)}$ be the same as in* (3.1). *Then the following are true:*

(1) *For $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \setminus \{(0,\dots,0)\}$,*

$$\sum_{\substack{(i_1,i_2,\dots,i_N)\in\{1,2\}^N \\ (i_1,i_2,\dots,i_N)\neq(1,1,\dots,1)}} \Phi_{(i_1,i_2,\dots,i_N)}(\xi) = 1.$$

(2) *For $(i_1, \dots, i_N) \in \{1,2\}^N$ and $(\alpha_1, \dots, \alpha_N) \in \mathbb{Z}_+^n \times \dots \times \mathbb{Z}_+^n$ there exists a constant $C^{\alpha_1,\dots,\alpha_N}_{(i_1,\dots,i_N)} > 0$ such that*

$$|\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_N}^{\alpha_N}\Phi_{(i_1,\dots,i_N)}(\xi)| \le C^{\alpha_1,\dots,\alpha_N}_{(i_1,\dots,i_N)}(|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1|+\dots+|\alpha_N|)}$$

for all $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \setminus \{(0,\dots,0)\}$.

(3) *If $i_j = 2$ for some $1 \le j \le N$ and $i_k = 1$ for all $1 \le k \le N$ with $k \neq j$, then $\operatorname{supp}\Phi_{(i_1,\dots,i_N)} \subset \{(\xi_1,\dots,\xi_N) : |\xi_k| \le |\xi_j|/N$ for $k \neq j\}$. If $i_j = i_{j'} = 2$ for some $1 \le j, j' \le N$ with $j \neq j'$, then $\operatorname{supp}\Phi_{(i_1,\dots,i_N)} \subset \{(\xi_1,\dots,\xi_N) : |\xi_j|/(4N) \le |\xi_{j'}| \le 4N|\xi_j|,\ |\xi_k| \le 4N|\xi_j|$ for $k \neq j, j'\}$.*

Proof. Let $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \setminus \{(0,\dots,0)\}$. Since

$$(\phi_1(|\xi_1|/|\xi|) + \phi_2(|\xi_1|/|\xi|)) \times \dots \times (\phi_1(|\xi_N|/|\xi|) + \phi_2(|\xi_N|/|\xi|)) = 1$$

and $\Phi_{(1,1,\dots,1)}(\xi) = 0$, we have

$$\sum_{\substack{(i_1,i_2,\dots,i_N)\in\{1,2\}^N \\ (i_1,i_2,\dots,i_N)\neq(1,1,\dots,1)}} \phi_{i_1}(|\xi_1|/|\xi|)\phi_{i_2}(|\xi_2|/|\xi|)\dots\phi_{i_N}(|\xi_N|/|\xi|) = 1,$$

that is, Lemma 3.1 (1). On the other hand, since $\Phi_{(i_1,\dots,i_N)}$ is homogeneous of degree 0 as a function on $\mathbb{R}^{Nn}$, we have Lemma 3.1 (2).

Let $i_j = 2$ for some $1 \le j \le N$ and $i_k = 1$ for all $1 \le k \le N$ with $k \neq j$, and we may assume that $j = 1$, that is,

$$\Phi_{(i_1,i_2,\dots,i_N)}(\xi) = \phi_2(|\xi_1|/|\xi|)\phi_1(|\xi_2|/|\xi|)\dots\phi_1(|\xi_N|/|\xi|).$$

If $\xi \neq 0$ and $|\xi_k| \le |\xi|/(2N)$ for $2 \le k \le N$, then $|\xi_k| \le |\xi_1|/N$ for all $2 \le k \le N$. Indeed, if $|\xi_k| > |\xi_1|/N$ for some $2 \le k \le N$, then

$$|\xi| \le |\xi_1| + \dots + |\xi_N| < |\xi_2| + \dots + (N+1)|\xi_k| + \dots + |\xi_N| \le \frac{2N-1}{2N}|\xi|,$$

and this is a contradiction. Hence, since $\operatorname{supp}\phi_1(|\xi_k|/|\xi|) \subset \{|\xi_k| \le |\xi|/(2N)\}$ for $2 \le k \le N$, we see that $\operatorname{supp}\Phi_{(2,1,\dots,1)} \subset \{|\xi_k| \le |\xi_1|/N \text{ for } 2 \le k \le N\}$.

Let $i_j = i_{j'} = 2$ for some $1 \le j, j' \le N$ with $j \ne j'$, and we may assume that $j = 1$ and $j' = 2$, that is,

$$\Phi_{(i_1,i_2,i_3,\dots,i_N)}(\xi) = \phi_2(|\xi_1|/|\xi|)\phi_2(|\xi_2|/|\xi|)\phi_{i_3}(|\xi_3|/|\xi|)\dots\phi_{i_N}(|\xi_N|/|\xi|),$$

where $i_3, \dots, i_N \in \{1, 2\}$. Since $|\xi_k| \le |\xi|$, if $|\xi_1| \ge |\xi|/(4N)$ and $|\xi_2| \ge |\xi|/(4N)$, then $|\xi_1|/(4N) \le |\xi_2| \le 4N|\xi_1|$ and $|\xi_k| \le 4N|\xi_1|$ for all $3 \le k \le N$. Therefore, it follows from the fact $\operatorname{supp}\phi_2(|\xi_k|/|\xi|) \subset \{|\xi_k| \ge |\xi|/(4N)\}$ for $1 \le k \le 2$ that $\operatorname{supp}\Phi_{(2,2,i_3,\dots,i_N)} \subset \{|\xi_1|/(4N) \le |\xi_2| \le 4N|\xi_1|,\ |\xi_k| \le 4N|\xi_1| \text{ for } 3 \le k \le N\}$. □

Lemma 3.2. *Let $s > Nn/2$, $\max\{1, Nn/s\} < q < 2$ and $r > 0$. Then there exists a constant $C > 0$ such that*

$$\left|\int_{\mathbb{R}^{Nn}} 2^{Njn}\mathcal{F}^{-1}m(2^j(x-y_1),\dots,2^j(x-y_N))f_1(y_1)\dots f_N(y_N)\,dy_1\dots dy_N\right| \le C\|m\|_{H^s}M(|f_1|^q)(x)^{1/q}\dots M(|f_N|^q)(x)^{1/q}$$

for all $j \in \mathbb{Z}$, $m \in H^s(\mathbb{R}^{Nn})$ with $\operatorname{supp} m \subset \{\sqrt{|\xi_1|^2 + \dots + |\xi_N|^2} \le r\}$ and $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform in all the variables.

Proof. Note that $1 < q < 2$. By Hölder's inequality and Lemma 2.1,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{Nn}} 2^{Njn}\mathcal{F}^{-1}m(2^j(x-y_1),\dots,2^j(x-y_N))f_1(y_1)\dots f_N(y_N)\,dy_1\dots dy_N\right| \\
&= 2^{Njn}\Big|\int_{\mathbb{R}^{Nn}} \mathcal{F}^{-1}m(2^j(x-y_1),\dots,2^j(x-y_N))(1+2^j|x-y_1|+\dots+2^j|x-y_N|)^s \\
&\quad\times(1+2^j|x-y_1|+\dots+2^j|x-y_N|)^{-s}f_1(y_1)\dots f_N(y_N)\,dy_1\dots dy_N\Big| \\
&\le 2^{Njn/q}\left\{\int_{\mathbb{R}^{Nn}} |\mathcal{F}^{-1}m(y_1,\dots,y_N)|^{q'}(1+|y_1|+\dots+|y_N|)^{sq'}\,dy_1\dots dy_N\right\}^{1/q'} \\
&\quad\times\left\{\int_{\mathbb{R}^{2n}} (1+2^j|x-y_1|+\dots+2^j|x-y_N|)^{-sq}|f_1(y_1)\dots f_N(y_N)|^q\,dy_1\dots dy_N\right\}^{1/q} \\
&\le C\left\{\int_{\mathbb{R}^{Nn}} |\widehat{m}(y_1,\dots,y_N)|^{q'}(1+|y_1|^2+\dots+|y_N|^2)^{sq'/2}\,dy_1\dots dy_N\right\}^{1/q'} \\
&\quad\times\left\{\int_{\mathbb{R}^n} \frac{2^{jn}|f_1(y_1)|^q}{(1+2^j|x-y_1|)^{sq/N}}\,dy_1\right\}^{1/q}\dots\left\{\int_{\mathbb{R}^n} \frac{2^{jn}|f_N(y_N)|^q}{(1+2^j|x-y_N|)^{sq/N}}\,dy_N\right\}^{1/q} \\
&\le C\|\widehat{m}\|_{L^{q'}(w_{sq'})}M(|f_1|^q)(x)^{1/q}\dots M(|f_N|^q)(x)^{1/q},
\end{aligned}$$

where we have used $sq/N > n$. Since $2 < q' < \infty$, it follows from Lemma 2.5 that $\|\widehat{m}\|_{L^{q'}(w_{sq'})} \le C\|m\|_{H^s}$. This completes the proof. □

4. Proof of Theorem 1.2 under assumption (i)

In this section, we prove Theorem 1.2 under assumption (i), and first consider the bilinear case for the sake of simplicity. Let $\Phi_{(2,1)}, \Phi_{(1,2)}, \Phi_{(2,2)}$ be the same as

in (3.1) with $N = 2$. By Lemma 3.1 (3),

$$\begin{aligned} &\operatorname{supp}\Phi_{(2,1)} \subset \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi_2| \le |\xi_1|/2\}, \\ &\operatorname{supp}\Phi_{(1,2)} \subset \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi_1| \le |\xi_2|/2\}, \\ &\operatorname{supp}\Phi_{(2,2)} \subset \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi_1|/8 \le |\xi_2| \le 8|\xi_1|\}. \end{aligned} \tag{4.1}$$

Proof of Theorem 1.2 *with* $N = 2$ *under assumption* (i). Let $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $n < s \le 2n$, $\min\{p_1, p_2\} > 2n/s$ and $w \in A_{\min\{p_1 s/(2n), p_2 s/(2n)\}}$. Assume that $m \in L^\infty(\mathbb{R}^{2n})$ satisfies $\sup_{k\in\mathbb{Z}} \|m_k\|_{H^s(\mathbb{R}^{2n})} < \infty$, where m_k is defined by (1.4). Since $1 \le 2n/s < \min\{2, p_1, p_2\}$, we can take $q > 1$ satisfying $2n/s < q < \min\{2, p_1, p_2\}$. Moreover, by Lemma 2.2 (2), q can be chosen so close to $2n/s$ that $w \in A_{\min\{p_1/q, p_2/q\}}$. We decompose m as follows:

$$\begin{aligned} m(\xi_1,\xi_2) &= \sum_{\substack{(i_1,i_2)\in\{1,2\}^2 \\ (i_1,i_2)\neq(1,1)}} \Phi_{(i_1,i_2)}(\xi_1,\xi_2) m(\xi_1,\xi_2) \\ &= m_{(2,1)}(\xi_1,\xi_2) + m_{(1,2)}(\xi_1,\xi_2) + m_{(2,2)}(\xi_1,\xi_2), \end{aligned} \tag{4.2}$$

where $\Phi_{(i_1,i_2)}$ are the same as in (3.1) with $N = 2$.

Estimates for $m_{(2,1)}$ *and* $m_{(1,2)}$. Let us consider $m_{(2,1)}$. Since $w \in A_\infty$, it follows from Remark 2.6 that

$$\|T_{m_{(2,1)}}(f_1,f_2)\|_{L^p(w)} \le C \left\| \left(\sum_{j\in\mathbb{Z}} |\psi(D/2^j) T_{m_{(2,1)}}(f_1,f_2)|^2 \right)^{1/2} \right\|_{L^p(w)}, \tag{4.3}$$

where $\psi \in \mathcal{S}(\mathbb{R}^n)$ is as in (2.2). By (2.2) and (4.1), we note that $\operatorname{supp}\psi(\cdot/2^k) \subset \{2^{k-1} \le |\xi| \le 2^{k+1}\}$ and $\operatorname{supp} m_{(2,1)} \subset \{|\xi_2| \le |\xi_1|/2\}$. Hence, if $\xi_1 \in \operatorname{supp}\psi(\cdot/2^k)$ and $(\xi_1,\xi_2) \in \operatorname{supp} m_{(2,1)}$, then $2^{k-2} \le |\xi_1 + \xi_2| \le 2^{k+2}$. This implies

$$\begin{aligned} &\psi(D/2^j) T_{m_{(2,1)}}(f_1,f_2)(x) \\ &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)} \psi((\xi_1+\xi_2)/2^j) m_{(2,1)}(\xi_1,\xi_2) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2)\, d\xi_1 d\xi_2 \\ &= \sum_{k=-2}^{2} \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)} \psi((\xi_1+\xi_2)/2^j) \psi(\xi_1/2^{j+k}) \\ &\qquad\qquad \times m_{(2,1)}(\xi_1,\xi_2) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2)\, d\xi_1 d\xi_2 \\ &= \sum_{k=-2}^{2} \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)} \psi((\xi_1+\xi_2)/2^j) \psi(\xi_1/2^{j+k}) \varphi(\xi_2/2^{j+k}) \\ &\qquad\qquad \times m_{(2,1)}(\xi_1,\xi_2) \widetilde{\psi}(\xi_1/2^{j+k}) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2)\, d\xi_1 d\xi_2 \\ &= \sum_{k=-2}^{2} \int_{\mathbb{R}^{2n}} 2^{2(j+k)n} \mathcal{F}^{-1} m_{(2,1)}^{j,k}(2^{j+k}(x-y_1), 2^{j+k}(x-y_2)) \\ &\qquad\qquad \times \widetilde{\psi}(D/2^{j+k}) f_1(y_1) f_2(y_2)\, dy_1 dy_2, \end{aligned} \tag{4.4}$$

where $\varphi, \widetilde{\psi} \in \mathcal{S}(\mathbb{R}^n)$ are as in (2.1) and (2.2) with $N = 2$, and

$$m_{(2,1)}^{j,k}(\xi_1,\xi_2) = m_{(2,1)}(2^{j+k}\xi_1, 2^{j+k}\xi_2) \psi(2^k(\xi_1+\xi_2)) \psi(\xi_1) \varphi(\xi_2). \tag{4.5}$$

Since $\operatorname{supp} m^{j,k}_{(2,1)} \subset \{\sqrt{|\xi_1|^2+|\xi_2|^2} \le 34\}$ and $2n/s < q < 2$, it follows from Lemma 3.2 that

$$
\begin{aligned}
(4.6)\qquad & |\psi(D/2^j)T_{m_{(2,1)}}(f_1,f_2)(x)| \\
&\le C\sum_{k=-2}^{2} \|m^{j,k}_{(2,1)}\|_{H^s} M(|\widetilde{\psi}(D/2^{j+k})f_1|^q)(x)^{1/q} M(|f_2|^q)(x)^{1/q}.
\end{aligned}
$$

By (4.3), (4.6) and Hölder's inequality,

$$
\begin{aligned}
\|T_{m_{(2,1)}}(f_1,f_2)\|_{L^p(w)} &\le C\sum_{k=-2}^{2}\left(\sup_{j\in\mathbb{Z}}\|m^{j,k}_{(2,1)}\|_{H^s}\right)\left\|M(|f_2|^q)^{1/q}\right\|_{L^{p_2}(w)} \\
&\qquad\times\left\|\left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^{j+k})f_1|^q)^{2/q}\right)^{1/2}\right\|_{L^{p_1}(w)} \\
&= C\sum_{k=-2}^{2}\left(\sup_{j\in\mathbb{Z}}\|m^{j,k}_{(2,1)}\|_{H^s}\right)\|M(|f_2|^q)\|^{1/q}_{L^{p_2/q}(w)} \\
&\qquad\times\left\|\left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^{j})f_1|^q)^{2/q}\right)^{q/2}\right\|^{1/q}_{L^{p_1/q}(w)}.
\end{aligned}
$$

Since $1 < 2/q, p_1/q, p_2/q < \infty$ and $w \in A_{p_1/q} \cap A_{p_2/q}$, we have by Lemmas 2.3 and 2.4

$$
\|T_{m_{(2,1)}}(f_1,f_2)\|_{L^p(w)} \le C\left(\sup_{j\in\mathbb{Z},|k|\le 2}\|m^{j,k}_{(2,1)}\|_{H^s}\right)\|f_1\|_{L^{p_1}(w)}\|f_2\|_{L^{p_2}(w)}.
$$

Note that $\operatorname{supp}\psi(\xi_1)\varphi(\xi_2) \subset \{1/2 \le \sqrt{|\xi_1|^2+|\xi_2|^2} \le 34\}$ and $\operatorname{supp}\Psi(\xi_1/2^\ell, \xi_2/2^\ell) \subset \{2^{\ell-1} \le \sqrt{|\xi_1|^2+|\xi_2|^2} \le 2^{\ell+1}\}$, where Ψ is as in (1.3) with $d = 2n$. Then, since $H^s(\mathbb{R}^{2n})$ is a multiplication algebra when $s > n$ ([18, Theorem 2.8.3]),

$$
\begin{aligned}
\|m^{j,k}_{(2,1)}\|_{H^s} &= \|\Phi_{(2,1)}(2^{j+k}\xi_1, 2^{j+k}\xi_2)m(2^{j+k}\xi_1, 2^{j+k}\xi_2)\psi(2^k(\xi_1+\xi_2))\psi(\xi_1)\varphi(\xi_2)\|_{H^s} \\
&\le \sum_{\ell=-1}^{6}\|\Phi_{(2,1)}(2^{j+k}\xi_1, 2^{j+k}\xi_2)m(2^{j+k}\xi_1, 2^{j+k}\xi_2) \\
&\qquad\qquad\times\psi(2^k(\xi_1+\xi_2))\psi(\xi_1)\varphi(\xi_2)\Psi(\xi_1/2^\ell, \xi_2/2^\ell)\|_{H^s} \\
&\le C\sum_{\ell=-1}^{6}\|\Phi_{(2,1)}(2^{j+k}\xi_1, 2^{j+k}\xi_2)\psi(2^k(\xi_1+\xi_2))\psi(\xi_1)\varphi(\xi_2)\|_{H^s} \\
&\qquad\qquad\times\|m(2^{j+k}\xi_1, 2^{j+k}\xi_2)\Psi(\xi_1/2^\ell, \xi_2/2^\ell)\|_{H^s}.
\end{aligned}
$$

Using the fact $\operatorname{supp}\psi(2^k(\xi_1+\xi_2))\psi(\xi_1)\varphi(\xi_2) \subset \{\sqrt{|\xi_1|^2+|\xi_2|^2} \ge 1/2\}$ and Lemma 3.1 (2), we can easily prove

$$
\sup_{j\in\mathbb{Z},|k|\le 2}\|\Phi_{(2,1)}(2^{j+k}\xi_1, 2^{j+k}\xi_2)\psi(2^k(\xi_1+\xi_2))\psi(\xi_1)\varphi(\xi_2)\|_{H^{[s]+1}} < \infty,
$$

where $[s]$ is the integer part of s (see, for example, [17, Lemma 3.2]). On the other hand, by a change of variables,

$$\begin{aligned}&\|m(2^{j+k}\xi_1, 2^{j+k}\xi_2)\Psi(\xi_1/2^\ell, \xi_2/2^\ell)\|_{H^s} \le C\|m(2^{j+k+\ell}\xi_1, 2^{j+k+\ell}\xi_2)\Psi(\xi_1,\xi_2)\|_{H^s}\\ &\le C\sup_{k\in\mathbb{Z}}\|m(2^k\xi_1, 2^k\xi_2)\Psi(\xi_1,\xi_2)\|_{H^s} = C\sup_{k\in\mathbb{Z}}\|m_k\|_{H^s}\end{aligned}$$

for all $j \in \mathbb{Z}$, $|k| \le 2$ and $-1 \le \ell \le 6$. Therefore,

$$\sup_{j\in\mathbb{Z},|k|\le 2}\|m^{j,k}_{(2,1)}\|_{H^s} \le C\sup_{k\in\mathbb{Z}}\|m_k\|_{H^s}, \tag{4.7}$$

and consequently

$$\|T_{m_{(2,1)}}\|_{L^{p_1}(w)\times L^{p_2}(w)\to L^p(w)} \le C\sup_{j\in\mathbb{Z},|k|\le 2}\|m^{j,k}_{(2,1)}\|_{H^s} \le C\sup_{k\in\mathbb{Z}}\|m_k\|_{H^s}. \tag{4.8}$$

By interchanging the roles of ξ_1 and ξ_2, we can prove

$$\|T_{m_{(1,2)}}\|_{L^{p_1}(w)\times L^{p_2}(w)\to L^p(w)} \le C\sup_{j\in\mathbb{Z},|k|\le 2}\|m^{j,k}_{(1,2)}\|_{H^s} \le C\sup_{k\in\mathbb{Z}}\|m_k\|_{H^s}, \tag{4.9}$$

where $m^{j,k}_{(1,2)}(\xi_1,\xi_2) = m_{(1,2)}(2^{j+k}\xi_1, 2^{j+k}\xi_2)\psi(2^k(\xi_1+\xi_2))\varphi(\xi_1)\psi(\xi_2)$.

Estimate for $m_{(2,2)}$. (4.1) gives $\operatorname{supp} m_{(2,2)} \subset \{|\xi_1|/8 \le |\xi_2| \le 8|\xi_1|\}$. This implies that if $\xi_1 \in \operatorname{supp}\psi(\cdot/2^k)$ and $(\xi_1,\xi_2) \in \operatorname{supp} m_{(2,2)}$, then $2^{k-4} \le |\xi_2| \le 2^{k+4}$, and consequently $\zeta(\xi_2/2^k) = 1$, where ψ, ζ are as in (2.2) and (2.3) with $N = 2$. Hence,

$$\begin{aligned}&T_{m_{(2,2)}}(f_1,f_2)(x)\\ &= \sum_{k\in\mathbb{Z}}\frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)}\psi(\xi_1/2^k)m_{(2,2)}(\xi_1,\xi_2)\widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)\,d\xi_1 d\xi_2\\ &= \sum_{k\in\mathbb{Z}}\frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)}\psi(\xi_1/2^k)\zeta(\xi_2/2^k)m_{(2,2)}(\xi_1,\xi_2)\\ &\qquad\qquad\times\widetilde{\psi}(\xi_1/2^k)\widehat{f}_1(\xi_1)\widetilde{\zeta}(\xi_2/2^k)\widehat{f}_2(\xi_2)\,d\xi_1 d\xi_2\\ &= \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^{2n}} 2^{2kn}\mathcal{F}^{-1}m^k_{(2,2)}(2^k(x-y_1), 2^k(x-y_2))\\ &\qquad\qquad\times\widetilde{\psi}(D/2^k)f_1(y_1)\widetilde{\zeta}(D/2^k)f_2(y_2)\,dy_1 dy_2,\end{aligned} \tag{4.10}$$

where $\widetilde{\psi}, \widetilde{\zeta}$ are as in (2.2) and (2.3) with $N = 2$, and

$$m^k_{(2,2)}(\xi_1,\xi_2) = m_{(2,2)}(2^k\xi_1, 2^k\xi_2)\psi(\xi_1)\zeta(\xi_2). \tag{4.11}$$

Since $\operatorname{supp} m^k_{(2,2)} \subset \{\sqrt{|\xi_1|^2+|\xi_2|^2} \le 34\}$, it follows from Lemma 3.2 that

$$\begin{aligned}&|T_{m_{(2,2)}}(f_1,f_2)(x)|\\ &\qquad\le C\sum_{k\in\mathbb{Z}}\|m^k_{(2,2)}\|_{H^s}M(|\widetilde{\psi}(D/2^k)f_1|^q)(x)^{1/q}M(|\widetilde{\zeta}(D/2^k)f_2|^q)(x)^{1/q}.\end{aligned} \tag{4.12}$$

Then, by Schwarz's inequality and Hölder's inequality,

$$\begin{aligned}
&\|T_{m_{(2,2)}}(f_1,f_2)\|_{L^p(w)} \\
&\quad \le C\left(\sup_{k\in\mathbb{Z}}\|m^k_{(2,2)}\|_{H^s}\right)\left\|\left(\sum_{k\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^k)f_1|^q)^{2/q}\right)^{1/2}\right\|_{L^{p_1}(w)} \\
&\qquad \times\left\|\left(\sum_{k\in\mathbb{Z}} M(|\widetilde{\zeta}(D/2^k)f_2|^q)^{2/q}\right)^{1/2}\right\|_{L^{p_2}(w)}.
\end{aligned}$$

The rest of the proof is similar to that of $m_{(2,1)}$, and we have

$$\|T_{m_{(2,2)}}\|_{L^{p_1}(w)\times L^{p_2}(w)\to L^p(w)} \le C\sup_{k\in\mathbb{Z}}\|m^k_{(2,2)}\|_{H^s} \le C\sup_{k\in\mathbb{Z}}\|m_k\|_{H^s}. \tag{4.13}$$

Therefore, Theorem 1.2 with $N=2$ under assumption (i) follows from (4.2), (4.8), (4.9) and (4.13). □

We end this section by giving a sketch of the proof of Theorem 1.2 with $N\ge 3$ under assumption (i). We shall only indicate the necessary modifications, because the proof is similar to the bilinear case.

Assume that $1<p_1,\dots,p_N<\infty$, $1/p_1+\cdots+1/p_N=1/p$, $Nn/2<s\le Nn$, $\min\{p_1,\dots,p_N\}>Nn/s$ and $w\in A_{\min\{p_1s/(Nn),\dots,p_Ns/(Nn)\}}$. Since $1\le Nn/s<\min\{2,p_1,\dots,p_N\}$, we can take $q>1$ satisfying $Nn/s<q<\min\{2,p_1,\dots,p_N\}$ and $w\in A_{\min\{p_1/q,\dots,p_N/q\}}$. Let $\Phi_{(i_1,\dots,i_N)}$ be the same as in (3.1), and decompose $m\in L^\infty(\mathbb{R}^{Nn})$ as follows:

$$m(\xi)=\sum_{\substack{(i_1,\dots,i_N)\in\{1,2\}^N\\(i_1,\dots,i_N)\ne(1,\dots,1)}}\Phi_{(i_1,\dots,i_N)}(\xi)m(\xi)=\sum_{\substack{(i_1,\dots,i_N)\in\{1,2\}^N\\(i_1,\dots,i_N)\ne(1,\dots,1)}}m_{(i_1,\dots,i_N)}(\xi),$$

where $\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^n\times\cdots\times\mathbb{R}^n\setminus\{(0,\dots,0)\}$. We also use the functions $\varphi,\psi,\widetilde{\psi},\zeta,\widetilde{\zeta}\in\mathcal{S}(\mathbb{R}^n)$ as in (2.1)-(2.3).

We first consider the case where $(i_1,\dots,i_N)$ satisfies $\sharp\{j:i_j=2\}=1$, and may assume that $i_1=2$ and $i_k=1$ for $2\le k\le N$, that is, $m_{(i_1,i_2,\dots,i_N)}(\xi)=\Phi_{(2,1,\dots,1)}(\xi)m(\xi)$. It follows from Lemma 3.1 (3) that $\operatorname{supp} m_{(i_1,\dots,i_N)}\subset\{|\xi_k|\le|\xi_1|/N \text{ for } 2\le k\le N\}$. This implies that if $(\xi_1,\dots,\xi_N)\in\operatorname{supp} m_{(i_1,\dots,i_N)}$, then $|\xi_1|/N\le|\xi_1+\cdots+\xi_N|\le(2N-1)|\xi_1|/N$. By the same argument as in (4.4),

$$\begin{aligned}
&\psi(D/2^j)T_{m_{(i_1,\dots,i_N)}}(f_1,\dots,f_N)(x) \\
&\quad=\sum_{k=-2}^{k_0+1}\frac{1}{(2\pi)^{Nn}}\int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\cdots+\xi_N)}m_{(i_1,\dots,i_N)}(\xi)\psi((\xi_1+\cdots+\xi_N)/2^j) \\
&\qquad\times\psi(\xi_1/2^{j+k})\varphi(\xi_2/2^{j+k})\dots\varphi(\xi_N/2^{j+k}) \\
&\qquad\qquad\times\widetilde{\psi}(\xi_1/2^{j+k})\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\dots\widehat{f_N}(\xi_N)\,d\xi_1\dots d\xi_N,
\end{aligned}$$

where k_0 is an integer satisfying $2^{k_0} \geq N$. Then, it follows from Lemma 3.2 that

$$\begin{aligned}
&|\psi(D/2^j)T_{m_{(i_1,\dots,i_N)}}(f_1,\dots,f_N)(x)| \\
&\quad \leq C\sum_{k=-2}^{k_0+1} \|m^{j,k}_{(i_1,\dots,i_N)}\|_{H^s} M(|\widetilde{\psi}(D/2^{j+k})f_1|^q)(x)^{1/q} \\
&\qquad\qquad \times M(|f_2|^q)(x)^{1/q}\dots M(|f_N|^q)(x)^{1/q},
\end{aligned}$$

where

$$m^{j,k}_{(i_1,\dots,i_N)}(\xi) = m_{(i_1,\dots,i_N)}(2^{j+k}\xi)\psi(2^k(\xi_1+\dots+\xi_N))\psi(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)$$

(see (4.4)-(4.6) for the bilinear case). The rest of the proof is similar to the bilinear case, and we omit it.

We next consider the case where $(i_1,\dots,i_N)$ satisfies $\sharp\{j : i_j = 2\} \geq 2$, and may assume that $i_1 = i_2 = 2$, that is, $m_{(i_1,i_2,i_3,\dots,i_N)}(\xi) = \Phi_{(2,2,i_3,\dots,i_N)}(\xi)m(\xi)$, where $i_3,\dots,i_N \in \{1,2\}$. It follows from Lemma 3.1 (3) that $\operatorname{supp} m_{(i_1,\dots,i_N)} \subset \{|\xi_1|/(4N) \leq |\xi_2| \leq 4N|\xi_1|,\ |\xi_k| \leq 4N|\xi_1| \text{ for } 3 \leq k \leq N\}$. By the same argument as in (4.10),

$$\begin{aligned}
&T_{m_{(i_1,\dots,i_N)}}(f_1,\dots,f_N)(x) \\
&\quad = \sum_{k\in\mathbb{Z}} \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\xi_2+\xi_3+\dots+\xi_N)} m_{(i_1,\dots,i_N)}(\xi) \\
&\qquad \times \psi(\xi_1/2^k)\zeta(\xi_2/2^k)\varphi(\xi_3/2^k)\dots\varphi(\xi_N/2^k) \\
&\qquad\quad \times (\widetilde{\psi}(\xi_1/2^k)\widehat{f_1}(\xi_1))(\widetilde{\zeta}(\xi_2/2^k)\widehat{f_2}(\xi_2))\widehat{f_3}(\xi_3)\dots\widehat{f_N}(\xi_N)\,d\xi_1\dots d\xi_N.
\end{aligned}$$

Then, it follows from Lemma 3.2 that

$$\begin{aligned}
&|T_{m_{(i_1,\dots,i_N)}}(f_1,\dots,f_N)(x)| \\
&\quad \leq C\sum_{k\in\mathbb{Z}} \|m^k_{(i_1,\dots,i_N)}\|_{H^s} M(|\widetilde{\psi}(D/2^k)f_1|^q)(x)^{1/q} M(|\widetilde{\zeta}(D/2^k)f_2|^q)(x)^{1/q} \\
&\qquad \times M(|f_3|^q)(x)^{1/q}\dots M(|f_N|^q)(x)^{1/q} \\
&\quad \leq C\left(\sup_{k\in\mathbb{Z}} \|m^k_{(i_1,\dots,i_N)}\|_{H^s}\right) M(|f_3|^q)(x)^{1/q}\dots M(|f_N|^q)(x)^{1/q} \\
&\qquad \times \left(\sum_{k\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^k)f_1|^q)(x)^{2/q}\right)^{1/2} \left(\sum_{k\in\mathbb{Z}} M(|\widetilde{\zeta}(D/2^k)f_2|^q)(x)^{2/q}\right)^{1/2},
\end{aligned}$$

where

$$m^k_{(i_1,\dots,i_N)}(\xi) = m_{(i_1,\dots,i_N)}(2^k\xi)\psi(\xi_1)\zeta(\xi_2)\varphi(\xi_3)\dots\varphi(\xi_N)$$

(see (4.10)-(4.12) for the bilinear case). The rest of the proof is similar to the bilinear case, and we omit it.

5. Proof of Theorem 1.2 under assumption (ii)

For $m \in L^\infty(\mathbb{R}^{Nn})$ and $1 \leq j \leq N$, we set

$$m^{*j}(\xi) = m(\xi_1,\dots,\xi_{j-1},-(\xi_1+\dots+\xi_N),\xi_{j+1},\dots,\xi_N),$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$. The following formula is used in the proof of Theorem 1.2 under assumption (ii):

$$\int_{\mathbb{R}^n} T_m(f_1, \dots, f_N) g \, dx = \int_{\mathbb{R}^n} T_{m^{*j}}(f_1, \dots, f_{j-1}, g, f_{j+1}, \dots, f_N) f_j \, dx \tag{5.1}$$

for all $f_1, \dots, f_N, g \in \mathcal{S}(\mathbb{R}^n)$ (see, for example, [9]). We first consider the bilinear case as in Section 4.

Proof of Theorem 1.2 *with* $N = 2$ *under assumption* (ii). Let $1 < p_1, p_2, p < \infty$, $1/p_1 + 1/p_2 = 1/p$, $n < s \le 2n$, $\min\{p_1, p_2\} < (2n/s)'$, $w^{1-p'} \in A_{p's/(2n)}$, and let $m \in L^\infty(\mathbb{R}^{2n})$ satisfy $\sup_{k\in\mathbb{Z}} \|m_k\|_{H^s} < \infty$. We consider the case $p_1 = \min\{p_1, p_2\}$. Since $p_1 < (2n/s)'$, we see that $\max\{1/p', 1/p_2\} < 1/p' + 1/p_2 = 1/p_1' < s/(2n)$, that is, $p', p_2 > 2n/s$. Hence, $2n/s < \min\{2, p', p_2\}$.

By a change of variables, we can prove

$$\sup_{k\in\mathbb{Z}} \|(m^{*1})_k\|_{H^s} \le C \sup_{k\in\mathbb{Z}} \|m_k\|_{H^s}, \tag{5.2}$$

where

$$(m^{*1})_k(\xi_1, \xi_2) = m(-2^k(\xi_1 + \xi_2), 2^k \xi_2)\Psi(\xi_1, \xi_2)$$

and Ψ is as in (1.3) with $d = 2n$ (see [17, Section 4]). Then, by duality, it is enough to prove

$$\|T_{m^{*1}}\|_{L^{p'}(w^{1-p'}) \times L^{p_2}(w) \to L^{p_1'}(w^{1-p_1'})} \le C \sup_{k\in\mathbb{Z}} \|(m^{*1})_k\|_{H^s}. \tag{5.3}$$

Indeed, since $L^{p'}(w^{1-p'})$ is the dual space of $L^p(w)$ under the duality

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x) g(x) \, dx \qquad \text{for } f \in L^p(w) \text{ and } g \in L^{p'}(w^{1-p'}),$$

we have by (5.1)-(5.3)

$$\begin{aligned}
\|T_m(f_1, f_2)\|_{L^p(w)} &= \sup |\langle T_m(f_1, f_2), g\rangle| = \sup |\langle T_{m^{*1}}(g, f_2), f_1\rangle| \\
&\le \sup \|T_{m^{*1}}(g, f_2)\|_{L^{p_1'}(w^{1-p_1'})} \|f_1\|_{L^{p_1}(w)} \\
&\le C \sup \left(\sup_{k\in\mathbb{Z}} \|(m^{*1})_k\|_{H^s} \right) \|f_1\|_{L^{p_1}(w)} \|f_2\|_{L^{p_2}(w)} \|g\|_{L^{p'}(w^{1-p'})} \\
&\le C \left(\sup_{k\in\mathbb{Z}} \|m_k\|_{H^s} \right) \|f_1\|_{L^{p_1}(w)} \|f_2\|_{L^{p_2}(w)}
\end{aligned}$$

for all $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$, where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ with $\|g\|_{L^{p'}(w^{1-p'})} = 1$.

Note that $1/p' + 1/p_2 = 1/p_1'$ and

$$w^{(1-p_1')/p_1'} = w^{-1/p_1} = w^{-1/p} w^{1/p_2}. \tag{5.4}$$

Since $p < p_1$ and $w^{1-p'} \in A_{p's/(2n)} \subset A_{p'}$, it follows from Lemma 2.2 (1) that $w \in A_p \subset A_{p_1}$, and consequently

$$w^{1-p_1'} \in A_{p_1'}. \tag{5.5}$$

The assumption $p_1 \le p_2$ gives $1/p = 1/p_1 + 1/p_2 \ge 2/p_2$, that is, $p \le p_2/2$. Thus, using $w^{1-p'} \in A_{p'}$ and $s/(2n) > 1/2$, we have by Lemma 2.2 (1)

$$w \in A_p \subset A_{p_2/2} \subset A_{p_2 s/(2n)}. \tag{5.6}$$

Take $q > 1$ satisfying $2n/s < q < \min\{2, p', p_2\}$. Moreover, since $w^{1-p'} \in A_{p's/(2n)}$ and $w \in A_{p_2 s/(2n)}$, by Lemma 2.2 (2), q can be chosen so close to $2n/s$ that

$$w^{1-p'} \in A_{p'/q} \qquad \text{and} \qquad w \in A_{p_2/q}. \tag{5.7}$$

Let us prove (5.3). By Lemma 2.4, Hölder's inequality, (4.6), (5.4) and (5.5),

$$\begin{aligned}
&\|T_{(m^{*1})_{(2,1)}}(f_1, f_2)\|_{L^{p_1'}(w^{1-p_1'})} \\
&\le C \left\| \left(\sum_{j\in\mathbb{Z}} |\psi(D/2^j) T_{(m^{*1})_{(2,1)}}(f_1, f_2)|^2 \right)^{1/2} \right\|_{L^{p_1'}(w^{1-p_1'})} \\
&\le C \sum_{k=-2}^{2} \left(\sup_{j\in\mathbb{Z}} \|(m^{*1})^{j,k}_{(2,1)}\|_{H^s} \right) \\
&\qquad \times \left\| \left\{ \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^j) f_1|^q)^{2/q} \right)^{1/2} w^{-1/p} \right\} \left\{ M(|f_2|^q)^{1/q} w^{1/p_2} \right\} \right\|_{L^{p_1'}(\mathbb{R}^n)} \\
&\le C \sum_{k=-2}^{2} \left(\sup_{j\in\mathbb{Z}} \|(m^{*1})^{j,k}_{(2,1)}\|_{H^s} \right) \\
&\qquad \times \left\| \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^j) f_1|^q)^{2/q} \right)^{1/2} \right\|_{L^{p'}(w^{1-p'})} \left\| M(|f_2|^q)^{1/q} \right\|_{L^{p_2}(w)},
\end{aligned}$$

where $(m^{*1})^{j,k}_{(2,1)}$ is defined by (4.2) and (4.5) with m replaced by m^{*1}. Therefore, since $1 < 2/q, p'/q, p_2/q < \infty$, it follows from (4.7), (5.7), Lemmas 2.3 and 2.4 that

$$\begin{aligned}
&\|T_{(m^{*1})_{(2,1)}}(f_1, f_2)\|_{L^{p_1'}(w^{1-p_1'})} \\
&\qquad \le C \left(\sup_{k\in\mathbb{Z}} \|(m^{*1})_k\|_{H^s} \right) \|f_1\|_{L^{p'}(w^{1-p'})} \|f_2\|_{L^{p_2}(w)}.
\end{aligned} \tag{5.8}$$

In the same way, we can prove

$$\begin{aligned}
&\|T_{(m^{*1})_{(1,2)}}(f_1, f_2)\|_{L^{p_1'}(w^{1-p_1'})} \\
&\qquad \le C \left(\sup_{k\in\mathbb{Z}} \|(m^{*1})_k\|_{H^s} \right) \|f_1\|_{L^{p'}(w^{1-p'})} \|f_2\|_{L^{p_2}(w)}.
\end{aligned} \tag{5.9}$$

On the other hand, by (4.12), (5.4) and Hölder's inequality,

$$\begin{aligned}
&\|T_{(m^{*1})_{(2,2)}}(f_1, f_2)\|_{L^{p_1'}(w^{1-p_1'})} \\
&\qquad \le C \left(\sup_{k\in\mathbb{Z}} \|(m^{*1})^{k}_{(2,2)}\|_{H^s} \right) \left\| \left(\sum_{k\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^k) f_1|^q)^{2/q} \right)^{1/2} \right\|_{L^{p'}(w^{1-p'})} \\
&\qquad\qquad \times \left\| \left(\sum_{k\in\mathbb{Z}} M(|\widetilde{\zeta}(D/2^k) f_2|^q)^{2/q} \right)^{1/2} \right\|_{L^{p_2}(w)},
\end{aligned}$$

where $(m^{*1})^k_{(2,2)}$ is defined by (4.2) and (4.11). In a way similar to $(m^{*1})_{(2,1)}$, we have

$$\|T_{(m^{*1})_{(2,2)}}(f_1,f_2)\|_{L^{p_1'}(w^{1-p_1'})} \le C\left(\sup_{k\in\mathbb{Z}}\|(m^{*1})_k\|_{H^s}\right)\|f_1\|_{L^{p'}(w^{1-p'})}\|f_2\|_{L^{p_2}(w)}. \tag{5.10}$$

Theorem 1.2 with $N=2$ under assumption (ii) now follows from (4.2), (5.8)-(5.10).

In the case $p_2=\min\{p_1,p_2\}$, using m^{*2} instead of m^{*1}, we can prove Theorem 1.2 with $N=2$ under assumption (ii). □

We end this section by giving a sketch of the proof of Theorem 1.2 with $N\ge 3$ under assumption (ii), and shall only indicate the necessary modifications.

Let $1<p_1,\dots,p_N,p<\infty$, $1/p_1+\dots+1/p_N=1/p$, $Nn/2<s\le Nn$, $\min\{p_1,\dots,p_N\}<(Nn/s)'$ and $w^{1-p'}\in A_{p's/(Nn)}$. We consider the case $p_1=\min\{p_1,\dots,p_N\}$. To obtain the boundedness of T_m from $L^{p_1}(w)\times L^{p_2}(w)\times\dots\times L^{p_N}(w)$ to $L^p(w)$, it is enough to prove the boundedness of $T_{m^{*1}}$ from $L^{p'}(w^{1-p'})\times L^{p_2}(w)\times\dots\times L^{p_N}(w)$ to $L^{p_1'}(w^{1-p_1'})$. Since $p_1<(Nn/s)'$, we see that $1/p',1/p_k<1/p'+1/p_2+\dots+1/p_N=1/p_1'<s/(Nn)$ for $2\le k\le N$. Hence, $Nn/s<\min\{2,p',p_2,\dots,p_N\}$. In the same way as in (5.4) and (5.5), we can prove

$$w^{(1-p_1')/p_1'}=w^{-1/p}w^{1/p_2}\dots w^{1/p_N}$$

and $w^{1-p_1'}\in A_{p_1'}$. Since $p_1\le p_k$ for $2\le k\le N$, we see that

$$1/p=1/p_1+\dots+1/p_k+\dots+1/p_N>1/p_1+1/p_k\ge 2/p_k,$$

that is, $p<p_k/2$ for $2\le k\le N$. Then, $w\in A_p\subset A_{p_k/2}\subset A_{p_ks/(Nn)}$ for $2\le k\le N$ (see (5.6) for the bilinear case). Therefore, we can take $q>1$ satisfying $Nn/s<q<\min\{2,p',p_2,\dots,p_N\}$, $w^{1-p'}\in A_{p'/q}$ and $w\in A_{p_k/q}$ for $2\le k\le N$. The rest of the proof is similar to the bilinear case (see also the proof of Theorem 1.2 with $N\ge 3$ under assumption (i)), and we omit it.

6. Improvement of Theorem 1.2 under assumption (i)

Let $(s_1,\dots,s_N)\in\mathbb{R}^N$. Then the Sobolev space of product type $H^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})$ consists of all $F\in\mathcal{S}'(\mathbb{R}^{Nn})$ such that

$$\|F\|_{H^{(s_1,\dots,s_N)}}=\left(\int_{\mathbb{R}^{Nn}}(1+|\xi_1|^2)^{s_1}\dots(1+|\xi_N|^2)^{s_N}|\widehat{F}(\xi)|^2\,d\xi_1\dots d\xi_N\right)^{1/2}<\infty,$$

where $\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^n\times\dots\times\mathbb{R}^n$. In order to replace H^s appearing in Theorem 1.2 by $H^{(s_1,\dots,s_N)}$, we use the following lemma instead of Lemma 3.2 (see Appendix A for its proof):

Lemma 6.1. *Let $r>0$, $s_j>n/2$ and $\max\{1,n/s_j\}<q_j<2$ for $1\le j\le N$. Then there exists a constant $C>0$ such that*

$$\left|\int_{\mathbb{R}^{Nn}}2^{Njn}\mathcal{F}^{-1}m(2^j(x-y_1),\dots,2^j(x-y_N))f_1(y_1)\dots f_N(y_N)\,dy_1\dots dy_N\right| \le C\|m\|_{H^{(s_1,\dots,s_N)}}M(|f_1|^{q_1})(x)^{1/q_1}\dots M(|f_N|^{q_N})(x)^{1/q_N}$$

for all $j\in\mathbb{Z}$, $m\in H^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})$ with $\operatorname{supp} m\subset\{\sqrt{|\xi_1|^2+\dots+|\xi_N|^2}\le r\}$ and $f_1,\dots,f_N\in\mathcal{S}(\mathbb{R}^n)$.

We improve Theorem 1.2 under assumption (i) as follows:

Theorem 6.2. *Let $1 < p_1, \dots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$ and $n/2 < s_j \le n$ for $1 \le j \le N$. Assume*

$$p_j > n/s_j \quad \text{and} \quad w_j \in A_{p_j s_j/n} \quad \text{for} \quad 1 \le j \le N.$$

If $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies

$$\sup_{k\in\mathbb{Z}} \|m_k\|_{H^{(s_1,s_2,\dots,s_N)}(\mathbb{R}^{Nn})} < \infty,$$

then T_m is bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2) \times \cdots \times L^{p_N}(w_N)$ to $L^p(w)$, where m_k is defined by (1.4) and $w = w_1^{p/p_1} w_2^{p/p_2} \dots w_N^{p/p_N}$.

Taking $w_j = w$ and $s_j = s/N$ for $1 \le j \le N$, we have Theorem 1.2 under assumption (i) as a corollary of Theorem 6.2. Here we note that if $s \ge 0$, then

$$H^s(\mathbb{R}^{Nn}) \hookrightarrow H^{(s/N,\dots,s/N)}(\mathbb{R}^{Nn}).$$

Let us give a sketch of the proof of Theorem 6.2 with $N = 2$, because the other cases can be proved in a similar way. We use the notation in the proof of Theorem 1.2 under assumption (i), and the proof is a slight modification of it except we use Lemma 6.1 instead of Lemma 3.2.

For example, let us consider $m_{(2,1)}$, because the other terms can be proved in a similar way. Since $n/s_j < \min\{2, p_j\}$ and $w_j \in A_{p_j s_j/n}$ for $1 \le j \le 2$, by Lemma 2.2 (2), we can take $n/s_j < q_j < \min\{2, p_j\}$ satisfying $w_j \in A_{p_j/q_j}$ for $1 \le j \le 2$. Using Lemma 6.1 instead of Lemma 3.2, we can prove

$$\begin{aligned}
&|\psi(D/2^j) T_{m_{(2,1)}}(f_1, f_2)(x)| \\
&\quad \le C \sum_{k=-2}^{2} \|m_{(2,1)}^{j,k}\|_{H^{(s_1,s_2)}} M(|\widetilde{\psi}(D/2^{j+k}) f_1|^{q_1})(x)^{1/q_1} M(|f_2|^{q_2})(x)^{1/q_2}
\end{aligned}$$

in the same way as in (4.4)-(4.6). Note that $w_1 \in A_{p_1}$ and $w_2 \in A_{p_2}$ imply $w = w_1^{p/p_1} w_2^{p/p_2} \in A_{2p}$ ([14, pp. 1232-1233]). Hence, since $w \in A_\infty$, it follows from Remark 2.6 that

$$\begin{aligned}
\|T_{m_{(2,1)}}(f_1, f_2)\|_{L^p(w)} &\le C \left\| \left(\sum_{j\in\mathbb{Z}} |\psi(D/2^j) T_{m_{(2,1)}}(f_1, f_2)|^2 \right)^{1/2} \right\|_{L^p(w)} \\
&\le C \sum_{k=-2}^{2} \left(\sup_{j\in\mathbb{Z}} \|m_{(2,1)}^{j,k}\|_{H^{(s_1,s_2)}} \right) \\
&\quad \times \left\| \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^{j+k}) f_1|^{q_1})^{2/q_1} \right)^{1/2} M(|f_2|^{q_2})^{1/q_2} \right\|_{L^p(w)}.
\end{aligned}$$

Then, by Hölder's inequality,

$$\begin{aligned}
&\left\| \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^{j+k})f_1|^{q_1})^{2/q_1} \right)^{1/2} M(|f_2|^{q_2})^{1/q_2} \right\|_{L^p(w)} \\
&= \left\| \left\{ \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^{j})f_1|^{q_1})^{2/q_1} \right)^{1/2} w_1^{1/p_1} \right\} \left\{ M(|f_2|^{q_2})^{1/q_2} w_2^{1/p_2} \right\} \right\|_{L^p(\mathbb{R}^n)} \\
&\le \left\| \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^{j})f_1|^{q_1})^{2/q_1} \right)^{1/2} \right\|_{L^{p_1}(w_1)} \left\| M(|f_2|^{q_2})^{1/q_2} \right\|_{L^{p_2}(w_2)}.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 1.2 under assumption (i), but note that $H^{(s_1,s_2)}(\mathbb{R}^{2n})$ is a multiplication algebra when $s_j > n/2$ for $1 \le j \le 2$ (see Proposition A.2 in Appendix A).

Appendix A

Let $(q_1,\dots,q_N) \in [1,\infty)^N$ and $(s_1,\dots,s_N) \in \mathbb{R}^N$. The weighted Lebesgue space of mixed type $L^{(q_1,\dots,q_N)}(w_{(s_1,\dots,s_N)})$ is defined by the norm

$$\begin{aligned}
&\|F\|_{L^{(q_1,\dots,q_N)}(w_{(s_1,\dots,s_N)})} \\
&= \Big[\int_{\mathbb{R}^n} \dots \Big\{ \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |F(x)|^{q_1} \langle x_1\rangle^{s_1} dx_1 \Big)^{q_2/q_1} \langle x_2\rangle^{s_2} dx_2 \Big\}^{q_3/q_2} \dots \langle x_N\rangle^{s_N} dx_N \Big]^{1/q_N},
\end{aligned}$$

where $x = (x_1,\dots,x_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$ and $\langle x_j\rangle = (1+|x_j|^2)^{1/2}$. In order to prove Lemma 6.1, we use the following lemma instead of Lemma 2.5:

Lemma A.1. *Let $r > 0$, $2 \le q_j < \infty$ and $s_j \ge 0$ for $1 \le j \le N$. Then there exists a constant $C > 0$ such that*

$$\|\widehat{F}\|_{L^{(q_1,\dots,q_N)}(w_{(s_1,\dots,s_N)})} \le C\|F\|_{H^{(s_1/q_1,\dots,s_N/q_N)}}$$

for all $F \in H^{(s_1/q_1,\dots,s_N/q_N)}(\mathbb{R}^{Nn})$ with $\operatorname{supp} F \subset \{\sqrt{|x_1|^2+\dots+|x_N|^2} \le r\}$.

Proof. We consider only the case $N = 2$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\phi = 1$ on $\{y \in \mathbb{R}^n : |y| \le r\}$ and $\operatorname{supp}\phi \subset \{y \in \mathbb{R}^n : |y| \le 2r\}$. Since $\operatorname{supp} F \subset \{|x_1| \le r, |x_2| \le r\}$, we have $F(x_1,x_2) = \phi(x_1)\phi(x_2)F(x_1,x_2)$, and consequently $\widehat{F}(\xi_1,\xi_2) = (2\pi)^{-2n}(\widehat{\phi}\otimes\widehat{\phi}) * \widehat{F}(\xi_1,\xi_2)$. By Minkowski's inequality for integrals,

$$\begin{aligned}
&\|\widehat{F}\|_{L^{(q_1,q_2)}(w_{(s_1,s_2)})} \\
&= \frac{1}{(2\pi)^{2n}} \left\| \left\| \int_{\mathbb{R}^{2n}} \widehat{\phi}(\xi_1-\eta_1)\widehat{\phi}(\xi_2-\eta_2)\widehat{F}(\eta_1,\eta_2)\, d\eta_1 d\eta_2 \right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})} \right\|_{L^{q_2}(\langle\xi_2\rangle^{s_2})} \\
&\le \frac{1}{(2\pi)^{2n}} \left\| \int_{\mathbb{R}^n} \left\| \int_{\mathbb{R}^n} \widehat{\phi}(\xi_1-\eta_1)\widehat{F}(\eta_1,\eta_2)\, d\eta_1 \right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})} |\widehat{\phi}(\xi_2-\eta_2)|\, d\eta_2 \right\|_{L^{q_2}(\langle\xi_2\rangle^{s_2})}.
\end{aligned}$$

By Schwarz's inequality and Young's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\widehat{\phi}(\xi_1-\eta_1)\widehat{F}(\eta_1,\eta_2)\,d\eta_1\right|^{q_1}\langle\xi_1\rangle^{s_1}\,d\xi_1\\
&\le C\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\langle\xi_1-\eta_1\rangle^{s_1/q_1}|\widehat{\phi}(\xi_1-\eta_1)|\langle\eta_1\rangle^{s_1/q_1}|\widehat{F}(\eta_1,\eta_2)|\,d\eta_1\right)^{q_1}d\xi_1\\
&\le C\left(\sup_{\xi_1\in\mathbb{R}^n}\int_{\mathbb{R}^n}\langle\xi_1-\eta_1\rangle^{s_1/q_1}|\widehat{\phi}(\xi_1-\eta_1)|\langle\eta_1\rangle^{s_1/q_1}|\widehat{F}(\eta_1,\eta_2)|\,d\eta_1\right)^{q_1-2}\\
&\qquad\times\left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\langle\xi_1-\eta_1\rangle^{s_1/q_1}|\widehat{\phi}(\xi_1-\eta_1)|\langle\eta_1\rangle^{s_1/q_1}|\widehat{F}(\eta_1,\eta_2)|\,d\eta_1\right)^{2}d\xi_1\right\}\\
&\le C\|\langle\xi_1\rangle^{s_1/q_1}\widehat{\phi}\|_{L^1}^2\|\langle\xi_1\rangle^{s_1/q_1}\widehat{\phi}\|_{L^2}^{q_1-2}\left(\int_{\mathbb{R}^n}\langle\xi_1\rangle^{2s_1/q_1}|\widehat{F}(\xi_1,\eta_2)|^2\,d\xi_1\right)^{q_1/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^n}\left\|\int_{\mathbb{R}^n}\widehat{\phi}(\xi_1-\eta_1)\widehat{F}(\eta_1,\eta_2)\,d\eta_1\right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})}|\widehat{\phi}(\xi_2-\eta_2)|\,d\eta_2\right\|_{L^{q_2}(\langle\xi_2\rangle^{s_2})}\\
&\le C\left\|\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\langle\xi_1\rangle^{2s_1/q_1}|\widehat{F}(\xi_1,\eta_2)|^2\,d\xi_1\right)^{1/2}|\widehat{\phi}(\xi_2-\eta_2)|\,d\eta_2\right\|_{L^{q_2}(\langle\xi_2\rangle^{s_2})}.
\end{aligned}$$

By the same argument,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\langle\xi_1\rangle^{2s_1/q_1}|\widehat{F}(\xi_1,\eta_2)|^2\,d\xi_1\right)^{1/2}|\widehat{\phi}(\xi_2-\eta_2)|\,d\eta_2\right\}^{q_2}\langle\xi_2\rangle^{s_2}\,d\xi_2\\
&\le C\|\langle\xi_2\rangle^{s_2/q_2}\widehat{\phi}\|_{L^1}^2\|\langle\xi_2\rangle^{s_2/q_2}\widehat{\phi}\|_{L^2}^{q_2-2}\\
&\qquad\times\left(\int_{\mathbb{R}^{2n}}\langle\xi_1\rangle^{2s_1/q_1}\langle\xi_2\rangle^{2s_2/q_2}|\widehat{F}(\xi_1,\xi_2)|^2\,d\xi_1d\xi_2\right)^{q_2/2},
\end{aligned}$$

and this completes the proof with $N=2$. □

Proof of Lemma 6.1. Using Lemma A.1 instead of Lemma 2.5 and

$$(1+2^j|x-y_1|)^{s_1}\dots(1+2^j|x-y_N|)^{s_N}$$

instead of $(1+2^j|x-y_1|+\dots+2^j|x-y_N|)^s$, we can prove Lemma 6.1 in the same way as in the proof of Lemma 3.2. □

The following proposition seems to be known to many people, but we shall give a proof for the reader's convenience.

Proposition A.2. *If $s_j>n/2$ for $1\le j\le N$, then $H^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})$ is a multiplication algebra.*

Proof. We consider only the case $N=2$. Since

$$\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}\le C\left\{\langle\xi_1-\eta_1\rangle^{s_1}+\langle\eta_1\rangle^{s_1}\right\}\left\{\langle\xi_2-\eta_2\rangle^{s_2}+\langle\eta_2\rangle^{s_2}\right\},$$

we see that

$$\begin{aligned}\|FG\|_{H^{(s_1,s_2)}} &= \frac{1}{(2\pi)^{2n}}\left\|\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}\int_{\mathbb{R}^{2n}}\widehat{F}(\xi_1-\eta_1,\xi_2-\eta_2)\widehat{G}(\eta_1,\eta_2)\,d\eta_1 d\eta_2\right\|_{L^2}\\ &\le C\sum_{i_1,i_2=0}^{1}\|\widehat{F}_{(i_1,i_2)}*\widehat{G}_{(1-i_1,1-i_2)}\|_{L^2},\end{aligned}$$

where $\widehat{F}_{(0,0)}(\xi_1,\xi_2)=|\widehat{F}(\xi_1,\xi_2)|$, $\widehat{F}_{(1,0)}(\xi_1,\xi_2)=\langle\xi_1\rangle^{s_1}|\widehat{F}(\xi_1,\xi_2)|$, $\widehat{F}_{(0,1)}(\xi_1,\xi_2)=\langle\xi_2\rangle^{s_2}|\widehat{F}(\xi_1,\xi_2)|$ and $\widehat{F}_{(1,1)}(\xi_1,\xi_2)=\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}|\widehat{F}(\xi_1,\xi_2)|$. It is not difficult to estimate $\widehat{F}_{(1,1)}*\widehat{G}_{(0,0)}$ and $\widehat{F}_{(0,0)}*\widehat{G}_{(1,1)}$. In fact, since $s_1,s_2>n/2$, by Young's inequality and Schwarz's inequality,

$$\begin{aligned}\|\widehat{F}_{(1,1)}*\widehat{G}_{(0,0)}\|_{L^2} &\le \|\widehat{F}_{(1,1)}\|_{L^2}\|\widehat{G}_{(0,0)}\|_{L^1}=\|F\|_{H^{(s_1,s_2)}}\|\widehat{G}\|_{L^1}\\ &\le C\|F\|_{H^{(s_1,s_2)}}\|\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}\widehat{G}\|_{L^2}=C\|F\|_{H^{(s_1,s_2)}}\|G\|_{H^{(s_1,s_2)}}.\end{aligned}$$

Let us consider $\widehat{F}_{(1,0)}*\widehat{G}_{(0,1)}$ and $\widehat{F}_{(0,1)}*\widehat{G}_{(1,0)}$. By Minkowski's inequality for integrals and Young's inequality,

$$\begin{aligned}&\left\|\int_{\mathbb{R}^{2n}}\widehat{F}_{(1,0)}(\xi_1-\eta_1,\xi_2-\eta_2)\widehat{G}_{(0,1)}(\eta_1,\eta_2)\,d\eta_1 d\eta_2\right\|_{L^2(\mathbb{R}^{2n}_{\xi_1,\xi_2})}\\ &\le\left\|\int_{\mathbb{R}^n}\left\|\int_{\mathbb{R}^n}\widehat{F}_{(1,0)}(\xi_1-\eta_1,\xi_2-\eta_2)\widehat{G}_{(0,1)}(\eta_1,\eta_2)\,d\eta_1\right\|_{L^2(\mathbb{R}^n_{\xi_1})}d\eta_2\right\|_{L^2(\mathbb{R}^n_{\xi_2})}\\ &\le\left\|\int_{\mathbb{R}^n}\left\|\widehat{F}_{(1,0)}(\xi_1,\xi_2-\eta_2)\right\|_{L^2(\mathbb{R}^n_{\xi_1})}\left\|\widehat{G}_{(0,1)}(\xi_1,\eta_2)\right\|_{L^1(\mathbb{R}^n_{\xi_1})}d\eta_2\right\|_{L^2(\mathbb{R}^n_{\xi_2})}\\ &\le\left\|\left\|\widehat{F}_{(1,0)}(\xi_1,\xi_2)\right\|_{L^2(\mathbb{R}^n_{\xi_1})}\right\|_{L^1(\mathbb{R}^n_{\xi_2})}\left\|\left\|\widehat{G}_{(0,1)}(\xi_1,\xi_2)\right\|_{L^1(\mathbb{R}^n_{\xi_1})}\right\|_{L^2(\mathbb{R}^n_{\xi_2})}.\end{aligned}$$

Then, by Schwarz's inequality,

$$\begin{aligned}&\Big\{\int\Big(\int\widehat{F}_{(1,0)}(\xi_1,\xi_2)^2\,d\xi_1\Big)^{1/2}d\xi_2\Big\}\Big\{\int\Big(\int\widehat{G}_{(0,1)}(\xi_1,\xi_2)\,d\xi_1\Big)^2d\xi_2\Big\}^{1/2}\\ &\le C\Big(\iint\langle\xi_2\rangle^{2s_2}\widehat{F}_{(1,0)}(\xi_1,\xi_2)^2\,d\xi_1 d\xi_2\Big)^{1/2}\Big(\iint\langle\xi_1\rangle^{2s_1}\widehat{G}_{(0,1)}(\xi_1,\xi_2)^2\,d\xi_1 d\xi_2\Big)^{1/2}.\end{aligned}$$

Therefore, $\|\widehat{F}_{(1,0)}*\widehat{G}_{(0,1)}\|_{L^2}\le C\|F\|_{H^{(s_1,s_2)}}\|G\|_{H^{(s_1,s_2)}}$. In the same way, we can estimate $\widehat{F}_{(0,1)}*\widehat{G}_{(1,0)}$. □

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