UNIFORM HOMEOMORPHISMS OF BANACH SPACES AND ASYMPTOTIC STRUCTURE

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Abstract. We give a general result on the behavior of spreading models in Banach spaces which coarse Lipschitz-embed into asymptotically uniformly convex spaces. We use this result to study the uniqueness of the uniform structure in $\ell_p$-sums of finite-dimensional spaces for $1 < p < \infty$; in particular we give some new examples of spaces with unique uniform structure.

1. Introduction

It is known that asymptotic smoothness is preserved under uniform homeomorphisms of Banach spaces [11]. In quantitative terms this is measured by the behavior of the convex Szlenk index (Theorem 5.5 of [11]); unfortunately it is not true that one has a precise result on the preservation of the modulus of asymptotic smoothness, even after renorming. Thus, for example, if $X$ and $Y$ are separable uniformly homeomorphic Banach spaces and

$$\bar{\rho}_Y(t) \leq ct^p, \quad 0 < t < 1,$$

we can only conclude that for any $q < p$ and some equivalent norm on $X$, one has an estimate

$$\bar{\rho}_X(t) \leq ct^q, \quad 0 < t < 1.$$

A recent example in [29] shows that we cannot improve this to the case $q = p$. There is a simple application of the ideas of [11] to spreading models in $Y$. If $(e_n)_{n=1}^\infty$ is the basis of a spreading model $S$ of a normalized weakly null sequence in $X$ we have an estimate

$$\|e_1 + \cdots + e_n\| \leq C\|e_1 + \cdots + e_n\|_{\ell_{\bar{\rho}_Y}},$$

where the right-hand side represents the norm in the Orlicz sequence space generated by the Orlicz function $\bar{\rho}_Y$. This can be obtained by combining Theorem 4.4 and Theorem 5.5 of [11]. In [30], using simpler arguments, this result is shown to hold more generally (Theorem 6.1) when $X$ coarse Lipschitz-embeds into $Y$, under the additional hypothesis that $Y$ is reflexive.

Although these results have applications in the nonlinear theory of Banach spaces, it has been a significant drawback that there has been no corresponding result giving a lower bound in terms of asymptotic convexity to the upper bound.
in (1.1). In a recent article [2], results are obtained that suggest one might hope for similar results for asymptotic convexity. Here we justify that hope in Theorem 7.4, where we show that if $Y$ is reflexive and $X$ coarse Lipschitz-embeds in $Y$, then for some constant $c > 0$ and any spreading model $S$ of a normalized weakly null sequence in $X$ we have an estimate

$$ (1.2) \quad c\|e_1 + \cdots + e_n\|_{T_Y} \leq \|e_1 + \cdots + e_n\|_S. $$

We then use these ideas to study $\ell_p$-sums of finite-dimensional spaces. Suppose $1 < p < \infty$. It is a result of Johnson, Lindenstrauss, Preiss and Schechtman [19] that a separable reflexive Banach space $X$ which has two renormings $X_1$ and $X_2$ with $\delta_{X_1}(t) \sim \delta_{X_2}(t) \sim ct^p$ is linearly isomorphic to a subspace of a space $(\sum_{n=1}^{\infty} E_n)\ell_p$ with each $E_n$ being finite-dimensional. Unfortunately it is shown in [29] that if we take $(G_n)_{n=1}^{\infty}$ to be a sequence dense in all finite-dimensional normed spaces for Banach-Mazur distance, then $(\sum_{n=1}^{\infty} G_n)\ell_p$ (see e.g. [22]) is uniformly homeomorphic to $(\sum_{n=1}^{\infty} G_n)T_p$, where $T_p$ is $p$-convexified Tsirelson space (see e.g. [8]). This means that being embeddable in an $\ell_p$-sum of finite-dimensional spaces is not, in general, invariant under uniform homeomorphisms.

However, under some additional hypotheses, (1.2) and (1.1) can be combined to get such a conclusion. For example it is shown in [11] that if $X$ is uniformly homeomorphic to a subspace (respectively, quotient) of $\ell_p$, then $X$ is itself linearly isomorphic to a subspace (respectively, quotient) of $\ell_p$ when $2 \leq p < \infty$. We show here in Theorem 8.3 that the same conclusion can be obtained when $1 < p < 2$. Let us remark that in [29] we give examples of subspaces $X$ and $Y$ of $\ell_p$ ($1 < p < \infty$, $p \neq 2$) which are uniformly homeomorphic but not linearly isomorphic.

In [20] it was shown that $\ell_p$ has unique uniform structure. We extend this result here by showing that $(\sum_{n=1}^{\infty} E_n)\ell_p$ has unique uniform structure if $r > \max(p, 2)$ or $1 < r < \min(p, 2)$. A crucial point in these proofs is the role of the uniform approximation property. This mirrors the examples of two uniformly homeomorphic but nonisomorphic subspaces of $\ell_p$ mentioned above from [29], where one space has the approximation property (but not the uniform approximation property) and the other fails the approximation property.

On the way to obtaining these nonlinear results we require some new results in the linear theory of Banach spaces. If $X$ is a reflexive Banach space, then the condition

$$ \|e_1 + \cdots + e_n\|_S \leq Cn^{1/p} $$

for every spreading model of a normalized weakly null sequence is simply the requirement that $X$ has the so-called $p$-Banach-Saks property. The dual notion that

$$ \|e_1 + \cdots + e_n\|_S \geq cn^{1/p} $$

for every spreading model of a normalized weakly null sequence, we call the $p$-co-Banach-Saks property. If $X$ is a subspace or quotient of $L_p$ when $p > 2$ and has the $p$-Banach-Saks property, then Johnson [17] showed that $X$ is then also a subspace of a quotient of $\ell_p$. If $X$ is a subspace of a quotient of $L_p$ ($p > 2$) and has the $p$-Banach-Saks property, then Johnson obtained that $X$ is a subspace of a quotient of $\ell_p$ only under the additional hypothesis that $X$ has the approximation property. We remove this restriction, answering a question of Johnson, and provide dual results for $1 < p < 2$. In fact we give a more general framework for results of this type.
Foreword: Nigel Kalton, author of this work, suddenly passed away on August 31, 2010. The present article was essentially ready at the time of his death, but some editing work had to be done before it could actually be submitted. Nigel’s friends and colleagues are grateful to Gilles Lancien who took care of this editing task with kindness and efficiency.

2. Preliminaries from linear Banach space theory

Our notation for Banach spaces is fairly standard (see e.g. [11,18,35]). If $X$ is a Banach space, $B_X$ denotes its closed unit ball and $\partial B_X$ the unit sphere $\{ x : \| x \| = 1 \}$.

We recall that if $U$ is a nonprincipal ultrafilter on $\mathbb{N}$ and $X$ is a Banach space, then the ultrapower $X_U$ is defined to be the quotient of $\ell_\infty (X)$ by the subspace of all sequences $(x_n)_{n=1}^\infty$ such that $\lim_{n \in U} \| x_n \| = 0$. A Banach space is super-reflexive if every ultraproduct is reflexive.

We recall that a separable Banach space $X$ has the approximation property (AP) if given any compact subset $K$ of $X$ and $\epsilon > 0$ there is a finite-rank operator $T : X \to X$ with $\|Tx-x\| < \epsilon$ for $x \in K$. $X$ has the metric approximation property (MAP) if we can also require $\|T\| \leq 1$. Any reflexive Banach space with (AP) has (MAP)(see [35] p. 39). $X$ is said to have the uniform approximation property (UAP) if there is a constant $K$ such that for every $m$ there exists $n$ so that if $F$ is a subspace of $X$ of dimension $m$ we can find an operator $T : X \to X$ with rank at most $n$, $\|T\| \leq K$ and $T x = x$ for $x \in F$. The uniform approximation property was first introduced by Pełczyński and Rosenthal [33]; rather few spaces have this property, but they include the $L_p$-spaces and reflexive Orlicz spaces [34].

$X$ has a finite-dimensional decomposition (FDD) if there is a sequence of finite-rank operators $P_n : X \to X$ such that $P_m P_n = 0$ when $m \neq n$ and $x = \sum_{n=1}^\infty P_n x$ for every $x \in X$. If each $P_n$ has rank one, then $X$ has a basis. The (FDD) is called shrinking if we also have $x^* = \sum_{n=1}^\infty P_n^* x^*$ for every $x^* \in X^*$. If, in addition, $x = \sum_{n=1}^\infty P_n x$ unconditionally for every $x \in X$, then $X$ has an unconditional finite-dimensional decomposition (UFDD). Finally if $\| \sum_{k=1}^n \eta_k P_k \| \leq 1$ for every $n \in \mathbb{N}$ and $\eta_k = \pm 1$ for $1 \leq k \leq n$, then we say that $X$ has a 1-(UFDD).

We shall say that a Banach space $X$ is $p$-uniformly smooth for $1 < p \leq 2$ (or $X$ has a modulus of smoothness of power type $p$) if for some constant $C$ we have the estimate

$$\frac{1}{2}(\|x_1 + x_2\|^p + \|x_1 - x_2\|^p) \leq \|x_1\|^p + C\|x_2\|^p, \quad x_1, x_2 \in X.$$ 

We say that $X$ is $p$-uniformly convex for $2 \leq p < \infty$ (or $X$ has a modulus of convexity of power type $p$) if for some constant $c > 0$ we have

$$\|x_1\|^p + c\|x_2\|^p \leq \frac{1}{2}(\|x_1 + x_2\|^p + \|x_1 - x_2\|^p), \quad x_1, x_2 \in X.$$ 

We shall frequently deal with $\ell_p$-sums of Banach spaces $(X_n)_{n=1}^\infty$. We denote by $(\sum_{n=1}^\infty X_n)_{\ell_p}$ the space of sequences $(x_n)_{n=1}^\infty$ with $x_n \in X_n$ and

$$\|(x_n)_{n=1}^\infty\| = (\sum_{n=1}^\infty \|x_n\|^p)^{1/p} < \infty.$$ 

If $X_n = X$ is a fixed Banach space, we use the notation $\ell_p(X)$.
3. ASYMPTOTIC MODULI IN BANACH SPACE THEORY

We now discuss asymptotic uniform smoothness and asymptotic uniform convexity. Let $X$ be a separable Banach space. We define the modulus of asymptotic uniform smoothness (due to Milman [35]) $\overline{\rho}(t) = \overline{\rho}_X(t)$ by

$$\overline{\rho}(t) = \sup_{x \in \partial B_X} \inf_{E} \sup_{y \in \partial E} \left\{ \|x + ty\| - 1 \right\},$$

where $E$ runs through all closed subspaces of $X$ of finite codimension.

The modulus of asymptotic uniform convexity is defined by

$$\overline{\delta}(t) = \inf_{x \in \partial B_X} \sup_{E} \inf_{y \in \partial E} \left\{ \|x + ty\| - 1 \right\},$$

where $E$ runs through all closed subspaces of $X$ of finite codimension. Similarly in $X^*$ there is a weak$^*$-modulus of asymptotic uniform convexity defined by

$$\overline{\delta}^*(t) = \inf_{x^* \in \partial B_{X^*}} \sup_{E} \inf_{y^* \in \partial E} \left\{ \|x^* + ty^*\| - 1 \right\},$$

where $E$ runs through all weak$^*$-closed subspaces of $X^*$ of finite codimension.

As shown in [19], if $\overline{\rho}(t) < t$ for some $0 < t \leq 1$, then $X^*$ is separable. On the other hand if $\overline{\rho}(t) = 0$ for some $t > 0$, then $X$ is isomorphic to a subspace of $c_0$ (see [10] and [19]). We say that $X$ is asymptotically uniformly smooth if $\lim_{t \to 0} \overline{\rho}(t)/t = 0$. If $X$ is asymptotically uniformly smooth this implies that $\overline{\rho}(t)/t \leq Ct^\theta$ for some $0 < \theta < 1$ (see [32] and [11]). The function $\overline{\rho}$ is clearly convex, while the function $\overline{\delta}$ satisfies the condition that $\overline{\delta}(t)/t$ is increasing so that if we define the convex function

$$\overline{\delta}(t) = \int_0^t \overline{\delta}(s) \frac{ds}{s},$$

then

$$\overline{\delta}(t/2) \leq \overline{\delta}(t) \leq \overline{\delta}(t), \quad 0 < t < \infty$$

so that $\overline{\delta}$ is equivalent to a convex function.

It is clear that we have that if $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$, $x \neq 0$ and $(x_n)_{n=1}^\infty$ is a weakly null sequence, then we have

$$\lim_{n \in \mathcal{U}} \overline{\delta}(\|x_n\|/\|x\|) \leq \lim_{n \in \mathcal{U}} \|x + x_n\| - \|x\| \leq \|x\| \lim_{n \in \mathcal{U}} \overline{\rho}(\|x_n\|/\|x\|).$$

This can alternatively be viewed as the statement that

$$\lim_{n \to \infty} \overline{\delta}(\|x_n\|/\|x\|) \leq \lim_{n \to \infty} \|x + x_n\| - \|x\| \leq \|x\| \lim_{n \to \infty} \overline{\rho}(\|x_n\|/\|x\|)$$

classically all the limits exist. It is clear that if $X^*$ is separable this is an equivalent formulation of the definition.

We remark that it is trivial that if $1 < p < \infty$, $\overline{\delta}_{L_p}(t) = \overline{\rho}_{L_p}(t) = (1 + t^p)^{1/p} - 1$.

We will need the fact that for the corresponding function spaces we have:

**Proposition 3.1** ([35]). Suppose $1 < p < \infty$. If $1 < p < 2$, then there is a constant $c = c_p > 0$ such that

$$\overline{\rho}_{L_p}(t) \leq (1 + c^p t^p)^{1/p} - 1.$$

If $2 < p < \infty$, then there is a constant $c = c_p > 0$ such that

$$\overline{\delta}_{L_p}(t) \geq (1 + c^p t^p)^{1/p} - 1.$$
Remark. See [36], p. 117. This proposition may be expressed in the following terms. If $1 < p < 2$, then

$$\lim_{n \in \mathcal{U}} \|f + g_n\|_p \leq \lim_{n \in \mathcal{U}} (\|f\|^p + C^p_n \|g_n\|^p)^{1/p}$$

whenever $(g_n)_{n=1}^\infty$ is a weakly null sequence in $L_p$ and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. Similarly if $2 < p < \infty$,

$$\lim_{n \in \mathcal{U}} \|f + g_n\|_p \geq \lim_{n \in \mathcal{U}} (\|f\|^p + C^p_n \|g_n\|^p)^{1/p}$$

whenever $(g_n)_{n=1}^\infty$ is a weakly null sequence in $L_p$.

We will also need the following proposition.

**Proposition 3.2 ([19]).** Let $X$ be a Banach space and suppose $Y = X/E$ is a quotient of $X$. Then $\tilde{\delta}_Y \geq \tilde{\delta}_X$ and $\overline{\nu}_Y \leq \overline{\nu}_X$.

There is a natural variant of $\tilde{\delta}$ which will be very useful in this paper. We define

$$\hat{\delta}_X(t) = \inf_{x \in \partial B_X} \sup_{E \in \mathcal{U}} \inf_{y \in \partial B_E} \{ -\frac{1}{2} (\|x + ty\| + \|x - ty\|) - 1 \},$$

where again $E$ runs through all closed subspaces of $X$ of finite codimension. As with $\overline{\nu}_X$ the function $\tilde{\delta}_X(t)/t$ is increasing and so $\hat{\delta}_X$ is equivalent to a convex function. Clearly $\hat{\delta}_X \leq \overline{\delta}_X$.

If $(x_n)_{n=1}^\infty$ is a bounded sequence we define $\text{sep}\{x_n\}_{n=1}^\infty = \inf_{m \neq n} \|x_m - x_n\|$.  

**Proposition 3.3.** Suppose $u, v \in X$ with $\|u - v\| = 1$. Let $(x_n)_{n=1}^\infty$ be a bounded sequence in $X$ and let $t = \text{sep}\{x_n\}_{n=1}^\infty$. Then

$$\lim_{n \to \infty} \inf \{\|u - x_n\| + \|v - x_n\|\} \geq 1 + \hat{\delta}_X(t).$$

**Proof.** It is clearly enough to show that for all $\nu > 0$ there exists an $m$ with

$$\|u - x_m\| + \|v - x_m\| > 1 + \hat{\delta}_X(t) - \nu.$$  

Choose a finite-codimensional subspace $E$ so that if $z \in \partial B_E$, then

$$\frac{1}{2} (\|u - v + tz\| + \|u - v - tz\|) \geq 1 + \hat{\delta}(t) - \frac{1}{2} \nu.$$  

Since $X/E$ is finite-dimensional we can find $m \neq n$ so that $d(x_m - x_n, E) < \nu/2$. Hence there exists $z \in \partial B_E$ and $\tau > t$ so that $\|x_m - x_n - \tau z\| < \nu$. Then

$$\frac{1}{2} (\|u - v + (x_m - x_n)\| + \|u - v - (x_m - x_n)\|) \geq 1 + \hat{\delta}(t) - \nu.$$  

Now

$$\|u - v + (x_m - x_n)\| \leq \|u - x_n\| + \|v - x_m\|$$

and

$$\|u - v - (x_m - x_n)\| \leq \|u - x_m\| + \|v - x_n\|$$

so that combining we have either

$$\|u - x_m\| + \|v - x_m\| > 1 + \hat{\delta}_X(t) - \nu$$

or

$$\|u - x_n\| + \|v - x_n\| > 1 + \hat{\delta}_X(t) - \nu.$$
We recall that every bounded sequence \((y_n)_{n=1}^{\infty}\) in a Banach space has a spreading subsequence \((x_n)_{n=1}^{\infty}\) so that
\[
\lim_{(n_1, \ldots, n_m) \to \infty} \left\| \sum_{j=1}^{m} a_j x_{n_j} \right\| = \left\| \sum_{j=1}^{m} a_j e_j \right\|_S
\]
eexists for all finite scalar sequences \((a_1, \ldots, a_m)\) and defines a seminorm on the space \(c_{00}\) of all finitely supported scalar sequences. By this notation we mean that for any \(\epsilon > 0\) and \((a_1, \ldots, a_m)\) there exists \(q\) so that if \(q < n_1 < n_2 < \cdots < n_m\), then
\[
\left\| \sum_{j=1}^{m} a_j x_{n_j} \right\| - \left\| \sum_{j=1}^{m} a_j e_j \right\| < \epsilon.
\]
As long as \((x_n)_{n=1}^{\infty}\) is not convergent in norm the seminorm \(\left\| \cdot \right\|_S\) is a norm. Then \((e_j)_{j=1}^{\infty}\) is the spreading model associated to \((x_n)_{n=1}^{\infty}\) and is a sequence in the Banach space \(S\) obtained by completing \(c_{00}\). If \((x_n)_{n=1}^{\infty}\) is weakly null we say that \((e_j)_{j=1}^{\infty}\) is a weakly null spreading model; this may not imply that \((e_j)_{j=1}^{\infty}\) is itself a weakly null sequence in \(S\).

We will be particularly interested in the possible growth rate of \(\left\| \sum_{j=1}^{n} e_j \right\|_S\), for a given normalized spreading sequence \((x_n)_{n=1}^{\infty}\). Note that if \(\lim_{n \to \infty} \left\| \sum_{j=1}^{n} e_j \right\| = \infty\), then given any \(\nu > 0\) and \(k \in \mathbb{N}\), using Ramsey arguments, we can pass to a subsequence and assume that
\[
(1 - \nu) \left\| \sum_{j=1}^{k} e_j \right\| \leq \left\| \sum_{j=1}^{k} x_{n_j} \right\| \leq (1 + \nu) \left\| \sum_{j=1}^{k} e_j \right\|, \quad n_1 < n_2 < \cdots < n_k.
\]

**Lemma 4.1.** Let \(X\) be a Banach space and suppose \((e_j)_{j=1}^{\infty}\) is a spreading model of a normalized sequence \((x_n)_{n=1}^{\infty}\). Then:
\[
\sup_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^{k} e_j e_j \right\| \leq \left\| \sum_{j=1}^{k} e_j \right\| \leq 3 \left\| \sum_{j=1}^{k} e_j \right\|
\]
and if \(X\) is super-reflexive and \((x_n)_{n=1}^{\infty}\) is weakly null,
\[
\left\| \sum_{j=1}^{k} e_j \right\| \leq 2E \left\| \sum_{j=1}^{k} e_j e_j \right\|,
\]
where \((\epsilon_j)_{j}\) denotes a sequence of independent Rademacher variables.

**Proof.** Let \(\alpha_k = \left\| \sum_{j=1}^{k} e_j \right\|\). Then \(\alpha_{k+l} \leq \alpha_k + \alpha_l\). Hence \(\lim_{k \to \infty} \alpha_k/k = \inf_k \alpha_k/k = \theta\) exists. Now for any integer \(m\) we have
\[
\left\| \sum_{j=1}^{k} e_j + \frac{1}{m} \sum_{j=k+1}^{k+m} e_j \right\| \leq \alpha_{k+l}
\]
so that
\[
\alpha_k \leq \alpha_{k+l} + \frac{1}{m} \alpha_{ml}.
\]
Thus letting $m \to \infty$,
\[
\alpha_k \leq \alpha_{k+1} + l\theta \leq \frac{k + 2l}{k + l}\alpha_{k+1}.
\]

For any $k$ it is clear that
\[
\sup_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^{k} \epsilon_j e_j \right\| \leq \max_{j \leq k} (\alpha_j + \alpha_{k-j}) \leq 3\alpha_k
\]
by the preceding equation.

For the second part we observe that $(\epsilon_n)_{n=1}^\infty$ is also weakly null and hence 2-unconditional. \qed

For $\mathbb{M}$ an infinite subset of $\mathbb{N}$, let $G_k(\mathbb{M})$ denote the space of $k$-subsets $\{n_1, \ldots, n_k\}$ (where $n_1 < n_2 < \cdots < n_k$) of $\mathbb{M}$ regarded as a graph in which $\{m_1, m_2, \ldots, m_k\}$ and $\{n_1, n_2, \ldots, n_k\}$ are adjacent if they interlace, i.e. $m_1 \leq n_1 \leq m_2 \leq \cdots \leq m_k \leq n_k \text{ or } n_1 \leq m_1 \leq \cdots \leq n_k \leq m_k$. Let $d$ be the associated least path metric. Let us recall [28] that a Banach space $X$ has property $Q$ if there is a constant $C$ so that whenever $f : G_k(\mathbb{N}) \to X$ has Lipschitz constant one, then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that
\[
\|f(m_1, \ldots, m_k) - f(n_1, \ldots, n_k)\| \leq C, \quad (n_1, \ldots, n_k), (m_1, \ldots, m_k) \in G_k(\mathbb{M}).
\]
It is shown in [28] that if either $X$ coarsely embeds in a reflexive space or $B_X$ uniformly embeds in a reflexive space, then $X$ must have property $Q$.

**Proposition 4.2.** Let $X$ be a Banach space with property $Q$. Then for each spreading model $(\epsilon_n)_{n=1}^\infty$ of $X$ there is a constant $C$ so that
\[
\| \sum_{j=1}^{n} \epsilon_j e_j \| \leq C\| \sum_{j=1}^{n} \epsilon_j e_j \|.
\]

**Proof.** We consider two cases. If $(\epsilon_n)_{n=1}^\infty$ is not weakly Cauchy, then $(\epsilon_n)_{n=1}^\infty$ is equivalent to the unit vector basis of $\ell_1$ by Rosenthal’s theorem [42], and the result is clear. If not, then the sequence $(e_{2j-1} - e_{2j})_{j=1}^{\infty}$ is 2-unconditional. Hence
\[
\| \sum_{j=1}^{k} (e_{2j-1} - e_{2j}) \| \leq 2\| \sum_{j=1}^{k} \epsilon_j (e_{2j-1} - e_{2j}) \| \leq 4\| \sum_{j=1}^{k} \epsilon_j e_j \|.
\]

Now passing to a suitable subsequence of $(x_n)_{n=1}^{\infty}$ we can assume that
\[
\frac{1}{2} \| \sum_{j=1}^{2k} a_j e_j \| \leq \| \sum_{j=1}^{2k} a_j x_{n_j} \| \leq 2\| \sum_{j=1}^{2k} a_j e_j \|
\]
whenever $n_1 < n_2 < \cdots < n_{2k}$ and $|a_j| = 1$.

Define $f : G_k \to X$ by $f(n_1, n_2, \ldots, n_k) = x_{n_1} + \cdots + x_{n_k}$. Then $f$, using the preceding calculation, has Lipschitz constant at most $8\| \sum_{j=1}^{k} \epsilon_j e_j \|$. Hence by property $Q$ for a suitable constant $C$ independent of $k$, we can find $n_1 < n_2 < \cdots < n_k < m_1 < \cdots < m_k$ with
\[
\| \sum_{j=1}^{k} x_{n_j} - \sum_{j=1}^{k} x_{m_j} \| \leq C\| \sum_{j=1}^{k} \epsilon_j e_j \|. 
\]
Hence

\[ \| \sum_{j=1}^{k} e_j - \sum_{j=k+1}^{2k} e_j \| \leq 2CE \| \sum_{j=1}^{k} e_j e_j \|. \]

In this case \((e_n)_{n=1}^\infty\) has basis constant one and so

\[ \| \sum_{j=1}^{k} e_j \| \leq 2CE \| \sum_{j=1}^{k} e_j e_j \|. \]

\[ \square \]

We say that a norm \(N\) on \(\mathbb{R}^2\) is absolute if

\[ N(a, b) = N(|a|, |b|), \quad a, b \in \mathbb{R}. \]

For any Lipschitz convex Orlicz function \(F\), the limit \(\lim_{t \to \infty} F(t)/t = \theta\) exists and there is a corresponding absolute norm defined by

\[ N_F(a, b) = \begin{cases} |a|(1 + F(|b|/|a|)), & a \neq 0, \\ \theta|b|, & a = 0. \end{cases} \]

Now suppose \(N\) is an absolute norm on \(\mathbb{R}^2\) with \(N(1, 0) = 1\). We define the sequence space \(\Lambda_N\) as the completion of \(c_{00}\) under the norm defined iteratively by

\[ \| e_1 \|_{\Lambda_N} = 1 \text{ and then} \]

\[ \| \sum_{j=1}^{n} a_j e_j \|_{\Lambda_N} = N(\| \sum_{j=1}^{n-1} a_j e_j \|_{\Lambda_N}, |a_n|), \quad n \geq 2. \]

Spaces of this type were first considered in [26]. The space \(\Lambda_N\) coincides with the space \(h_F\), where \(F(t) = N(1, t) - 1\); here \(h_F\) denotes the closure of \(c_{00}\) in the Orlicz sequence space \(\ell_F\). In fact we have

\[ \text{Lemma 4.3. If } a \in c_{00}, \text{ then} \]

\[ \frac{1}{2}\|a\|_{\ell_F} \leq \|a\|_{\Lambda_N} \leq \|a\|_{\ell_F}. \]

\[ \text{Proof. Assume } \|a\|_{\ell_F} \leq 1. \text{ Then} \]

\[ \|e_1 + \sum_{j=1}^{n} a_j e_{j+1} \|_{\Lambda_N} \leq \prod_{j=1}^{n} (1 + F(|a_j|)) \leq e. \]

Conversely if \(\|a\|_{\Lambda_N} \leq 1\) we have

\[ \|e_1 + \sum_{j=1}^{n} a_j e_{j+1} \|_{\Lambda_N} \geq \prod_{j=1}^{n} (1 + F(|a_j|/2)) \geq 1 + \sum_{j=1}^{n} F(|a_j|/2) \]

so that \(\|a\|_{\ell_F} \leq 2. \]

\[ \square \]

We will need the following proposition:

\[ \text{Proposition 4.4. Let } X \text{ be a Banach space with separable dual. Then there exist constants } 0 < c < C < \infty \text{ so that for any spreading model } (e_j)_{j=1}^\infty \text{ of a normalized weakly null sequence we have} \]

\[ c\| \sum_{j=1}^{n} a_j e_j \|_{\ell_\pi} \leq \| \sum_{j=1}^{n} a_j e_j \|_{S} \leq C\| \sum_{j=1}^{n} a_j e_j \|_{\ell_\pi}. \]
Remark. Of course, the function $\tilde{\delta}$ is not necessarily convex but is equivalent to the convex function $\tilde{\delta}$.

Proof. It is easy to check that

$$\|a_1e_1 + \cdots + a_ne_n\|_S \geq \|a_1e_1 + \cdots + a_ne_n\|_{\Lambda_N},$$

where $N(1,t) = 1 + \tilde{\delta}(t)$. Similarly

$$\|a_1e_1 + \cdots + a_ne_n\|_S \leq \|a_1e_1 + \cdots + a_ne_n\|_{\Lambda_N'},$$

where $N'(1,t) = 1 + \tilde{\rho}(t)$. Then apply Lemma 4.3. □

The left-hand side of (4.3) can be improved:

**Proposition 4.5.** Let $X$ be any Banach space. Then there exists a constant $0 < c < \infty$ so that for any spreading model $(e_j)_{j=1}^{\infty}$ of a normalized sequence we have

(4.4) \[ c\|\sum_{j=1}^{n} a_je_j\|_{\ell_\tilde{\delta}} \leq \EE\|\sum_{j=1}^{n} \epsilon_je_je_j\|_S. \]

Proof. Let $N$ be the absolute norm such that

$$N(1,t) = 1 + \int_{0}^{t} \tilde{\delta}(s)\frac{ds}{s}, \quad t \geq 0.$$

Then we prove that

$$\|\sum_{j=1}^{n} a_je_j\|_{\Lambda_N} \leq \EE\|\sum_{j=1}^{n} \epsilon_je_je_j\|_S$$

by induction on $n$. Assume $n \geq 2$ and the result is known for $n - 1$. It is clear that

$$\EE\|\sum_{j=1}^{n} \epsilon_je_je_j\|_S \geq \EE N(\|\sum_{j=1}^{n-1} \epsilon_je_je_j\|, |a_n|)$$

$$\geq N(\EE\|\sum_{j=1}^{n-1} \epsilon_je_j\|, |a_n|)$$

$$\geq N(\|\sum_{j=1}^{n-1} a_je_j\|_{\Lambda_N}, |a_n|)$$

$$= \|\sum_{j=1}^{n} a_je_j\|_{\Lambda_N}.$$

This concludes the proof. □

We say that a Banach space $X$ not containing $\ell_1$ has the $p$-Banach-Saks property ($1 < p < \infty$) if there is a constant $C$ so that for every spreading model $(e_j)_{j=1}^{\infty}$ of a normalized weakly null sequence we have

$$\|\sum_{j=1}^{k} e_j\|_S \leq Ck^{1/p}, \quad k = 1, 2, \ldots.$$
This is equivalent to the requirement that there is a constant $C'$ so that every normalized weakly null sequence $(x_n)_n$ has a subsequence $(x_{n_j})_{j=1}^\infty$ such that

$$\|\sum_{j=1}^k x_{n_j}\| \leq C'k^{1/p}, \quad k \in \mathbb{N}, \; n_1 < \cdots < n_k.$$ 

We say that $X$ has the $p$-co-Banach-Saks property ($1 < p < \infty$) if there is a constant $c > 0$ so that for every spreading model $(e_j)_{j=1}^\infty$ of a normalized weakly null sequence we have

$$\|\sum_{j=1}^k e_j\|_S \geq ck^{1/p}, \quad k = 1, 2, \ldots.$$ 

The following proposition follows from Proposition 4.4:

**Proposition 4.6.** If $\bar{p}(t) \leq Ct^p$ for $0 \leq t \leq 1$, then $X$ has the $p$-Banach-Saks property. If $\bar{\delta}(t) \geq ct^p$ for $0 \leq t \leq 1$, then $X$ has the $p$-co-Banach-Saks property.

There is a simple duality relationship between these concepts, which we will need:

**Proposition 4.7.** Let $X$ be a reflexive space with the $p$-Banach-Saks property, where $1 < p < \infty$. Then $X^*$ has the $q$-co-Banach-Saks property, where $q = p/(p-1)$.

**Proof.** Let $C$ be the constant of the $p$-Banach-Saks property for $X$. Let $(x_n^*_n)_{n=1}^\infty$ be a normalized weakly null sequence in $X^*$. We may pick a normalized sequence $(x_n)_{n=1}^\infty$ in $X$ with $x_n^*(x_n) = 1$. Passing to a subsequence we can assume that

$$\lim_{n \to \infty} x_n = x \text{ weakly.}$$

Then $\|x_n - x\| \leq 2$ and so passing to a further subsequence we can assume that for any $k$,

$$\lim_{(n_1, n_2, \ldots, n_k) \to \infty} \|x_{n_1} + \cdots + x_{n_k} - kx\| \leq 2Ck^{1/p}$$

for any $k$. However

$$\lim_{n_1 \to \infty} \cdots \lim_{n_k \to \infty} \langle x_{n_1} + \cdots + x_{n_k} - kx, x_{n_1}^* + \cdots + x_{n_k}^* \rangle = k,$$

which implies that in any spreading model $(e_j)_{j=1}^n$ of $(x_j^*)_{j=1}^\infty$ we must have

$$\|\sum_{j=1}^k e_j\|_S \geq (1/(2C))k^{1/q}.$$ 

Finally let us also introduce a version of the $p$-co-Banach-Saks property for $p = 1$. We will say that $X$ has the anti-Banach-Saks property if there is a constant $c > 0$ so that for every spreading model $(e_j)_{j=1}^\infty$ of a normalized sequence, we have

$$\|\sum_{j=1}^k e_j\|_S \geq ck, \quad k = 1, 2, \ldots.$$ 

We make some simple observations about this property.
Proposition 4.8. Let $X$ be any Banach space. The following conditions on $X$ are equivalent:

(i) $X$ has the anti-Banach-Saks property.

(ii) There is a constant $c'$ so that for any spreading model of a normalized sequence $(x_n)_{n=1}^\infty$ we have

$$E\|\sum_{j=1}^k \epsilon_j e_j\|_S \geq c' \theta k,$$

where $\theta = \text{sep}\{x_n\}_{n=1}^\infty$.

Proof. (i) $\implies$ (ii). Let us first note that if $X$ has the anti-Banach-Saks property (with constant $c$) and $(e_j)_{j=1}^\infty$ is a spreading model of a normalized sequence, then any weak*-cluster point $x^{**}$ of $(e_j)_{j=1}^\infty$ in $S^{**}$ has norm at least $c$. Indeed, given $\nu > 0$, by Goldstine’s theorem we can find $a_1,\ldots,a_m \geq 0$ with $\sum_{j=1}^m a_j = 1$ and $\|\sum_{j=1}^m a_j e_j\|_S < \|x^{**}\| + \nu$. Hence for any $k > m$,

$$\|\sum_{i=1}^k \sum_{j=1}^m a_j e_{i+j}\|_S \leq k(\|x^{**}\| + \nu).$$

Rewriting this we obtain

$$c(k-m) \leq \|\sum_{i=m+1}^k e_i\|_S \leq k(\|x^{**}\| + \nu) + 2m.$$

Letting $k \to \infty$ gives $c \leq \|x^{**}\| + \nu$, where $\nu > 0$ is arbitrary.

Hence under the conditions of (ii), applying the above reasoning to $(e_{2j-1} - e_{2j})$, we find a weak*-cluster point $z^{**}$ of this sequence with $\|z^{**}\| \geq c \theta$. It follows that there exists $\varphi \in S^*$ with $\|\varphi\| = 1$ and $\lim_{j \to \infty} \varphi(e_{2j}) = \alpha$ and $\lim_{j \to \infty} \varphi(e_{2j-1}) = \beta$, where $\beta - \alpha \geq c \theta$. By considering translates we deduce the existence of $\psi \in S^*$ with $\|\psi\| \leq 1$ and $\psi(e_j) = \frac{1}{2} (\beta - \alpha)(-1)^j$. From this it is clear using the properties of the spreading model that for any choice of sign $\epsilon_j$ we have:

$$\|\sum_{j=1}^k \epsilon_j e_j\|_S \geq \frac{1}{2} c \theta k.$$

(ii) $\implies$ (i). Let $(e_j)_{j=1}^\infty$ be a spreading model of an arbitrary normalized sequence. If $\|e_1 - e_2\|_S \leq 1/2$, then

$$\|e_1 + \cdots + e_k\|_S \geq k/2, \quad k = 1,2,\ldots.$$

Otherwise

$$E\|\sum_{j=1}^k \epsilon_j e_j\|_S \geq \frac{1}{2} c' k$$

and so, using Lemma 4.1

$$\|\sum_{j=1}^k e_j\|_S \geq \frac{1}{6} c' k.$$

\[\square\]
5. $\ell_p$-Sums of Finite-Dimensional Spaces

The special properties of $\ell_p$-sums of finite-dimensional spaces have been studied in detail by many authors. Many of the ideas in this section originated in the early work of Johnson and Zippin on the spaces $C_p$ ([16], [22] and [23]). See also [35].

For $1 \leq p < \infty$, we shall say that a separable Banach space $X$ has property $(\tilde{m}_p)$ if it is isomorphic to a closed subspace of a space $(\sum_{n=1}^{\infty} E_n)_{\ell_p}$, where $(E_n)_{n=1}^{\infty}$ is a sequence of finite-dimensional subspaces of $X$. This terminology is motivated by the definition of property $(m_p)$ for $1 < p < \infty$. We recall that a Banach space $X$ has property $(m_p)$ [31] if for every $x \in X$ and every weakly null sequence $(x_n)_{n=1}^{\infty}$ such that the limits exist we have

$$\lim_{n \to \infty} \|x + x_n\|^p = \|x\|^p + \lim_{n \to \infty} \|x_n\|^p.$$  

Condition $(m_p)$ is exactly equivalent to the conditions $\overline{p}_X(t) = \overline{\delta}_X(t) = (1 + t^p)^{1/p} - 1$. It is clear that if $X$ has $(\tilde{m}_p)$ for $1 < p < \infty$ it has an equivalent norm with property $(m_p)$.

There are several characterizations of property $(\tilde{m}_p)$ for $1 < p < \infty$. The following result is due to Johnson, Lindenstrauss, Preiss and Schechtman [19], Proposition 2.11 (see also [37] for another isomorphic version).

**Theorem 5.1.** Suppose $1 < p < \infty$ and let $X$ be a separable reflexive Banach space. In order that $X$ has $(\tilde{m}_p)$ it is necessary and sufficient that it is isomorphic to a space $Y$ with $\overline{p}_Y(t) \leq Ct^p$ for $0 \leq t \leq 1$ and to a space $Z$ with $\overline{\delta}_Z(t) \geq ct^p$ for $0 \leq t \leq 1$, where $0 < c, C < \infty$.

On the other hand we have the following theorem. Part (i) is proved in [31], Theorem 3.2 and its proof; part (ii) follows from (i) by duality.

**Theorem 5.2.** Suppose $1 < p < \infty$. If $X$ is a separable Banach space with property $(m_p)$, then:

(i) $X$ is linearly isomorphic to a quotient of a space $(\sum_{n=1}^{\infty} E_n)_{\ell_p}$, where $(E_n)_{n=1}^{\infty}$ is a sequence of finite-dimensional subspaces of $X$.

(ii) $X$ is linearly isomorphic to a subspace of a space $(\sum_{n=1}^{\infty} E_n)_{\ell_p}$, where $(E_n)_{n=1}^{\infty}$ is a sequence of finite-dimensional quotients of $X$.

Note that it is actually shown in [31], Theorem 3.2 that if $X$ has property $(m_p)$, then for every $\epsilon > 0$, $X$ is $(1 + \epsilon)$–isomorphic to a subspace of a space $(\sum_{n=1}^{\infty} E_n)_{\ell_p}$ with each dim $E_n < \infty$.

The fact that in Theorem 5.2 (ii) one requires that the $(E_n)_{n=1}^{\infty}$ be quotients rather than subspaces is an inconvenience which can be rectified if $X$ has the approximation property. Results of this nature go back to the early work of Johnson and Zippin [23], who proved such a result for the special case of $C_p = (\sum_{n=1}^{\infty} G_n)_{\ell_p}$, where $(G_n)_{n=1}^{\infty}$ is a sequence dense in all finite-dimensional spaces in the sense of Banach–Mazur distance.

**Proposition 5.3.** Suppose $1 < p < \infty$ and $X$ is a separable Banach space with property $(\tilde{m}_p)$ and the approximation property. Then there is a sequence of finite-rank operators $A_n : X \to X$ such that $A_j A_k = 0$ for $|k - j| > 1$,

$$x = \sum_{n=1}^{\infty} A_n x, \quad x \in X$$
and for some constant $C$ we have

$$C^{-1}\|x\| \leq \left( \sum_{k=1}^{\infty} \|A_k x\|^p \right)^{1/p} \leq C \|x\|, \quad x \in X.$$ 

Proof. We may assume that $X$ is a subspace of a space $Z = (\sum_{n=1}^{\infty} G_n)_{e_p}$. Let $S_n : Z \to Z$ be the partial sum operators associated to the canonical (FDD) of $Z$. Let $S_0 = 0$. It follows from [16] that $X$ has the commuting metric approximation property and so (see [7]), we may find a sequence of finite-rank operators $R_n : X \to Z$ that are finite-rank operators such that $x = \lim_{n \to \infty} R_n x$ for $x \in X$ and $x^* = \lim_{n=1}^{\infty} R^*_n x^*$ for $x^* \in X^*$ and $R_m R_n = R_n R_m = R_n$ if $m > n$. Consider the operators $S_n, R_n$ in $\mathscr{K}(X, Z)$. Then we have $\lim_{n \to \infty} x^*(S^*_n - R^*_n)z^* = 0$ for $x^* \in X$ and $z^* \in Z^*$. This implies by Corollary 3 of [25] that $(S_n - R_n)$ converges weakly to 0.

Now fix $(\epsilon_k)_{k=1}^{\infty}$ with $\epsilon_k > 0$ and such that $\sum_{k=1}^{\infty} \epsilon_k < 1/8$. It follows from Mazur’s Theorem that we can find an increasing sequence of integers $(m_k)_{k=0}^{\infty}$ with $m_0 = 0$ and nonnegative $(a_j)_{j=1}^{\infty}$ with

$$\sum_{j=m_{k-1}+1}^{m_k} a_j = 1, \quad k = 1, 2, \ldots$$

and

$$\left\| \sum_{j=m_{k-1}+1}^{m_k} a_j (S_j - R_j) \right\|_{X \to Z} < \epsilon_k, \quad k = 1, 2, \ldots.$$ 

We define $V_k = \sum_{j=m_{k-1}+1}^{m_k} a_j R_j$ and $T_k = \sum_{j=m_{k-1}+1}^{m_k} a_j S_j$ with $V_0 = T_0 = 0$. Let $A_k = V_k - V_{k-1}$ and $B_k = T_k - T_{k-1}$. Then $A_j A_k = 0$ if $|j - k| \geq 1$,

$$\|A_k - B_k\| \leq \epsilon_k + \epsilon_{k-1}.$$ 

Hence for $x \in X$,

$$\left| \left( \sum_{k=1}^{\infty} \|A_k x\|^p \right)^{1/p} - \left( \sum_{k=1}^{\infty} \|B_k x\|^p \right)^{1/p} \right| \leq \frac{1}{4} \|x\|.$$ 

On the other hand,

$$\frac{1}{2} \|x\| \leq \left( \sum_{k=1}^{\infty} \|B_k x\|^p \right)^{1/p} \leq 2 \|x\|, \quad x \in X.$$

\[\square\]

**Theorem 5.4.** Suppose $1 < p < \infty$. Suppose $X$ is a separable Banach space with property $(\tilde{m}_n)$ and the approximation property. If $X$ is a complemented subspace in a Banach space $Y$ and $(E_i)_{i \in I}$ is a directed family of finite-dimensional subspaces of $Y$ with $\bigcup_{i \in I} E_i$ dense in $Y$, then $X$ is isomorphic to a complemented subspace of a space $(\sum_{i \in I} E_i)_{\ell_p}$ for some sequence $(i_n)_{n=1}^{\infty}$ in $I$.

In particular there is a sequence of finite-dimensional subspaces, $(F_n)_{n=1}^{\infty}$ of $X$ such that $X$ is linearly isomorphic to a complemented subspace of $(\sum_{n=1}^{\infty} F_n)_{\ell_p}$.

Proof. Let $(A_n)$ be the finite rank operators given by the previous proposition. We may embed $A_n(X)$ in a finite-dimensional subspace of $Y$, $H_n$, say, such that
d(H_n, E_{i_n}) \leq 2$ for a suitable choice of $i_n$. Let $P : Y \to X$ be a bounded projection and define $Q : (\sum_{n=1}^\infty H_n)_{\ell_p} \to X$ by

$$Q(h_k)_{k=1}^\infty = \sum_{j=1}^\infty \sum_{|j-k| \leq 1} A_j Ph_k.$$  

Notice that if $(x_k)_k$ is a finitely nonzero sequence with $x_k \in A_k(X)$ we have an estimate for $j = 0, 1, 2$:

$$\|QId\| \leq 9C(\sum_{j=1}^\infty \|A_j(\sum_{|k-j| \leq 1} Ph_k)\|^{1/p})^p \leq 3^3 C^2 \|P\|\|h\|$$

so that $Q$ extends to a bounded operator.

Define $J : X \to (\sum_{n=1}^\infty H_n)_{\ell_p}$ by $Jx = (A_nx)_{n=1}^\infty$. Then $J$ is bounded and $QJ = Id_X$. 

Our final result will be useful when studying uniform homeomorphisms.

**Theorem 5.5.** Suppose $1 < p < \infty$ and that $X$ is a separable Banach space with property ($\tilde{m}_p$). Let $(E_n)_{n=1}^\infty$ be a sequence of finite-dimensional subspaces of $X$ such that for some constant $\lambda \geq 1$ and every $m, n$ there is a subspace $F_{m,n}$ of $X$ such that $F_{m,n}$ is $\lambda$–complemented in $X$ and $d(F_{m,n}, \ell^m_p(E_n)) \leq \lambda$. Then $(\sum_{n=1}^\infty E_n)_{\ell_p}$ is isomorphic to a complemented subspace of $X$.

**Proof.** We can assume $X$ has $(m_p)$ (and so $X^*$ has $(m_q)$). We first show that given any finite-dimensional subspaces $G \subset X$, $H \subset X^*$ and $n \in \mathbb{N}$ there exist operators $A : E_n \to X$ and $B : X \to E_n$ with $BA = I_{E_n}$, $\|A\|, \|B\| \leq 2\lambda$, and $A(E_n) \subset H^\perp$, $B^*(E^*_n) \subset G^\perp$.

Let $d_c = \dim E_n$, $d_g = \dim G$ and $d_h = \dim H$. Fix $m > 2^\delta \lambda^4 d_h(d_g + d_h)d_c$. By hypothesis there exist operators $S : \ell^m_p(E_n) \to X$ and $T : X \to \ell^m_p(E_n)$ with $TS = I_{\ell^m_p(E_n)}$ and $\|S\|, \|T\| \leq \lambda$. If we write $S(u_j)_{j=1}^m = \sum_{j=1}^m S_j u_j$ and $T x = (T_j x)_{j=1}^m$, then $T_j S_j = I_{E_n}$.

We clearly have $\|\sum_{j=1}^m \theta_j T_j\|_{X \to E_n} \leq \lambda$ for all $\theta_j = \pm 1$. Since $\mathcal{L}(G, E_n)$ is $\sqrt{d_g d_c}$-isometric to a Hilbert space we have

$$\sum_{j=1}^m \|T_j\|_{G \to E_n}^2 \leq d_g d_c \lambda^2.$$

Similarly

$$\sum_{j=1}^m \|S_j^*\|_{H \to E_n^*}^2 \leq d_h d_c \lambda^2.$$
Thus there exists \( j \) so that
\[
\|T_j\|_{\ell^p \to E_n}^2 + \|S_j\|_{H \to E_n^*}^2 \leq m^{-1}(d_g + d_h)d_e\lambda^2.
\]

Now we can find two projections, \( P \) and \( Q \) on \( X \) with \( \|P\| \leq \sqrt{d_g} \) and \( \|Q\| \leq 2\sqrt{d_h} \) so that \( P(X) = G \) and \( Q^*(X^*) = H \). Now consider the operator \( T_j(I - P)(I - Q)S_j \). We have
\[
\|QS_j\| = \|S_j^*Q^*\| \leq 2\sqrt{d_h}\|S_j\|_{H \to E_n^*} \leq 2\lambda d_e^{1/2}d_h^{1/2}m^{-1/2}(d_g + d_h)^{1/2} \leq 1/(8\lambda)
\]
and
\[
\|T_jP\| \leq \lambda d_e^{1/2}d_h^{1/2}m^{-1/2}(d_g + d_h) \leq 1/(8\lambda).
\]
Hence
\[
\|I_{E_n} - T_j(I - P)(I - Q)S_j\| \leq 1/4 + 1/(64\lambda^2) < 1/3.
\]
Hence there is an operator \( D : E_n \to E_n \) with \( \|D\| \leq 3/2 \) so that
\[
T_j(I - P)(I - Q)S_jD = I_{E_n}.
\]
Let \( B = T_j(I - P) \) and \( A = (I - Q)S_jD \); then \( \|A\|, \|B\| \leq 2\lambda \). This completes the proof of our claim.

Since \( X \) has \( (m_p) \) and \( X^* \) has \( (m_q) \), it now follows that we can use an inductive construction to find two sequences of operators \( A_n : E_n \to X \) and \( B_n : X \to E_n \) so that
\[
\left\| \sum_{n=1}^{\infty} A_n u_n \right\| \leq 4\lambda \left( \sum_{n=1}^{\infty} \|u_n\|^p \right)^{1/p}, \quad u_n \in E_n, \ n = 1, 2, \ldots
\]
and
\[
\left\| \sum_{n=1}^{\infty} B_n u_n^* \right\| \leq 4\lambda \left( \sum_{n=1}^{\infty} \|u_n^*\|^q \right)^{1/q}, \quad u_n^* \in E_n^*, \ n = 1, 2, \ldots
\]
and \( B_n A_n = I_{E_n} \).

Hence we may define \( A : \left( \sum_{n=1}^{\infty} E_n \right)_{\ell_p} \to X \) and \( B : X \to \left( \sum_{n=1}^{\infty} E_n \right)_{\ell_p} \) by \( A((u_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} A_n u_n \) and \( Bx = \sum_{n=1}^{\infty} B_n x \) and we have \( BA = I_{\left( \sum_{n=1}^{\infty} E_n \right)_{\ell_p}} \) and \( \|A\|, \|B\| \leq 4\lambda \).

6. Subspaces and Quotients of \( L_p \)

We now introduce a definition which will be useful to us later. This idea was first used in the work of Haydon, Raynaud and Levy on ultraproducts (\[33\] and \[13\]).

Let us say that a Banach space \( Y \) has a random \( L_p \)-norm if there is a (nonlinear) map \( V : Y \to Z \), where \( Z \) is an abstract \( L_p \)-space such that:
\[
Vy \geq 0, \quad y \in Y,
\]
\[
V(\alpha y) = |\alpha|Vy, \quad y \in Y, \ \alpha \in \mathbb{R},
\]
\[
V(y_1 + y_2) \leq Vy_1 + Vy_2, \quad y \in Y,
\]
and
\[
\|Vy\|_p = \|y\|, \quad y \in Y.
\]

\( V \) is then called the random \( L_p \)-norm on \( Y \). \( V \) is easily verified to be continuous and hence if \( Y \) is separable we can replace \( Z \) by \( L_p[0,1] \). If \( r > p \) we say that \( V \) is of type \( r \) if there is a constant \( C \) such that for any \( y_1, y_2 \in Y \) we have
\[
\frac{1}{2}(V(y_1 + y_2) + V(y_1 - y_2)) \leq ((Vy_1)^r + C^r(Vy_2)^r)^{1/r}.
\]
Theorem 6.1. Suppose \( 1 < p < r \leq 2 \). Let \( Y \) be a separable Banach space with a random \( L_p \)-norm of type \( r \), and let \( X \) be any quotient of \( Y \). Then if \( X \) has the \( p \)-co-Banach-Saks property, then \( X \) has property \((\tilde{m}_p)\).

Proof. Let \( V : Y \to L_p[0,1] \) be the random \( L_p \)-norm. Let us use \(|E|\) to denote the Lebesgue measure of a measurable set \( E \). First we define for any \( 0 < \theta < 1 \),

\[
\|f\|_{p,\theta} = \sup_{|E| \leq \theta} \|\chi_E f\|_p, \quad f \in L_p.
\]

We also let \( Q : Y \to X \) be the quotient map.

Next observe that \( Y \) (and hence \( X \)) is super-reflexive. Indeed if \( C \) is the constant in the definition of a random \( L_p \)-norm of type \( r \),

\[
\frac{1}{2}(\|y_1+y_2\|^{p}+\|y_1-y_2\|^{p}) \leq \int_0^1 ((Vy_1(s))^{r}+C^{r}(Vy_2(s))^{r})^{p/r} \, ds \leq \|y_1\|^{p}+C^{p}\|y_2\|^{p}.
\]

This implies that \( Y \) is \( p \)-uniformly smooth. Further if \( \|y\| = 1 \) and \( \|z\| = t \) is such that \( y^*(z) = 0 \), where \( \|y^*\| = y^*(y) = 1 \), then

\[
\|y+tz\| - 1 \leq \|y+tz\| + \|y-tz\| - 2 \leq \|y+tz\|^{p} + \|y-tz\|^{r} - 2 \leq 2C^{p}t^{r}.
\]

Hence \( \bar{p}_{Y}(t) \leq 2C^{p}t^{r} \) for \( 0 \leq t \leq 1 \). This implies that also by Proposition 3.2 \( \bar{p}_{X}(t) \leq 4C^{p}t^{r} \). To prove the theorem it therefore suffices by Theorem 5.1 to show that \( \bar{p}_{X}(t) \geq at^{r} \) for \( 0 \leq t \leq 1 \) for some \( a > 0 \).

Let us suppose that \( X \) has the \( p \)-co-Banach-Saks property with constant \( c > 0 \). Suppose \( \|x\| = 1 \) and \( (x_n)_{n=1}^{\infty} \) is a weakly null sequence with \( \|x_n\| = t \leq 1 \). We will show that

\[
(6.5) \quad \liminf_{n \to \infty} \|x + x_n\|^{p} \geq 1 + 2^{-p}C^{-p}c^{p}t^{p}.
\]

If this false we can pass to a subsequence and suppose that

\[
\lim_{n \to \infty} \|x + x_n\|^{p} = 1 + b^{p},
\]

where \( b < \frac{c}{2^{r}}t \). If \( 0 < \lambda < 1 \) is chosen so that \( b < \frac{\lambda c}{2^{r}} \), we can pass to a further subsequence and suppose that

\[
\|x_{n_1} + \cdots + x_{n_k}\| \geq \lambda ck^{1/p}t, \quad n_1 < n_2 < \cdots < n_k
\]

and that (using Lemma 4.1)

\[
\mathbb{E} \left\| \sum_{j=1}^{k} \epsilon_j x_{n_j} \right\| \geq \frac{\lambda c}{2} \cdot k^{1/p}t, \quad n_1 < n_2 < \cdots < n_k.
\]

Pick \( z_n \in Y \) so that \( Qz_n = x + x_n \) and \( \|z_n\| = \|x + x_n\| \). Passing to a yet further subsequence we can suppose that \( (z_n)_{n=1}^{\infty} \) converges weakly to some \( y \in Y \); then let \( y_n = z_n - y \). Thus \( Qy = x \) and \( Qy_n = x_n \). In particular

\[
(\mathbb{E} \left\| \sum_{j=1}^{k} \epsilon_j y_{n_j} \right\|^{p})^{1/p} \geq \frac{\lambda c}{2} k^{1/p}t, \quad n_1 < n_2 < \cdots < n_k.
\]

Now suppose \( 0 < \theta < 1 \). Note that we have a crude estimate \( \|y\|, \|z_n\| \leq 2 \) and hence \( \|y_n\| \leq 4 \). For each \( n \), let \( E_n \) be a measurable subset of \([0,1]\) with measure \( |E_n| = \theta \) and \( \|V_{y_n} 1_{E_n} \|_{p,\theta} = \|\chi_{E_n} V_{y_n} 1_{E_n} \|_p \). Let us denote \( \bar{E}_n = [0,1] \setminus E_n \). Then

\[
\|V_{y_n} 1_{E_n}\|_{\infty} \leq 4 \theta^{-1/p} \quad \text{and so} \quad \|V_{y_n} 1_{E_n}\|_r \leq 4 \theta^{-1/p} (1 - \theta)^{1/r}.
\]
Now
\[
(\mathbb{E}\|\sum_{j=1}^{k} \epsilon_j y_{n_j}\|^p)^{1/p} = (\mathbb{E}\|V(\sum_{j=1}^{k} \epsilon_j y_{n_j})^p\|^{1/p})^{1/p} \leq C\|\sum_{j=1}^{k} (V y_{n_j})^r\|^{1/r}_p.
\]

We first estimate:
\[
\|\sum_{j=1}^{k} (V y_{n_j})^r \chi_{E_{n_j}}\|_p^{1/r} \leq \|\sum_{j=1}^{k} (V y_{n_j})^p \chi_{E_{n_j}}\|_p^{1/p}
\]
\[
\leq 4 \theta^{-1/p} (1 - \theta)^{1/r} k^{1/r}.
\]

On the other hand
\[
\|\sum_{j=1}^{k} (V y_{n_j})^r \chi_{E_{n_j}}\|_p^{1/r} \leq \|\sum_{j=1}^{k} (V y_{n_j})^p \chi_{E_{n_j}}\|_p^{1/p}
\]
\[
\leq \left(\sum_{j=1}^{k} \|V y_{n_j}\|_{p,\theta}^{p}\right)^{1/p}.
\]

Combining we have that
\[
\frac{1}{2} \lambda c k^{1/p} t \leq C\left(\sum_{j=1}^{k} \|V y_{n_j}\|_{p,\theta}^{p}\right)^{1/p} + 4C\theta^{-1/p} (1 - \theta)^{1/r} k^{1/r}.
\]

Since this holds for any \(n_1 < \cdots < n_k\) we conclude that (for any \(0 < \theta < 1\)),
\[
(6.6) \quad \liminf_{n \to \infty} \|V y_n\|_{p,\theta} \geq \frac{\lambda c t}{2C}.
\]

Choose \(b_1\) so that \(b < b_1 < \frac{\lambda c t}{2C}\). Then we pick for each \(n\) a set \(G_n\) of minimal measure so that \(\|\chi_{G_n} V y_n\|_p = b_1\). Then from (6.6) we have \(\lim_{n \to \infty} |G_n| = 0\).

On \(Y\) consider the seminorm \(z \mapsto \int (V y(s))^{p-1} V z(s)\, ds\). Then by the Hahn-Banach theorem there is a linear functional \(y^* \in Y^*\) with \(y^*(y) = \|V y\|^p = \|y\|^p\) and
\[
y^*(z) \leq \int (V y(s))^{p-1} V z(s)\, ds, \quad z \in Y.
\]

In particular
\[
\|y\|^p = \lim_{n \to \infty} y^*(y + y_n) \leq \liminf_{n \to \infty} \int (V y(s))^{p-1} V (y + y_n)(s)\, ds.
\]

Now
\[
\int_{G_n} (V y(s))^{p-1} V (y + y_n)(s)\, ds \leq \left(\int_{G_n} (V y(s))^p\, ds\right)^{1-1/p} \|y + y_n\|
\]
so that
\[
\lim_{n \to \infty} \int_{G_n} (V y(s))^{p-1} V (y + y_n)(s)\, ds = 0.
\]

Hence
\[
\liminf_{n \to \infty} \int_{G_n} (V y(s))^{p-1} V (y + y_n)(s)\, ds \geq \|y\|^p.
\]
This implies by Hölder’s inequality that
\[ \liminf_{n \to \infty} \| \chi G_n V(y + y_n) \|_p \geq \| y \| \geq 1. \]
On the other hand
\[ \| \chi G_n V y \|_p + \| \chi G_n V(y + y_n) \|_p \geq \| \chi G_n V y_n \|_p = b_1 \]
so that
\[ \liminf_{n \to \infty} \| \chi G_n V(y + y_n) \|_p \geq b_1. \]
Hence
\[ \liminf_{n \to \infty} \| V(y + y_n) \|_p \geq b_1. \]
However
\[ \lim_{n \to \infty} \| V(y + y_n) \|_p = \lim_{n \to \infty} \| x + x_n \|_p = 1 + b^p. \]
This contradiction shows that \( \delta_X(t) \geq (1 + 2^{-p}C^{-p}t^p) - 1 \) for \( 0 \leq t \leq 1 \) and concludes the proof. □

**Corollary 6.2.** (i) Suppose \( 1 < p < 2 \) and that \( X \) is a subspace (respectively quotient space) of \( L_p \) with the \( p \)-co-Banach-Saks property. Then \( X \) is a subspace (respectively quotient space) of \( \ell_p \).

(ii) Suppose \( 2 < p < \infty \) and that \( X \) is a subspace (respectively quotient space) of \( L_p \) with the \( p \)-Banach-Saks property. Then \( X \) is a subspace (respectively quotient space) of \( \ell_p \).

**Proof.** (ii) is due to Johnson [17]. For (i) we observe that \( X \) has \( (\tilde{m}_p) \) by Theorem 6.1 and so \( X^* \) has \( (\tilde{m}_q) \), where \( 1/p + 1/q = 1 \). Hence we can apply (ii) to deduce that \( X^* \) is a quotient (respectively a subspace) of \( \ell_q \) and then use duality. □

The second part of the next theorem was proved by Johnson [17] with an additional hypothesis that \( X \) is the quotient of a subspace of \( L_p \) with the approximation property. The theorem answers a question raised by Johnson (Problem IV.2) in that paper.

**Theorem 6.3.** Suppose \( 1 < p < \infty \) and that \( X \) is a subspace of a quotient of \( L_p \).

(i) If \( 1 < p < 2 \) and \( X \) has the \( p \)-co-Banach-Saks property, then \( X \) is isomorphic to a subspace of a quotient of \( \ell_p \).

(ii) If \( 2 < p < \infty \) and \( X \) has the \( p \)-Banach-Saks property, then \( X \) is isomorphic to a subspace of a quotient of \( \ell_p \).

**Proof.** (i) By Theorem 6.1 \( X \) has property \( (\tilde{m}_p) \) and hence by Theorem 5.2 \( X \) embeds into \( (\sum_{n=1}^{\infty} E_n)_{\ell_p} \), where the \( E_n \)'s are finite-dimensional subspaces of quontients of \( L_p \) and hence also of \( \ell_p \). Thus \( X \) is a subspace of a quotient of \( \ell_p \).

(ii) By Theorem 4.7 \( X^* \) has the \( q \)-co-Banach-Saks property, where \( 1/p + 1/q = 1 \); hence by (i), \( X^* \) is a subspace of a quotient of \( \ell_q \) and the result follows by duality. □

Let us now consider the analogue of these results when \( p = 1 \). Let us recall that a Banach space \( X \) has the strong Schur property if there is a constant \( c > 0 \) so that if \( (x_n)_{n=1}^{\infty} \) is a sequence in \( X \) with \( \text{sep} \{x_n\}_{n=1}^{\infty} = \delta > 0 \), then there is a subsequence with
\[ \| \sum_{j=1}^{k} a_j x_{n_j} \| \geq c \sum_{j=1}^{k} |\alpha_j|. \]
This concept was considered first (implicitly) by Johnson and Odell [21] and then by Bourgain and Rosenthal [5], who gave examples of subspaces of $L_1$ with the strong Schur property but failing to have the Radon-Nikodým Property.

An alternative formulation of the strong Schur property is given in [27]. $X$ has the strong Schur property if there is a constant $c$ so that for every bounded sequence $(x_n)_{n=1}^\infty$ there exists $x^* \in X^*$ with $\|x^*\| = 1$ and

$$\limsup_{n \to \infty} x^*(x_n) \geq c \limsup_{n \to \infty} \|x_n\|.$$

**Theorem 6.4.** Let $X$ be a closed subspace of $L_1$. The following conditions on $X$ are equivalent:

(i) $X$ has the anti-Banach-Saks property.

(ii) $X$ has the strong Schur property.

(iii) For some $c > 0$ we have $\hat{\delta}_X(t) \geq ct$.

*Proof.* That (iii) implies (ii) follows from Proposition 4.5. It is clear that (ii) implies (i). It remains to show that (i) implies (iii). The argument is a variation on Theorem 6.1. By Proposition 4.8 there is a constant $c > 0$ so that for every normalized sequence $(g_n)_{n=1}^\infty$ with $\text{sep} \{g_n\}_{n=1}^\infty = \alpha$ in $X$ we can pass to a subsequence $(f_n)_{n=1}^\infty$ with

$$E\| \sum_{j=1}^k \epsilon_j f_n_j \| \geq c\alpha k.$$

Let us fix such a sequence $(f_n)_{n=1}^\infty$. Suppose $0 < \theta < 1$ and pick $E_n \subset [0, 1]$ so that $|E_n| = \theta$ and $\|\chi_{E_n} f_n\|_1 = \|f_n\|_{1,\theta}$. Then for any $n_1 < n_2 < \cdots < n_k$,

$$\|\left( \sum_{j=1}^k \chi_{\tilde{E}_{nj}} |f_n_j|^2 \right)^{1/2}\|_1 \leq k^{1/2} \theta^{-1}$$

so that

$$E\| \sum_{j=1}^k \epsilon_j \chi_{\tilde{E}_{nj}} f_n_j \|_1 \leq k^{1/2} \theta^{-1}.$$

Hence

$$E\| \sum_{j=1}^k \epsilon_j \chi_{E_{nj}} f_n_j \|_1 \geq c\alpha k - \theta^{-1} k^{1/2}$$

so that

$$\sum_{j=1}^k \|f_n_j\|_{1,\theta} \geq c\alpha k - \theta^{-1} k^{1/2}.$$

In particular $\liminf_{n \to \infty} \|f_n\|_{1,\theta} \geq c\alpha$.

Now if $f \in L_1$ with $\|f\|_1 = 1$ and $t > 0$, we have

$$\frac{1}{2}(\|f + tf_n\|_1 + \|f - tf_n\|_1) \geq t \int_{E_n} |f_n(s)| ds + \int_{E_n} |f(s)| ds$$

$$= 1 + t\|f_n\|_{1,\theta} - \int_{E_n} |f(s)| ds.$$
Hence
\[
\liminf_{n \to \infty} \frac{1}{2}(\|f + tf_n\|_1 + \|f - tf_n\|_1) \geq 1 + \text{cat} \sup_{|E| \leq \theta} \int_E |f| \, ds.
\]

As \( \theta > 0 \) is arbitrary we have \( \hat{\delta}_X(t) \geq ct. \) \( \square \)

7. MAPPINGS ON ORLICZ SPACES AND APPLICATIONS

We refer to [3] for background on nonlinear theory. However, we need to recall some definitions and notation. Let \((M, d)\) and \((N, \delta)\) be two unbounded metric spaces. We define for \(f: M \to N\):
\[
\forall t > 0 \quad \omega_f(t) = \sup \left\{ \delta(f(x), f(y)) : x, y \in M, \ d(x, y) \leq t \right\}.
\]

We say that \(f\) is uniformly continuous if \(\lim_{t \to 0} \omega_f(t) = 0\). The map \(f\) is said to be coarsely continuous if \(\omega_f(t) < \infty\) for some \(t > 0\).

Let us now introduce
\[
\text{Lip}_s(f) = \sup_{t \geq s} \frac{\omega_f(t)}{t}, \quad \text{for } s > 0
\]
and
\[
L(f) = \sup_{s > 0} \text{Lip}_s(f), \quad \text{Lip}_\infty(f) = \inf_{s > 0} \text{Lip}_s(f).
\]

A map is Lipschitz if and only if \(L(f) < \infty\). We will say that it is coarse Lipschitz if \(\text{Lip}_\infty(f) < \infty\). Clearly, a coarse Lipschitz map is coarsely continuous. If \(f\) is bijective, we will say that \(f\) is a uniform homeomorphism (respectively, coarse homeomorphism, Lipschitz homeomorphism, coarse Lipschitz homeomorphism) if \(f\) and \(f^{-1}\) are uniformly continuous (respectively, coarsely continuous, Lipschitz, coarse Lipschitz). Finally we say that \(f\) is a coarse Lipschitz embedding if it is a coarse Lipschitz homeomorphism from \(M\) onto \(f(M)\).

It is well known that if \(X\) and \(Y\) are Banach spaces, then for any map \(f: X \to Y\), \(\omega_f\) is a subadditive function. It follows that any uniform homeomorphism \(f: X \to Y\) is a coarse Lipschitz homeomorphism.

Given a metric space \(X\), two points \(x, y \in X\), and \(\nu > 0\), the approximate metric midpoint set between \(x\) and \(y\) with error \(\nu\) is the set:
\[
\text{Mid}(x, y, \nu) = \left\{ z \in X : \max\{d(x, z), d(y, z)\} \leq (1 + \nu)\frac{d(x, y)}{2} \right\}.
\]

The use of metric midpoints in the study of nonlinear geometry is due to Enflo in an unpublished paper and has since been used elsewhere, e.g. [4], [12] and [20].

The following version of the Midpoint Lemma was formulated in [30] (see also [3], Lemma 10.11). Note that completeness of \(X\) is not needed.

**Lemma 7.1.** Let \(X\) be a normed space and suppose \(M\) is a metric space. Let \(f: X \to M\) be a coarse Lipschitz map. If \(\text{Lip}_\infty(f) > 0\), then for any \(t, \epsilon > 0\) and any \(0 < \nu < 1\), there exist \(x, y \in X\) with \(\|x - y\| > t\) and
\[
f(\text{Mid}(x, y, \nu)) \subset \text{Mid}(f(x), f(y), (1 + \epsilon)\nu).
\]
Lemma 7.2. Let $X$ be a normed space and suppose $(x_n)_{n=1}^{\infty}$ is a normalized sequence in $X$. Define $T : c_00 \to X$ by $T \xi = \sum_{k=1}^{\infty} \xi_k x_k$. Let

$$\sigma_k = \sup \left\{ \| \sum_{j=1}^{k} \theta_j x_{n_j} \| : n_1 < n_2 < \cdots < n_k, \, \theta_j = \pm 1 \right\}.$$

For each $k \in \mathbb{N}$ define

$$(7.7) \quad F_k(t) = \begin{cases} \sigma_k t/k, & 0 \leq t \leq 1/\sigma_k, \\ t + 1/k - 1/\sigma_k, & 1/\sigma_k \leq t < \infty. \end{cases}$$

Then for any $\xi \in c_00$ we have

$$\| T \xi \| \leq 2\| \xi \|_{F_k}.$$

Proof. Note first that for any set $A \subset \mathbb{N}$ with $|A| = m$ we have

$$\| \sum_{j \in A} \xi_j x_j \| \leq \max_{j \in A} |\xi_j| \sigma_m.$$

Let $a \in \ell_1$ with $\sum_{j=1}^{\infty} F_k(|\xi_j|) \leq 1$ and let $(\xi^*_j)_{j=1}^{\infty}$ be the decreasing rearrangement of $(|\xi_j|)_{j=1}^{\infty}$. Now $F_k(\xi^*_j) \leq 1/k$ and hence $\xi^*_k \leq 1/\sigma_k$. Then

$$\| T \xi \| \leq \sum_{j=1}^{\infty} (\xi^*_j - \xi^*_{j+1}) \sigma_j$$

$$\leq \sum_{j=1}^{k} (\xi^*_j - \xi^*_{j+1}) \sigma_j + \frac{\sigma_k}{k} \sum_{j=k+1}^{\infty} j(\xi^*_j - \xi^*_{j+1})$$

$$= \sum_{j=1}^{k} \xi^*_j (\sigma_j - \sigma_{j-1}) + \frac{\sigma_k}{k} \sum_{j=k+1}^{\infty} \xi^*_j$$

$$\leq \sum_{j=1}^{k} (F_k(\xi^*_j) + \sigma_k^{-1})(\sigma_j - \sigma_{j-1}) + \sum_{j=k+1}^{\infty} F_k(\xi^*_j)$$

$$\leq 1 + \sum_{j=1}^{\infty} F_k(\xi^*_j)$$

$$\leq 2.$$

Hence $\| T \|_{\ell_{F_k} \to X} \leq 2$. \hfill \Box

Theorem 7.3. Let $X$ and $Y$ be two Banach spaces such that $X$ coarse Lipschitz-embeds into $Y$. Then there is a constant $c > 0$ so that given any normalized sequence $(x_n)_{n=1}^{\infty}$ with $\text{sep}\{x_n\}_{n=1}^{\infty} = \theta > 0$ in $X$ and any integer $k$ there exist $n_1 < \cdots < n_k$ so that

$$c\theta \| e_1 + \cdots + e_k \|_{\ell_{kY}} \leq \mathbb{E}\| e_1 x_{n_1} + \cdots + e_k x_{n_k} \|.$$

Proof. We may assume that for some constant $K$ we have a map $f : X \to Y$ such that $f(0) = 0$ and

$$\| x - z \| - 1 \leq \| f(x) - f(z) \| \leq K \| x - z \| + 1, \quad x, z \in X.$$
Let
\[ \sigma_k = \sup \left\{ \mathbb{E} \left\| \sum_{j=1}^k \epsilon_j x_{n_j} \right\|, \quad n_1 < n_2 < \cdots < n_k \right\}. \]

Let \( \sigma_0 = 0 \). Then \( (\sigma_k)_{k=0}^\infty \) is a monotone increasing sequence.

For each \( k \), define the Orlicz function \( F_k \) by (7.7). We let \( N_k \) be the absolute norm on \( \mathbb{R}^2 \) such that
\[ N_k(1,t) = 1 + F_k(t), \quad t \geq 0. \]

We also define an absolute norm on \( \mathbb{R}^2 \) by
\[ N_Y(1,t) = 1 + \int_0^t \delta_Y(s) \frac{ds}{s}, \quad t \geq 0. \]

Let us note, for future use, the following property of \( N_Y \). If \( y, z \in Y \) and \( (y_n)_{n=1}^{\infty} \) is any bounded sequence in \( Y \), then
\[ \liminf_{n \to \infty} (\| y - y_n \| + \| z - y_n \|) \geq N_Y(\| y - z \|, \text{sep} \{y_n\}_{n=1}^{\infty}). \]

This is an immediate consequence of Proposition 3.3.

We define an operator \( T : c_{00} \to L_1(\Delta; X) \) by
\[ T(\xi) = \sum_{j=1}^\infty \xi_j \epsilon_j \otimes x_j. \]

Combining Lemmas 4.3 and 7.2 we have
\[ \|T\xi\| \leq 4\|\xi\|_{\Lambda_{N_k}}. \]

We then consider the map \( g : c_{00} \to L_1(\Delta; Y) \) defined by \( \xi \to f \circ T\xi \). This is well-defined because \( f(0) = 0 \) and \( T\xi \) is a simple function so that there are no measurability problems. We have an estimate
\[ \|g(\xi) - g(\eta)\| \leq 4K\|\xi - \eta\|_{\Lambda_{N_k}} + 1, \quad \xi, \eta \in c_{00}. \]

We also have \( \|g(te_1)\| = \frac{1}{2}(\| f(tx_1)\| + \| f(-tx_1)\|) \geq t - 1 \) so that \( \text{Lip}_\infty(g) \geq 1. \)

We apply the Midpoint Lemma (Lemma 7.1 to \( g : (c_{00}, \cdot \|_{\Lambda_{N_k}}) \to L_1(\Delta, Y) \) with \( \nu = 1/k \). For any \( \tau_0 > 0 \) we can find \( \tau > \tau_0 \) and points \( \eta, \zeta \in c_{00} \) with \( \|\eta - \zeta\|_{\Lambda_{N_k}} = 2\tau \) such that
\[ g(\text{Mid}(\eta, \zeta, 1/k)) \subset \text{Mid}(g(\eta), g(\zeta), 2/k). \]

Let \( \xi = \frac{1}{2}(\eta + \zeta) \).

There exists \( m \in \mathbb{N} \) so that \( \eta, \zeta \in \{e_1, \ldots, e_{m-1}\} \). Thus if \( j \geq m \) we have, from the iterative nature of the norm on \( \Lambda_{N_k} \), \( \xi + \tau \sigma_k^{-1} e_j \in \text{Mid}(\eta, \zeta, 1/k) \).

Thus the functions
\[ h_j = f \left( \sum_{i=1}^{m-1} \xi_i e_i \otimes x_i + \tau \sigma_k^{-1} e_j \otimes x_j \right) \]

all belong to \( \text{Mid}(g(\eta), g(\zeta), 2/k) \) for \( j \geq m \). Since both \( g(\eta) \) and \( g(\zeta) \) depend only on the first \( m - 1 \) coordinates of \( \Delta \), this implies that the same is true for the functions
\[ h'_j = f \left( \sum_{i=1}^{m-1} \xi_i e_i \otimes x_i + \tau \sigma_k^{-1} e_m \otimes x_j \right). \]
The functions $h_j'$ now depend on the first $m$ coordinates of $\Delta$. In particular
\[(7.9) \quad \|g(\eta) - h_j'\| + \|g(\zeta) - h_j'\| - \|g(\eta) - g(\zeta)\| \leq 2k^{-1}\|g(\eta) - g(\zeta)\|.
\]
Note that for any $s \in \Delta$ we have
\[
\|h_i'(s) - h_j'(s)\| \geq \theta \tau \sigma_k^{-1} - 1, \quad i > j \geq m.
\]
Hence, using (7.8), we have
\[
\liminf_{j \to \infty} \|g(\eta)(s) - h_j'(s)\| + \|g(\zeta)(s) - h_j'(s)\| \geq N_Y(\|g(\eta)(s) - g(\zeta)(s)\|, \theta \tau \sigma_k^{-1} - 1)
\]
as long as $\tau > \sigma_k/\theta$.

Integrating (note the integral is simply a finite sum in this case),
\[
\liminf_{j \to \infty} (\|g(\eta) - h_j'\| + \|g(\zeta) - h_j'\|) \geq \int_{\Delta} N_Y(\|g(\eta)(s) - g(\zeta)(s)\|, \theta \tau \sigma_k^{-1} - 1) ds
\]
\[
\geq N_Y(\|g(\eta) - g(\zeta)\|, \theta \tau \sigma_k^{-1} - 1).
\]

Now $\|g(\eta) - g(\zeta)\| \leq 8K\tau + 1$ and since $N_Y(t, 1) - t$ is a decreasing function we conclude that
\[
\liminf_{j \to \infty} (\|g(\eta) - h_j'\| + \|g(\zeta) - h_j'\| - \|g(\eta) - g(\zeta)\|) \geq N_Y(8K\tau + 1, \theta \tau \sigma_k^{-1} - 1) - (8K\tau + 1).
\]
Hence, by (7.9),
\[
N_Y(8K\tau + 1, \theta \tau \sigma_k^{-1} - 1) - (8K\tau + 1) \leq \frac{2}{k} \|g(\eta) - g(\zeta)\| \leq 2(8K\tau + 1)\sigma_k^{-1}.
\]
We simplify this as
\[
N_Y\left(1, \frac{\theta - \sigma_k^{-1}}{\sigma_k(8K + \tau^{-1})}\right) \leq 1 + \frac{2}{k}.
\]
Now we can let $\tau \to \infty$ and deduce that
\[
N_Y(1, \frac{\theta}{8K\sigma_k}) \leq 1 + \frac{2}{k}.
\]
This implies that
\[
\delta\left(\frac{\theta}{16K\sigma_k}\right) \leq \frac{2}{k}
\]
and hence
\[
\delta\left(\frac{\theta}{32K\sigma_k}\right) \leq \frac{1}{k}
\]
or
\[
\|e_1 + \cdots + e_k\|_{\ell_Y} \leq 32K\theta^{-1}\sigma_k.
\]
\[\square\]

Our next theorem combines Theorem 7.3 with Theorem 6.1 from [30]. Note of course that reflexivity of $Y$ is not used for the left-hand inequality, and the right-hand inequality could be improved to
\[
\|a_1 e_1 + \cdots + a_k e_k\|_S \leq C\|a_1 e_1 + \cdots + a_k e_k\|_{\ell_Y}
\]
for any $a_1, \ldots, a_k$. However the theorem as stated shows that we have both an upper and lower estimate for the behavior of weakly null spreading sequences in $X$. 
Theorem 7.4. Suppose $X$ and $Y$ are Banach spaces. Suppose there is a coarse Lipschitz embedding of $X$ into $Y$ and $Y$ is reflexive. Then, there is a constant $C$ so that for any spreading model of a weakly null sequence in $X$ we have:

\begin{equation}
\frac{1}{C} \|e_1 + \cdots + e_k\|_{\ell^\infty_Y} \leq \|e_1 + \cdots + e_k\|_S \leq C \|e_1 + \cdots + e_k\|_{\ell^\infty_Y}, \quad k \in \mathbb{N}.
\end{equation}

In particular if $\overline{\delta}_Y(t) > 0$ for any $t > 0$, then there is a constant $C$ so that every normalized weakly null sequence $(x_n)_{n=1}^{\infty}$ has a subsequence $(x_m)_{m \in \mathbb{N}}$ so that

\begin{equation}
\frac{1}{C} \|e_1 + \cdots + e_k\|_{\ell^\infty_Y} \leq \|x_{m_1} + \cdots + x_{m_k}\| \leq C \|e_1 + \cdots + e_k\|_{\ell^\infty_Y}, \quad k \in \mathbb{N}.
\end{equation}

Proof. The left-hand side follows from Theorem 7.3 and Proposition 4.2. For the right-hand side, suppose $f : X \rightarrow Y$ is a coarse Lipschitz embedding. We may assume that

\[ \|x - y\| - 1 \leq \|f(x) - f(y)\| \leq K \|x - y\| + 1. \]

Consider the space $\mathcal{P}_k$ of $k$-subsets of $\mathbb{N}$ with the metric

\[ d\{\{m_1, \ldots, m_k\}, \{n_1, \ldots, n_k\}\} = |\{k : m_k \neq n_k\}|. \]

Let $(x_n)_{n=1}^{\infty}$ be a normalized spreading sequence generating the spreading model $(e_n)_{n=1}^{\infty}$. Then, for any $\lambda > 0$, the map

\[ F_\lambda(\{n_1, \ldots, n_k\}) = f(\lambda x_{n_1} + \cdots + x_{n_k}) \]

is Lipschitz with constant at most $2(\lambda + 1)$. Hence, if $\nu > 0$ we can find an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that if $\{m_1, \ldots, m_k, n_1, \ldots, n_k\} \subset \mathbb{M}$ we have

\[ \|F_\lambda(m_1, \ldots, m_k) - F_\lambda(n_1, \ldots, n_k)\| \leq 2e(\lambda + 1)\|e_1 + \cdots + e_k\|_{\ell^\infty_Y} + \nu. \]

Hence

\[ \|x_{m_1} + \cdots + x_{m_k} - x_{n_1} - \cdots - x_{n_k}\| \leq 2e(1 + 1/\lambda)\|e_1 + \cdots + e_k\|_{\ell^\infty_Y} + (2 + \nu)/\lambda \]

and thus, letting $n_1, \ldots, n_k \rightarrow \infty$, $\lambda \rightarrow \infty$ and $\nu \rightarrow 0$, we have

\[ \|x_{m_1} + \cdots + x_{m_k}\| \leq 2e\|e_1 + \cdots + e_k\|_{\ell^\infty_Y} \]

so that the right-hand side follows.

The second part (7.11) is an equivalent statement.

Remark. If $X$ and $Y$ are uniformly homeomorphic one can relax the assumption that $Y$ is reflexive. This follows from results in [11]. If we assume $\lim_{t \rightarrow 0} \overline{\delta}_Y(t)/t = 0$, then the Szlenk Index of $Y$ is $\omega_0$ and hence by Theorem 5.5 so is the Szlenk index of $X$; furthermore the convex Szlenk indices of these spaces are equivalent and the argument is similar to that of Theorem 5.8 of [11], which treats the special case $\overline{\delta}_Y(t)/t \leq ct^p$.

8. APPLICATIONS TO UNIFORM AND COARSE HOMEOMORPHISMS

The first proposition is well known and goes back to work of Ribe [40] and [41] (who considered only the uniform case).

Proposition 8.1. Let $X$ and $Y$ be separable Banach spaces and suppose there is a coarse Lipschitz embedding of $X$ into $Y$. Then $X$ is finitely representable in $Y$ and hence isomorphic to a subspace of any ultraproduct $Y_\mathcal{U}$.

Proof. There is a Lipschitz embedding of $X$ into $Y_\mathcal{U}$ and hence a linear embedding into $Y_\mathcal{U}^{**}$ ([3], p.176).
In order to apply our results we need to use the ultraproduct technique which goes back to the classic paper of Heinrich and Mankiewicz [15]. The next result summarizes these ideas.

**Theorem 8.2.** Let $X$ and $Y$ be separable Banach spaces which are coarsely (or uniformly) homeomorphic. Assume $Y$ is super-reflexive. Then given any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we can find separable closed subspaces $X_1$ of $X_\mathcal{U}$ and $Y_1$ of $Y_\mathcal{U}$, such that:

(i) $X \subset X_1$, $Y \subset Y_1$.

(ii) $X_1$ is complemented in $X_\mathcal{U}$ and $Y_1$ is complemented in $Y_\mathcal{U}$.

(iii) $X_1$ and $Y_1$ are linearly isomorphic.

**Proof.** The argument is standard. Since $Y$ is super-reflexive, so is $X$ ([30], [3]). First one notes that $X_\mathcal{U}$ and $Y_\mathcal{U}$ are Lipschitz isomorphic (and both are reflexive). Then it is possible (using the separable complementation property) to find separable 1-complemented subspaces, $X \subset X_1 \subset X_\mathcal{U}$ and $Y \subset Y_0 \subset Y_\mathcal{U}$ so that $X_1$ and $Y_0$ are Lipschitz isomorphic. But this implies that $X_1$ is isomorphic to a complemented subspace $Y_1$ of $Y_0$ (see [15], [3]).

**Theorem 8.3.** Suppose $1 < p < \infty$ and that $X = \left( \sum_{n=1}^{\infty} E_n \right)_{\ell_p}$, where $(E_n)^{\infty}_{n=1}$ is a sequence of finite-dimensional spaces. Suppose either that:

(i) $1 < p < r \leq 2$ and the spaces $(E_n)^{\infty}_{n=1}$ are uniformly $r$-uniformly smooth, or

(ii) $2 \leq r < p < \infty$ and the spaces $(E_n)^{\infty}_{n=1}$ are uniformly $r$-uniformly convex.

Suppose $Y$ coarse Lipschitz-embeds into a quotient of $X$. Then $Y$ has property $\tilde{m}_p$.

**Proof.** In either case $X$ (and hence $Y$) is super-reflexive.

(i) We start with the observation that if $V : X \to \ell_p$ is defined by $V((x_n)_{n=1}^\infty) = (\|x_n\|)_{n=1}^{\infty}$, then $V$ is a random $L_p$-norm. By our assumptions this is a random $L_p$-norm of type $r$. It follows that we can induce a random $L_p$-norm of type $r$, $\tilde{V} : X_\mathcal{U} \to (\ell_p)_\mathcal{U}$. Now $Y$ embeds into a quotient of $X_\mathcal{U}$, and hence is a quotient of a subspace of $X_\mathcal{U}$. However, by Theorem 7.3 $Y$ has the $p$-co-Banach-Saks property. By Theorem 6.1 this implies that $Y$ has property $\tilde{m}_p$.

(ii) In this case we argue similarly that $(X^*)_\mathcal{U} = (X_\mathcal{U}^*)$ has a random $L_q$-norm of type $s$ where $1/p + 1/q = 1/r + 1/s = 1$. In this case $Y^*$ is a quotient of a separable subspace) of $(X^*)_\mathcal{U}$. We again use Theorem 7.3 to deduce that $Y$ has the $p$-Banach-Saks property. By Proposition 4.7 this means that $Y^*$ has the $q$-co-Banach-Saks property. By Theorem 6.1 $Y^*$ has property $\tilde{m}_q$, and so $Y$ has property $(\tilde{m}_p)$.

**Theorem 8.4.** Suppose $1 < p < \infty$. Then

(i) If $X$ is a Banach space which can be coarse Lipschitz-embedded in $\ell_p$, then $X$ is linearly isomorphic to a closed subspace of $\ell_p$.

(ii) If $X$ is a Banach space which is coarsely homeomorphic to a quotient of $\ell_p$, then $X$ is linearly isomorphic to a quotient of $\ell_p$.

(iii) If $X$ can be coarse Lipschitz-embedded into a quotient of $\ell_p$, then $X$ is linearly isomorphic to a subspace of a quotient of $\ell_p$.

**Proof.** Suppose first that $X$ can be coarse Lipschitz-embedded into a quotient of $\ell_p$. Then it is a special case of Theorem 8.3 that $X$ has property $(\tilde{m}_p)$.

(i) In this case for $2 \leq p < \infty$ the result is proved in [11]. If $1 < p < 2$, then $X^*$ is isomorphic to a quotient of $L_q$ where $1/p + 1/q = 1$ and has property $(\tilde{m}_q)$;
in particular it has the \( q \)-Banach-Saks property and by [17], \( X^* \) is isomorphic to a quotient of \( \ell_q \); i.e. \( X \) is isomorphic to a subspace of \( \ell_p \).

(ii) Again this is proved for \( 2 \leq p < \infty \) in [11]. If \( 1 < p < 2 \), then \( X^* \) is isomorphic to a subspace of \( L_q \) which has property \((\tilde{m}_q)\) and hence contains no subspace isomorphic to \( \ell_2 \). By the classical result of Kadets and Pelczyński [24] this implies that \( X^* \) is isomorphic to a subspace of \( \ell_q \). Hence \( X \) is isomorphic to a quotient of \( \ell_p \).

(iii) In this case, \( X \) is isomorphic to a subspace of a quotient of \( \ell_p \). Since \( X \) has property \((m_p)\) we can use Theorem 5.2 to embed \( X \) in an \( \ell_p \)-sum, \( (\sum_{n=1}^{\infty} E_n)_{\ell_p} \), where the spaces \( E_n \) are finite-dimensional and uniformly quotients of \( X \) and hence into a subspace of a quotient of \( \ell_p \). Thus \( X \) is isomorphic to a subspace of a quotient of \( \ell_p \).

\[ \square \]

Remark. Of course if \( X \) is uniformly homeomorphic to \( \ell_p \), then \( X \) is linearly isomorphic to \( \ell_p \) [20]. In [29] we show that for every \( 1 < p < \infty \) there are two uniformly homeomorphic subspaces (respectively, quotients) of \( \ell_p \) which are not isomorphic. We do not know if Theorem 8.4 holds for subspaces or quotients of \( \ell_p \). In the Lipschitz category there are corresponding results proved in [10] and [9] (except note in [9] for the case of quotients one needs an extra hypothesis that \( X^* \) has the approximation property).

**Theorem 8.5.** (i) Suppose \( 1 < p < r \leq 2 \) and that \( Z \) is an \( r \)-uniformly smooth Banach space with the (UAP). Suppose \((E_n)_{n=1}^{\infty}\) is an increasing sequence of uniformly complemented finite-dimensional subspaces of \( Z \). Then \( X = (\sum_{n=1}^{\infty} E_n)_{\ell_p} \) has unique coarse (or uniform) structure.

(ii) Suppose \( 2 \leq r < p < \infty \) and that \( Z \) is an \( r \)-uniformly convex Banach space with the (UAP). Suppose \((E_n)_{n=1}^{\infty}\) is an increasing sequence of uniformly complemented finite-dimensional subspaces of \( Z \). Then \( X = (\sum_{n=1}^{\infty} E_n)_{\ell_p} \) has unique coarse (or uniform) structure.

**Proof.** Let us start by observing that, in both cases (i) and (ii), \( X \) is linearly isomorphic to \( \ell_p(X) \). Indeed if \((n_k)_{k=1}^{\infty}\) is any sequence of natural numbers such that \( n_k = j \) is infinite for each \( j \) and \( n_k \leq k \), then \( (\sum_{k=1}^{\infty} E_{n_k})_{\ell_p} \) is complemented in \( X \); hence \( \ell_p(X) \) is isomorphic to a complemented subspace of \( X \). Hence for some Banach space \( W \), we have \( X \approx \ell_p(X) \oplus W \approx \ell_p(X) \oplus \ell_p(Y) \oplus W \approx \ell_p(Y) \). Now we observe that \( X \) is isomorphic to a complemented subspace of \( \ell_p(Z) \) and so has the (UAP) by Theorem 9.4 of [14].  

Now suppose \( Y \) is coarsely homeomorphic to \( X \). Since \( X \) is super-reflexive we can apply Theorem 8.2 to deduce that \( Y \) is super-reflexive and has the approximation property. By Theorem 8.3 \( Y \) has property \((\tilde{m}_p)\). We can therefore apply Theorem 5.5. It follows that \( Y \) is isomorphic to a complemented subspace of a space \((\sum_{n=1}^{\infty} F_n)_{\ell_p}\), where each \( F_n \) can be assumed to be of the form \((\sum_{j=1}^{k} E_j)_{\ell_p}\) for some \( k \). This implies that \( Y \) is isomorphic to a complemented subspace of \( X \).

To complete the proof we use Theorem 6.3. Since \( X \) is isomorphic to a complemented subspace of an ultraproduct \( Y_{\ell} \) of \( Y \) it follows that there is a constant \( \lambda \) so that for each \( m, n \) the finite-dimensional subspace \( \ell_m^p(E_n) \) is \( \lambda \)-isomorphic to a \( \lambda \)-complemented subspace of \( Y \). Hence \( X = (\sum_{n=1}^{\infty} E_n)_{\ell_p} \) is isomorphic to a complemented subspace of \( Y \). Now by the standard Pełczyński decomposition trick, this means (since \( X \approx \ell_p(X) \)) that \( X \) is isomorphic to \( Y \).  

\[ \square \]
The following corollary extends the result of Johnson, Lindenstrauss and Schechtman [20] of the uniqueness of the uniform structure of $\ell_p$ for $1 < p < \infty$.

**Corollary 8.6.** Suppose $1 < p, r < \infty$. The spaces $(\sum_{n=1}^{\infty} \ell_p^n)\ell_p$ have unique uniform structure if either $1 < p < \min(r, 2)$ or $p > \max(r, 2)$.

Note that for the case $r = 2$, Corollary 8.6 reduces to the result of Johnson, Lindenstrauss and Schechtman [20] since $(\sum_{n=1}^{\infty} \ell_2^n)\ell_2 \approx \ell_2$ (see [38]). As pointed out in the Introduction for every $1 < p < \infty$ we can find two nonisomorphic subspaces (respectively, quotients) of $\ell_p$ which are uniformly homeomorphic (see [29]).

**Theorem 8.7.** Let $X$ be a subspace of $L_1$ with the strong Schur property. Suppose $Y$ coarse Lipschitz-embeds into $X$; then $Y$ also has the strong Schur property.

**Proof.** By Theorem 7.3 and Proposition 4.8 it is clear that $Y$ has the anti-Banach-Saks property. We also have that $Y$ Lipschitz-embeds into an ultraproduct of $X$ and hence into $L_1$. Thus $Y$ linearly embeds into $L_1^{**}$ and hence into $L_1$. Finally we apply Theorem 6.4.

If $X$ is uniformly homeomorphic to a subspace of $\ell_1$, then $X$ is linearly isomorphic to a subspace of $L_1$; the above theorem implies that $X$ has the strong Schur property, but we do not know if $X$ linearly embeds into $\ell_1$. If $X$ is Lipschitz isomorphic to a subspace of $\ell_1$, then one can deduce that $X$ linearly embeds into $\ell_1$ by exploiting the Radon-Nikodým property and differentiability arguments (see [3]).

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