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COFINALITY AND MEASURABILITY OF THE FIRST THREE UNCOUNTABLE CARDINALS

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Abstract. This paper discusses models of set theory without the Axiom of Choice. We investigate all possible patterns of the cofinality function and the distribution of measurability on the first three uncountable cardinals. The result relies heavily on a strengthening of an unpublished result of Kechris: we prove (under AD) that there is a cardinal κ such that the triple $(\kappa, \kappa^+, \kappa^{++})$ satisfies the strong polarized partition property.

1. Introduction

In ZFC, small cardinals such as $\aleph_1$, $\aleph_2$, and $\aleph_3$ cannot be measurable, as measurability implies strong inaccessibility; they cannot be singular either, as successor cardinals are always regular. So, in ZFC, these three cardinals are non-measurable regular cardinals. But both of the mentioned results use the Axiom of Choice, and there are many known situations in set theory where these small cardinals are either singular or measurable: in the Feferman-Lévy model, $\aleph_1$ has countable cofinality (cf. [Jec03, Example 15.57]), in the model constructed independently by Jech and Takeuti, $\aleph_1$ is measurable (cf. [Jec03, Theorem 21.16]), and in models of AD, both $\aleph_1$ and $\aleph_2$ are measurable and $\mathrm{cf}(\aleph_3) = \aleph_2$ (cf. [Kan94, Theorem 28.2, Theorem 28.6, and Corollary 28.8]). Simple adaptations of the Feferman-Lévy and Jech and Takeuti arguments show that one can also make $\aleph_2$ or $\aleph_3$ singular or measurable, but is it possible to control these properties simultaneously for the three cardinals $\aleph_1$, $\aleph_2$ and $\aleph_3$?

In this paper, we investigate all possible patterns of measurability and cofinality for the three mentioned cardinals. Combinatorially, there are exactly 60 such patterns of which 13 are impossible for trivial reasons (e.g., if $\aleph_1$ is singular, then $\aleph_2$ cannot have cofinality $\aleph_1$). In this paper, we prove that the remaining 47 patterns are all consistent relative to large cardinals.

Our 47 consistency results will be proved by reducing all cases to eight base cases from which the other patterns can be obtained by standard methods. Three of the base cases will be proved consistent by techniques from forcing, and five of them will require the existence of a model of AD. In particular, we will be using the existence of a polarized partition property under AD which generalizes an

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unpublished theorem of Kechris from the 1980s (cf. [AH86, p. 600]). This polarized partition property is of independent interest and its proof will take up the larger part of this paper.

We begin by giving some basic definitions and terminology (§§ 2.1 and 2.2). We then present a proof of (a slightly stronger version of) Kechris' result (§ 2.3) and generalize it to higher exponents (§§ 2.4 and 2.5).

We then move towards our application of the polarized partition property. We start by listing some basic tools for forcing in the ZF context, some of them using both polarized and ordinary partition properties, in §§ 3.1 and 3.2. These tools will allow us to reduce the 47 consistent patterns to eight base cases in § 3.3. In §§ 3.4 and 3.5, we prove the consistency of all base cases. We use consequences of the polarized partition property established in § 2.5 in our applications in § 3.5. Finally, in § 4, we summarize upper and lower consistency strength bounds of all 60 patterns and discuss open questions.

Our notation is mostly standard. We will make frequent use of coding and decoding maps on ω and ω^ω, especially in § 2. We fix a recursive bijection $i \mapsto (i_0, i_1)$ from ω to ω^2 and its inverse function $(i, j) \mapsto \langle i, j\rangle$ (and similarly, for n-tuples). For $x \in \omega^\omega$, we define $(x)_i(j) := x(\langle i, j\rangle)$.

2. Polarized partition properties under determinacy

2.1. **Definitions.** Fix a strictly increasing triple $(\kappa_0, \kappa_1, \kappa_2)$ of cardinals and an ordinal $\delta \leq \kappa_0$. We say a function $f\colon 3 \times \delta \to \mathrm{On}$ is a **block function** if $\kappa_{i-1} < f(i, \alpha) < \kappa_i$ for $i \in 3$ (and $\kappa_{-1} := 0$), and we say it is **increasing** if $f(i, \alpha) < f(i, \beta)$ whenever $\alpha < \beta$. We denote the set of increasing block functions by IBF_δ. If $\vec{H} = (H_0, H_1, H_2)$ is a triple such that $H_i \subseteq \kappa_i$ (for $i \in 3$), we define a subset $F_{\vec{H},\delta} \subseteq \mathrm{IBF}_\delta$ by

$$f \in F_{\vec{H},\delta} :\Longleftrightarrow \text{ for all } \alpha \in \delta \text{ and } i \in 3, \text{ we have } f(i, \alpha) \in H_i.$$

If $P \subseteq \mathrm{IBF}_\delta$ is a partition of all increasing block functions into two disjoint sets, we call a triple $\vec{H}$ **δ-homogeneous for** P if either $F_{\vec{H},\delta} \subseteq P$ or $F_{\vec{H},\delta} \cap P = \varnothing$. For Ramsey-type partition properties, we also define the set $\mathrm{IBF}_{<\delta} := \bigcup_{\alpha<\delta} \mathrm{IBF}_\alpha$, and for a partition $P \subseteq \mathrm{IBF}_{<\delta}$, we say that a triple $\vec{H}$ is **$<\delta$-homogeneous for** P if for all $\alpha < \delta$, either $F_{\vec{H},\alpha} \subseteq P$ or $F_{\vec{H},\alpha} \cap P = \varnothing$.

Definition 1. The **polarized partition property**

$$(\kappa_0, \kappa_1, \kappa_2) \to (\kappa_0, \kappa_1, \kappa_2)^\delta$$

is the statement that for every partition P, there is a δ-homogeneous triple $\vec{H}$ with $|H_i| = \kappa_i$. If $\delta = \kappa_0$, we call it the **strong polarized partition property**.[1] The **Ramsey-type polarized partition property**

$$(\kappa_0, \kappa_1, \kappa_2) \to (\kappa_0, \kappa_1, \kappa_2)^{<\delta}$$

is the statement that for every partition P, there is a $<\delta$-homogeneous triple $\vec{H}$ with $|H_i| = \kappa_i$. Polarized partition-properties with pairs of cardinals instead of triples are defined analogously.[2]

[1] Note that this terminology differs from that of [AHJ00].

[2] Note that as in [AHJ00, Definition 4.11], these definitions are equivalent to the partition theoretic ones found in [AHJ00, Definition 4.1].

As is the case for ordinary partition relations, the polarized partition property is equivalent to a c.u.b. version. A block function f is said to be of **uniform cofinality** ω if there is a function $g\colon 3\times\delta\times\omega\to\mathrm{On}$ such that $f(i,\alpha)=\sup\{g(i,\alpha,n)\,;\,n\in\omega\}$, and g is strictly increasing in the last argument. We say that a block function f is **of the correct type** if it is increasing, everywhere discontinuous, and of uniform cofinality ω. We write CTF_δ for the functions of the correct type. If $P\subseteq\mathrm{CTF}_\delta$, we call a triple $\vec{H}$ **δ-c.u.b.-homogeneous** if either $F_{\vec{H},\delta}\cap\mathrm{CTF}_\delta\subseteq P$ or $F_{\vec{H},\delta}\cap P=\varnothing$.

Definition 2. We say $(\kappa_0,\kappa_1,\kappa_2)\xrightarrow{\text{c.u.b.}}(\kappa_0,\kappa_1,\kappa_2)^\delta$ if for every partition $P\subseteq\mathrm{CTF}_\delta$, there is a triple $\vec{C}=(C_0,C_1,C_2)$ such that C_i is a closed unbounded set in κ_i (for $i\in 3$) and $\vec{C}$ is δ-c.u.b.-homogeneous.

Fact 3. For any $\delta\le\kappa_0<\kappa_1<\kappa_2$, we have that $(\kappa_0,\kappa_1,\kappa_2)\xrightarrow{\text{c.u.b.}}(\kappa_0,\kappa_1,\kappa_1)^\delta$ implies $(\kappa_0,\kappa_1,\kappa_2)\to(\kappa_0,\kappa_1,\kappa_1)^\delta$. Also, $(\kappa_0,\kappa_1,\kappa_2)\to(\kappa_0,\kappa_1,\kappa_1)^{\omega\cdot\delta}$ implies $(\kappa_0,\kappa_1,\kappa_2)\xrightarrow{\text{c.u.b.}}(\kappa_0,\kappa_1,\kappa_1)^\delta$.

Proof. The easy argument can be found in [Jac08, Lemma 3.3]. □

2.2. **The axiom of determinacy and Suslin cardinals.** Let us recall some basic definitions from descriptive set theory. By a **boldface pointclass $\mathbf{\Gamma}$** we mean a collection of sets of reals closed under continuous preimages. For a pointclass $\mathbf{\Gamma}$ we let $\breve{\mathbf{\Gamma}}$ denote the **dual pointclass** $\breve{\mathbf{\Gamma}}:=\{A\,;\,\omega^\omega\setminus A\in\mathbf{\Gamma}\}$. A pointclass $\mathbf{\Gamma}$ is called **selfdual** if $\mathbf{\Gamma}=\breve{\mathbf{\Gamma}}$, and **non-selfdual** otherwise. If $\mathbf{\Gamma}$ is non-selfdual, we can define $\mathbf{\Delta}:=\mathbf{\Gamma}\cap\breve{\mathbf{\Gamma}}$. We say that a non-selfdual pointclass $\mathbf{\Gamma}$ has the **separation property** (in symbols: $\mathrm{Sep}(\mathbf{\Gamma})$) if any two disjoint sets in $\mathbf{\Gamma}$ can be separated by a set in $\mathbf{\Delta}$. In general, at most one of $\mathbf{\Gamma}$ and $\breve{\mathbf{\Gamma}}$ can have the separation property. In [Ste81b], Steel proved that AD implies that one of the two does. From now on in this section, we shall assume AD.

The class of pairs of non-selfdual pointclasses $(\mathbf{\Gamma},\breve{\mathbf{\Gamma}})$ (i.e., $\mathbf{\Gamma}\neq\breve{\mathbf{\Gamma}}$) such that one of them is closed under $\exists^{\omega^\omega}$ has order type under the Wadge ordering

$$\Theta:=\sup\{\alpha;\ \text{there is a surjection from } \omega^\omega \text{ onto } \alpha\}.$$

We call these the **Lévy** pointclasses. Let $(\mathbf{\Gamma}_\alpha,\breve{\mathbf{\Gamma}}_\alpha)$ be the αth such pair. If one of them is not closed under $\forall^{\omega^\omega}$, the other one is. If this is the case, we let $\mathbf{\Sigma}^1_\alpha$ be the one that isn't. If both of them are closed under $\forall^{\omega^\omega}$, let $\mathbf{\Sigma}^1_\alpha$ be the one with the separation property. As usual, $\mathbf{\Pi}^1_\alpha:=(\mathbf{\Sigma}^1_\alpha)\breve{}$. In [KSS81, §4], the authors proposed a classification of these pointclasses. They fall into four types of "projective-like hierarchies" which are distinguished by the closure properties of the pointclass at the base of the hierarchy (this is recalled after Proposition 6 below). For example, if $\mathrm{cf}(\alpha)=\omega$, then $\mathbf{\Sigma}^1_\alpha$ is at the base of a type I hierarchy. In this case, $\mathbf{\Sigma}^1_\alpha$ is the collection of sets which can written as a countable union of sets each of which is in $\mathbf{\Sigma}^1_\beta$ for some $\beta<\alpha$. These pointclasses play a particularly important role in the arguments of §§ 2.3–2.5.

As usual, a set of reals A is called **λ-Suslin** if there is a tree $T\subseteq(\omega\times\lambda)^{<\omega}$ such that $A=\mathrm{p}[T]:=\{x\,;\,\exists y\in\lambda^\omega((x,y)\in[T])\}$. We write $\mathbf{S}(\lambda)$ for the pointclass of all λ-Suslin sets. These pointclasses are closed under $\exists^{\omega^\omega}$, and thus show up in our list mentioned in the last paragraph. A cardinal κ is called a **Suslin cardinal** if $\mathbf{S}(\kappa)\setminus\bigcup_{\lambda<\kappa}\mathbf{S}(\lambda)\neq\varnothing$.

We give the well-known Kunen-Martin theorem (cf. [Kec78, Theorem 3.11]) with its proof, as the general idea of this proof will be used repeatedly in our results of §§2.3–2.5.

Theorem 4 (Kunen-Martin)**.** *Let $\prec$ be a κ-Suslin well-founded relation on ω^ω. Then the rank of $\prec$ is less than κ^+.*

Proof. Let T be a tree on $\omega \times \omega \times \kappa$ with $\prec = \mathrm{p}[T]$. Let U be the well-founded tree consisting of finite $\prec$-decreasing sequences $(x_0, \dots, x_n)$, that is, $x_n \prec \cdots \prec x_1 \prec x_0$. It is easy to see that $\prec$ and U have the same rank. To each $\vec{x} = (x_0, \dots, x_n) \in U$, assign $\pi(\vec{x}) = (x_0 \restriction n+1, \dots, x_n \restriction n+1, \ell(x_1, x_0) \restriction n+1, \dots \ell(x_n, x_{n-1}) \restriction n+1)$, where $\ell(y, z) \in \kappa^\omega$ is the leftmost branch of $T_{y,z}$. If $\vec{y}$ extends $\vec{x}$, we view $\pi(\vec{y})$ as extending $\pi(\vec{x})$ in a natural way. The map π is order-preserving from U into a well-founded relation on a set in bijection with κ. Thus the rank of $\prec$ must be less than κ^+. □

In [Ste83, Theorem 4.3], Steel identifies (assuming $\mathbf{V} = \mathbf{L}(\mathbb{R})$) the pointclasses $\mathbf{S}(\kappa)$ in the list of $\mathbf{\Sigma}^1_\alpha$s and $\mathbf{\Pi}^1_\alpha$s and calculates their Suslin cardinals. For instance, if $\kappa^\mathbb{R}$ is the least non-hyperprojective ordinal,[3] we have $\mathbf{S}(\kappa^\mathbb{R}) = \mathbf{\Pi}^1_{\kappa^\mathbb{R}} = \mathbf{IND}$, and $\kappa^\mathbb{R}$ is a Suslin cardinal as witnessed by the inductive sets. If V is an arbitrary model of AD, then V contains $\mathbf{L}(\mathbb{R})^V$ to which the Steel analysis applies. Moreover, $\kappa < \Theta^{\mathbf{L}(\mathbb{R})^V}$ is a Suslin cardinal in $\mathbf{L}(\mathbb{R})^V$ if and only if κ is a Suslin cardinal in V.[4]

Proposition 5. *If AD holds, then there are weakly inaccessible Suslin cardinals.*

Proof. As just mentioned, Steel's analysis of scales in $\mathbf{L}(\mathbb{R})$ shows that $\kappa^\mathbb{R}$ is a Suslin cardinal in $\mathbf{L}(\mathbb{R})$, and thus in V. In [KKMW81, Theorem 3.1], the authors show that it is in fact weakly Mahlo. Note that $\kappa^\mathbb{R}$ is by no means the only (or smallest) weakly inaccessible Suslin cardinal (cf. [Ste81a, Theorem 3.1]). □

By the work of [KKMW81] mentioned in the proof of Proposition 5, the analysis of the scale property of pointclasses is closely connected to partition properties. Our results from §2.4 and §2.5 can be seen as an extension of this work. In the following overview, we follow [Jac08, pp. 295–297]:

For a selfdual pointclass $\mathbf{\Delta}$, we let $\mathrm{o}(\mathbf{\Delta}) := \sup\{|A|_\mathrm{W}\,;\, A \in \mathbf{\Gamma}\}$ and $\delta(\mathbf{\Delta}) := \sup\{\alpha\,;\, \text{there is a } \mathbf{\Delta}\text{-prewellordering of length } \alpha\}$. Note that (under AD) if $\mathbf{\Delta}$ is closed under $\exists^{\omega^\omega}$ and finite intersections, then $\mathrm{o}(\mathbf{\Delta}) = \delta(\mathbf{\Delta})$ [KSS81, Theorem 2.3.1]. Let us fix the increasing enumeration of all Suslin cardinals $\langle \kappa_\alpha\,;\, \alpha < \Xi \rangle$. Note that $\mathsf{ZF} + \mathsf{AD}$ does not fix the value of Ξ: the Suslin cardinals could be unbounded below Θ (in this case, every set has a scale and thus by a result of Woodin [Kan94, Theorem 32.23], $\mathsf{AD}_\mathbb{R}$ holds) or there could be a largest Suslin cardinal. It is enough for the results of this paper to consider the Suslin cardinals in $\mathbf{L}(\mathbb{R})$, i.e., $\langle \kappa_\alpha\,;\, \alpha \leq (\boldsymbol{\delta}^2_1)^{\mathbf{L}(\mathbb{R})} \rangle$. Here [Ste83] gives a complete analysis of the Suslin cardinals. We note though that the main partition result we prove in § 2.5 only uses AD.

We recall some facts about the classification of projective-like hierarchies and how this pertains to Suslin cardinals. The facts we review below are sufficient for the results of this paper. The reader can consult [Ste81a] and [Ste83] for more

[3] Here, $\mathbf{IND}$ is the pointclass of inductive sets, and the pointclass $\mathbf{HYP} := \mathbf{IND} \cap \mathbf{IND}^\smile$ is the class of hyperprojective sets. All three mentioned pointclasses are closed under $\exists^{\omega^\omega}$ and $\forall^{\omega^\omega}$, and we have $\mathrm{Sep}(\mathbf{IND}^\smile)$.

[4] If κ is a Suslin cardinal of $\mathbf{L}(\mathbb{R})^V$, then the scale which witnesses a new Suslin representation is trivially also in V. The other direction follows from the coding lemma.

details. We note that the latter paper assumes $\mathbf{V} = \mathbf{L}(\mathbb{R})$. Although we don't need it for the results of this paper, [Jac09] presents the theory of the Suslin cardinals from just AD.

Following [Ste81a] consider

$$D := \{\mathrm{o}(\mathbf{\Delta})\,;\, \mathbf{\Delta} \text{ is selfdual and closed under } \wedge,\ \exists^{\omega^\omega}\}.$$

Clearly, D consists of limit ordinals and is closed unbounded in Θ. Each $\alpha \in D$ corresponds to the base of a projective-like hierarchy. If $\mathrm{cf}(\alpha) = \omega$, this is called a type I hierarchy. In this case the Wadge degree of rank α is selfdual and consists of a countable join of sets of lower Wadge rank. We let $\mathbf{\Sigma}^\alpha_0$ in this case be the collection of countable unions of sets of Wadge rank below α. We let $\mathbf{\Pi}^\alpha_0$ be the dual class, and define $\mathbf{\Sigma}^\alpha_n$, $\mathbf{\Pi}^\alpha_n$ for $n > 0$ as usual. This defines the projective-like hierarchy. In this case, $\mathbf{\Sigma}^\alpha_0$, $\mathbf{\Pi}^\alpha_1$, etc. have the prewellordering property. These classes will be particularly important for the arguments of §§2.3–2.5. If $\mathrm{cf}(\alpha) > \omega$, the Wadge pair $(\mathbf{\Gamma}, \breve{\mathbf{\Gamma}})$ of rank α is non-selfdual. By [Ste81a], exactly one of $\mathbf{\Gamma}$, $\breve{\mathbf{\Gamma}}$, say $\breve{\mathbf{\Gamma}}$, has the separation property, and this class is closed under $\exists^{\omega^\omega}$. If this pointclass is not also closed under $\forall^{\omega^\omega}$ we are in type II if $\mathbf{\Gamma}$ is not closed under $\vee$ and in type III if $\mathbf{\Gamma}$ is closed under $\vee$. We call $\mathbf{\Gamma}$ in these cases the **Steel pointclass** at the base of the hierarchy. If $\mathbf{\Gamma}$ is closed under real quantification, then we are in type IV (in this case the projective-like hierarchy is built up by applying quantifiers to $\mathbf{\Gamma} \wedge \breve{\mathbf{\Gamma}}$). This analysis of the projective-like hierarchies does not depend on the scale property and assumes just AD.

We now specialize to the Suslin cardinals and Suslin pointclasses. We say the λth Suslin cardinal κ_λ is a limit Suslin cardinal if λ is a limit ordinal, and otherwise a successor Suslin cardinal (so a successor Suslin cardinal may be a limit cardinal). First we recall that [Ste83] shows that the Suslin cardinals form a closed set in $\mathbf{L}(\mathbb{R})$ (which as we mentioned above has largest element $(\boldsymbol{\delta}^2_1)^{\mathbf{L}(\mathbb{R})}$). More generally, Steel and Woodin have shown that the Suslin cardinals are closed below Θ assuming AD^+, and closed below their supremum assuming just AD. So we have:

Proposition 6. *If λ is a limit ordinal, then κ_λ is a limit of Suslin cardinals.*

If $\kappa = \kappa_\lambda$ is a limit Suslin cardinal, then $\mathbf{\Delta} := \bigcup_{\rho<\kappa} \mathbf{S}(\rho)$ is selfdual and closed under $\wedge$, $\exists^{\omega^\omega}$, and so $\mathrm{o}(\mathbf{\Delta}) \in D$. As discussed above, $\mathbf{\Delta}$ sits at the base of a projective-like hierarchy in one of four possible types. In [Ste83], Steel identifies the pointclasses $\mathbf{S}(\kappa)$ among the $\mathbf{\Sigma}^1_\alpha$ and $\mathbf{\Pi}^1_\alpha$ and in fact determines the scaled Lévy classes among the $\mathbf{\Sigma}^1_\alpha$, $\mathbf{\Pi}^1_\alpha$ (assuming $\mathbf{V} = \mathbf{L}(\mathbb{R})$). We recall some of the consequences in terms of the possible hierarchy types. In all cases, $\kappa = \mathrm{o}(\mathbf{\Delta}) = \delta(\mathbf{\Delta})$.

- **Type I:** In this case $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda) = \omega$. If we let, as above, $\mathbf{\Sigma}^\alpha_0$ (where $\alpha = \mathrm{o}(\mathbf{\Delta}) = \kappa$) be the collection of countable unions of sets in $\mathbf{\Delta}$, then $\mathbf{\Sigma}^\alpha_0$, $\mathbf{\Pi}^\alpha_1$, etc. have the scale property. κ^+ is a Suslin cardinal, and a $\mathbf{\Pi}^\alpha_1$ scale on a $\mathbf{\Pi}^\alpha_1$-complete set has norms of length κ^+. We have $\mathbf{S}(\kappa) = \mathbf{\Sigma}^\alpha_1$ and $\mathbf{S}(\kappa^+) = \mathbf{\Sigma}^\alpha_2$.
- **Type II or III:** In this case, let $\mathbf{\Gamma}$ be the Steel class defined above. So, $\mathbf{\Delta} = \mathbf{\Gamma} \cap \breve{\mathbf{\Gamma}}$, and $\mathbf{\Gamma}$ is closed under $\wedge$, $\forall^{\omega^\omega}$. Then $\mathbf{S}(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$, and $\mathrm{Scale}(\mathbf{\Gamma})$, $\mathrm{Scale}(\mathbf{S}(\kappa))$ hold.

Type IV: In this case, the pointclasses $\mathbf{\Gamma}$, $\breve{\mathbf{\Gamma}}$ of Wadge degree κ are closed under real quantification. Let $\mathbf{\Gamma}$ be such that $\breve{\mathbf{\Gamma}}$ has the separation property. Then Scale($\mathbf{\Gamma}$) holds, and $\mathbf{S}(\kappa) = \mathbf{\Gamma}$.

If κ is a regular limit Suslin cardinal, then [Ste81a, Theorem 2.1] shows that $\mathbf{\Gamma}$ (as in the above hierarchy descriptions) is closed under $\vee$. Thus, we are in type III or IV. Finally, the analysis of [Ste83, Theorem 4.3] shows that a successor Suslin cardinal is either a successor cardinal or has cofinality ω. Thus, a weakly inaccessible Suslin cardinal κ_λ must be a limit Suslin cardinal (and so $\lambda = \kappa$).

Summarizing, our inaccessible Suslin cardinal κ is a limit of Suslin cardinals, and $\mathbf{S}(\kappa)$ has the scale property. In fact, $\mathbf{S}(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is a non-selfdual pointclass with $\mathrm{o}(\mathbf{\Gamma}) = \kappa$, $\mathbf{\Gamma}$ is closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$, and Scale($\mathbf{\Gamma}$). It is possible that $\mathbf{\Gamma} = \mathbf{S}(\kappa)$ if we are in the case of a Type IV hierarchy. We again note that these results can be obtained from just AD (cf. [Jac09]).

We fix a $\mathbf{\Gamma}$-complete set P (which exists by Wadge's Lemma for all non-selfdual pointclasses under AD) and let $\{\varphi_n\}_{n\in\omega}$ be a (regular) $\mathbf{\Gamma}$-scale on P. An inspection of the standard argument shows that we have the following boundedness condition (as $\mathbf{\Gamma}$ is a boldface pointclass with the prewellordering property and closed under $\forall^{\omega^\omega}$ and finite unions): any $\mathbf{\Delta} = \mathbf{\Gamma} \cap \breve{\mathbf{\Gamma}}$ subset A of P is bounded in the codes, that is, $\sup\{\varphi_x(x) : n \in \omega, x \in A\} < \kappa$.

Our results from §§2.4 and 2.5 have to be understood in the context of proofs of partition properties for $\delta(\mathbf{\Delta})$ for highly closed pointclasses. For instance, consider the following example theorem as listed in [Jac08, Theorem 3.10]:

Theorem 7. *Let $\mathbf{\Gamma}$ be non-selfdual, closed under $\forall^{\omega^\omega}$ and finite unions, and with the prewellordering property. Define $\mathbf{\Delta} := \mathbf{\Gamma} \cap \breve{\mathbf{\Gamma}}$. If $\exists^{\omega^\omega}\mathbf{\Delta} \subseteq \mathbf{\Delta}$, then $\delta(\mathbf{\Delta})$ has the strong partition property.*

Note that if κ is an inaccessible Suslin cardinal and $\mathbf{\Gamma}$ is the pointclass defined as above, then $\mathbf{\Gamma}$ satisfies all of the requirements of Theorem 7, and therefore $\delta(\mathbf{\Delta}) = \kappa$ has the strong partition property. Our results are extensions of this observation.

The fact that κ has the strong partition property immediately implies that the ω-cofinal measure $\mu := \mathcal{C}^\omega_\kappa$ on κ is a normal ultrafilter (cf. [Kle70, Theorem 2.1]).

Finally, we recall one more result, due to Martin (cf. [Kec78, Theorem 3.7]) in the AD theory of pointclasses which will be used frequently later. For the sake of completeness, we sketch the proof.

Theorem 8 (Martin)**.** *Let $\mathbf{\Gamma}$ be a non-selfdual pointclass closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$, and assume pwo($\mathbf{\Gamma}$). Let $\delta = \delta(\mathbf{\Delta})$ (where $\mathbf{\Delta} = \mathbf{\Gamma} \cap \breve{\mathbf{\Gamma}}$). Then $\mathbf{\Delta}$ is closed under unions and intersections of length $< \delta$.*

Proof. Assume the contrary, and let $\rho < \delta$ be least such that there exists some ρ union, say $A = \bigcup_{\alpha<\rho} A_\alpha$, of sets in $\mathbf{\Delta}$ that is not in $\mathbf{\Delta}$. Easily, ρ is regular. We may assume the A_α are strictly increasing. Since $\rho < \delta$, there is a $\mathbf{\Delta}$-prewellordering of length ρ. The coding lemma then shows that $A \in \breve{\mathbf{\Gamma}}$ (since $\breve{\mathbf{\Gamma}}$ is closed under $\exists^{\omega^\omega}$). By assumption, $A \in \breve{\mathbf{\Gamma}} - \mathbf{\Delta}$. Define a norm φ of length ρ on A by $\rho(x) =$ least α such that $x \in A_\alpha$. To see that φ is a $\breve{\mathbf{\Gamma}}$-norm, notice that the corresponding norm relation $<^*$ can be written as $x <^* y \leftrightarrow \exists\alpha < \rho\ (x \in A_\alpha \wedge y \notin A_\alpha)$, a ρ-union of $\mathbf{\Delta}$ sets which is therefore in $\breve{\mathbf{\Gamma}}$. A similar computation works for $\leq^*$, showing φ is a $\breve{\mathbf{\Gamma}}$-prewellordering on A. This shows that $\breve{\mathbf{\Gamma}}$ has the prewellordering property since A is $\breve{\mathbf{\Gamma}}$-complete and so every $\breve{\mathbf{\Gamma}}$ set is a ρ-union of $\mathbf{\Delta}$ sets. This is

a contradiction since for pointclasses $\mathbf{\Gamma}_0$ closed under $\wedge$, $\vee$ we have the chain of implications $\mathrm{pwo}(\mathbf{\Gamma}_0) \Rightarrow \mathrm{Red}(\mathbf{\Gamma}_0) \Rightarrow \mathrm{Sep}(\breve{\mathbf{\Gamma}}_0)$ and the separation property cannot hold on both sides of a non-selfdual pointclass. $\square$

2.3. **Kechris' Theorem.** In this section, we shall prove Kechris' theorem, announced in the 1980s, but not published. The proofs of our extensions of this theorem in §§2.4 and 2.5 build on this proof and will use definitions from this section.

Theorem 9. *Assume* AD *and let* κ *be a weakly inaccessible Suslin cardinal. Then for all* $\vartheta < \omega_1$ *we have* $(\kappa, \kappa^+, \kappa^{++}) \to (\kappa, \kappa^+, \kappa^{++})^\vartheta$.

Throughout this section, κ will be a weakly inaccessible Suslin cardinal (which exists by Proposition 5). Note that by Fact 3 it doesn't matter whether we are using the standard or the c.u.b. version of the partition property, and we shall freely switch between them.

Partition property proofs under AD always follow the same lines as developed by Tony Martin (cf. [Kec78, Lemma 11.1] and [Jac09, Theorem 2.3.4]): to show $\kappa \to \kappa^\lambda$ we must find a sufficiently good coding of the functions $f\colon \lambda \to \kappa$. This involves identifying a Lévy pointclass $\mathbf{\Gamma}$ and a coding map $\varphi\colon \omega^\omega \to \mathcal{P}(\lambda \times \kappa)$ with certain coding relations being in $\mathbf{\Delta}$. In this paper we shall use Martin's method directly, so the reader need not be familiar with these general results.

In our setting, $\mathbf{\Gamma}$ is the (Steel) pointclass forming the lowest level of the projective-like hierarchy containing $\mathbf{S}(\kappa)$. We have seen in §2.2 that this pointclass has the required properties: $\mathbf{S}(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$ has the scale property. The pointclass $\mathbf{\Gamma}$ (possibly $\mathbf{\Gamma} = \mathbf{S}(\kappa)$) is scaled, non-selfdual, closed under $\forall^{\omega^\omega}$ and finite intersections and unions. We fixed a $\mathbf{\Gamma}$-complete set P and a regular $\mathbf{\Gamma}$-scale $\{\varphi_n\}_{n\in\omega}$ on P which allows boundedness arguments. In the following, P and φ will be used to code ordinals less than κ. Since we also want to code higher ordinals, we shall have to come up with a means of coding for these (in §2.3.2).

By μ, we denote the ω-cofinal measure on κ (which is an ultrafilter by Theorem 7). We shall show that $[\alpha \mapsto \alpha^+]_\mu = \kappa^+$ and that $\delta := [\alpha \mapsto \alpha^{++}]_\mu = \kappa^{++}$. The first claim can be proved directly (Lemma 14), after which we shall show the following auxiliary theorem:

Theorem 10. $(\kappa, \kappa^+, \delta) \to (\kappa, \kappa^+, \delta)^\vartheta$ *for all* $\vartheta < \omega_1$.

It follows immediately from Theorem 10 that $\delta \to (\delta)^\vartheta$ for all $\vartheta < \omega_1$. In particular, δ is regular. By showing that $\kappa^+ < \delta \leq \kappa^{++}$ (Claim 19), we establish that $\delta = \kappa^{++}$, thus proving Theorem 9.

2.3.1. *Countable unions of* $<\alpha$*-Suslin sets.* An ordinal $\alpha < \kappa$ is called **$\vec{\varphi}$-strongly reliable** if for all $\beta < \alpha$, we have $\sup\{\varphi_n(x)\,;\, n \in \omega \wedge \varphi_0(x) \leq \beta\} < \alpha$. Let $C \subseteq \kappa$ be a c.u.b. set contained in the $\vec{\varphi}$-strongly reliable ordinals. Without loss of generality, we may assume that C is contained in the Suslin cardinals. The relation $R(x, y) \leftrightarrow x, y \in P \wedge \varphi_0(x) \leq \varphi_0(y)$ is in $\mathbf{\Gamma}$, and so admits a $\mathbf{\Gamma}$-scale $\vec{\sigma}$ (with norms into κ). By boundedness, we may assume C has the property that for all $\alpha \in C$ and $\beta < \alpha$, if $R(x, y)$ and $\varphi_0(x) \leq \varphi_0(y) \leq \beta$, then $\sup_n \sigma_n(x, y) < \alpha$. Let C_ω denote the elements of C of cofinality ω.

As in §2.2, for $\alpha \in C_\omega$, let $\mathbf{\Sigma}^\alpha_0$ denote the pointclass of countable unions of sets which are in $\bigcup\{\mathbf{S}(\beta)\,;\, \beta < \alpha\}$. Thus, $\mathrm{Scale}(\mathbf{\Sigma}^\alpha_0)$ holds. Define $\mathbf{\Sigma}^\alpha_n$, $\mathbf{\Pi}^\alpha_n$ from $\mathbf{\Sigma}^\alpha_0$ as usual. Then $\mathrm{Scale}(\mathbf{\Pi}^\alpha_1)$ holds, and $\mathbf{\Sigma}^\alpha_1$ is the pointclass of α-Suslin sets. From

the Coding Lemma and the Kunen-Martin Theorem 4 it follows that α^+ is the supremum of the lengths of the $\mathbf{\Sigma}^\alpha_1$ prewellorderings, and is the supremum of the lengths of the $\mathbf{\Sigma}^\alpha_1$ well-founded relations. In particular, α^+ is regular.

The pointclass $\mathbf{\Sigma}^\alpha_1$ behaves sufficiently similar to $\mathbf{\Sigma}^1_1$ to allow proving the following result (by essentially the same argument as is used in the countable partition property of $\aleph_1$; cf. [Kec78, Theorem 11.2]):

Theorem 11. *For all $\vartheta < \omega_1$, we have $\alpha^+ \to (\alpha^+)^\vartheta$.*

Applying [Kle70, Theorem 2.1] again, we get that the ω-cofinal normal measure $\mu_\alpha := \mathcal{C}^\omega_{\alpha^+}$ on α^+ is an ultrafilter. We shall use this measure in Lemma 18 and its proof.

As we shall need to do arguments about α uniformly, we check in the next result that the assignment of scales and universal sets is uniform:

Lemma 12. *There is a function $\alpha \mapsto (A^\alpha, \vec{\rho}^\alpha)$ which assigns to each $\alpha \in C_\omega$ a universal $\mathbf{\Sigma}^\alpha_0$ set A^α and a $\mathbf{\Sigma}^\alpha_0$ scale $\vec{\rho}^\alpha = \{\rho^\alpha_n\}_{n\in\omega}$ on A^α with norms into α. Furthermore, there is a function $\alpha \mapsto (B^\alpha, V^\alpha)$ which assigns to such an α a universal $\mathbf{\Sigma}^\alpha_1$ set B^α and a tree V^α on $\omega \times \alpha$ with $B^\alpha = \mathrm{p}[V^\alpha]$. Finally, there is a function $\alpha \mapsto (Q^\alpha, \vec{\psi}^\alpha)$ which assigns to such an α a universal $\mathbf{\Pi}^\alpha_1$ set and a $\mathbf{\Pi}^\alpha_1$-scale $\vec{\psi}$ on Q^α.*

Proof. For $\alpha \in C_\omega$, let $R^\alpha = \{\langle x, y\rangle\,;\, x, y \in P \wedge \varphi_0(x) \leq \varphi_0(y) < \alpha\}$. From the definition of C, R^α can be written as an α union of sets each of which is $<\alpha$-Suslin. Thus, $R^\alpha \in \mathbf{\Sigma}^\alpha_0$. Since R^α is a prewellordering of length α, it cannot be $<\alpha$-Suslin. Define $A^\alpha = \{\langle \tau, z\rangle\,;\, \exists n((\tau(z))_n \in R^\alpha)\}$, where we view every real τ as a strategy for player II in an integer game in some standard manner, and $\tau(z)$ is the result of applying τ to z. Clearly $A^\alpha \in \mathbf{\Sigma}^\alpha_0$. Moreover, since R^α has Wadge degree at least α, it follows easily that A^α is $\mathbf{\Sigma}^\alpha_0$-universal. We define, uniformly in α, a tree U^α with $A^\alpha = \mathrm{p}[U^\alpha]$. Define $(s, (\alpha_0, \ldots, \alpha_n)) \in U^\alpha$ if and only if

(i) $\alpha_0 > \max\{\alpha_1, \ldots, \alpha_m\}$, and $\alpha_0 < \alpha$.
(ii) $\alpha_1 \in \omega$.
(iii) There is a $\langle \tau, z\rangle \in \omega^\omega$ extending s such that if $(\tau(z))_{\alpha_1} = \langle x, y\rangle$, then $\sigma_i(x, y) = \alpha_{i+2}$ for all $i \leq n-2$ ($\vec{\sigma}$ is the scale on R as above).

From the closure properties of C it follows that $A^\alpha = \mathrm{p}[U^\alpha]$. Then let $\{\bar{\rho}^\alpha_n\}$ be the semi-scale derived from the Suslin representation U^α, and let $\{\rho^\alpha_n\}$ be the corresponding scale. Since each $\bar{\rho}^\alpha_n$ maps into α, so does ρ^α_n, using property (i) in the definition of U_α. [In passing from the semi-scale to the scale we can take $\rho^\alpha_n(w) = |(\bar{\rho}^\alpha_0(w), \ldots, \bar{\rho}^\alpha_n(w))|_\prec$, where $\prec$ is lexicographic ordering on the set of $n+1$ tuples satisfying (i).] To see that $\vec{\rho}^\alpha$ is a $\mathbf{\Sigma}^\alpha_0$-scale, it is enough to show that the semi-scale $\{\bar{\rho}^\alpha_n\}$ is a $\mathbf{\Sigma}^\alpha_0$ semi-scale, since $\mathbf{\Sigma}^\alpha_0$ is closed under $\wedge, \vee$. However, each of the norm relations $<^*_n$, $\leq^*_n$ corresponding to the norm $\bar{\rho}^\alpha_n$ can be written as an α union of $<\alpha$-Suslin sets. For example, for $<^*_0$ we have: $z <^*_0 w$ if and only if there is a $\beta < \alpha$ such that $(U^\alpha{\restriction}\beta)_z$ is ill-founded and $(U^\alpha{\restriction}\beta)_w$ is well-founded. Since $U^\alpha{\restriction}\beta$ and its complement are $<\alpha$-Suslin (since α is a limit of Suslin cardinals), the claim follows.

Define B^α by $B^\alpha(\langle \tau, z\rangle)$ if and only if there is a w such that for all n, we have $R^\alpha(\tau(z, w, n))$. Since $\mathbf{\Sigma}^\alpha_1$ is closed under $\exists^{\omega^\omega}$ and countable unions and intersections, $B^\alpha \in \mathbf{\Sigma}^\alpha_1$. Since R^α has Wadge degree at least α, it is easy to check that B^α is universal for $\mathbf{\Sigma}^\alpha_1$. The tree U^α projecting to A^α easily gives a tree V^α projecting

to B^α (as in the proof that Suslin representations are closed under $\exists^{\omega^\omega}$ and $\forall^\omega$). Finally, we can define Q^α by $Q^\alpha(w)$ if and only if for all z, we have $A^\alpha(\langle w, z\rangle)$, and use periodicity to transfer the $\mathbf{\Sigma}^\alpha_0$ scale on A^α to a $\mathbf{\Pi}^\alpha_1$ scale on Q^α. $\square$

2.3.2. *Coding of ordinals below κ, κ^+ and δ.* The coding of elements of κ is completely standard: A real x will code an ordinal below κ if and only if $x \in P$. In this case, x codes the ordinal $|x| = \varphi_0(x)$. We let $P_0 = P$ be the set of codes of ordinals below κ.

In order to code ordinals less than κ^+, we need a tree T^+ on $\omega \times \kappa$ which we shall use in our coding of the ordinals. For the definition of T^+, we need a number of auxiliary objects: W, T_2, and U.

Let $W = \{w \in \omega^\omega\,;\, \forall n\ (w)_n \in P\}$. Define the norm ψ on W by $\psi(w) = \sup\{\varphi_0((w)_n)\,;\, n \in \omega\}$. It is easy to see that ψ is a $\mathbf{\Gamma}$-norm on the set $W \in \mathbf{\Gamma}$. If we define a tree T_2 on $\omega \times \kappa$ by $(s, (\alpha_0, \dots, \alpha_n)) \in T_2$ if and only if there is a w extending s such that

$$w \in W \text{ and for all } i \le n, \text{ we have } (\varphi_{i_0}((w)_{i_1}) = \alpha_i),$$

then $\mathrm{p}[T_2] = W$. For $\alpha \in C$ with $\mathrm{cf}(\alpha) = \omega$ we also have that $\mathrm{p}[T_2 \restriction \alpha] = W_\alpha := \{w \in W\,;\, \psi(w) \le \alpha\}$.

Furthermore, we define a tree U on $(\omega)^4 \times \kappa \times \kappa$ as follows (we recycle the notation here; U has nothing to do with the trees U^α above). As a motivation, it is helpful to think of the first two coordinates of U in the definition that follows as defining reals x, y with $x \in W$ and y defining a $\mathbf{\Sigma}^{\psi(x)}_1$ relation via the universal set $B^{\psi(x)}$ from Lemma 12. Define $(s, t, u, v, \vec{\alpha}, \vec{\beta}) \in U$ if and only if there are $x, y, z, w \in \omega^\omega$ extending s, t, u, v such that:

(i) z codes the reals $z_0, z_1, \dots$, w codes $w_0, w_1, \dots$, and for each $i, n \in \omega$ the subsequence $\gamma_j = \alpha_{\langle i,n,j\rangle}$ of the $\vec{\alpha}$ satisfies the following. View y as coding a Lipschitz integer strategy for player II, and set $r_i = \langle y(\langle z_i, z_{i+1}\rangle), w_i\rangle$. Let $b = (r_i)_n$. Then $(b, \vec{\gamma}) \in [T_{\vec{\sigma}}]$, where $T_{\vec{\sigma}}$ is the tree of the scale $\vec{\sigma}$ on R.

(ii) For each $i, n \in \omega$ we have (using the notation immediately above) if $\delta_j = \beta_{\langle i,n,j\rangle}$, then $\delta_0 \in \omega$ and $(\langle (b)_1, x_{\delta_0}\rangle, \vec{\delta}') \in [T_{\vec{\sigma}}]$, where $\vec{\delta}' = (\delta_1, \delta_2, \dots)$.

Let us explain the idea behind the definition of U: the objects w, $\vec{\alpha}$, $\vec{\beta}$ are attempting to witness that the $z_0, z_1, \dots$ form a decreasing sequence in the $\mathbf{\Sigma}^{\psi(x)}_1$ relation coded by y, as in the proof of the Kunen-Martin Theorem 4. The relation coded by y is the set of (c, d) such that there is a w such that for all n, we have $\langle y(\langle c, d\rangle), w\rangle_n = (e, f) \in R^{\psi(x)}$ (where R is as in the proof of Lemma 12). The ordinals $\vec{\beta}$ witness that the various f reals satisfy $\varphi_0(f) < \varphi_0((x)_n)$ for some n, and so $(e, f) \in R^{\psi(x)}$.

Lemma 13. *Suppose that $x \in W$, $\psi(x) \in C$, and y codes a well-founded relation A in $\mathbf{\Sigma}^{\psi(x)}_1$. Then $U_{x,y}$ is well-founded. Furthermore, $|U_{x,y} \restriction \psi(x)| \ge |A|$.*

Proof. If $(z, w, \vec{\alpha}, \vec{\beta}) \in [U_{x,y}]$, then for each i, $A(z_i, z_{i+1})$, where A is the $\mathbf{\Sigma}^{\psi(x)}_1$ relation coded by y: $A(c, d)$ if and only if there is a w such that for all n, we have $\langle y(\langle c, d\rangle), w\rangle_n = (e, f) \in R^{\psi(x)}$. The $\vec{\beta}$ witness that the various (e, f) are in $R^{\psi(x)}$. So, as in the Kunen-Martin Theorem 4, this produces an infinite decreasing chain through A, a contradiction.

For any c, d such that $A(c, d)$, we can find a w such that for all n, if $\langle y(\langle c, d\rangle), w\rangle_n = (e, f)$, then $\sup_j \sigma_j(e, f) < \psi(x)$ and $\sup_j \sigma_j(f, x_k) < \psi(x)$ for any k such that

$\varphi_0(x_k) > \varphi_0(f)$. This follows from the closure properties of C and the fact that $\psi(x) \in C$. The proof of the Kunen-Martin Theorem 4 then shows that $U_{x,y}{\restriction}\psi(x)$ has rank at least $|A|$. $\square$

We note that it is important for the following argument that for x, y as in the statement of Lemma 13 that the entire tree $U_{x,y}$ is well-founded (not just $U_{x,y}{\restriction}\psi(x)$).

Finally, we can now define the tree T^+ on $(\omega)^2 \times \kappa \times (\omega)^4 \times \kappa \times \kappa$ such that $(a, b, \vec{\gamma}, s, t, u, v, \vec{\alpha}, \vec{\beta}) \in T^+$ if and only if $(b, \vec{\gamma}) \in T_2$, $(s, t, u, v, \vec{\alpha}, \vec{\beta}) \in U$, and there are σ, r, x, y extending a, b, s, t, respectively, such that $\sigma(r) = \langle x, y\rangle$. We may identify T^+ with a tree on $\omega \times \kappa$ by identifying the last coordinates with a single coordinate (i.e., taking a reasonable bijection between $\omega \times \kappa \times (\omega)^4 \times \kappa \times \kappa$ and κ). We furthermore fix a reasonable bijection between κ and $\kappa^{<\omega}$. We assume (without loss of generality) that our c.u.b. set C is closed under both of these bijections.

Coding ordinals below κ^+. A code for an ordinal below κ^+ will be a pair (x, σ) where $x \in P$ and thus codes an ordinal $|x| = \varphi_0(x)$ below κ, and $\sigma \in \omega^\omega$ such that T^+_σ is well-founded. Using our bijection between κ and $\kappa^{<\omega}$, we can ask for the rank of an ordinal $\xi < \kappa$ in the tree T^+_σ; we write $|T^+_\sigma(\xi)|$ for this. Given a pair (x, σ), we now consider the map $f\colon \kappa \to \kappa$ defined by $\alpha \mapsto |(T^+_\sigma{\restriction}\alpha)(|x|)|$. Using the ω-cofinal normal measure $\mu := \mathcal{C}^\omega_\kappa$ on κ, we now define $|(x, \sigma)| := [f]_\mu$. We let P_1 be the set of codes of ordinals below κ^+. The next lemma shows that this works.

Lemma 14. $\{|(x, \sigma)|\,;\, (x, \sigma) \in P_1\} = \kappa^+$. *Also,* $\kappa^+ = [\alpha \mapsto \alpha^+]_\mu$.

Proof. Suppose $(x, \sigma) \in P_1$, and let $f = f_{x,\sigma}$ be given by $f(\alpha) = |T^+_\sigma{\restriction}\alpha(|x|)|$ for all $\alpha > |x|$. If $g\colon \kappa \to \kappa$ is such that $\forall^*_\mu\alpha\ g(\alpha) < f(\alpha)$, then $\forall^*_\mu\alpha\ \exists\beta < \alpha\ g(\alpha) = |T^+_\sigma{\restriction}\alpha(\beta)|$. By normality of μ we may fix $\beta < \kappa$, and fix $x' \in P_0$ coding β, such that $\forall^*_\mu\alpha\ g(\alpha) = |(T^+_\sigma{\restriction}\alpha)(|x'|)|$. So, $[g]_\mu = |(x', \sigma)|$. So, the ordinals coded by P_1 form an initial segment of the ordinals. This argument also shows that there is a map from $\kappa = [\alpha \mapsto \alpha]_\mu$ onto $|(x, \sigma)|$, namely $\beta \mapsto [\alpha \mapsto |T^+_\sigma{\restriction}\alpha(\beta)|]_\mu$. So, $\{|(x, \sigma)|\,;\, (x, \sigma) \in P_1\} \leq \kappa^+$.

If $\prec$ is a wellorder of κ, let $f_\prec$ be given by $f_\prec(\alpha) = |\prec{\restriction}\alpha| < \alpha^+$. If $|\prec| = |\prec'|$, then there is a c.u.b. subset of κ on which $f_\prec$ and $f_{\prec'}$ agree. Also, if $|\prec| < |\prec'|$, then there is a c.u.b. set on which $f_\prec(\alpha) < f_{\prec'}(\alpha)$. This gives an order-preserving map from κ^+ into $[\alpha \mapsto \alpha^+]_\mu$. So, $[\alpha \mapsto \alpha^+]_\mu \geq \kappa^+$.

Suppose $\gamma = [f]_\mu$, where $f(\alpha) < \alpha^+$ for all $\alpha < \kappa$. Consider the following game G_f:

Player I	$r(0)$		$r(1)$		$r(2)$		...
Player II		$x(0)$		$x(1)$		$x(2)$	...
		$y(0)$		$y(1)$		$y(2)$	...

where player I plays out a real $r \in \omega^\omega$ and player II plays out reals x, y. We interpret the real r as coding countably many reals $\{(r)_i\,;\, i \in \omega\}$ and x as coding $\{(x)_i\,;\, i \in \omega\}$. Let i be least, if it exists, such that $(r)_i \notin P_0$ or $(x)_i \notin P_0$. Player I loses if $r_i \notin P_0$. Assume then that $(r)_i, (x)_i \in P_0$ for all i. Thus, $x \in W$. Recall $\psi(x) = \sup_i |(x)_i|$. Player II then wins if and only if $\psi(x) \geq \psi(r)$ and y codes a $\mathbf{\Sigma}^{\psi(x)}_1$ well-founded relation A_y of rank $> f(\psi(x))$.

Player I cannot have a winning strategy σ, for suppose σ were winning for player I. Note that $\{(r)_0\,;\, r \in \sigma[\omega^\omega \times \omega^\omega]\} \subseteq P_0$. By boundedness, let $\alpha_0 \in C$ be such that $\alpha_0 > \sup\{|(r)_0|\,;\, r \in \sigma[\omega^\omega \times \omega^\omega]\}$. Fix a real $x_0 \in P_0$ with $|x_0| = \alpha_0$.

Note then that $\{r_1\,;\, r \in \sigma[\{(x,y)\,;\,(x)_0 = x_0\} \subseteq P_0$. By boundedness, fix $\alpha_1 > \alpha_0$ in C with $\alpha_1 > \sup\{|(r)_1|\,;\, r \in \sigma[\{(x,y)\,;\,(x)_0 = x_0\}$. Continuing, we define α_i and x_i for all i. Let $x \in \omega^\omega$ be the real with $(x)_i = x_i$ for all i. Let $\alpha = \sup_i \alpha_i$. Clearly, $\alpha \in C$. By the coding lemma, there is a $\mathbf{\Sigma}^\alpha_1$ well-founded relation, say coded by the real y, of length greater than $f(\alpha)$. If player II plays x and y, then player II defeats σ, a contradiction.

Now let σ be a winning strategy for player II in G_f. First note that T^+_σ is well-founded, for a branch $(r, \vec{\gamma}, x, y, z, w, \vec{\alpha}, \vec{\beta}) \in [T^+_\sigma]$ would give $r \in W$ (witnessed by $\vec{\gamma}$) and so $x \in W$ and $\psi(x) \geq \psi(r)$ as $\sigma(r) = \langle x, y\rangle$ and σ is winning for player II. Also, y codes a $\mathbf{\Sigma}^{\psi(x)}_1$ well-founded relation A_y of rank at least $f(\psi(x))$. z would however give a decreasing chain in A_y, a contradiction. Next note that there is a c.u.b. $D \subseteq \kappa$ such for all $\alpha \in D$ and $r \in W$ with $\psi(r) = \alpha$, if $\sigma(r) = \langle x, y\rangle$, then $\psi(x) = \psi(r)$. This follows by a boundedness argument similar to the above. We may assume $D \subseteq C$. For $\alpha \in D$ with $\mathrm{cf}(\alpha) = \omega$, let $r \in W$ with $\psi(r) = \alpha$. If $\sigma(r) = \langle x, y\rangle$, then $w \in W$ and $\psi(x) = \psi(r) = \alpha$. Thus, y codes a $\mathbf{\Sigma}^\alpha_1$ well-founded relation A_y of rank at least $f(\alpha)$. From Lemma 13 it follows that $|T^+_\sigma \restriction \alpha| > f(\alpha)$. So, $[f]_\mu \leq [\alpha \mapsto |T^+_\sigma \restriction \alpha|]_\mu$. It follows that for some $x \in P$ that $[f]_\mu = |(x, \sigma)|$. So, $[\alpha \mapsto \alpha^+]_\mu \leq \{|(x,\sigma)|\,;\,(x,\sigma) \in P_1\} \leq \kappa^+$. □

Coding ordinals below δ. We use a variation T^{++} of the tree T^+ above. The tree T^{++} will be defined exactly as T^+ except we use a tree U' in place of U. In order to define U', we need an auxiliary tree V_2. In Lemma 16, we shall construct a tree V which is then refined to V_2 in Lemma 17.

First, recall from Lemma 12 that there is a function $\alpha \mapsto (Q^\alpha, \vec{\psi}^\alpha)$, for $\alpha \in C_\omega$, where Q^α is a universal $\mathbf{\Pi}^\alpha_1$ set, and $\vec{\psi}^\alpha$ is a (regular) $\mathbf{\Pi}^\alpha_1$-scale on Q^α. Note that the norms ψ^α_n map into α^+, since α^+ is the supremum of the lengths of the $\mathbf{\Sigma}^\alpha_1$ well-founded relations.

We need the following technical lemma.

Lemma 15. *There is a continuous function $c\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ such that for all $x \in W$, $\beta \geq \psi(x)$, and $y \in \omega^\omega$ we have that $c(x,y) \in Q^\beta$ if and only if there is no z such that for all n, we have $B^{\psi(x)}(\langle y, \langle z_n, z_{n+1}\rangle\rangle)$. Here, $B^{\psi(x)}$ and Q^β are as in Lemma 12.*

Proof. Let $E(x,y)$ if and only if there is no z such that for all i, we have $B^{\psi(x)}(\langle y, \langle z_i, z_{i+1}\rangle\rangle)$. Then:

$$\begin{aligned} E(x,y) &\leftrightarrow \neg\exists z\ \forall i\ \exists w\ \forall n\ R^{\psi(x)}(y(\langle z_i, z_{i+1}\rangle, w, n)) \\ &\leftrightarrow \neg\exists u\ \forall m\ R^{\psi(x)}(y(\langle (u_0)_{m_0}, (u_0)_{m_0+1}\rangle, (u_1)_{m_0}, m_1)) \\ &\leftrightarrow \neg\exists \langle u, v\rangle\ \forall m\ (R(a,b) \wedge R(b, x_{v(m)})), \end{aligned}$$

where

$$\begin{aligned} a = a(y,u,m) &= (y(\langle (u_0)_{m_0}, (u_0)_{m_0+1}\rangle, (u_1)_{m_0}, m_1))_0, \\ b = b(y,u,m) &= (y(\langle (u_0)_{m_0}, (u_0)_{m_0+1}\rangle, (u_1)_{m_0}, m_1))_1. \end{aligned}$$

Here we use the fact that (for all j) $R(a,b)\wedge R(b,x_j)$ holds if and only if $R^{\psi(x)}(a,b)\wedge R^{\psi(x)}(b,x_j)$ if and only if $R^\beta(a,b)\wedge R^\beta(b,x_j)$. We therefore have

$$\begin{aligned} E(x,y) &\leftrightarrow \neg\exists t=\langle u,v\rangle\ \forall m\ (R^\beta(a,b)\wedge R^\beta(b,x_{v(m)})) \\ &\leftrightarrow \neg\exists t\ \forall m\ R^\beta(c_0(x,y)(c_1(x,y),t,m)) \\ &\leftrightarrow Q^\beta(c(x,y)), \end{aligned}$$

where $c(x,y)=\langle c_0(x,y),c_1(x,y)\rangle$ and c_0, c_1 are continuous functions such that $c_0(x,y)$ is a strategy for player II satisfying $c_0(x,y)(c_1(x,y),t,m)=\langle a(y,u,\frac{m}{2}), b(y,u,\frac{m}{2})\rangle$ if m is even and $c_0(x,y)(c_1(x,y),t,m)=\langle b(y,u,\frac{m-1}{2}),x_{v(m)}\rangle$ if m is odd. We may take $c_1(x,y)=\langle x,y\rangle$, and then easily get a continuous c_0 satisfying this equation. □

Lemma 16. *There is a tree V on $\omega\times\omega\times\kappa\times\kappa$ such that $(x,y)\in \mathrm{p}[V]$ if and only if $x\in W$ and y codes a $\mathbf{\Sigma}^{\psi(x)}_1$ well-founded relation. Furthermore, if $x\in W$, $\psi(x)\in C$, and y codes a well-founded $\mathbf{\Sigma}^{\psi(x)}_1$ relation, then there is a $\beta<\psi(x)^+$ such that $V_{x,y}\restriction\beta$ is ill-founded. In fact, for any $\alpha\in C$ with $\mathrm{cf}(\alpha)=\omega$, and any $A\in\mathbf{\Sigma}^\alpha_1$ consisting of pairs (x,y) such that $x\in W$, $\psi(x)\leq\alpha$, and y codes a well-founded $\mathbf{\Sigma}^{\psi(x)}_1$ relation, there is a $\beta<\alpha^+$ such that $A\subseteq \mathrm{p}[V\restriction\beta]$.*

Proof. Define $(s,t,\vec\alpha,\vec\beta)\in V$ if and only if

(i) $\beta_0\in C$ and $\beta_0>\max\{\vec\alpha,\vec\beta\}$.
(ii) $(s,\vec\alpha)\in T_2$.
(iii) There are x, y extending s, t such that $\beta_i=\psi^{\beta_0}_i(c(x,y))$ for $1\leq i<\mathrm{lh}(s)$, where $\{\psi^{\beta_0}_i\}$ is the scale on Q^{β_0} from Lemma 12 and c is the continuous function of Lemma 15.

It is clear that V has the desired properties from Lemma 15. For the last property claimed, we use the fact that $\{\psi^\alpha_i\}$ is a $\mathbf{\Pi}^\alpha_1$-scale, and so every $\mathbf{\Sigma}^\alpha_1$ subset of Q^α is bounded in these norms. □

The next lemma is a small variation of the previous one, allowing y to code countably many well-founded relations instead of just one.

Lemma 17. *There is a tree V_2 on $\omega\times\omega\times\kappa\times\kappa$ such that $(x,y)\in\mathrm{p}[V]$ if and only if $x\in W$ and for all n, $(y)_n$ codes a $\mathbf{\Sigma}^{\psi(x)}_1$ well-founded relation. Furthermore, for each $\alpha\in C$ with $\mathrm{cf}(\alpha)=\omega$ there is a c.u.b. $D\subseteq\alpha^+$ such that for all $\gamma\in D$, all $x\in W$ with $\psi(x)\leq\alpha$, and all y such that for all n, $|U_{x,(y)_n}\restriction\alpha|<\gamma$, we have that $(V_2)_{x,y}\restriction\gamma$ is ill-founded.*

Proof. The tree V_2 is constructed as V except that in (iii) we require that $\beta_i=\psi^{\beta_0}_{i_0}(c(x,(y)_{i_1}))$. Given $\alpha\in C$ with $\mathrm{cf}(\alpha)=\omega$, define $D\subseteq\alpha^+$ as follows. For $\beta<\alpha^+$, let $B_{\alpha,\beta}=\{(x,y)\,;\,x\in W\wedge\psi(x)\leq\alpha\wedge\forall n\ |U_{x,(y)_n}\restriction\alpha|\leq\beta\}$.[5] Since $\mathbf{\Delta}^\alpha_1$ is closed under $<\alpha^+$ unions and intersections (by Theorem 8), it follows that $B_{\alpha,\beta}\in\mathbf{\Delta}^\alpha_1$. Let $g(\alpha,\beta)=\sup\{\psi^\alpha_n(c(x,y))\,;\,(x,y)\in B_{\alpha,\beta}\}<\alpha^+$ by boundedness, as in Lemma 16. Let D be the c.u.b. sets of points below α^+ which are closed under g. □

[5] Using the set $B^*_{\alpha,\beta}=\{(x,y)\,;\,x\in W\wedge\psi(x)\leq\alpha\wedge\forall n\ (y)_n$ codes a $\mathbf{\Sigma}^\alpha_1$ well-founded relation of length $\leq\beta\}$ might seem more natural, but it is not clear that this definies a $\mathbf{\Sigma}^\alpha_1$ set.

We define $(s, t, u, \vec{\alpha}, \vec{\beta}, v, a, b, \vec{\gamma}, \vec{\delta}) \in U'$ if and only if there are x, τ, w, y, z, w extending s, t, u, v, a, b, respectively, such that

(i) $(s, u, \vec{\alpha}, \vec{\beta}) \in V_2$.
(ii) $\tau(w) = y$.
(iii) $(s, v, a, b, \vec{\gamma}, \vec{\delta}) \in U$. Here U is as in Lemma 13.

We now define the tree T^{++} on $\omega^2 \times \kappa \times \omega^3 \times \kappa^2 \times \omega^3 \times \kappa^2$ consisting of tuples $(c, d, \vec{\eta}, s, t, u, \vec{\alpha}, \vec{\beta}, v, a, b, \vec{\gamma}, \vec{\delta})$ such that there are σ, r, x, τ extending c, d, s, t such that:

(i) $\sigma(r) = \langle x, \tau \rangle$.
(ii) $(d, \vec{\eta}) \in T_2$.
(iii) $(s, t, u, \vec{\alpha}, \vec{\beta}, v, a, b, \vec{\gamma}, \vec{\delta}) \in U'$.

Let us explain the definition of T^{++}: The first coordinate of the tree T^{++} produces a strategy σ. We intend for σ to be a strategy such that when player I plays $r \in W$, then $\sigma(r) = \langle x, \tau \rangle$, where $x \in W$ and $\psi(x) \geq \psi(r)$. The object τ is also a strategy which we intend to do the following. If player I plays a w such that for all n, $(w)_n$ codes a $\mathbf{\Sigma}^{\psi(x)}_1$ well-founded relation (so w codes the ordinal $\sup_n\{|U_{x,(w)_n} \restriction \psi(x)|\}$), then $\tau(w) = y$ codes a well-founded relation in $\mathbf{\Sigma}^{\psi(x)}_1$. Finally, T^{++} attempts to produce an infinite decreasing chain in the relation coded by y, as in the Kunen-Martin Theorem 4.

For the following result, remember that μ_α is the ω-cofinal measure on α^+ which exists by Theorem 11.

Lemma 18. *For all $\alpha \in C_\omega$, we have $j_{\mu_\alpha}(\alpha^+) = \alpha^{++}$.*

Proof. The proof follows by a Kunen tree argument, as in the proof for the odd projective ordinals. It is also a special case of the argument given below. Briefly, define the tree K on $\omega^2 \times \alpha^+ \times \omega^3 \times \alpha^2$ by: $(s, t, \vec{\alpha}, u, v, w, \vec{\beta}, \vec{\gamma}) \in K$ if and only if there are τ, w, and y extending s, t, u such that $\tau(w) = y$, $(t, \vec{\alpha}) \in \bar{V}$, and $(t, u, v, w, \vec{\beta}, \vec{\gamma}) \in U$. Here $\bar{V}$ is a tree such that $\mathrm{p}[\bar{V}]$ is the set of w such that for all n, $(w)_n$ codes a $\mathbf{\Sigma}^\alpha_1$ well-founded relation, and there is a c.u.b. $D \subseteq \alpha^+$ such that for all $\beta \in D$ there is a $w \in \mathrm{p}[\bar{V} \restriction \beta]$ such that for all n, $(w)_n$ has rank $|(w)_n| < \beta$ and $\sup_n |(w)_n| = \beta$.[6]

If $F\colon \alpha^+ \to \alpha^+$, consider the game where player I plays out w, player II plays out y, and player II wins if and only if whenever for all n, $(w)_n$ codes a $\mathbf{\Sigma}^\alpha_1$ well-founded relation. Then y codes a $\mathbf{\Sigma}^\alpha_1$ well-founded relation of length $> f(|w|)$, where $|w|$ is the supremum of the lengths of the relations coded by the $(w)_n$. By boundedness, player II has a winning strategy σ for the game. For any $\beta \in D$ with $\mathrm{cf}(\beta) = \omega$ we have $f(\beta) < |\bar{V}_\sigma \restriction \beta|$. This shows $[f]_{\mu_\alpha} < \alpha^+$, and so $j_{\mu_\alpha}(\alpha^+) \leq \alpha^{++}$. The lower bound follows from the embedding argument given earlier (the second paragraph of the proof of Lemma 14). $\square$

Claim 19. $[\alpha \mapsto \alpha^{++}]_\mu \leq \kappa^{++}$.

[6]Take any $x \in W$ with $\psi(x) = \alpha$ and use the section $\bar{V} = (V_2)_x$. Note that there is a c.u.b. $E \subseteq \alpha^+$ such that for $\beta \in E$ and any $\gamma < \beta$, there is a w such that $|U_{x,w} \restriction \alpha| < \beta$. So the distinction of footnote 5 is irrelevant here.

Proof. Fix $f\colon \kappa \to \kappa$ such that for all α, $f(\alpha) < \alpha^{++}$. Consider the game G_f defined as follows:

Player I	$r(0)$		$r(1)$		$r(2)$		...
Player II		$x(0)$		$x(1)$		$x(2)$	...
		$\tau(0)$		$\tau(1)$		$\tau(2)$	...

If there is a least i such that $(r)_i$ or $(x)_i$ is not in P_0, then player I wins if and only if $(r)_i \in P_0$. Suppose then that $r, x \in W$, that is, for all n, $(r)_n \in P_0 \wedge (x)_n \in P_0$. Let $\alpha = \psi(x) = \sup_n \varphi_0((x)_n)$. Then player II wins provided τ is a strategy with the following properties. There is a $g\colon \alpha^+ \to \alpha^+$ such that if for all n, $(w)_n$ codes a $\mathbf{\Sigma}^\alpha_1$ well-founded relation, then $\tau(w)$ codes a $\mathbf{\Sigma}^\alpha_1$ well-founded relation of length $> g(\sup_n |U_{x,(w)_n} \restriction \alpha|)$, and also $[g]_{\mu_\alpha} \geq f(\alpha)$.

The usual boundedness argument and Lemma 18 (and its proof) show that player I cannot have a winning strategy for G_f. Let σ be a winning strategy for player II in G_f. Inspecting the definition of T^{++} shows that T^{++}_σ is well-founded. There is a c.u.b. $C_2 \subseteq C$ such that if player I plays $r \in W$ with $\psi(r) = \alpha \in C_2$, then $\sigma(r) = \langle x, \tau\rangle$, where $x \in W$ and $\psi(x) = \alpha$. Let $\alpha \in C_2$ with $\mathrm{cf}(\alpha) = \omega$. Fix $r \in W$ with $\psi(x) = \alpha$, and let $\sigma(r) = \langle x, \tau\rangle$. Then by Lemma 17 and the comment in footnote 6 there is a c.u.b. $D \subseteq \alpha^+$ such that for $\beta \in D$ with $\mathrm{cf}(\beta) = \omega$, there is a w such that for all n, $|U_{x,(w)_n} \restriction \alpha| < \beta$ and we also have $\sup_n |U_{x,(w)_n} \restriction \alpha| = \beta$ and $(V_2)_{x,w} \restriction \beta$ is ill-founded. Also, for such a β and w, $\tau(w) = y$ codes a $\mathbf{\Sigma}^\alpha_1$ well-founded relation of length $> g(\beta)$, where $[g]_{\mu_\alpha} > f(\alpha)$. It follows that for such an α that $[\beta \mapsto T^{++}_\sigma \restriction \beta]_{\mu_\alpha} > f(\alpha)$.

From the normality of the measures μ_α it follows that if $[f']_\mu < [f]_\mu$, then there is a function h such that $h(\alpha) < \alpha^+$ and $f'(\alpha) = [\beta \mapsto T^{++}_\sigma \restriction \beta(h(\alpha))|$ for almost all α. This shows that $[f]_\mu$ is a well-ordering of $[\alpha \mapsto \alpha^+]_\mu = \kappa^+$. So, $[f]_\mu < \kappa^{++}$. So, $[\alpha \mapsto \alpha^{++}]_\mu \leq \kappa^{++}$. □

Let $\delta = [\alpha \mapsto \alpha^{++}]_\mu$. We have shown $\delta \leq \kappa^{++}$. The lower bound will follow from the fact that δ is regular, which follows from the partition property $\delta \to (\delta)^\vartheta$ for $\vartheta < \omega_1$, which follows from the polarized partition property we show below.

We are finally in the position to code ordinals below δ. Such a code is a triple of the form (x, σ_1, σ_2), where $x \in P_0$, $T^+_{\sigma_1}$ is well-founded, and $T^{++}_{\sigma_2}$ is well-founded. Let P_2 be the set of codes for ordinals below δ.

Since $(x, \sigma_1) \in P_1$, it determines a function h with $h(\alpha) < \alpha^+$ almost everywhere. The triple then codes the ordinal $[f]_\mu$, where $f(\alpha) = [\beta \mapsto |T^{++}_{\sigma_2} \restriction \beta(h(\alpha))|]_{\mu_\alpha}$.

Lemma 20. *Every ordinal below δ is coded by a triple in P_2.*

Proof. This is clear from the proof of Claim 19. □

2.3.3. *Proof of Theorem* 10. Let $\mathcal{P}$ be a partition of the block functions from $3 \times \omega \cdot \vartheta$ to $(\kappa, \kappa^+, \delta)$. Fix a bijection $\pi\colon \omega \cdot \vartheta \to \omega$. Let $\prec_\pi$ be the corresponding well-ordering of ω. An ordinal $j < \omega \cdot \vartheta$ can be identified with a pair $j = (i, n)$, where $i < \vartheta$ and $n < \omega$, using lexicographic ordering on the pairs. We shall frequently pass back and forth from this identification.

Consider the following game G, where player I plays out a real $\langle x, y, z\rangle$, and player II plays out the real $\langle x', y', z'\rangle$. If there is a $j < \omega \cdot \vartheta$ such that $(x)_{\pi(j)} \notin P_0$ or $(x')_{\pi(j)} \notin P_0$, then player I wins if and only if for the least such j we have that $(x)_{\pi(j)} \in P_0$. Suppose then that for all $j < \omega \cdot \vartheta$, $(x)_{\pi(j)}$ and $(x')_{\pi(j)}$ are in P_0. In this case, x and x' each determine a function from $\omega \cdot \vartheta$ to κ, viz.

$\bar{F}_x(j) := |(x)_{\pi(j)}|$ and $\bar{F}_{x'} := |(x')_{\pi(j)}|$, respectively. So, together they produce a function $F = F_{x,x'} \colon \vartheta \to \kappa$ given by $F(i) = \sup_n \max\{\bar{F}_x(i,n), \bar{F}_{x'}(i,n)\}$.

Suppose next that there is an $\alpha < \kappa$ such that one of the following holds.

(a) There is a $j < \omega \cdot \vartheta$ such that either $T^+_{(y)_{\pi(j)}} \restriction \alpha$ or $T^+_{(y')_{\pi(j)}} \restriction \alpha$ is ill-founded.

(b) There is a $\beta < \alpha^+$ and a $j < \omega \cdot \vartheta$ such that either $T^{++}_{(z)_{\pi(j)}} \restriction \beta$ or $T^{++}_{(z')_{\pi(j)}} \restriction \beta$ is ill-founded.

Let $\alpha < \kappa$ be least such that (a) or (b) above holds. If (a) holds, let j be least such that (a) holds for α and this j. In this case, player I wins provided $T^+_{(y)_{\pi(j)}}$ is well-founded. If (a) does not hold at α, but (b) does, let (β, j) be lexicographically least such that (b) holds. Player I wins in this case provided $T^{++}_{(z)_{\pi(j)}} \restriction \beta$ is well-founded.

Suppose finally that neither (a) nor (b) hold for all $\alpha < \kappa$. Then each of y, y' determine a block function from $(\omega \cdot \vartheta) \times \kappa$ to κ. Namely, for $\alpha \in C$ and $j < \omega \cdot \vartheta$, let $\bar{g}_y = |T^+_{(y)_{\pi(j)}} \restriction \alpha|$. Likewise, y' determines the block function $\bar{g}_{y'}$. Together, they determine the block function $g \colon \vartheta \times \kappa \to \kappa$ by $g(\alpha, i) = \sup_n \max\{\bar{g}_y(\alpha, j), \bar{g}_{y'}(\alpha, j)\}$, where $j = (i, n)$. Finally, g determines a function $G = G_{y,y'} \colon \vartheta \to \kappa^+$ by $G(i) = [\alpha \mapsto g(\alpha, i)]_\mu$.

In a similar fashion, each of z, z' determine block functions $\bar{h}_z$, $\bar{h}_{z'}$. For $\alpha \in C$, $\beta < \alpha^+$, and $j < \omega \cdot \vartheta$, let $\bar{h}_z(\alpha, \beta, j) = |T^{++}_{(z)_{\pi(j)}} \restriction \beta|$. Similarly define $\bar{h}_{z'}$. Together they determine a block function h defined by: for $\alpha \in C$, $\beta < \alpha^+$, and $i < \vartheta$, let $h(\alpha, \beta, i) = \sup_n \max\{\bar{h}_z(\alpha, \beta, j), \bar{h}_{z'}(\alpha, \beta, j)\}$, where $j = (i, n)$. Finally, h determines a function $H = H_{z,z'} \colon \vartheta \to \delta$ given by $H(i) = [\alpha \mapsto [\beta \mapsto h(\alpha, \beta, i)]_{\mu_\alpha}]_\mu$.

Finally in this case we say Player II wins the run of the game if and only if $\mathcal{P}(F, G, H) = 1$.

Suppose without loss of generality that Player II has a winning strategy τ (the case where player I has a winning strategy is slightly easier). We define first a c.u.b. set $C_0 \subseteq \kappa$. For each $\eta < \kappa$ and $j < \omega \cdot \vartheta$, let

$$A_{\eta,j} = \{(x, y, z) \,;\, \forall j' \leq j \ ((x)_{\pi(j')} \in P_0 \wedge \varphi_0((x)_{\pi(j')}) \leq \eta)\}.$$

Clearly $A_{\eta,j} \in \mathbf{\Delta}$ (recall $\mathbf{\Gamma}$ is the Steel pointclass of Wadge rank κ). Since τ is winning for Player II, if $(x, y, z) \in A_{\eta,j}$, and $\tau(x, y, z) = (x', y', z')$, then for all $j' \leq j$, we have $((x')_{\pi(j')} \in P_0)$ and by boundedness

$$\rho_0(\eta, j) := \sup\{\varphi_0((x')_{\pi(j')}) \,;\, (x', y', z') \in \tau[A_{\eta,j}] \wedge j' \leq j\} < \kappa.$$

Let $C_0 \subseteq C$ be c.u.b. and closed under ρ_0.

We next define a c.u.b. $C_1 \subseteq \kappa^+$. For $\alpha \in C_0$ with $\mathrm{cf}(\alpha) = \omega$, $\eta < \alpha^+$, and $j < \omega \cdot \vartheta$, let

$$\begin{aligned} A_{\alpha,\eta,j} = \{&(x, y, z) \,;\, \forall j \ ((x)_{\pi(j)} \in P_0 \wedge \varphi_0((x)_{\pi(j)}) < \alpha) \\ &\wedge \forall \alpha' < \alpha \ \forall \beta \ < (\alpha')^+ \ \forall j \ (T^+_{(y)_{\pi(j)}} \restriction \alpha \text{ and } T^{++}_{(z)_{\pi(j)}} \restriction \beta \text{ are well-founded}) \\ &\wedge \forall j' \leq j \ (|T^+_{(y)_{\pi(j')}} \restriction \alpha| \leq \eta)\}. \end{aligned}$$

Since τ is winning for player II, if $(x, y, z) \in A_{\alpha,\eta,j}$ and $\tau(x, y, z) = (x', y', z')$, then $x' \in W$ and $\varphi_0((x')_{\pi(j)}) < \alpha$ for all j. Furthermore, since $A_{\alpha,\eta,j} \in \mathbf{\Delta}^\alpha_1$, we have by boundedness that

$$\rho_1(\alpha, \eta, j) := \sup\{|T^+_{(y')_{\pi(j')}} \restriction \alpha| \,;\, j' \leq j \wedge (x', y', z') \in \tau[A_{\alpha,\eta,j}]\} < \alpha^+.$$

Construct (uniformly in α) sets $D^\alpha \subseteq \alpha^+$ which are c.u.b. and closed under $(\eta, j) \mapsto \rho_1(\alpha, \eta, j)$. Let $E_1 \subseteq \kappa^+$ be the set of $[f]_\mu$ such that $\forall^*_\mu \alpha\ f(\alpha) \in D^\alpha$. Let $F_1 \subseteq \kappa^+$ be the set of limit points of ordinals of the form $[\alpha \mapsto |T^+_x \restriction \alpha|]_\mu$, where T^+_x is well-founded. F_1 is c.u.b. in κ^+ from Lemma 14. Clearly E_1 is also c.u.b. in κ^+. Let $C_1 = E_1 \cap F_1$.

Finally, we define a c.u.b. $C_2 \subseteq \delta$. For $\alpha \in C_0$ with $\mathrm{cf}(\alpha) = \omega$, $\beta, \eta < \alpha^+$, and $j < \omega \cdot \vartheta$, let

$$\begin{aligned} A_{\alpha,\beta,\eta,j} = \{(x,y,z)\,;\, \forall j\ ((x)_{\pi(j)} \in P_0 \wedge \varphi_0((x)_{\pi(j)}) < \alpha) \\ \wedge \forall \alpha' < \alpha\ \forall \beta\ < (\alpha')^+\ \forall j\ (T^+_{(y)_{\pi(j)}} \restriction \alpha \text{ and } T^{++}_{(z)_{\pi(j)}} \restriction \beta \text{ are well-founded}) \\ \wedge \forall j\ |T^+_{(y)_{\pi(j)}} \restriction \alpha| < \beta \wedge \forall (\beta', j') \leq_{\mathrm{lex}} (\beta, j)\ (|T^{++}_{(z)_{\pi(j)}} \restriction \beta| \leq \eta)\}. \end{aligned}$$

We have $A_{\alpha,\beta,\eta,j} \in \mathbf{\Delta}^\alpha_1$. Since τ is winning for Player II, for each $(x,y,z) \in A_{\alpha,\beta,\eta,j}$, if $\tau(x,y,x) = (x',y',z')$, then $\forall (\beta', j') \leq_{\mathrm{lex}} (\beta, j)\ T^{++}_{(z')_{\pi(j')}} \restriction \beta$ is well-founded. By boundedness,

$$\rho_2(\alpha, \beta, \eta, j) := \sup\{|T^{++}_{(z')_{\pi(j')}} \restriction \beta|\,;\, (x', y', z') \in \tau[A_{\alpha,\beta,\eta,j}] \wedge j' \leq j\} < \alpha^+.$$

Let E^α be a c.u.b. subset of α^+ closed under ρ_2. Let $E^\alpha_2 \subseteq \alpha^{++}$ be the c.u.b. set of all $[f]_{\mu_\alpha}$, where $\mathrm{ran}(f) \subseteq E^\alpha$. Let $E_2 \subseteq \delta$ be the c.u.b. set of all $[g]_\mu$, where $g(\alpha) \in E^\alpha_2$ for $\alpha < \kappa$. E_2 is c.u.b. in δ from the definition of δ. Let F_2 be the c.u.b. subset of δ consisting of limits of points of the form $[\alpha \mapsto [\beta \mapsto |T^{++}_x \restriction \beta|]_{\mu_\alpha}]_\mu$, where T^{++}_x is well-founded. From Claim 19, F_2 is c.u.b. in δ. Let $C_2 = E_2 \cap F_2$.

Let C'_0 be the set of limit points of C_0, and likewise for C'_1, C'_2. To finish, we show the following.

Claim 21. (C'_0, C'_1, C'_2) is homogeneous for $\mathcal{P}$.

Proof. Suppose (F, G, H) is a block function from $3 \times \vartheta$ into (C'_0, C'_1, C'_2) of the correct type (since ϑ is countable, this just means that F, G, H are increasing, discontinuous, and have range in points of cofinality ω).

Let $\bar{F} \colon \omega \cdot \vartheta \to \kappa$ be increasing and induce F, that is, $F(i) = \sup_{j<\omega\cdot(i+1)} \bar{F}(j)$ for all $i < \vartheta$. Let $x \in \omega^\omega$ be such that for all $j < \omega \cdot \vartheta$, $(x)_{\pi(j)} \in P_0$ and $\varphi_0((x)_{\pi(j)}) = \bar{F}(j)$.

Let $\bar{G} \colon \omega \cdot \vartheta \to \kappa^+$ be increasing and induce G, that is, $G(i) = \sup_{j<\omega\cdot(i+1)} \bar{G}(j)$ for all $i < \vartheta$. We may assume $\bar{G}$ has range in C_1. Since $C_1 \subseteq F_1$, for each $j < \omega \cdot \vartheta$ we may get a $y_j \in \omega^\omega$ (using countable choice) such that $T^+_{y_j}$ is well-founded and $\bar{G}(j) = [\alpha \mapsto |T^+_{y_j} \restriction \alpha|]_\mu$. Let $y \in \omega^\omega$ be such that for all $j < \omega\cdot\vartheta$ we have $(y)_{\pi(j)} = y_j$.

Let $\bar{H} \colon \omega \cdot \vartheta \to \delta$ be increasing and induce H. We may assume H has range in C_2. Since $C_2 \subseteq F_2$, for each $j < \omega \cdot \vartheta$ there is a $z_j \in \omega^\omega$ such that $T^{++}_{z_j}$ is well-founded and $\bar{H}(j) = [\alpha \mapsto [\beta \mapsto |T^{++}_{z_j} \restriction \beta|]_{\mu_\alpha}]_\mu$. Let $z \in \omega^\omega$ be such that for all $j < \omega \cdot \vartheta$ we have $(z)_{\pi(j)} = z_j$.

Let $(x', y', z') = \tau(x, y, z)$. Recall that we identify $\omega \cdot \vartheta$ with lexicographic order on pairs (i, n) where $i < \vartheta$ and $n \in \omega$. Since $x \in W$ and τ is winning for Player II, it follows that $x' \in W$ as well. Let $\bar{F}_{x'} \colon \omega \cdot \vartheta \to \kappa$ be the function determined by x', that is, $\bar{F}_{x'}(j) = |(x')_{\pi(j)}|$. Since $\mathrm{ran}(\bar{F}_x) \subseteq C_0$, it follows from the definition of C_0 that $\bar{F}_{x'}(i, n) < \bar{F}_x(i, n+1) < \bar{F}(i)$ for all $i < \vartheta$. So, for each $i < \vartheta$ we have $\sup_n \max\{\bar{F}_x(i, n), \bar{F}_{x'}(i, n)\} = F(i)$. Thus the function $F_{x,x'}$ jointly produced by x and x' is equal to F.

Consider next y and y'. For all $j < \omega \cdot \vartheta$ we have that $T^+_{(y)_{\pi(j)}}$ and $T^{++}_{(z)_{\pi(j)}}$ are well-founded. Let $\alpha_0 = \sup_{i<\vartheta} F(i) = \sup_{j<\omega\cdot\vartheta} \bar{F}(j)$. Let $\bar{g}_y$ be the block function determined by y. That is, for $\alpha \in C_0$ and $j < \omega \cdot \vartheta$, $\bar{g}_y(\alpha, j) = |T^+_{(y)_{\pi(j)}} \restriction \alpha|$. For μ almost all α the function $\bar{g}^\alpha_y(j) = \bar{g}_y(\alpha, j)$ is increasing. Let g^α_y be the function induced by $\bar{g}^\alpha_y$, that is, $g^\alpha_y(i) = \sup_n \bar{g}^\alpha_y(j)$, where $j = (i, n)$. For μ almost all α, g^α_y has range in the limit points of D^α (as defined above in the construction of C_1). Say $M_1 \subseteq C_0$ is this measure one set. Consider any $\alpha \in M_1$ with $\alpha > \alpha_0$. Then for any $j < \omega \cdot \vartheta$ we have that $(x, y, z) \in A_{\alpha,\eta,j}$, where $\eta = \bar{g}^\alpha_y(j) < g^\alpha_y(i)$ (where again $j = (i, n)$). It follows from the definition of D^α that $|T^+_{(y')_{\pi(j)}} \restriction \alpha| < g^\alpha_y(i)$ as well. So, if $\bar{g}^\alpha_{y'}$ is the function determined by y' (i.e., $\bar{g}^\alpha_{y'}(j) = |T^+_{(y')_{\pi(j)}} \restriction \alpha|$), then $\bar{g}^\alpha_{y'}$ and $\bar{g}^\alpha_y$ both induce the function g^α_y. It follows that the function $G_{y,y'}$ they jointly produce is equal to G.

Consider finally z and z'. For $\alpha \in M_1$, $\beta < \alpha^+$, and $j < \omega \cdot \vartheta$, let $\bar{h}^\alpha_z(\beta, j) = |T^{++}_{(z)_{\pi(j)}} \restriction \beta|$. Since $\bar{H}$ is increasing and induces H, it follows from the definition of z that if $j' < j$, then $\forall^*_\mu \alpha\ \forall^*_{\mu_\alpha} \beta\ \bar{h}^\alpha_z(\beta, j') < \bar{h}^\alpha_z(\beta, j)$. This implies that there is a μ measure one set $M'_2 \subseteq M_1$ such that for $\alpha \in M'_2$ there is a c.u.b. $C \subseteq \alpha^+$ which is closed under $\bar{h}^\alpha_z$ and such that the map $(\beta, j) \mapsto \bar{h}^\alpha_z(\beta, j)$ is order-preserving when restricted to pairs with $\beta \in C$, $\mathrm{cf}(\beta) = \omega$. Also, from the definition of E_2 there is a μ measure one set $M_2 \subseteq M'_2$ such that for $\alpha \in M_2$ we have that there is a c.u.b. $C \subseteq \alpha^+$ such that for $\beta \in C$ with $\mathrm{cf}(\beta) = \omega$ we have in addition that for all $i < \vartheta$ that $\sup_n h^\alpha_z(\beta, j) \in E^\alpha_2$, where $j = (i, n)$.

Now consider $\alpha \in M_2$ with $\alpha > \alpha_0$ (α_0 as above). Fix a c.u.b. $C \subseteq \alpha^+$ as with the two properties specified immediately above. Let $\beta \in C$ with $\mathrm{cf}(\beta) = \omega$ and $\beta > \sup g^\alpha_y(j)$. For such a β, if $j < \omega \cdot \vartheta$ and $\eta = h^\alpha_z(\beta, j)$, then from the definition of $A_{\alpha,\beta,\eta,j}$ we see that $(x, y, z) \in A_{\alpha,\beta,\eta,j}$. From the definition of E^α_2 it follows that $h^\alpha_{z'}(\beta, j) < \sup h^\alpha_z(\beta, j')$, where $j = (i, n)$, and the supremum ranges over $j' = (i, n')$. It follows that the function $\bar{H}_{z'}$ induces the function H, that is, $H = H_{z,z'}$. Since τ is winning for Player II, we have that $\mathcal{P}(F, G, H) = 1$ and we are done. q.e.d. (Claim 21)

We have proved the claim, and this finishes the proof of Theorem 10 (and thus the proof of Theorem 9).

2.4. **A polarized partition property with higher exponents.** We now improve Theorem 9 from countable exponents to arbitrary exponents $\vartheta < \kappa$. The setup is the same as in the proof of Theorem 9: we have the (Steel) pointclass $\mathbf{\Gamma} \subseteq \mathbf{S}(\kappa)$ forming the lowest level of the projective-like hierarchy containing $\mathbf{S}(\kappa)$ which is scaled, non-selfdual, closed under $\forall^{\omega^\omega}$ and finite intersections and unions, and let $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$.

Theorem 22. *Assume* AD. *Let κ be a weakly inaccessible Suslin cardinal. Then for all $\vartheta < \kappa$ we have $(\kappa, \kappa^+, \kappa^{++}) \to (\kappa, \kappa^+, \kappa^{++})^\vartheta$.*

Proof. We fix $\vartheta < \kappa$, and fix a prewellordering $\preceq \in \mathbf{\Delta}$ of length $\omega \cdot \vartheta$. We shall use the coding lemma to code functions from $\omega \cdot \vartheta$ to κ. We again identify the ordinals below $\omega \cdot \vartheta$ with the pairs (i, n) ordered lexicographically, where $i < \vartheta$ and $n < \omega$. We shall also use the trees T^+ and T^{++} from the proof of Theorem 9.

Fix a pointclass $\mathbf{\Gamma}_0 \subseteq \mathbf{\Delta}$ which is non-selfdual, is closed under $\exists^{\omega^\omega}$, has the prewellordering property, and contains the prewellordering $\preceq$. Fix a $\mathbf{\Gamma}_0$-universal set U. Without loss of generality we may assume $\mathrm{dom}(\preceq) = \omega^\omega$. Let $|a|$ denote the rank of $a \in \omega^\omega$ in $\preceq$. For $x \in \omega^\omega$, we say x codes a function at $j < \omega \cdot \vartheta$ provided

(i) $\forall a\ (|a| = j \to \exists b\ U(x, a, b))$;
(ii) $\forall a, a', b, b'\ (U(x, a, b) \wedge U(x, a', b') \wedge |a| = |a'| = j \to (b, b' \in P_0 \wedge \varphi_0(b) = \varphi_0(b'))$.

We say x codes a function from $\omega \cdot \vartheta$ to κ (or just say x codes a function) if x codes a function at j for all $j < \omega \cdot \vartheta$.

Recall that if S is a tree, we write $S(\gamma)$ to denote the subtree of S consisting of points in S below γ (we are identifying ordinals with finite tuples here for convenience).

Now fix a partition $\mathcal{P}$ of the block functions from $3 \times \vartheta$ to $(\kappa, \kappa^+, \kappa^{++})$. Consider the game G where player I plays out the real $\langle x, y, u, z, v, w\rangle$ and player II plays out the real $\langle x', y', u', z', v', w'\rangle$. Suppose first that there is a least $j < \omega \cdot \vartheta +$ such that x or x' does not code a function at j. In this case, player II wins if and only if x does not code a function at j. Suppose next that both x and x' code functions. If there is a least j such that y or y' does not code a function at j, player II again wins if and only if y does not code a function at j. Likewise, if all of x, y, x', y' code functions and there is a least j such that z or z' doesn't code a function at j, player II wins if and only if z doesn't code a function at j.

Suppose next that x, y, z, x', y', z' all code functions from $\omega \cdot \vartheta$ to κ. We let $\bar{f}_x, \bar{f}_y$, etc. denote these functions. Let $\alpha \in C_\omega$ be the least ordinal, if it exists, such that one of the following holds.

(a) There is a $j < \omega \cdot \vartheta$ such that either $T_u^+ \restriction \alpha(\bar{f}_y(j))$ or $T_{u'}^+ \restriction \alpha(\bar{f}_{y'}(j))$ is ill-founded.
(b) There is a $j < \omega \cdot \vartheta$ such that either $T_v^+ \restriction \alpha(\bar{f}_z(j))$ or $T_{v'}^+ \restriction \alpha(\bar{f}_{z'}(j))$ is ill-founded.
(c) There is a $\beta < \alpha^+$ and a $j < \omega \cdot \vartheta$ such that either $T_w^{++} \restriction \beta(|T_v^+ \restriction \alpha(\bar{f}_z(j))|)$ or $T_{w'}^{++} \restriction \beta(|T_{v'}^+ \restriction \alpha(\bar{f}_{z'}(j))|)$ is ill-founded.

Suppose first that such an α exists. If (a) holds, let j be least such that either $T_u^+ \restriction \alpha(\bar{f}_y(j))$ or $T_{u'}^+ \restriction \alpha(\bar{f}_{y'}(j))$ is ill-founded. Player II then wins if and only if $T_u^+ \restriction \alpha(\bar{f}_y(j))$ is ill-founded. If (a) does not hold, but (b) holds, then let j be least such that either $T_v^+ \restriction \alpha(\bar{f}_z(j))$ or $T_{v'}^+ \restriction \alpha(\bar{f}_{z'}(j))$ is ill-founded. Player II then wins if and only if $T_v^+ \restriction \alpha(\bar{f}_z(j))$ is ill-founded. If (a) and (b) do not hold but (c) holds, then let (β, j) be the lexicographically least pair witnessing (c). Player II then wins if and only if $T_w^{++} \restriction \beta(T_v^+ \restriction \alpha(\bar{f}_z(j)))$ is ill-founded.

Finally, if no such α exists, then let $F = F_{x,x'}$ be the function jointly produced by $\bar{f}_x$ and $\bar{f}_{x'}$. That is, $\bar{F}(i) = \sup_n \max\{\bar{f}_x(i, n), \bar{f}_{x'}(i, n)\}$ for all $i < \vartheta$.

Let $\bar{g}_{y,u}$ be the block function defined as follows. For $\alpha \in C_\omega$ and $j < \omega \cdot \vartheta$, let $\bar{g}_{y,u}(\alpha, j) = |T_u^+ \restriction \alpha(f_y(j))|$. Likewise define $\bar{g}_{y',u'}$ using y' and u'. Let $\bar{G}_{y,u}$ be the function from $\omega \cdot \vartheta$ to κ^+ represented by $\bar{g}_{y,u}$. That is, $\bar{G}_{y,u}(j) = [\alpha \mapsto \bar{g}_{y,u}(\alpha, j)]_\mu$. Likewise define $\bar{G}_{y',u'}$. Finally, let $G = G_{y,u,y',u'} \colon \vartheta \to \kappa^+$ be the function they jointly produce: $G(i) = \sup_n \max\{\bar{G}_{y,u}(i, n), \bar{G}_{y',u'}(i, n)\}$.

Let $\bar{h}_{z,v,w}$ be the block function defined as follows. For $\alpha \in C_\omega$, $\beta < \alpha^+$, and $j < \omega \cdot \vartheta$, let $\bar{h}_{z,v,w}(\alpha, \beta, j) = |T_w^+ \restriction \beta(|T_v^+ \restriction \alpha(\bar{f}_z(j))|)|$. Notice we may write this as $\bar{h}_{z,v,w}(\alpha, \beta, j) = |T_w^+ \restriction \beta(\bar{g}_{z,v}(\alpha, j))|$. Similarly define $\bar{h}_{z',v',w'}$ using z', v',

w'. Let $\bar{H}_{z,v,w}\colon \omega\cdot\vartheta \to \kappa^{++}$ be the function represented by $\bar{h}_{z,v,w}$. That is, $\bar{H}_{z,v,w}(j) = [\alpha \mapsto [\beta \mapsto \bar{h}_{z,v,w}(\alpha,\beta,j)]_{\mu_\alpha}]_\mu$. Let $H = H_{z,v,w,z',v',w'}\colon \vartheta \to \kappa^{++}$ be the function they jointly produce: $H(i) = \sup_n \max\{\bar{H}_{z,v,w}(i,n), \bar{H}_{z',v',w'}(i,n)\}$.

Player II then wins the run of the game G provided $\mathcal{P}(F,G,H) = 1$.

Suppose without loss of generality that player II has a winning strategy τ for G. We define c.u.b. sets C_0', C_1', C_2' in κ, κ^+, κ^{++} respectively which are homogeneous for $\mathcal{P}$. The argument is similar to that of Theorem 9, so we shall concentrate on the differences.

We first define C_0 (C_0' will be the set of limit points of C_0). For each $\eta < \kappa$ and $j < \omega\cdot\vartheta$, let

$$A_{\eta,j} = \{c = \langle x,y,u,z,v,w\rangle\,;\, \forall j' \le j\ (x \text{ codes a function at } j' \wedge f_x(j') \le \eta)\}.$$

Clearly $A_{\eta,i} \in \mathbf{\Delta}$. Since τ is winning for player II, if $c \in A_{\eta,i}$, and $\tau(c) = \langle x',y',u',z',v',w'\rangle$, then for all $j' \le j$, $f_{x'}$ codes a function at j'. By boundedness,

$$\rho(\eta,j) := \sup\{f_{x'}(j')\,;\, \langle x',y',u',z',v'\rangle \in \tau[A_{\eta,j}] \wedge j' \le j\} < \kappa.$$

Let $C_0 \subseteq C$ be c.u.b. and closed under f.

We next define C_1. For $\eta < \kappa$, $\eta \in C_0$, with $\mathrm{cf}(\eta) > \vartheta$ (for convenience), $\alpha \in C_\omega$, $\alpha > \eta$, $j < \omega\cdot\vartheta$, and $\delta < \alpha^+$, let

$$\begin{aligned} A_{\eta,\alpha,j,\delta} = \{&c = \langle x,y,u,z,v\rangle\,;\, x,y,z \text{ code functions } \wedge (\forall j\ \bar{f}_x(j), \bar{f}_y(j), \bar{f}_z(j) \le \eta)\\ &\wedge \forall(\alpha',j') \le_{\mathrm{lex}} (\alpha,j)\ |T_u^+\restriction\alpha'(\bar{f}_y(j'))| \le \delta\\ &\wedge \forall\alpha' < \alpha\ \forall\beta' < (\alpha')^+\ \forall j' < \omega\cdot\vartheta,\\ &\quad T_w^{++}\restriction\beta'(|T_v^+\restriction\alpha'(\bar{f}_z(j'))|) \text{ is well-founded}\}. \end{aligned}$$

Let $c' = \langle x',y',u',z',v',w'\rangle = \tau(c)$, where $c = \langle x,y,u,z,v,w\rangle \in A_{\eta,\alpha,j,\delta}$. From the second and third conjuncts in the above definition it follows that the least α' such that (a), (b), or (c) holds for c, c' is at least α. Furthermore, the second conjunct gives that (a) does not hold at α for all $j' \le j$. It follows that $T_{u'}^+\restriction\alpha'(f_{y'}(j'))$ is well-founded for all $(\alpha',j') \le_{\mathrm{lex}} (\alpha,j)$. For η, α, j, δ as above, define

$$\rho_1(\eta,\alpha,j,\delta) = \sup\{T_{u'}^+\restriction\alpha(f_{y'}(j'))\,;\, j' \le j \wedge c' = \langle x',y',u',z',v'\rangle \in \tau[A_{\eta,\alpha,j,\delta}]\}.$$

Claim 23. $\rho_1(\eta,\alpha,j,\delta) < \alpha^+$.

Proof. We define a $\mathbf{\Sigma}_1^\alpha$ well-founded relation of length at least $\rho_1(\eta,\alpha,j,\delta)$. Since $\mathbf{\Sigma}_1^\alpha = \mathbf{S}(\alpha)$, it then follows that $\rho_1(\eta,\alpha,j,\delta) < \alpha^+$. In defining this relation we again for convenience think of the elements of T^+ as ordinals rather than tuples of ordinals. Define a relation S by letting

$$\begin{aligned} &(b,u,s)S(b',u',s') \leftrightarrow b = b' \wedge u = u' \wedge \exists c \in A_{\eta,\alpha,j,\delta}\ \exists y\ \exists j' \le j\ \exists a\\ &\quad [\tau(c)_1 = y \wedge \tau(c)_2 = u \wedge |a| = j' \wedge U(y,a,b) \wedge (b \in P_0 \wedge \varphi_0(b) \le \eta) \wedge s,s' \in P_0\\ &\quad \wedge \varphi_0(s), \varphi_0(s') < \alpha \wedge (\varphi_0(s) <_{T_u^+\restriction\alpha} \varphi_0(s') <_{T_u^+\restriction\alpha} \varphi_0(b))], \end{aligned}$$

where $<_{T_u^+\restriction\alpha}$ refers to the Kleene-Brouwer ordering on the tree $T_u^+\restriction\alpha$. From the above remarks we have that S is well-founded. From the coding lemma, $T^+\restriction\alpha$ is $\mathbf{\Sigma}_1^\alpha$ in the codes (relative to $P_0\restriction\alpha$). Also, $A_{\eta,\alpha,j,\delta} \in \mathbf{\Delta}_1^\alpha$ using the closure of $\mathbf{\Delta}_1^\alpha$ under $<\alpha^+$ unions and intersections (for the last conjunct in the definition of $A_{\eta,\alpha,j,\delta}$ note that if well-founded $|T_w^{++}\restriction\beta'(|T_v^+\restriction\alpha'(\bar{f}_z(j'))|)|$ must have rank less than $(\alpha')^+ < \alpha$). Since also $\eta < \alpha$, it follows that $S \in \mathbf{\Sigma}_1^\alpha$, and we are done. $\square$

We let C_1^α be the set of ordinals below α^+ closed under the ρ_1 function. Let $C_1 \subseteq \kappa^+$ be the set of $[G]_\mu$, where $G(\alpha) \in C_1^\alpha$ for μ almost all α.

The definition of C_2 is similar to that of C_1. For $\eta < \kappa$, $\eta \in C_0$, with $\mathrm{cf}(\eta) > \vartheta$, $\alpha \in C_\omega$, $\alpha > \eta$, $j < \omega \cdot \vartheta$, and $\beta, \delta < \alpha^+$, let

$$\begin{aligned} A_{\eta,\alpha,j,\beta,\delta} = \{ & c = \langle x, y, u, z, v, w \rangle \,;\, x, y, z \text{ code functions } \wedge (\forall j \ \bar{f}_x(j), \bar{f}_y(j), \bar{f}_z(j) \leq \eta) \\ & \wedge \forall \alpha' \leq \alpha \ \forall j' < \omega \cdot \vartheta \ |T_u^+ \restriction \alpha'(\bar{f}_y(j'))| \leq \delta \\ & \wedge \forall \alpha' \leq \alpha \ \forall j' < \omega \cdot \vartheta \ |T_v^+ \restriction \alpha'(\bar{f}_z(j'))| \leq \delta \\ & \wedge \forall \alpha' < \alpha \ \forall \beta' < (\alpha')^+ \ \forall j' < \omega \cdot \vartheta, \\ & \quad T_w^{++} \restriction \beta'(|T_v^+ \restriction \alpha'(\bar{f}_z(j'))|) \text{ is well-founded} \\ & \wedge \forall (\beta', j') \leq_{\mathrm{lex}} (\beta, j) \ |T_w^{++} \restriction \beta'(|T_v^+ \restriction \alpha(\bar{f}_z(j'))|)| \leq \delta \}. \end{aligned}$$

It again follows that $A_{\eta,\alpha,j,\beta,\delta} \in \mathbf{\Delta}_1^\alpha$. Suppose $c' \langle x', y', u', z', v', w' \rangle = \tau(c)$, where $c = \langle x, y, u, z, v, w \rangle \in A_{\eta,\alpha,j,\beta,\delta}$. Since x, y, z all define functions taking values below η, and since $\eta \in C_0$ it follows that x', y', z' also all code functions below η (for y', z' we also use that $\mathrm{cf}(\eta) > \vartheta$ so that $\sup(\bar{f}_y)$, $\sup(\bar{f}_z)$ are also below η). From the second, third, and fourth conjuncts in the definition of $A_{\eta,\alpha,j,\beta,\delta}$ and the fact that τ is winning for player II it follows that the least α' such that (a), (b), or (c) holds at α' is $\geq \alpha$. Also, from the second conjunct it follows that (a) cannot hold at α. From the third conjunct it follows that (b) cannot hold at α. From the last conjunct it then follows that for all $\beta' < \alpha^+$ and j' with $(\beta', j') \leq_{\mathrm{lex}} (\beta, j)$ that $T_{w'}^{++} \restriction \beta'(|T_{v'}^+ \restriction \alpha(\bar{f}_{z'}(j'))|)$ is well-founded.

For η, α, β, j, δ as in the definition of $A_{\eta,\alpha,j,\beta,\delta}$ define

$$\begin{aligned} \rho_2(\eta, \alpha, \beta, j, \delta) = \sup\{ & |T_{w'}^{++} \restriction \beta'(|T_{v'}^+ \restriction \alpha(\bar{f}_{z'}(j'))|)| \,:\, (\beta', j') \leq_{\mathrm{lex}} (\beta, j) \\ & \wedge c' = \langle x', y', u', z', v' \rangle \in \tau[A_{\eta,\alpha,\beta,j,\delta}] \}. \end{aligned}$$

Analogous to Claim 23 we have:

Claim 24. $\rho_2(\eta, \alpha.\beta.j, \delta) < \alpha^+$.

Proof. The proof follows from a computation as in Claim 23, using the fact that $|\beta| = \alpha$, so $T^{++} \restriction \beta$ is isomorphic to a tree on α. □

For $\alpha \in C_\omega$, now let C_2^α be the c.u.b. subset of α^+ consisting of points closed under the ρ_2 function. Let $D^\alpha \subseteq \alpha^{++}$ be the c.u.b. set of ordinals of the form $[h]_{\mu\alpha}$, where $\mathrm{ran}(h) \subseteq D_2^\alpha$. Finally, let $C_2 \subseteq \kappa^{++}$ be the c.u.b. set of ordinals of the form $[H]_\mu$, where $H(\alpha) \in D^\alpha$ for μ almost all α.

We now show that the c.u.b. sets C_0', C_1', C_2' are homogeneous for the given partition $\mathcal{P}$. Fix a block function (F, G, H) from $3 \times \vartheta$ to (C_0', C_1', C_2') of the correct type. Let $\bar{F}$, $\bar{G}$, $\bar{H}$ from $3 \times \omega \cdot \vartheta$ to $(\kappa, \kappa^+, \kappa^{++})$ be increasing and induce (F, G, H).

From the coding lemma, let x be such that x codes a function at j for all $j < \omega \cdot \vartheta$ and such that $\bar{f}_x = \bar{F}$.

Since κ^+ is regular by Theorem 9, $\sup(G) < \kappa^+$. Let u be such that T_u^+ is well-founded and $[\alpha \mapsto |T_u^+ \restriction \alpha|]_\mu > \sup(G)$. Let $\tilde{g} \colon \omega \cdot \vartheta \to \kappa$ be defined as follows. For $j < \omega \cdot \vartheta$, let $\bar{g}(j)$ be the ordinal less than κ such that $\forall_\mu^* \alpha \ [\alpha \mapsto |T_u \restriction \alpha(\bar{g}(j))|]_\mu = \bar{G}(j)$. This is well-defined by the normality of μ and the definition of u. Let y be such that y codes the function $\tilde{g}$ (i.e., for all $j < \omega \cdot \vartheta$, y codes a function at j, and the value coded at j is $\tilde{g}(j)$).

Since κ^{++} is also regular by Theorem 9, $\sup(H) < \kappa^{++}$. From the previous section there is a real w such that T_w^{++} is well-founded and such that $[\alpha \mapsto [\beta \mapsto |T_w^{++} \restriction \beta]_{\mu_\alpha}]_\mu > \sup(H)$. Define a function $\ell\colon \omega \cdot \vartheta \to \kappa^+$ as follows. For $j < \omega \cdot \vartheta$, let $\ell(j) < \kappa^+$ be the ordinal represented by $\alpha \mapsto \ell(j, \alpha)$ with respect to μ, where for almost all $\alpha \in C_\omega$ we have:

$$H(j) = [\alpha \mapsto [\beta \mapsto |T_w^{++} \restriction \beta(\ell(j, \alpha))|]_{\mu_\alpha}]_\mu.$$

This is well-defined from the definition of w and the fact that each μ_α is normal. Let z, v be the reals corresponding to ℓ just as y, u correspond to $\bar{g}$. So, z codes a function $\bar{f}_z$ from $\omega \cdot \vartheta$ to κ, and T_v^+ is well-founded.

Consider the run of the game where player I plays $c = \langle x, y, u, z, v, w \rangle$, and player II responds with $c' = \tau(c) = \langle x', y', u', z', v', w' \rangle$. Let η be the least point in C_0 greater than $\max\{\sup(f_x), \sup(f_y), \sup(f_z)\}$ which has cofinality greater than ϑ. Since $\bar{f}_x = \bar{F}$ has range in C_0, it follows that x' also codes a function $\bar{f}_{x'} : \omega \cdot \vartheta \to \kappa$, and the first function they jointly produce, namely,

$$F_{x.x'}(i) = \sup_n \max\{\bar{f}_x(i, n), \bar{f}_{x'}(i, n)\}$$

is equal to F.

Now consider $\alpha > \eta$ in C_ω. For such an α and any $j = (i, n) < \omega \cdot \vartheta$, let $\delta = |T_u^+ \restriction \alpha(\bar{f}_y(j)| < \alpha^+$. An easy argument shows, as in the proof of the last section, that there is a μ measure one set of α such that the map $j \mapsto |T_u^+ \restriction \alpha(\bar{f}_y(j))|$ is increasing, and we may assume α is in this set. By definition we have $c \in A_{\eta,\alpha,j,\delta}$. Thus, $c' \in \tau[A_{\eta,\alpha,j,\delta}]$. Hence $T_{u'}^+ \restriction \alpha(\bar{f}_{y'}(j))$ is well-founded and $|T_{u'}^+ \restriction \alpha(\bar{f}_{y'}(j))| \leq \rho_1(\eta, \alpha, j, \delta)$.

Claim 25. For μ almost all α and all $i < \vartheta$, $\sup_n \bar{g}_{y,u}(i, n) \in C_1^\alpha$.

Proof. We have $\operatorname{ran}(G) \subseteq C_1'$. If for almost all α there is an $i < \vartheta$ for which this fails, them by the κ-completeness of μ we could fix a i which witnessed the failure for almost all α. Now, $G(i) = \sup_n \bar{G}(i, n)$ is represented with respect to μ by the function $\alpha \mapsto \sup_n \bar{g}_{y,u}(\alpha, (i, n))$.

So, for μ almost all α, $\sup_n \bar{g}_{y,uu}(\alpha, (i, n)) \in (C_1^\alpha)'$. □

It follows that $\rho_1(\eta, \alpha, j, \delta) < \sup_n \bar{g}_{y,u}(\alpha, (i, n)) = g_{y,u}(\alpha, i)$, where $j = (i, m)$ for some m. Thus, for μ almost all α we have that for all $i < \vartheta$ that

$$\sup_n \max\{\bar{g}_{y,u}(\alpha, (i, n)), \bar{g}_{y',u'}(\alpha, (i, n))\} = g_{y,u}(\alpha, i).$$

It follows that the second function $G_{y,u,y',u'}$ that the players jointly produce is equal to G.

By a similar argument, the third function $H_{z,v,w,z',v',w'}$ they jointly produce is equal to H. Here we consider $\alpha > \eta$ and $\beta < \alpha^+$ such that $\beta > \sup_j \bar{g}_{y,u}(\alpha, j)$, and $\beta > \sup_j \bar{g}_{z,v}(\alpha, j)$. For such α, β we have that $c \in A_{\eta,\alpha,j,\beta,\delta}$, where $\delta = \bar{h}_{z,v,w}(\alpha, \beta, j)$. We assume here that α is in the μ measure one set such that the function $(\beta, j) \mapsto \bar{h}_{z,v,w}(\alpha, \beta, j)$ is order-preserving when restricted to a c.u.b. $C \subseteq \alpha^+$ (as in the proof of the previous section). An easy argument as above shows that we may assume that for μ almost all α, and for μ_α almost all β, and all $i < \vartheta$ that $\sup_n \bar{h}_{z,v,w}(\alpha, \beta, (i.n)) \in C_2^\alpha$. Thus, $\forall_\mu^* \alpha\ \forall_{\mu_\alpha}^* \beta\ \forall i < \vartheta\ (\sup_n \bar{h}_{z',v',w'}(\alpha, \beta, (i, n)) = \sup_n \bar{h}_{z,v,w}(\alpha, .\beta, (i, n)))$. It follows that $H_{z,v,w,z',v',w'} = H$. Since τ is winning for Player II it follows that $\mathcal{P}(F, G, H) = 1$, and we are done. □

2.5. **The strong polarized partition property.** In this section, we now prove the optimal result, consequences of which will be used in the applications in §3.5.

Theorem 26. *Assume* AD. *Let κ be a weakly inaccessible Suslin cardinal. Then* $(\kappa, \kappa^+, \kappa^{++}) \to (\kappa, \kappa^+, \kappa^{++})^\kappa$.

The proof of Theorem 26 is again similar to that of the previous results, and we again use the trees T^+ and T^{++} from before. Let $\mathcal{P}$ denote the given partition of the block functions of the correct type from (κ, κ, κ) to $(\kappa, \kappa^+, \kappa^{++})$.

We now code functions from κ to κ using the uniform coding lemma (cf. [KKMW81]). Let $U \subseteq \omega^\omega \times \omega^\omega$ be universal for the syntactic class $\mathbf{\Sigma}_1(Q)$, where Q is a binary predicate symbol. Recall $A \in \mathbf{\Sigma}_1(Q)$ if $A(x) \leftrightarrow A'(x_0, x) \leftrightarrow \exists y\ (B(x,y) \wedge \forall n\ Q((y)_n))$, where $B \in \mathbf{\Sigma}^1_1$. So, we may define the universal set U by: $U_z(x,y) \leftrightarrow \exists w\ (S(z, \langle x, y\rangle, w) \wedge \forall n\ Q((w)_n))$, where S is universal for $\mathbf{\Sigma}^1_1$.

Recall that P is our $\mathbf{\Gamma}$-complete set, and $\{\varphi_m\}$ a $\mathbf{\Gamma}$-scale on P (with norms onto κ). Let $P_\alpha = \{x \in P\,;\, |x| = \alpha\}$ be the set of codes for $\alpha < \kappa$. Also, R is the prewellordering on P given by φ_0, so $R \in \mathbf{\Gamma}$. For $\alpha < \kappa$, recall also that $R_\alpha = \{(x,y) \in R\,;\, \varphi_0(y) < \alpha\}$ is the restriction of R to reals of norm less than α. Let R'_α be the restriction of R to reals of norm $\leq \alpha$.

Let $\{\rho_n\}$ be a $\mathbf{\Gamma}$-scale on R, and let ρ_α (or ρ'_α) denote its restriction to R_α (or R'_α). For any $\alpha < \kappa$, ρ_α is a $\mathbf{\Delta}$-scale on R_α (similarly for ρ'_α). Uniformly in α, the scale ρ'_α induces a scale on $U(R'_\alpha)$. This gives, uniformly in α, a uniformizing relation $\overline{U}(R'_\alpha) \in \mathbf{\Delta}$ (uniformizing on the last coordinate) of $U(R'_\alpha)$. In fact, $\overline{U}(R'_\alpha)$ is in the projective hierarchy containing R'_α.

For $\alpha < \kappa$, let $\alpha' < \kappa$ be the least reliable ordinal $\geq \alpha$ (with respect to the scale $\{\varphi_n\}$ on P). We let $G\colon \kappa^\omega \to \omega^\omega$ be the Lipschitz continuous generic coding function from the Kechris-Woodin theory of generic codes for uncountable ordinals (cf. [KW08] for the theory of generic codes). This means G has the following properties. For all $s \in \kappa^\omega$, $G(s) \in P$. Also, for all $\alpha < \kappa$, and any $s \in (\alpha')^\omega$ enumerating an honest set $S \subseteq \alpha'$, $|G(\alpha^\frown s)| = \alpha$. Here (and throughout this section) $|z|$ denotes $\varphi_0(z)$. For $\alpha < \kappa$, we say that comeager many $x \in P_\alpha$ have property B (where $B \subseteq \omega^\omega$), written $\forall^* x \in P_\alpha\ B(x)$, if player II has a winning strategy in the game where players I and II play $s_i \in (\alpha')^{<\omega}$ and player II wins the run if and only if $G(\alpha^\frown s_0^\frown s_1^\frown \cdots) \in B$. If B is Suslin and co-Suslin, then this game is Suslin and co-Suslin as well, and hence determined (cf. [KKMW81, Theorem 2.5]). We also write $\forall^* s \in (\alpha')^\omega$ to denote that player II has a winning strategy in the game where players I and II play $s_i \in (\alpha')^{<\omega}$ to produce $s = s_0^\frown s_1^\frown s_2 \cdots$.

We code functions from κ to κ using the uniform coding lemma as follows. For any $f\colon \kappa \to \kappa$ there is a real x such that for all $\alpha < \kappa$, the set $U_x(R_\alpha)$ codes $f{\restriction}\alpha$. That is, $U_x(R_\alpha)(a,b)$ if and only if $\varphi_0(a) < \alpha$, $b \in P$, and $\varphi_0(b) = f(\varphi_0(a))$. We say x **codes a function** at α if $U_x(R'_\alpha)$ satisfies:

(i) For all a, $|a| = \alpha$ implies that there is some b with $U_x(R'_\alpha)(a,b)$.
(ii) For all a, a', b, and b', we have that $(U_x(R'_\alpha)(a,b) \wedge U_z(R'_\alpha)(a',b') \wedge |a| = |a'| = \alpha \to |b| = |b'|)$ holds.

We code functions from κ to κ^+ as follows. Given $y \in \omega^\omega$, and given $\delta < \alpha \in C_0$, we say y is good at (δ, α) if for comeager many $a \in P_\delta$, there is a (unique) b such that $\overline{U}_y(R'_\delta)(a,b)$, and for this b we have that $T^+_b{\restriction}\alpha$ is well-founded. We let $g_y(\delta, \alpha)$ be the least ordinal $<\alpha^+$ such that for comeager many $a \in P_\delta$, and b as above,

$|T_b^+ \restriction \alpha| \leq g_y(\delta, \alpha)$. This is well-defined using the fact that $\mathrm{cf}(\alpha^+) > \omega$ and the additivity of category.

We say y is good at $\alpha \in C_\omega$ if y is good at (δ, α) for all $\delta < \alpha$. If y is good at α for μ almost all $\alpha \in C_\omega$, then y codes the function $G_y \colon \kappa \to \kappa^+$ defined by $G_y(\delta) = [\alpha \mapsto g_y(\delta, \alpha)]_\mu$.

We code functions from κ to κ^{++} as follows. Given $z \in \omega^\omega$, $\delta < \alpha \in C_\omega$ and $\beta < \alpha^+$, we say z is good at (δ, α, β) if for comeager many $a \in P_\delta$, if $\overline{U}_z(R'_\delta)(a, b)$, then $T_b^{++} \restriction \beta$ is well-founded. We let $h_z(\delta, \alpha, \beta)$ be the least ordinal $< \alpha^+$ such that for comeager many $a \in P_\delta$, and b as above, $|T_b^{++} \restriction \beta| \leq h_z(\delta, \alpha, \beta)$. We say z is good at (α, β) if for all $\delta < \alpha$ we have that z is good at (δ, α, β). We say z is good at α if for all $\delta < \alpha$ and all $\beta < \alpha^+$, z is good at (δ, α, β). If for μ almost all $\alpha \in C_\omega$ and μ_α almost all $\beta < \alpha^+$ we have that z is good at (α, β), then z codes the function $H_z \colon \kappa \to \kappa^{++}$ given by

$$H_z(\delta) = [\alpha \mapsto [\beta \mapsto g_z(\delta, \alpha, \beta)]_{\mu_\alpha}]_\mu.$$

Consider the game G where player I plays out reals (x, y, z) and player II plays out (x', y', z'). Let $\alpha < \kappa$ be the least ordinal, if one exists, such that one of the following holds.

(1) For some $\delta < \alpha$ we have that y or y' is not good (δ, α).
(2) For some $\delta < \alpha$ and $\beta < \alpha^+$ we have that z or z' is not good at (δ, α, β).
(3) x or x' does not code an ordinal at α.

Suppose first that an $\alpha < \kappa$ satisfying (1), (2), or (3) exists, and let α be the least such. First we check to see if case (1) holds at α. If so, then player II wins the run if and only if for the least δ as in (1) we have that y is not good at (δ, α). Suppose next that case (1) does not hold at α. Then we check to see if case (2) holds at α. If so, and if (β, δ) is the lexicographically least pair as in (2), then player II wins the run if and only if z is not good at (δ, α, β). Suppose next that (1) and (2) do not hold at α, but case (3) holds. Player II then wins provided x does not code an ordinal at α.

Finally, suppose that there is no $\alpha < \kappa$ satisfying (1), (2), or (3). So, x, x', both code functions f_x, $f_{x'}$ from κ to κ. Let $F \colon \kappa \to \kappa$ be defined from f_x and $f_{x'}$ as usual, that is, $F(\beta) = \sup_{j < \omega \cdot (\beta+1)} \max\{f_x(j), f_{x'}(j)\}$. Similarly, y and y' code functions G_y, $G_{y'}$ from κ to κ^+. These determine the function $G \colon \kappa \to \kappa^+$ in the usual way. Likewise, z and z' determine H_z, $H_{z'} \colon \kappa \to \kappa^{++}$ which then determine $H \colon \kappa \to \kappa^{++}$.

Player II then wins the run of the game if and only if $\mathcal{P}(F, G, H) = 1$. Suppose without loss of generality that player II has a winning strategy τ for the game, and we define homogeneous sets $C_0 \subseteq \kappa$, $C_1 \subseteq \kappa^+$, and $C_2 \subseteq \kappa^{++}$.

For $\eta_1, \eta_2 < \kappa$, let $A(\eta_1, \eta_2)$ be the set of (x, y, z) satisfying the following:

(a) y is good at α for all $\alpha \leq \eta_1$.
(b) z is good at α for all $\alpha \leq \eta_1$.
(c) x codes a function at all $\alpha \leq \eta_1$ and $f_x(\alpha) \leq \eta_2$.

A straightforward computation using the closure of $\mathbf{\Delta}$ under quantifiers shows that $A(\eta_1, \eta_2) \in \mathbf{\Delta}$. From the definition of G, if $(x, y, z) \in A(\eta_1, \eta_2)$ and $(x', y', z') = \tau(x, y, z)$, then x' codes a function at all $\alpha \leq \eta_1$. By boundedness (since $\mathbf{\Gamma}$ is closed under $\wedge$, $\vee$), it follows that

$$\rho_0(\eta_1, \eta_2) := \sup\{f_{x'}(\alpha)\,;\, \alpha \leq \eta_1 \wedge (x', y', z') \in \tau[A(\eta_1, \eta_2)]\} < \kappa.$$

Let $C_0 \subseteq \kappa$ be a c.u.b. subset closed under ρ_0.

For $\alpha \in C_\omega$, $\delta < \alpha$, and $\eta < \alpha^+$, let $A(\delta, \alpha, \eta)$ be the set of (x, y, z) satisfying:

(a) y and z are good at all $\alpha' < \alpha$, and for all $\alpha' < \alpha$, x codes a function at α' with $f_x(\alpha') < \alpha$.

(b) For all $\delta' \leq \delta$, y is good at (δ', α) and $g_y(\delta', \alpha) \leq \eta$.

Lemma 27. *For $\delta < \alpha \in C_\omega$, if $X(x, y) \in \mathbf{\Sigma}^\alpha_1$, then $X'(x) \leftrightarrow \forall^* s \in (\delta')^\omega\ X(G(s), y)$ is also in $\mathbf{\Sigma}^\alpha_1$.*

Proof. Write $X(x, y) \leftrightarrow \exists z\ Y(x, y, z)$, where $Y \in \mathbf{\Pi}^\alpha_0$. Fix a non-selfdual pointclass $\mathbf{\Gamma}_0$ closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$ contained within $<\alpha$-Suslin and which has prewellorderings of length at least δ' (the least reliable $\geq \delta$). Using the coding lemma, we code strategies on δ' by reals. We then have:

$$z \in X' \leftrightarrow \exists w[w \text{ codes a strategy } \tau_w \colon (\delta'^{<\omega})^\omega \to (\delta'^{<\omega})^\omega \times \omega^\omega \times \omega^\omega$$
$$\wedge\, \forall (s, y, z) \text{ a run according to } \tau_w,\ Y(G(s), y, z)].$$

Saying w codes a strategy is projective over $\mathbf{\Gamma}_0$, as is coding a run according to τ_w. Since $Y \in \mathbf{\Pi}^\alpha_0$, it follows that X' is $\mathbf{\Sigma}^\alpha_1$. □

Claim 28. $A(\delta, \alpha, \eta) \in \mathbf{\Delta}^\alpha_1$.

Proof. The set $B = \{w\,;\, |T^+_w \restriction \alpha| \leq \eta\}$ is in $\mathbf{\Delta}^\alpha_1$. Since $\mathbf{\Delta}^\alpha_1$ is closed under $<\alpha^+$ unions and intersections (Theorem 8), it is enough to show that $A' = A'(\delta, \alpha, \eta)$ is $\mathbf{\Delta}^\alpha_1$, where

$$z \in A' \leftrightarrow \forall^* a \in P_\delta\ \exists b\ (\overline{U}_z(R'_\delta)(a, b) \wedge |T^+_b \restriction \alpha| \leq \delta).$$

It is enough, by a symmetrical argument, to show that $A' \in \mathbf{\Sigma}^\alpha_1$. This follows immediately from Lemma 27. □

If $y \in A(\delta, \alpha, \eta)$ and $y' = \tau(y)$, then y' is good at (δ, α). This follows from the winning conditions for player II in the game G, specifically the fact that case (1) is considered at stage α first.

Claim 29. $\sup\{g_{y'}(\alpha, \delta)\,;\, y' \in \tau[A(\delta, \alpha.\eta)]\} < \alpha^+$.

Proof. The supremum in question has length bounded by the length of the following ordering:

$$y_1 \prec y_2 \leftrightarrow (y_1, y_2 \in \tau[A(\delta, \alpha, \eta)]) \wedge \exists s_0 \in (\delta')^{<\omega}\ \forall^* s_1 \in (\delta')^\omega\ \forall^* s_2 \in (\delta')^\omega$$
$$\exists b_1, b_2\ (\overline{U}_{y_1}(R'_\delta)(G(s), b_1) \wedge (\overline{U}_{y_2}(R'_\delta)(G(s_0^\frown s), b_2) \wedge |T^+_{b_1} \restriction \alpha| < |T^+_{b_2} \restriction \alpha|))$$

It follows from Lemma 27 that $\preceq\, \in \mathbf{\Sigma}^\alpha_1$ provided we show that there is a $\mathbf{\Sigma}^\alpha_1$ relation $S(b_1, b_2)$ which when restricted to pairs such that $T^+_{b_1} \restriction \alpha$ and $T^+_{b_2} \restriction \alpha$ are well-founded correctly computes the relation $|T^+_{b_1} \restriction \alpha| \leq |T^+_{b_2} \restriction \alpha|$. To see this, let $\mathbf{\Gamma}_n$ be a sequence of non-selfdual pointclasses closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$ of Wadge ranks cofinal in α. Let U_n be universal sets for $\mathbf{\Gamma}_n$. Let ψ_n be a $\mathbf{\Gamma}_n$ prewellordering of length α_n, where $\sup_n \alpha_n = \alpha$. Each real z codes the relation $R_z \subseteq \alpha \times \alpha$ given by $R_z = \bigcup_n R^n_z$, where R^n_z is the relation on α_n defined by $(\alpha, \beta) \in R^n_z$ if and only if there are u and v such that $\psi_n(u) = \alpha$, $\psi_n(v) = \beta$, and $U_n((z)_n, u, v)$. From the coding lemma, every relation on α is coded in this manner by some z. We can then say that $S(b_1, b_2)$ holds if and only if there is a z such that R_z is an order-preserving map from $T^+_{b_1} \restriction \alpha$ to $T^+_{b_1} \restriction \alpha$. Using the closure of $\mathbf{\Delta}^\alpha_1$ under $<\alpha$ unions and intersections (Theorem 8), it is straightforward to verify that $S \in \mathbf{\Sigma}^\alpha_1$. q.e.d. (Claim 29)

Now let

$$\rho_1(\delta,\alpha,\eta) = \sup\{g_{y'}(\alpha,\delta)\,;\, y' \in \tau[A(\delta,\alpha.\eta)]\}.$$

Let $C_1'(\alpha)$ be the c.u.b. subset of α^+ of points closed under ρ_1. Let $C_1' = [\alpha \mapsto C_1'(\alpha)]_\mu$, so C_1' is a c.u.b. subset of κ^+. Let C_1'' be a c.u.b. subset of κ^+ so that between any two elements $\rho_1 = [f]_\mu < [g]_\mu = \rho_2$ of C_1'', there is a $b \in \omega^\omega$ such that T_b^+ is well-founded and $\forall_\mu^* \alpha\ f(\alpha) < |T_b^+ \restriction \alpha| < g(\alpha)$. Let $C_1 = C_1' \cap C_1''$.

Lastly, we define $C_2 \subseteq \kappa^{++}$. For $\delta < \alpha \in C_\omega$, and $\beta < \eta < \alpha^+$, let $A(\delta,\alpha,\beta,\eta)$ be the set of (x,y,z) satisfying:

(a) y and z are good at all $\alpha' < \alpha$, and for all $\alpha' < \alpha$, x codes a function at α' with $f_x(\alpha') < \alpha$.
(b) y is good at (δ',α) for all $\delta' < \alpha$, and $g_y(\delta',\alpha) < \beta$.
(c) For all $(\beta',\delta') \leq_{\text{lex}} (\beta,\delta)$, z is good at (β',δ') and $h_z(\delta',\alpha,\beta') \leq \eta$.

A computation as in the proof of Claim 28 shows that $A(\delta,\alpha,\beta,\eta) \in \mathbf{\Delta}_1^\alpha$. If $(x,y,z) \in A(\delta,\alpha,\beta,\eta)$ and $(x',y',z') = \tau(x,y,z)$, then from the winning conditions on G it follows that z' is good at (δ,α,β). A computation as in Claim 29 shows that

$$\rho_2(\delta,\alpha,\beta,\eta) := \sup\{h_{z'}(\alpha,\delta,\beta)\,;\, z' \in \tau[A(\delta,\alpha.\beta,\eta)]\} < \alpha^+.$$

Let $C_2'(\alpha)$ be c.u.b. in α^+ and closed under ρ_2. Let C_2' be those $\rho < \kappa^{++}$ such that $\rho = [\alpha \mapsto [\beta \mapsto \ell(\alpha,\beta)]_{\mu_\alpha}]_\mu$, where $\ell(\alpha,\beta) \in C_2'(\alpha)$. An easy argument shows that C_2' is c.u.b. in κ^{++}. Let C_2'' be c.u.b. in κ^{++} such that between any two ordinals $\rho_1 < \rho_2$ of C_2'', there is a z such that T_z^{++} is well-founded and $\rho_1 < [\alpha \mapsto [\beta \mapsto |T_z^{++} \restriction \beta|]] < \rho_2$. From the proof of Claim 19 it follows that such a C_2'' exists. Let $C_2 = C_2' \cap C_2''$.

Now suppose that (F,G,H) are block functions of the correct type into the block c.u.b. sets (C_0,C_1,C_2), and we show that $\mathcal{P}(F,G,H) = 1$.

Let $\bar{F}\colon \omega\cdot\kappa \to C_0$ induce F, and let x code the function $\bar{F}$, that is, x is good at all $\alpha < \kappa$ and $f_x = \bar{F}$.

Let $\bar{G}\colon \omega\cdot\kappa \to C_1$ induce G. There is a function $\bar{g}$ which induces $\bar{G}$ in the following sense. $\bar{g}(\delta,\alpha)$ is defined for all $\delta < \alpha \in C_\omega$, and $\bar{g}(\delta,\alpha) < \alpha^+$. Also, $\bar{G}(\delta) = [\alpha \mapsto \bar{g}(\delta,\alpha)]_\mu$ for all $\delta < \kappa$. From the normality of μ, we may assume without loss of generality that $\bar{g}(\delta,\alpha) \in C_1'(\alpha)$ for all $\alpha \in C_\omega$ and that for all $\alpha \in C_\omega$ that $\delta \mapsto \bar{g}(\delta,\alpha)$ is strictly increasing. We code (some) c.u.b. subsets of κ by reals as follows. Say σ is a code if for all $w \in P$, $\sigma(w) \in P$. In this case, let C_σ be the c.u.b. subset of κ closed under σ, that is, $C_\sigma = \{\alpha\,;\, \forall w\ (w \in P_{<\alpha} \to \sigma(w) \in P_{<\alpha})\}$. An easy boundedness argument shows that C_σ is actually c.u.b. in κ. Also, an easy Solovay game argument shows that every c.u.b. $C \subseteq \kappa$ contains a subset of the form C_σ. Since $\bar{G}$ has range in C_1'', for each $\delta < \kappa$ there are reals w such that T_w^+ is well-founded and $\forall_\mu^* \alpha\ \bar{g}(\delta,\alpha) \leq |T_w^+ \restriction \alpha| < \bar{g}(\delta+1,\alpha)$. For each $\delta < \kappa$, let $\bar{G}'(\delta)$ be the least ordinal between $\bar{G}(\delta)$ and $\bar{G}(\delta+1)$ which is of the form $[\alpha \mapsto |T_w^+ \restriction \alpha]_\mu$ for some w with T_w^+ well-founded. There is a function $\bar{g}'$, with $\bar{g}'(\delta,\alpha)$ defined for $\delta < \alpha \in C_\omega$, such that for all $\delta < \kappa$ we have $\bar{G}'(\delta) = [\alpha \mapsto \bar{g}'(\delta,\alpha)]_\mu$ and for all α, $\delta \mapsto \bar{g}'(\delta,\alpha)$ is increasing. Also, we may assume $\bar{g}(\delta,\alpha) \leq \bar{g}'(\delta,\alpha) < \bar{g}(\delta+1,\alpha)$ for all $\delta < \alpha \in C_\omega$.

From the uniform coding lemma, let $y \in \omega^\omega$ be such that:

(1) For all $\delta < \kappa$ and all $a \in P_\delta$, there is a (unique) $\langle b, c\rangle$ such that $\overline{U}_y(R'_\delta)(a, \langle b, c\rangle)$.
(2) For all $a \in P_\delta$ and $\langle b, c\rangle$ such that $\overline{U}_y(R'_\delta)(a, \langle b, c\rangle)$, T_b^+ is well-founded and $[\alpha \mapsto |T_b^+ \restriction \alpha|]_\mu = \bar{G}'(\delta)$.
(3) For such a, b, c we have that C_c codes a c.u.b. subset of κ such that for all $\alpha \in C_c \cap C_\omega$ we have that $|T_b^+ \restriction \alpha| = \bar{g}'(\delta, \alpha)$ (where $\delta = |a|$ as above).

For $\delta < \alpha \in C_\omega$, let

$$\ell(\delta, \alpha) = \sup\{|c(x)|\,;\, \exists a, b, x\ (a \in P_\delta \wedge \overline{U}_y(R'_\delta)(a, \langle b, c\rangle) \wedge x \in P_{<\alpha}\}.$$

By boundedness, $\ell(\delta, \alpha) < \kappa$. Let $E \subseteq \kappa$ be a c.u.b. subset closed under ℓ. For all $\alpha \in E \cap C_\omega$ we have that y is (δ, α) good and $g_y(\delta, \alpha) = \bar{g}'(\delta, \alpha)$ for all $\delta < \alpha$.

In a similar manner, we let $\bar{H} \colon \kappa \to \kappa^{++}$ induce H, and let $\bar{h}$ be defined for $\delta < \alpha \in C_\omega$ and $\beta < \alpha^+$ and such that for all $\delta < \omega \cdot \kappa$, $\bar{H}(\delta) = [\alpha \mapsto [\beta \mapsto \bar{h}(\delta, \alpha, \beta)]_{\mu_\alpha}]_\mu$. We may assume that for all $\alpha < \kappa$ that $(\beta, \delta) \mapsto \bar{h}(\delta, \alpha, \beta)$ is increasing (with respect to lexicographic order), and $\bar{h}(\delta, \alpha, \beta) \in C_2''(\alpha)$ for all $\delta < \alpha \in C_\omega$ and $\beta < \alpha^+$. For $\delta < \omega \cdot \kappa$, let $\bar{H}'(\delta) < \bar{H}(\delta + 1)$ be least such that for some $w \in \omega^\omega$, T_w^{++} is well-founded and $\bar{H}'(\delta) = [\alpha \mapsto [\beta \mapsto [|T_w^{++} \restriction \beta|]]_{\mu_\alpha}]_\mu$. An easy argument shows that there is an $\bar{h}'$ such that for all $\delta < \alpha \in C_\omega$ and $\beta < \alpha^+$ we have $\bar{h}(\delta, \alpha, \beta) < \bar{h}'(\delta, \alpha, \beta) < \bar{h}(\delta + 1, \alpha, \beta)$, and for each δ there is a real w such that T_w^{++} is well-founded and $\forall_\mu^* \alpha\ \forall_{\mu_\alpha}^* \beta\ \bar{h}'(\delta, \alpha, \beta) = |T_w^{++} \restriction \beta|$. We say a pair (σ, w) codes a measure one set if σ codes a c.u.b. subset C_σ of κ (as above) and T_w^{++} is well-founded. We let $A_{(\sigma, w)} = \{(\alpha, \beta)\,;\, \alpha \in C_\sigma \wedge \forall \gamma < \beta\ (|T_w^{++} \restriction \gamma| < \beta)\}$. Using the tree T^{++} it follows that if A has measure one in the sense that $\forall_\mu \alpha\ \forall_{\mu_\alpha}^*\ \beta\ (\alpha, \beta) \in A$, then there is a (σ, w) with $A_{(\sigma, w)} \subseteq A$. From the uniform coding lemma, fix a real z such that:

(1) For all $\delta < \omega \cdot \kappa$ and all $a \in P_\delta$, there is a (unique) $\langle b, c, d\rangle$ such that $\overline{U}_z(R'_\delta)(a, \langle b, c, d\rangle)$.
(2) For all $a \in P_\delta$ and $\langle b, c, d\rangle$ such that $\overline{U}_z(R'_\delta)(a, \langle b, c, d\rangle)$, T_b^{++} is well-founded and $[\alpha \mapsto [\beta \mapsto |T_b^{++} \restriction \beta|]_{\mu_\alpha}]_\mu = \bar{H}'(\delta)$.
(3) For such a, b, c, d we have that (c, d) codes a measure one set $A_{(c,d)}$ such that for all $(\alpha, \beta) \in A_{(c,d)}$ we have that $|T_b^+ \restriction \beta| = \bar{h}'(\delta, \alpha, \beta)$ (where again $\delta = |a|$).

For $\delta < \alpha \in D_0$ define:

$$\ell_1(\delta, \alpha) = \sup\{|c(x)|\,;\, \exists a, b, d\ (a \in P_\delta \wedge \overline{U}_z(R'_\delta)(a, \langle b, c, d\rangle) \wedge x \in P_{<\alpha}\}.$$

For $\delta < \alpha \in D_0$ and $\beta < \alpha^+$ define:

$$\ell_2(\delta, \alpha, \beta) = \sup\{|T_d^{++} \restriction \eta|\,;\, \exists a, b, c\ (a \in P_\delta \wedge \overline{U}_z(R'_\delta)(a, \langle b, c, d\rangle) \wedge \eta < \beta\}.$$

A boundedness argument as before shows that $\ell_1(\delta, \alpha) < \kappa$. Likewise, a tree argument shows that $\ell_2(\delta, \alpha, \beta) < \alpha^+$.

Let $D \subseteq \kappa$ be the c.u.b. set of points closed under ℓ_1. For $\alpha \in D \cap C_\omega$, let E_α be the c.u.b. subset of α^+ closed under ℓ_2. So, if $\alpha \in D \cap C_\omega$ and $\beta \in E_\alpha$, then $(\alpha, \beta) \in A_{(c,d)}$ for all (c, d) such that for some $a \in P_\delta$, $\delta < \alpha$, we have $\overline{U}_z(R'_\delta)(a, \langle b, c, d\rangle)$.

Now consider the run of the game G where player I plays out (x, y, z) and player II responds with $\tau(x, y, z) = (x', y', z')$. First note that x, y, and z are all fully good (with the obvious meaning). In particular x' codes a function $f_{x'} \colon \kappa \to \kappa$. For

any $\alpha < \omega \cdot \kappa$, $(x, y, z) \in A(\alpha, \bar{F}(\alpha))$. Also, $A(\alpha, \bar{F}(\alpha)) \in \mathbf{\Delta}$. So, $(x', y', z') \in \tau[A(\alpha, \bar{F}(\alpha)]$ and therefore $f_{x'}(\alpha) < \rho_0((\alpha), \bar{F}(\alpha)) < \bar{F}(\alpha+1)$ since $\bar{F}$ has range in C_0 which is closed under ρ_0. So, the function $F_{x,x'}$ jointly produced from f_x and $f_{x'}$ is equal to F.

Let E_y be the c.u.b. subset of κ as defined above, the set of closure points of $\ell = \ell_y$. Let $\bar{g}$, $\bar{g}'$ be as in the definition of y, so $\bar{g}(\delta, \alpha) \leq \bar{g}'(\delta, \alpha) < \bar{g}(\delta + 1, \alpha)$ for all $\delta < \alpha \in C_\omega$. Also let g_y be the function coded by y, since y is good. That is, $g_y(\delta, \alpha)$ is the least $\gamma < \alpha^+$ such that for comeager many $a \in P_\delta$, if $\overline{U}_y(R'_\delta)(a, \langle b, c\rangle)$, then $|T_b^+ \restriction \alpha| = g_y(\delta, \alpha)$. So, for all $\alpha \in E_y \cap C_\omega$ and $\delta < \alpha$ we have $g_y(\delta, \alpha) = \bar{g}'(\delta, \alpha)$. If α is in addition closed under the function F, then we have that $(x, y, z) \in A(\delta, \alpha, \bar{g}'(\delta, \alpha))$. For such δ, α it follows that $g_{y'}(\delta, \alpha) < \ell(\delta, \alpha, \bar{g}'(\delta, \alpha)) < \bar{g}(\delta+1, \alpha)$. So, for all $\delta < \omega\cdot\kappa$, $\bar{G}_y(\delta) = [\alpha \mapsto g_y(\delta, \alpha)]_\mu$ and $G_{y'}(\delta)$ are both less than $\bar{G}_y(\delta + 1, \alpha)$. So, $\sup_{\delta' < \omega\cdot(\delta+1)} \max\{\bar{G}_y(\delta'), \bar{G}_{y'}(\delta')\} = G(\delta)$. So, the function jointly produced by y and y' is equal to G.

The argument for z, z' is similar. Recall that $H\colon \kappa \to \kappa^{++}$, $\bar{H}\colon \omega\cdot\kappa \to \kappa^{++}$, and $\bar{h}(\delta, \alpha, \beta)$ induces $\bar{H}$, that is, $\bar{H}(\delta) = [\alpha \mapsto [\beta \mapsto \bar{h}(\delta, \alpha, \beta)]_{\mu_\alpha}]_\mu$. Also, $\bar{h}'$ is fixed and $\bar{h}(\delta, \alpha, \beta) \leq \bar{h}'(\delta, \alpha, \beta) < \bar{h}(\delta + 1, \alpha, \beta)$. Let $D_z \subseteq \kappa$ be the c.u.b. set of points closed under ℓ_1 as above. For $\alpha \in D_z \cap C_\omega$, let $E_z^\alpha \subseteq \alpha^+$ be the c.u.b. set of points closed under ℓ_2 (more precisely, the function $(\delta, \beta) \mapsto \ell_2(\delta, \alpha, \beta)$). Consider (α, β) such that $\alpha \in E_y$, $\alpha \in D_z \cap C_\omega$, $\beta \in E_z^\alpha$, α is closed under F, $\beta > \sup_{\delta<\alpha}\{\bar{g}'(\delta, \alpha)\}$, and for all $\beta' < \beta$ and $\delta < \alpha$, $h_z(\delta, \alpha, \beta') < \beta$. This set of pairs A has measure one set with respect to the iterated measure, that is, $\forall^*_\mu \alpha\ \forall^*_{\mu_\alpha} \beta\ (\alpha, \beta) \in A$. For $(\alpha, \beta) \in A$, z is in the set $A(\delta, \alpha, \beta, \bar{h}'(\delta, \alpha, \beta))$ for all $\delta < \alpha$. Since $\bar{h}$ has its range in the $C_2'(\alpha)$, $h_{z'}(\delta, \alpha, \beta) < \bar{h}(\delta + 1, \alpha, \beta)$. Thus, for all $\delta < \omega \cdot \kappa$, $\bar{H}_z(\delta)$ and $\bar{H}_{z'}(\delta)$ are both less than $\bar{H}(\delta + 1)$. It follows that the function jointly produced by z and z' is equal to H.

Since τ is winning for player II, it follows that $\mathcal{P}(F, G, H) = 1$, and we are done.

3. Application to choiceless set theory

As mentioned in § 1, our main application and the motivation for proving Theorem 26 is the determination of the consistent patterns of cofinality and measurabilty for the first three uncountable cardinals.

We shall use the labels $\mathbf{M}$ and $\aleph_n$, standing for "measurable" and "non-measurable and cofinality $\aleph_n$", respectively, and write

$$[\,\mathsf{x}_1 \,/\, \mathsf{x}_2 \,/\, \mathsf{x}_3\,]$$

for the statement "$\aleph_1$ has property x_1, $\aleph_2$ has property x_2, and $\aleph_3$ has property x_3". There are exactly 60 ($= 3 \times 4 \times 5$) such patterns: $\aleph_1$ can be measurable, regular non-measurable, or singular (3 possibilities); $\aleph_2$ can be measurable, regular non-measurable, or have cofinality $\aleph_1$ or $\aleph_0$ (4 possibilities); and $\aleph_3$ can be measurable, regular non-measurable, or have cofinality $\aleph_2$, $\aleph_1$, or $\aleph_0$ (5 possibilities).

A pattern $[\,\mathsf{x}_1 \,/\, \mathsf{x}_2 \,/\, \mathsf{x}_3\,]$ is called **trivially inconsistent** if there are $0 \leq k < i < j \leq 3$ such that $\mathsf{x}_i = \aleph_k$ and $\mathsf{x}_j = \aleph_i$. For example, $[\,\aleph_0 \,/\, \aleph_1 \,/\, \mathbf{M}\,]$ is trivially inconsistent. This is because $\aleph_1$ is singular, but $\mathrm{cf}(\aleph_2) = \aleph_1$, which is obviously impossible. A simple combinatorial calculation shows that there are 13 trivially inconsistent patterns. These are the patterns 13, 18, 33, 38, 44, 49, 51, 52, 53, 54, 55, 58, and 59 in our table in Figures 1 and 2.

3.1. **Forcing facts.**

Theorem 30. *If* $\mathbf{V} \models \mathsf{ZF}+$ *"κ is a measurable cardinal", $\mathbb{P}_\kappa$ is Příkrý forcing for κ, and G is $\mathbb{P}_\kappa$-generic over $\mathbf{V}$, then in the generic extension $\mathbf{V}[G]$, $\mathrm{cf}(\kappa) = \aleph_0$, any cardinal having cofinality κ in $\mathbf{V}$ now has cofinality $\aleph_0$, and the cofinalities and measurability of all other cardinals are unchanged.*

Proof. This is [Apt96, Lemmas 1.2, 1.3 and 1.5]. □

Theorem 31. *If* $\mathbf{V} \models \mathsf{ZF}+$ *"$\aleph_1$ is measurable", $\mathrm{Add}(\omega, \omega_1)$ is the partial order for adding ω_1 many Cohen reals, and G is $\mathrm{Add}(\omega, \omega_1)$-generic over $\mathbf{V}$, then in the generic extension $\mathbf{V}[G]$, $\aleph_1$ is regular but non-measurable and the cofinalities and measurability of all other cardinals are unchanged.*

Proof. The fact that $\aleph_1$ becomes non-measurable is a special case of the general ZF-result (due to Ulam) that if κ injects into 2^λ for some $\lambda < \kappa$, then there cannot be a κ-complete ultrafilter on κ (cf. [Kan94, Theorem 2.8]). Obviously, the ω_1-sequence of Cohen reals produces an injection of ω_1 into 2^ω. Since $\mathrm{Add}(\omega, \omega_1)$ is canonically well-orderable and $|\mathrm{Add}(\omega, \omega_1)| = \aleph_1$, the proof that all cardinals and cofinalities are preserved is the same as when AC is true. Since $|\mathrm{Add}(\omega, \omega_1)| = \aleph_1$, the argument given in the proof of [AH86, Lemma 2.1] shows that the measurability of all cardinals greater than $\aleph_1$ is preserved. □

Theorem 32. *If* $\mathbf{V} \models \mathsf{ZFC}+$ *"$\kappa < \lambda$ are measurable cardinals", then for $\mathsf{x}_3 \in \{\aleph_0, \aleph_1, \aleph_2, \aleph_3, \mathbf{M}\}$, there is a symmetric submodel N_{x_3} satisfying $[\,\mathbf{M} \,/\, \aleph_2 \,/\, \mathsf{x}_3\,]$. If $\mathsf{x}_3 \neq \mathbf{M}$, only one measurable cardinal is needed in $\mathbf{V}$.*

Proof. We sketch the proof of Theorem 32. Without loss of generality, we assume that GCH holds in $\mathbf{V}$. Let G_0 be $\mathrm{Col}(\omega, {<}\kappa)$-generic over $\mathbf{V}$, where for $\rho < \zeta$, ρ a regular cardinal, ζ a cardinal, $\mathrm{Col}(\rho, {<}\zeta)$ is the Lévy collapse of all cardinals less than ζ to ρ. For H which is $\mathrm{Col}(\rho, {<}\zeta)$-generic over $\mathbf{V}$ and $\xi \in (\rho, \zeta)$ a cardinal, let $H{\restriction}\xi$ be all elements of H which are members of $\mathrm{Col}(\rho, {<}\xi)$. Let G_1 be $\mathrm{Col}(\kappa^+, {<}\gamma)$-generic over $\mathbf{V}$, where γ is either $\kappa^{+\omega}$, $\kappa^{+\kappa}$, $\kappa^{+\eta}$ for $\eta = \kappa^+$, or λ. We write $\mathrm{HD}_{\mathbf{V}}(X)$ for the class of sets hereditarily $\mathbf{V}$-definable with a parameter from X. Consider the symmetric model $N_{\mathsf{x}_3} := \mathrm{HD}_{\mathbf{V}}(\{G_0{\restriction}\delta\,;\, \delta \in (\omega, \kappa) \text{ and } \delta \text{ is a cardinal}\} \cup \{G_1{\restriction}\delta\,;\, \delta \in (\kappa^+, \gamma) \text{ and } \delta \text{ is a cardinal}\})$. Since in $\mathbf{V}$, there is a κ^+ sequence of subsets of κ, standard arguments show that N_{x_3} is a model for $[\,\mathbf{M} \,/\, \aleph_2 \,/\, \aleph_0\,]$, $[\,\mathbf{M} \,/\, \aleph_2 \,/\, \aleph_1\,]$, $[\,\mathbf{M} \,/\, \aleph_2 \,/\, \aleph_2\,]$, or $[\,\mathbf{M} \,/\, \aleph_2 \,/\, \mathbf{M}\,]$, for γ either $\kappa^{+\omega}$, $\kappa^{+\kappa}$, $\kappa^{+\eta}$ for $\eta = \kappa^+$, or λ respectively. Since in $\mathbf{V}$, there is a κ^+ sequence of subsets of κ and a κ^{++} sequence of subsets of κ^+, $N_{\mathsf{x}_3} := \mathrm{HD}_{\mathbf{V}}(\{G_0{\restriction}\delta\,;\, \delta \in (\omega, \kappa) \text{ and } \delta \text{ is a cardinal}\})$ is a model for $[\,\mathbf{M} \,/\, \aleph_2 \,/\, \aleph_3\,]$. Clearly, the only time a second measurable cardinal is needed in the construction is for the pattern $[\,\mathbf{M} \,/\, \aleph_2 \,/\, \mathbf{M}\,]$. □

Theorem 33. *Suppose $i \in \omega$. Let $\mathbf{V} \models \mathsf{ZF}$ + "κ is a limit cardinal" + "$\lambda := \kappa^{+i}$". Let G be $\mathrm{Col}(\omega, {<}\kappa)$-generic over $\mathbf{V}$. Consider the model M obtained by symmetrically collapsing κ to $\aleph_1$, i.e., the model $M := \mathrm{HD}_{\mathbf{V}}(\{G{\restriction}\delta\,;\, \delta \in (\omega, \kappa)$ and δ is a cardinal$\})$. Then the following hold:*

(i) *If $\mathbf{V} \models$ "λ is measurable", then $M \models$ "$\lambda = \aleph_{i+1}$ is measurable".*
(ii) *If $\mathbf{V} \models$ "$\mathrm{cf}(\lambda) = \kappa^{+j}$ for some $j \leq i$", then $M \models$ "$\mathrm{cf}(\lambda) = \aleph_{j+1}$".*

Proof. Since $\mathbf{V} \models$ "$\mathrm{Col}(\omega, <\kappa)$ is canonically well-orderable and $|\mathrm{Col}(\omega, <\kappa)| = \kappa$", (i) follows from the argument given in the proof of [AH86, Lemma 2.1], the fact that $\kappa = \aleph_1$ in M, and the fact that cardinals at and above κ are preserved to M; (ii) follows from the fact that $\kappa = \aleph_1$ in M and the fact that cardinals and cofinalities at and above κ are preserved to M. □

3.2. **Magidor-like forcing.** In [Hen83], Henle introduced Magidor-like forcing for controlling the cofinalities of cardinals in choiceless contexts in the presence of partition properties. Assuming that $\kappa \to (\kappa)^{<\delta}$ and that δ is a regular, uncountable cardinal, Magidor-like forcing changes the cofinality of κ to δ without adding any bounded subsets to κ (thereby preserving the fact that κ is a cardinal; cf. [Hen83, Proposition 1.3]). We define the set $\mathcal{P}_{\delta,\kappa}$ by

$$\mathcal{P}_{\delta,\kappa} = \{\langle s, x\rangle \ ; \ s \in [\kappa]^{<\delta}, x \in [\kappa]^{\kappa}, \bigcup s < \bigcap x\}.$$

We use $\langle x\rangle$ to denote $\{{}_\omega q \ ; \ q \in [x]^\kappa\}$, where ${}_\omega q = \{\bigcup_{n<\omega} q(\alpha + n) \ ; \ \alpha < \kappa\}$.

The partial ordering for $\mathcal{P}_{\delta,\kappa}$ is now defined by saying that $\langle s', x'\rangle$ extends $\langle s, x\rangle$ if and only if $s \subseteq s'$, $\langle x'\rangle \subseteq \langle x\rangle$, and $s' \setminus s = {}_\omega t$ for some $t \in [x]^{<\delta}$. For $p \in \mathcal{P}_{\delta,\kappa}$, we denote the coordinates of p by p_0 and p_1, i.e., $p = \langle p_0, p_1\rangle$.

This was generalized in [AHJ00, §6] to the context of polarized partition properties. In the following, we shall need a preservation result from [AHJ00]:

Lemma 34 (Countable final segment preservation)**.** *If $(\kappa_0, \kappa_1) \to (\kappa_0, \kappa_1)^{<\delta}$ and δ is regular and uncountable, then after forcing with $\mathcal{P}_{\delta,\kappa_0}$, the relation $\kappa_1 \to (\kappa_1)^{<\omega_1}$ remains true.*

Proof. This follows from the proof of [AHJ00, Proposition 6.4]. □

3.3. **Reducing to base cases.** Of the 60 combinatorially possible patterns, we have already excluded 13 as trivially inconsistent. The remaining 47 patterns will be split into graphs according to the following rules:

- If $P = [\,\mathsf{x}_1 \,/\, \mathsf{x}_2 \,/\, \mathsf{x}_3\,]$ is a pattern with $\mathsf{x}_i = \mathbf{M}$, and $P' = [\,\mathsf{y}_1 \,/\, \mathsf{y}_2 \,/\, \mathsf{y}_3\,]$ is a pattern with $\mathsf{y}_i = \aleph_0$ and for $j \neq i$,

$$\mathsf{y}_j = \begin{cases} \mathsf{x}_j & \text{if } \mathsf{x}_j \neq \aleph_i, \text{ and} \\ \aleph_0 & \text{if } \mathsf{x}_j = \aleph_i, \end{cases}$$

  then there is an edge from P to P'. This corresponds to a forcing extension with Příkrý forcing according to Theorem 30.
- There is an edge from $[\,\mathbf{M} \,/\, \mathsf{x}_2 \,/\, \mathsf{x}_3\,]$ to $[\,\aleph_1 \,/\, \mathsf{x}_2 \,/\, \mathsf{x}_3\,]$. This corresponds to a forcing extension adding ω_1 many Cohen reals according to Theorem 31.

Because of Theorems 30 and 31, if P is consistent and there is an edge from P to P', then P' is consistent. This allows us to reduce the consistency of patterns to

the patterns that are top elements in the graph. We shall now list all components of this graph:

Base Case #1: [M/M/M].

$$\begin{tikzcd}
 & & & [\mathbf{M}/\mathbf{M}/\mathbf{M}] \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & & \\
[\aleph_0/\mathbf{M}/\mathbf{M}] \arrow[d] \arrow[dr] & [\mathbf{M}/\aleph_0/\mathbf{M}] \arrow[dl] \arrow[dr] & [\mathbf{M}/\mathbf{M}/\aleph_0] \arrow[dl] \arrow[d] \arrow[dr] & [\aleph_1/\mathbf{M}/\mathbf{M}] \arrow[d] \arrow[dr] & & \\
[\aleph_0/\aleph_0/\mathbf{M}] \arrow[dr] & [\aleph_0/\mathbf{M}/\aleph_0] \arrow[d] & [\mathbf{M}/\aleph_0/\aleph_0] \arrow[dl] & [\aleph_1/\aleph_0/\mathbf{M}] \arrow[d] & [\aleph_1/\mathbf{M}/\aleph_0] \arrow[dl] & \\
 & [\aleph_0/\aleph_0/\aleph_0] & & [\aleph_1/\aleph_0/\aleph_0] & &
\end{tikzcd}$$

The component of the graph reachable from the pattern $[\mathbf{M}/\mathbf{M}/\mathbf{M}]$ covers 12 of our patterns, the ones numbered 1, 5, 16, 20, 21, 25, 36, 40, 41, 45, 56, and 60 in our table.

Base Case #2: [M/M/$\aleph_2$].

$$\begin{tikzcd}
 & & [\mathbf{M}/\mathbf{M}/\aleph_3] \arrow[dl] \arrow[d] \arrow[dr] & & \\
[\aleph_0/\mathbf{M}/\aleph_3] \arrow[d] & & [\mathbf{M}/\aleph_0/\aleph_3] \arrow[dl] \arrow[dr] & & [\aleph_1/\mathbf{M}/\aleph_3] \arrow[d] \\
[\aleph_0/\aleph_0/\aleph_3] & & & & [\aleph_1/\aleph_0/\aleph_3]
\end{tikzcd}$$

The component of the graph reachable from the pattern $[\mathbf{M}/\mathbf{M}/\aleph_3]$ covers 6 of our patterns, the ones numbered 2, 17, 22, 37, 42, and 57. None of these was included in the component of Base Case #1.

Base Case #3: [M/M/$\aleph_2$].

$$\begin{tikzcd}
 & & [\mathbf{M}/\mathbf{M}/\aleph_2] \arrow[dl] \arrow[d] \arrow[dr] & & \\
[\aleph_0/\mathbf{M}/\aleph_2] \arrow[d] & & [\mathbf{M}/\aleph_0/\aleph_0] \arrow[dl] \arrow[dr] & & [\aleph_1/\mathbf{M}/\aleph_2] \arrow[d] \\
[\aleph_0/\aleph_0/\aleph_0] & & & & [\aleph_1/\aleph_0/\aleph_0]
\end{tikzcd}$$

The component of the graph reachable from the pattern $[\mathbf{M}/\mathbf{M}/\aleph_2]$ covers 6 of our patterns, the ones numbered 3, 20, 23, 40, 43, and 60. Of these, three were not included in the components of Base Cases #1 and #2.

Base Case #4: [M/M/$\aleph_1$].

$$\begin{tikzcd}
 & {[\mathbf{M}/\mathbf{M}/\aleph_1]} \arrow[dl] \arrow[d] \arrow[dr] & \\
{[\aleph_0/\mathbf{M}/\aleph_0]} \arrow[d] & {[\mathbf{M}/\aleph_0/\aleph_1]} \arrow[dl] \arrow[dr] & {[\aleph_1/\mathbf{M}/\aleph_1]} \arrow[d] \\
{[\aleph_0/\aleph_0/\aleph_0]} & & {[\aleph_1/\aleph_0/\aleph_1]}
\end{tikzcd}$$

The component of the graph reachable from the pattern $[\mathbf{M}/\mathbf{M}/\aleph_1]$ covers 6 of our patterns, the ones numbered 4, 19, 24, 39, 45, and 60. Of these, four were not included in the components of Base Cases #1 through #3.

Base Cases #5a-d: [M/$\aleph_2$/$\times_3$].

$$\begin{tikzcd}
 & & & {[\mathbf{M}/\aleph_2/\mathbf{M}]} \arrow[dl] \arrow[d] \arrow[dr] & & \\
{[\mathbf{M}/\aleph_2/\aleph_1]} \arrow[d] \arrow[dr] & {[\aleph_0/\aleph_2/\mathbf{M}]} \arrow[d] & {[\mathbf{M}/\aleph_2/\aleph_0]} \arrow[dl] \arrow[dr] & & {[\aleph_1/\aleph_2/\mathbf{M}]} \arrow[d] & \\
{[\aleph_1/\aleph_2/\aleph_1]} & {[\aleph_0/\aleph_2/\aleph_0]} & & & {[\aleph_1/\aleph_2/\aleph_0]} & \\
{[\mathbf{M}/\aleph_2/\aleph_2]} \arrow[d] \arrow[dr] & & & & {[\mathbf{M}/\aleph_2/\aleph_3]} \arrow[dl] \arrow[d] & \\
{[\aleph_0/\aleph_2/\aleph_2]} & {[\aleph_1/\aleph_2/\aleph_2]} & {[\aleph_0/\aleph_2/\aleph_3]} & & {[\aleph_1/\aleph_2/\aleph_3]} &
\end{tikzcd}$$

This base case splits into four subcases, Base Case #5a $[\mathbf{M}/\aleph_2/\mathbf{M}]$, Base Case #5b $[\mathbf{M}/\aleph_2/\aleph_3]$, Base Case #5c $[\mathbf{M}/\aleph_2/\aleph_2]$, and Base Case #5d $[\mathbf{M}/\aleph_2/\aleph_1]$. The components of the graph reachable from the patterns $[\mathbf{M}/\aleph_2/\times_3]$ cover 14 of our patterns, the ones numbered 6, 7, 8, 9, 10, 26, 27, 28, 29, 30, 46, 47, 48, and 50, none of which was included in the components of Base Cases #1 through #4.

Base Case #6: [M/$\aleph_1$/M].

$$\begin{tikzcd}
 & {[\mathbf{M}/\aleph_1/\mathbf{M}]} \arrow[dl] \arrow[d] \arrow[dr] & \\
{[\aleph_0/\aleph_0/\mathbf{M}]} \arrow[d] & {[\mathbf{M}/\aleph_1/\aleph_0]} \arrow[dl] \arrow[dr] & {[\aleph_1/\aleph_1/\mathbf{M}]} \arrow[d] \\
{[\aleph_0/\aleph_0/\aleph_0]} & & {[\aleph_1/\aleph_1/\aleph_0]}
\end{tikzcd}$$

The component of the graph reachable from the pattern $[\mathbf{M}/\aleph_1/\mathbf{M}]$ covers 6 of our patterns, the ones numbered 11, 15, 31, 35, 56, and 60. Of these, four were not included in the components of Base Cases #1 through #5.

Base Case #7: $[\mathbf{M}/\aleph_1/\aleph_3]$.

$$\begin{tikzcd}[column sep=small] & {[\,\mathbf{M}\,/\,\aleph_1\,/\,\aleph_3\,]} \arrow[dl] \arrow[dr] & \\ {[\,\aleph_0\,/\,\aleph_0\,/\,\aleph_3\,]} & & {[\,\aleph_1\,/\,\aleph_1\,/\,\aleph_3\,]} \end{tikzcd}$$

The component of the graph reachable from the pattern $[\,\mathbf{M}\,/\,\aleph_1\,/\,\aleph_3\,]$ covers 3 of our patterns, the ones numbered 12, 32, and 57. Of these, two were not included in the components of Base Cases #1 through #6.

Base Case #8: $[\mathbf{M}/\aleph_1/\aleph_1]$.

$$\begin{tikzcd}[column sep=small] & {[\,\mathbf{M}\,/\,\aleph_1\,/\,\aleph_1\,]} \arrow[dl] \arrow[dr] & \\ {[\,\aleph_1\,/\,\aleph_1\,/\,\aleph_1\,]} & & {[\,\aleph_0\,/\,\aleph_0\,/\,\aleph_0\,]} \end{tikzcd}$$

The component of the graph reachable from the pattern $[\,\mathbf{M}\,/\,\aleph_1\,/\,\aleph_1\,]$ covers 3 of our patterns, the ones numbered 14, 34, and 60. Of these, two were not included in the components of any of the other base cases.

By our earlier remarks, it is enough to show the consistency of the eight base cases in order to prove the consistency of all patterns that are not trivially inconsistent. Note that in some cases, the graph will not give us the optimal consistency strength upper bounds. For instance, the ZFC-pattern $[\,\aleph_1\,/\,\aleph_2\,/\,\aleph_3\,]$ shows up in Base Case #5b and is obtained from the large cardinal pattern $[\,\mathbf{M}\,/\,\aleph_2\,/\,\aleph_3\,]$ by forcing. For more on upper and lower bounds, cf. § 4.

3.4. **Base Cases #2, #5, and #7.** In this section, we handle three of the base cases. These three are proved consistent with techniques from forcing with large cardinals and do not rely on either polarized partition properties or AD.

Base Cases #5a-d are just Theorem 32 and do not need any large cardinals beyond the ones explicitly mentioned in the pattern that is created. The other cases in this section will be proved consistent from large cardinal assumptions by forcing in the following two theorems. None of these proofs is new. They all use published techniques and essentially consist of proof inspection to check that the relevant properties hold in the situation in which we are interested.

Theorem 35 (Woodin)**.** *If there are $\kappa < \lambda$ such that κ is supercompact and λ is measurable, then there is a model in which Base Case #2 holds (i.e., $[\,\mathbf{M}\,/\,\mathbf{M}\,/\,\aleph_3\,]$).*

Proof. This theorem is discussed in [AH86, p. 591]. Theorem 1 of that paper is a generalization of Woodin's result. Suppose $\mathbf{V} \models \mathsf{ZFC}$ + "$\kappa < \lambda$ are such that κ is supercompact and λ is measurable". Let P_0 be supercompact Radin forcing as defined in [AH86, p. 592f], with κ playing the role of κ_1 and λ playing the role of κ_2. Let $P_1 = \mathrm{Col}(\omega, {<}\kappa)$, and let $P = P_0 \times P_1$. Let G be P-generic over $\mathbf{V}$, and take N as the choiceless inner model of [AH86, Theorem 1] defined with respect to G. By suitably modified versions of [AH86, Lemmas 1.1 through 1.4], $N \models \mathsf{ZF}$+ "$\kappa = \aleph_1$ is measurable via the club filter" + "$\lambda = \aleph_2$ is measurable". By the appropriate version of [AH86, Lemma 1.2], $N \models$ "$\lambda^+ = \aleph_3 = (\lambda^+)^{\mathbf{V}}$". Therefore, since $\mathbf{V} \subseteq N$

and $\mathbf{V}$ contains a λ^+ sequence of subsets of λ, N does as well. This means that $N \models$ "$\lambda^+ = \aleph_3$ is not measurable". □

Theorem 36. *If there is a supercompact cardinal, then there is a model of Base Case #7 (i.e.,* $[\,\mathbf{M}\,/\,\aleph_1\,/\,\aleph_3\,]$*).*

Proof. This construction is essentially the same as in the proof of Theorem 35. Suppose $\mathbf{V} \models \mathsf{ZFC} + \mathsf{GCH} +$ "$\kappa < \lambda$ are such that κ is supercompact and $\lambda = \kappa^{+\kappa}$". Again, let P_0 be supercompact Radin forcing as in the proof of Theorem 35. Let $P_1 = \mathrm{Col}(\omega, {<}\kappa)$, and let $P = P_0 \times P_1$. A similar argument as in the proof of Theorem 35, using GCH to show that λ can be symmetrically collapsed to become $\aleph_2$, yields that the symmetric model N is such that $N \models \mathsf{ZF} +$ "$\kappa = \aleph_1$ is measurable via the club filter" + "$\lambda = \aleph_2$ is singular of cofinality $\kappa = \aleph_1$" + "$\lambda^+ = \aleph_3$ is not measurable". GCH is used to infer that λ is a strong limit cardinal, which is the key fact required in order to preserve that λ remains a cardinal after the symmetric collapse.

□

3.5. **Base Cases #1, #3, #4, #6, and #8.** In this section, we shall handle Base Cases #1, #3, #4, #6, and #8, all under the assumption that there is a model of AD. Among these, Base Case #3 is a special case since this is the famous AD-pattern:

Theorem 37 (Solovay-Martin)**.** *Assume* AD*. Then* $\aleph_1$ *and* $\aleph_2$ *are measurable cardinals and* $\mathrm{cf}(\aleph_3) = \aleph_2$*.*

Proof. Cf. [Kan94, Theorems 28.2, 28.6 and Corollary 28.8]. □

In order to construct models for Base Cases #1, #4, and #6, we first need the following lemma.

Lemma 38. *Suppose that* $(\kappa_0, \kappa_1, \kappa_2) \to (\kappa_0, \kappa_1, \kappa_2)^{\kappa_0}$ *and* $\kappa_0 > \omega_1$*. Then the following hold:*

(i) $\kappa_i \to (\kappa_i)^{<\kappa_0}$ *for* $i = 1, 2$ *(so* $\kappa_i \to (\kappa_i)^{<\omega_1}$ *for* $i = 1, 2$*),*
(ii) $(\kappa_1, \kappa_2) \to (\kappa_1, \kappa_2)^{<\kappa_0}$*, and*
(iii) $\kappa_0 \to (\kappa_0)^{<\omega_1}$*.*

Proof. Since $(\kappa_0, \kappa_1, \kappa_2) \to (\kappa_0, \kappa_1, \kappa_2)^{\kappa_0}$ and $\kappa_0 > \omega_1$, it is trivially true that $\kappa_i \to (\kappa_i)^{\kappa_0}$ for $i = 1, 2$, $(\kappa_1, \kappa_2) \to (\kappa_1, \kappa_2)^{\kappa_0}$, and $\kappa_0 \to (\kappa_0)^{\omega_1}$. Claims (i)–(iii) now follow from [AHJ00, Proposition 4.10]. □

Theorem 39. *If there is a model of* AD*, then there is a model of Base Case #1 (i.e.,* $[\,\mathbf{M}\,/\,\mathbf{M}\,/\,\mathbf{M}\,]$*) and Base Case #4 (i.e.,* $[\,\mathbf{M}\,/\,\mathbf{M}\,/\,\aleph_1\,]$*).*

Proof. Using Theorem 26, we start with a limit cardinal κ such that $(\kappa, \kappa^+, \kappa^{++}) \to (\kappa, \kappa^+, \kappa^{++})^\kappa$. By Lemma 38, for $\gamma = \kappa$, $\gamma = \kappa^+$, or $\gamma = \kappa^{++}$, $\gamma \to (\gamma)^{<\omega_1}$. From this, it easily follows that $\gamma \to (\gamma)^{\omega+\omega}$, so by [Kle70, Theorem 2.1], κ, κ^+, and κ^{++} are all measurable. This yields a model of Base Case #1 (cf. [AH86, Theorem 2]). By Lemma 38, we know that $\kappa^{++} \to (\kappa^{++})^{<\kappa}$. Therefore, by forcing with Magidor-like forcing $\mathcal{P}_{\kappa,\kappa^{++}}$, we obtain a model in which $\mathrm{cf}(\kappa^{++}) = \kappa$ and no new bounded subsets of κ^{++} are added. In particular, κ and κ^+ remain measurable after this forcing. Now we can collapse κ symmetrically to become $\aleph_1$ and apply Theorem 33 to obtain our model for Base Case #4. □

We explicitly note that the full power of Theorem 26 was not used in establishing Theorem 39. For Base Case #1, all that is required are cardinals κ, κ^+, and κ^{++} such that each of κ, κ^+, and κ^{++} is measurable and κ is a limit cardinal (which is a fairly weak consequence of Theorem 26 and actually follows from Theorem 9). For Base Case #4, we only use the existence of cardinals κ, κ^+, and κ^{++} such that both κ and κ^+ are measurable and κ^{++} satisfies the Ramsey-type partition relation $\kappa^{++} \to (\kappa^{++})^{<\kappa}$ (another consequence of Theorem 26; in this case, the weaker Theorems 9 and 22 are—as far as we know—not enough to establish these assumptions). The partition relation ensures that forcing with $\mathcal{P}_{\kappa,\kappa^{++}}$ changes the cofinality of κ^{++} to κ without adding any bounded subsets to κ^{++}. Without the partition relation, it does not seem possible to be able to show that the forcing $\mathcal{P}_{\kappa,\kappa^{++}}$ satisfies the aforementioned properties.

Theorem 40. *If there is a model of* AD*, then there is a model of Base Case #6 (i.e.,* $[\,\mathbf{M}\,/\,\aleph_1\,/\,\mathbf{M}\,]$*).*

Proof. Again, from Theorem 26, we start with a limit cardinal κ such that $(\kappa, \kappa^+, \kappa^{++}) \to (\kappa, \kappa^+, \kappa^{++})^\kappa$, and we get with Lemma 38 the Ramsey-type polarized partition property $(\kappa^+, \kappa^{++}) \to (\kappa^+, \kappa^{++})^{<\kappa}$ and the ordinary partition relation $\kappa^+ \to (\kappa^+)^{<\kappa}$. Because $\kappa^+ \to (\kappa^+)^{<\kappa}$, after forcing with Magidor-like forcing $\mathcal{P}_{\kappa,\kappa^+}$, the measurability of κ is preserved (as the forcing does not add bounded subsets of κ^+), and $\mathrm{cf}(\kappa^+) = \kappa$. Furthermore, since $(\kappa^+, \kappa^{++}) \to (\kappa^+, \kappa^{++})^{<\kappa}$, by the countable final segment preservation from Lemma 34, $\kappa^{++} \to (\kappa^{++})^{<\omega_1}$ remains true. Hence, κ^{++} stays measurable. Now, we can collapse κ symmetrically to become $\aleph_1$ and apply Theorem 33 to obtain our result. □

As is the case with Theorem 39, the full power of Theorem 26 is not required in order to establish Theorem 40. For Base Case #6, we are using the existence of cardinals κ, κ^+, and κ^{++} such that κ is measurable, $\kappa^+ \to (\kappa^+)^{<\kappa}$, and the pair (κ^+, κ^{++}) satisfies the Ramsey-type polarized partition property $(\kappa^+, \kappa^{++}) \to (\kappa^+, \kappa^{++})^{<\kappa}$. The partition relation $\kappa^+ \to (\kappa^+)^{<\kappa}$ allows us to deduce that forcing with $\mathcal{P}_{\kappa,\kappa^+}$ changes the cofinality of κ^+ to κ without adding any bounded subsets to κ^+ (thereby preserving the measurability of κ). The polarized partition property ensures that forcing with $\mathcal{P}_{\kappa,\kappa^+}$ preserves the ordinary partition relation $\kappa^{++} \to (\kappa^{++})^{<\omega_1}$, which we then use to infer that κ^{++} remains a measurable cardinal. Without the ordinary partition relation and something along the lines of the polarized partition relation, it does not seem possible to be able to show that the forcing $\mathcal{P}_{\kappa,\kappa^+}$ satisfies the aforementioned properties.

For Base Case #8, we rely on the methods of [AHJ00].

Theorem 41. *If* $\mathbf{L}(\mathbb{R}) \models \mathsf{AD}$*, then there is a model of Base Case #8 (i.e.,* $[\,\mathbf{M}\,/\,\aleph_1\,/\,\aleph_1\,]$*).*

Proof. Suppose V is a model of $\mathbf{V} = \mathbf{L}(\mathbb{R})$ and AD. We use the model N constructed and investigated in [AHJ00, §8] (in particular, [AHJ00, Theorem 8.1]) and applied in [AHJ00, Theorem 11.1]. In this model, which is a symmetric submodel of a forcing extension of V, $\aleph_2$ and $\aleph_3$ have cofinality $\aleph_1$. Further, by [AHJ00, Proposition 6.2 and Lemma 8.2], N and V have the same bounded subsets of $\aleph_1$. Thus, since $V \models$"$\aleph_1$ is measurable", $N \models$"$\aleph_1$ is measurable" as well. This means that N is as desired. □

			upper bound	lower bound
Base Case #1:	1	$[\mathbf{M}/\mathbf{M}/\mathbf{M}]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
Base Case #2:	2	$[\mathbf{M}/\mathbf{M}/\aleph_3]$	$\mathsf{ZFC}+\mathsf{SC}+\mathsf{M}$	$\mathsf{ZFC}+\mathsf{WC}$
Base Case #3:	3	$[\mathbf{M}/\mathbf{M}/\aleph_2]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
Base Case #4:	4	$[\mathbf{M}/\mathbf{M}/\aleph_1]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#1)	5	$[\mathbf{M}/\mathbf{M}/\aleph_0]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
Base Case #5a:	6	$[\mathbf{M}/\aleph_2/\mathbf{M}]$	$\mathsf{ZFC}+2\mathsf{MC}$	$\mathsf{ZFC}+2\mathsf{MC}$
Base Case #5b:	7	$[\mathbf{M}/\aleph_2/\aleph_3]$	$\mathsf{ZFC}+\mathsf{MC}$	$\mathsf{ZFC}+\mathsf{MC}$
Base Case #5c:	8	$[\mathbf{M}/\aleph_2/\aleph_2]$	$\mathsf{ZFC}+\mathsf{MC}$	$\mathsf{ZFC}+\mathsf{MC}$
Base Case #5d:	9	$[\mathbf{M}/\aleph_2/\aleph_1]$	$\mathsf{ZFC}+\mathsf{MC}$	$\mathsf{ZFC}+\mathsf{MC}$
(#5a)	10	$[\mathbf{M}/\aleph_2/\aleph_0]$	$\mathsf{ZFC}+\mathsf{MC}$	$\mathsf{ZFC}+\mathsf{MC}$
Base Case #6:	11	$[\mathbf{M}/\aleph_1/\mathbf{M}]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
Base Case #7:	12	$[\mathbf{M}/\aleph_1/\aleph_3]$	$\mathsf{ZFC}+\mathsf{SC}$	$\mathsf{ZFC}+\mathsf{WC}$
	13	$[\mathbf{M}/\aleph_1/\aleph_2]$	$0=1$	$0=1$
Base Case #8:	14	$[\mathbf{M}/\aleph_1/\aleph_1]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#6)	15	$[\mathbf{M}/\aleph_1/\aleph_0]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#1)	16	$[\mathbf{M}/\aleph_0/\mathbf{M}]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#2)	17	$[\mathbf{M}/\aleph_0/\aleph_3]$	$\mathsf{ZFC}+\mathsf{SC}+\mathsf{M}$	$\mathsf{ZFC}+\mathsf{WC}$
	18	$[\mathbf{M}/\aleph_0/\aleph_2]$	$0=1$	$0=1$
(#4)	19	$[\mathbf{M}/\aleph_0/\aleph_1]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#1,#3)	20	$[\mathbf{M}/\aleph_0/\aleph_0]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#1)	21	$[\aleph_1/\mathbf{M}/\mathbf{M}]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#2)	22	$[\aleph_1/\mathbf{M}/\aleph_3]$	$\mathsf{ZFC}+\mathsf{MC}$	$\mathsf{ZFC}+\mathsf{MC}$
(#3)	23	$[\aleph_1/\mathbf{M}/\aleph_2]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#4)	24	$[\aleph_1/\mathbf{M}/\aleph_1]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#1)	25	$[\aleph_1/\mathbf{M}/\aleph_0]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#5a)	26	$[\aleph_1/\aleph_2/\mathbf{M}]$	$\mathsf{ZFC}+\mathsf{MC}$	$\mathsf{ZFC}+\mathsf{MC}$
(#5b)	27	$[\aleph_1/\aleph_2/\aleph_3]$	ZFC	ZFC
(#5c)	28	$[\aleph_1/\aleph_2/\aleph_2]$	ZFC	ZFC
(#5d)	29	$[\aleph_1/\aleph_2/\aleph_1]$	ZFC	ZFC
(#5a)	30	$[\aleph_1/\aleph_2/\aleph_0]$	ZFC	ZFC

FIGURE 1. Lower and upper bounds for the consistency strength of patterns 1 to 30.

4. Summary, lower bounds and open questions

Figures 1 and 2 list all of the sixty patterns of measurability and cofinality for the first three uncountable cardinals. In the first column, we list "**Base Case #n**" if a pattern is one of our base cases. We list numbers in parentheses to indicate in which of the diagrams of § 3.3 the pattern shows up (if at all: of course, the 13 inconsistent patterns do not show up in the diagrams).

For the purpose of listing the upper and lower consistency strength bounds of our patterns, we define the following theories: $\mathsf{ZFC}+\mathsf{SC}+\mathsf{M}$ stands for ZFC together with the statement "There are $\kappa<\lambda$ where κ is supercompact and λ is measurable"; $\mathsf{ZFC}+\mathsf{SC}$ stands for ZFC together with the statement "There is a supercompact cardinal"; $\mathsf{ZFC}+\mathsf{MC}$ stands for ZFC together with the statement "There is a measurable cardinal"; $\mathsf{ZFC}+2\mathsf{MC}$ stands for ZFC together with the statement "There are two measurable cardinals"; $\mathsf{ZFC}+\mathsf{WC}$ stands for ZFC together with the statement "There is a Woodin cardinal".

Upper bounds. Most of the upper bounds come directly from our consistency proofs in Theorems 35, 37, 39, 32, 40, 36, and 41 (corresponding to the eight base cases) and the reduction diagrams as listed in § 3.3. In a few cases, the upper bound for the consistency strength obtained by our reduction diagrams is patently not optimal. In our table, we have given the optimal bounds and briefly list these exceptional cases in the following: patterns 22 and 26 can be obtained from a measurable cardinal by symmetrically collapsing it to the desired cardinal. Patterns 27, 28, 29, 30, 47, 48, and 50 all share the feature that $\aleph_2$ is regular but non-measurable and do not involve any measurable cardinals; consequently, the methods of Theorem 32 allow us to obtain them from ZFC. Patterns 32 and 37 only involve one singular cardinal, and can thus be obtained from ZFC by symmetrically collapsing a strong limit of the desired cofinality. Finally, pattern 46 is another application of the methods of Theorem 32 that only requires one measurable cardinal.

Lower bounds. For the purpose of calculating lower bounds, we shall define "κ is measurable" by "there is a normal κ-complete ultrafilter on κ". Usually, "normal" is not required; in $\mathsf{ZF}+\mathsf{DC}$, it is possible to construct a normal ultrafilter from a κ-complete one (cf. [Jec03, Theorem 10.20]); i.e., our stronger definition is equivalent to the usual definition.

There are a number of trivial lower bounds: any pattern involving a measurable or two measurables necessarily has $\mathsf{ZFC}+\mathsf{MC}$ or $\mathsf{ZFC}+2\mathsf{MC}$ as a lower bound (by the standard $\mathbf{L}[U]$ argument). For other lower bounds, our main tool is the following theorem:

Theorem 42 (Schindler / Jensen-Steel)**.** *Suppose $\delta < \delta^+$ are singular. Then there is an inner model with a Woodin cardinal.*

Proof. [Sch99, Theorem 1] proved this claim under the additional assumption that there is some $\Omega > \delta^+$ that is inaccessible and measurable in $\mathbf{HOD}$. Schindler needed this assumption to build the core model. In the meantime, Jensen and Steel have eliminated this assumption from the construction of the core model (cf. [JS07a, JS07b]). □

Theorem 42 allows us to deal immediately with those patterns that have two consecutive singular cardinals (patterns 14, 15, 19, 20, 34, 35, 39, 40, 56, 57, and 60) and get a lower bound of a Woodin cardinal. Patterns that involve κ and κ^+ such that either both are measurable or one of them is measurable and the other is singular have to be transformed into those that have two consecutive singulars by Příkrý forcing via Theorem 30. This argument uses our slightly non-standard definition of measurable cardinal (guaranteeing the existence of a normal ultrafilter). This allows us to transform patterns 1, 2, 3, 4, 5, 11, 12, 16, 17, 21, 23, 24, 25, 31, 36, 41, 42, 43, and 45 into a pattern with two consecutive singulars and thus apply Theorem 42 to get a lower bound of a Woodin cardinal. Without the additional normality assumption, we do not know how to derive more strength than a measurable out of, say, $[\,\aleph_0 \,/\, \mathbf{M} \,/\, \aleph_3\,]$.

Open questions. We end the paper by listing some remaining open questions. Six of the eight base cases can be obtained from a model of $\mathsf{ZF}+\mathsf{AD}$, but Base Case

			upper bound	lower bound
(#6)	31	$[\aleph_1 / \aleph_1 / \mathbf{M}]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#7)	32	$[\aleph_1 / \aleph_1 / \aleph_3]$	ZFC	ZFC
	33	$[\aleph_1 / \aleph_1 / \aleph_2]$	$0=1$	$0=1$
(#8)	34	$[\aleph_1 / \aleph_1 / \aleph_1]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#6)	35	$[\aleph_1 / \aleph_1 / \aleph_0]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#1)	36	$[\aleph_1 / \aleph_0 / \mathbf{M}]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#2)	37	$[\aleph_1 / \aleph_0 / \aleph_3]$	ZFC	ZFC
	38	$[\aleph_1 / \aleph_0 / \aleph_2]$	$0=1$	$0=1$
(#4)	39	$[\aleph_1 / \aleph_0 / \aleph_1]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#1,#3)	40	$[\aleph_1 / \aleph_0 / \aleph_0]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#1)	41	$[\aleph_0 / \mathbf{M} / \mathbf{M}]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#2)	42	$[\aleph_0 / \mathbf{M} / \aleph_3]$	$\mathsf{ZFC}+\mathsf{SC}+\mathsf{M}$	$\mathsf{ZFC}+\mathsf{WC}$
(#3)	43	$[\aleph_0 / \mathbf{M} / \aleph_2]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
	44	$[\aleph_0 / \mathbf{M} / \aleph_1]$	$0=1$	$0=1$
(#1,#4)	45	$[\aleph_0 / \mathbf{M} / \aleph_0]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#5a)	46	$[\aleph_0 / \aleph_2 / \mathbf{M}]$	$\mathsf{ZFC}+\mathsf{MC}$	$\mathsf{ZFC}+\mathsf{MC}$
(#5b)	47	$[\aleph_0 / \aleph_2 / \aleph_3]$	ZFC	ZFC
(#5c)	48	$[\aleph_0 / \aleph_2 / \aleph_2]$	ZFC	ZFC
	49	$[\aleph_0 / \aleph_2 / \aleph_1]$	$0=1$	$0=1$
(#5a,#5d)	50	$[\aleph_0 / \aleph_2 / \aleph_0]$	ZFC	ZFC
	51	$[\aleph_0 / \aleph_1 / \mathbf{M}]$	$0=1$	$0=1$
	52	$[\aleph_0 / \aleph_1 / \aleph_3]$	$0=1$	$0=1$
	53	$[\aleph_0 / \aleph_1 / \aleph_2]$	$0=1$	$0=1$
	54	$[\aleph_0 / \aleph_1 / \aleph_1]$	$0=1$	$0=1$
	55	$[\aleph_0 / \aleph_1 / \aleph_0]$	$0=1$	$0=1$
(#1,#6)	56	$[\aleph_0 / \aleph_0 / \mathbf{M}]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$
(#2,#7)	57	$[\aleph_0 / \aleph_0 / \aleph_3]$	$\mathsf{ZFC}+\mathsf{SC}$	$\mathsf{ZFC}+\mathsf{WC}$
	58	$[\aleph_0 / \aleph_0 / \aleph_2]$	$0=1$	$0=1$
	59	$[\aleph_0 / \aleph_0 / \aleph_1]$	$0=1$	$0=1$
(#1,#3,#4,#6,#8)	60	$[\aleph_0 / \aleph_0 / \aleph_0]$	$\mathsf{ZF}+\mathsf{AD}$	$\mathsf{ZFC}+\mathsf{WC}$

FIGURE 2. Lower and upper bounds for the consistency strength of patterns 31 to 60.

#2 appears to need (assumptions on the order of) $\mathsf{ZFC}+\mathsf{SC}+\mathsf{M}$ and Base Case #7 appears to need (assumptions on the order of) $\mathsf{ZFC}+\mathsf{SC}$.[7]

Question 43. Is it possible to force Base Case #2 and Base Case #7 from $\mathsf{ZF}+\mathsf{AD}$ (thus reducing the consistency strength upper bound)?

The two mentioned patterns are among 30 (out of our 60) patterns for which the upper bound and the lower bound in consistency strength do not coincide.

Question 44. Can we determine the precise consistency strength in the cases where upper and lower bounds do not coincide?

There are other large cardinal properties that can be exhibited by small cardinals, such as "κ is κ^+-supercompact" (under AD, $\aleph_1$ exhibits this property (cf. [DPH78])).

[7] As the proof of [AH86, Theorem 1] shows, slightly weaker supercompactness hypotheses (which are still well beyond the consistency strength of AD) actually suffice to establish Base Case #2 and Base Case #7.

Let us add another label for this property to our patterns, resulting in $4\times5\times6 = 120$ patterns.

Question 45. Which of the 120 patterns involving cofinalities $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, measurability and κ^+-supercompactness are consistent?

Note that a 1975 result of Martin (cf. [DPH78, § 2] for details) about the κ^+-supercompactness of κ under the assumption that both κ and κ^+ carry a normal measure produces some non-trivial restrictions for Question 45.

Now, after considering all measurability and cofinality patterns for the cardinals $\aleph_1$, $\aleph_2$, and $\aleph_3$, one could ask what happens if the same question is posed for the first four uncountable cardinals. There are $3 \times 4 \times 5 \times 6 = 360$ such patterns for the first four uncountable cardinals.

Question 46. Which of the 360 measurability and cofinality patterns for the first four uncountable cardinals are consistent?

Of course, a complete answer to Question 46 would require (among other things) a solution of one of the big open questions of the field of large cardinals without the Axiom of Choice, viz. whether it is consistent to have four consecutive measurable cardinals. As a consequence, we do not expect an answer to Question 46 very soon.

Slightly less ambitious would be to ask the same question not for four consecutive cardinals, but for a different selection of three consecutive cardinals, e.g., the cardinals $\aleph_2$, $\aleph_3$, and $\aleph_4$. Here we would have $4 \times 5 \times 6 = 120$ patterns.

Question 47. Which of the 120 measurability and cofinality patterns for the cardinals $\aleph_2$, $\aleph_3$, and $\aleph_4$ are consistent?

However, most of the methods used in this paper to handle the case of the first three uncountable cardinals will not work in this setting. The main reason is that most of the proofs require symmetrically collapsing some large cardinal to be $\aleph_1$. This collapse is canonically well-orderable, and thus at our disposal in the choice-free situation. The collapse of a cardinal to be $\aleph_2$, however, is not canonically well-orderable; consequently, the obvious analogues of our proofs will not work in the setting of Question 47.

At this point, it might be useful to mention that some of the patterns have alternative consistency proofs that are more likely to transfer to the situation of $\aleph_2$, $\aleph_3$, and $\aleph_4$. We would like to give one example: if there is a strongly compact cardinal κ, it is possible to obtain the pattern $[\,\aleph_1 \,/\, \aleph_0 \,/\, \aleph_1\,]$ by using strongly compact Příkrý forcing. Obviously, this proof is not optimal in terms of consistency strength (as we can get it from $\mathsf{ZF} + \mathsf{AD}$ via Theorem 39). However, this proof lifts to give a consistency proof of the pattern "$\aleph_2$ is regular but not measurable, $\aleph_3$ has cofinality $\aleph_0$, and $\aleph_4$ has cofinality $\aleph_2$".[8]

[8] A sketch of the proof is as follows. Let $\kappa < \lambda$ be such that in our ground model $\mathbf{V}$, κ is strongly compact and λ is the least singular strong limit cardinal of cofinality $\aleph_2$ greater than κ. Force over $\mathbf{V}$ with $P_1 \times P_2$, where $P_1 = \mathrm{Col}(\aleph_2, {<}\kappa)$ and P_2 is strongly compact Příkrý forcing based on κ and λ as defined in the proof of [AH91, Theorem 1]. Let $G = G_1 \times G_2$ be the resulting generic, with $r = \langle r_n \,;\, n < \omega\rangle$ the generic sequence through $P_\kappa(\lambda)$ generated by G_2. For $\delta \in (\kappa, \lambda)$ a cardinal, define $r{\restriction}\delta = \langle r_n \cap \delta \,;\, n < \omega\rangle$. Consider the symmetric model $N := \mathrm{HD}_{\mathbf{V}}(\{G_1{\restriction}\delta \,;\, \delta \in (\aleph_2, \kappa)$ and δ is a cardinal$\} \cup \{r{\restriction}\delta \,;\, \delta \in (\kappa, \lambda)$ and δ is a cardinal$\})$. The arguments found in the proofs of [AH91, Theorem 1] and Theorems 31 and 32 of this paper then show that N is as desired. Note that if P_1 is redefined as $\mathrm{Col}(\aleph_1, {<}\kappa)$, λ is redefined as the least singular strong limit cardinal of cofinality $\aleph_1$ greater than κ, P_2 remains strongly compact Příkrý forcing based on κ and λ, and

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N is redefined as $N := \mathrm{HD}_{\mathbf{V}}(\{G_1 \restriction \delta\,;\, \delta \in (\aleph_1, \kappa)$ and δ is a cardinal$\} \cup \{r \restriction \delta\,;\, \delta \in (\kappa, \lambda)$ and δ is a cardinal$\})$, then N is a model of the pattern $[\,\aleph_1 \,/\, \aleph_0 \,/\, \aleph_1\,]$.

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