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ON THE EXISTENCE OF ATTRACTORS

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Abstract. On every compact 3-manifold, we build a non-empty open set $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ such that, for every $r \geq 1$, every C^r-generic diffeomorphism $f \in \mathcal{U} \cap \mathrm{Diff}^r(M)$ has no topological attractors. On higher-dimensional manifolds, one may require that f has neither topological attractors nor topological repellers. Our examples have finitely many *quasi-attractors*. For flows, we may require that these quasi-attractors contain singular points. Finally we discuss alternative definitions of attractors which may be better adapted to generic dynamics.

1. Introduction

The aim of dynamical systems is to describe the asymptotic behavior of the orbits when the time tends to infinity. For simple dynamical systems, the behavior of the orbits looks like the gradient flow of a Morse function: most of the orbits tend to a sink, and the union of the basins of the sink is a dense open set in the ambient manifold.

However, many dynamical systems present a more complicated behavior and many orbits do not tend to periodic orbits; their ω-limit set may be chaotic. In the sixties and seventies, many people tried to give a definition to attracting sets, allowing them to describe most of the possible behaviors of dynamical systems. An attractor Λ of a diffeomorphism f needs to satisfy two kinds of properties:

- it attracts "many orbits". According to the authors, this means: the basin of Λ contains a neighborhood of Λ, an open set, a residual subset of an open set, a set with positive Lebesgue measure, $\ldots$;
- it is indecomposable; that is, it cannot split into the union of smaller attractors. Many notions of indecomposability are used: transitivity (generic orbits of the attractor are dense in the attractor), chain recurrence (for every $\delta > 0$, one can go from any point of the attractor to any point of the attractor by δ-pseudo orbits inside the attractor), uniqueness of the SRB measure, $\ldots$.

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None of these notions can cover all the possible behaviors of dynamical systems. For every notion of (indecomposable) attractors, one can find examples of dynamical systems without attractors.[1]

A natural idea for bypassing this difficulty is to restrict the study to generic dynamical systems in order to avoid the most pathological and fragile behaviors. A property is C^r-generic if it holds on a residual subset of the space of C^r-diffeomorphisms $\mathrm{Diff}^r(M)$ endowed with the C^r-topology.

This viewpoint was considered very early by Smale and Thom, with the hope that generic dynamical systems would have a simple behavior. For instance, one can read in [T, Chapter 4.1 B]: *Il n'est pas certain qu'un champ X donné dans M présente des attracteurs, a fortiori des attracteurs structurellement stables. Toutefois, selon certaines idées récentes de Smale, si la variété M est compacte, presque tout champ présenterait un nombre fini d'attracteurs isolément structurellement stables; (...).*[2] Thom's idea was renewed and formalized in 1975 as *Thom's conjecture* by Palis and Pugh [PP, Problem 26]: *There is a dense open set in* $\mathrm{Diff}^r(M)$ *such that for almost every point $x \in M$, the ω-limit set $\omega(x)$ is a topological attractor, and each attractor is topologically stable.*

After thirty years of progress in the field, this conjecture can look naively optimistic. Indeed, Thom's original idea was disproved in most of its aspects: finiteness and stability

- there are open sets of systems without structurally stable attractors, as the robustly transitive non-hyperbolic diffeomorphisms built by Shub in [Sh];
- there are C^r-locally generic diffeomorphisms having infinitely many sinks (see [N1, N2] for $r \geq 2$ and [BD] for $r = 1$).

However, the existence of at least one attractor remains an open question. In this paper, we will give a negative answer to this question, showing that the usual notion of topological attractor is too strong and not adapted to generic dynamical systems. Let us now be somewhat more precise.

A *topological attractor* of a diffeomorphism $f\colon M \to M$ is a compact subset $\Lambda \subset M$ with the following properties:

- Λ is *invariant* (i.e. $f(\Lambda) = \Lambda$);
- Λ admits a compact neighborhood U which is *an attracting region* (i.e. the image $f(U)$ is contained in the interior of U) such that all the orbits in U converge to Λ: $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(U)$;
- Λ is *transitive* (i.e. the positive orbits of generic points in Λ are dense in Λ) or at least *chain recurrent*[3] (i.e. any two points in Λ can be joined by ε-pseudo orbits, for all $\varepsilon > 0$). We will speak on the *transitive topological attractor* and on the *chain recurrent topological attractor* if we need to emphasize the kind of indecomposability we require.

[1] If one removes the indecomposability hypothesis, there are many notions of attractors which ensure their existence: for instance, the limit set or the non-wandering set can be considered as (non-indecomposable) attractors. In this paper, we pay some attention to Conley's theory of chain recurrence points; this theory also uses a notion of attractors, without assuming indecomposability.

[2] *"It is not clear if a given vector field in M has an attractor,* a fortiori *a structurally stable attractor; however, according to recent ideas by Smale, if the manifold is compact, almost all vector fields would admit finitely many attractors, each of them structurally stable; (...)."*

[3] Some authors say "*chain transitive*".

A *topological repeller* of f is, by definition, a topological attractor of f^{-1}. In 2004 [BC] and in 2005 [BDV, Problem 10.35] still asked:

Problem 1. Do C^1-generic diffeomorphisms admit at least one (chain recurrent or transitive)[4] topological attractor? Does the union of the basins of the topological attractors cover a dense open subset of the manifold?

The answer is "no". The fact that C^0-generic homeomorphisms have no attractors is known from Hurley's work [H]: every attracting region of a C^0-generic homeomorphism contains infinitely many repelling regions and infinitely many disjoint attracting regions. Theorems A and B show that, for every $r \geq 1$, the property of having (at least) one topological attractor is not C^r-generic.

Our results use the notion of *quasi-attractors*, introduced by Hurley: a chain recurrence class of a homeomorphism is a *quasi-attractor*[5] if it is the intersection of a sequence of attracting regions. A *quasi-repeller* for h is a quasi-attractor for h^{-1}.

Theorem A. *For every three-dimensional manifold M^3, there is a non-empty C^1-open subset $\mathcal{U} \subset \mathrm{Diff}^1(M^3)$ such that:*

- *there are hyperbolic periodic saddles $p_{1,f}, \dots, p_{k,f}$ varying continuously with $f \in \mathcal{U}$, whose chain recurrence classes $\Lambda_{1,f}, \dots, \Lambda_{k,f}$ are the unique quasi-attractors of f;*
- *the set $\{f \in \mathcal{U}, f \text{ has no attractors}\}$ is C^r-residual in $\mathcal{U} \cap \mathrm{Diff}^r(M)$ for every $r \geq 1$.*

Remark. We will discuss the case of generic surface diffeomorphisms in Section 6.

The space of analytic diffeomorphisms with the usual topology is not complete, thus it is not a good setting for considering generic problems. However, as asked by one of the referees of this paper, is it important to notice that the lack of an attractor is not due to a lack of regularity? In answering this important question, we adapted our result in that setting, replacing genericity by density.

More precisely, one first notices that analytic diffeomorphisms are C^r-dense in $\mathrm{Diff}^r(M^3)$, for any $r \geq 1$. In particular, analytic diffeomorphisms are C^r-dense in the open set $\mathcal{U}$ in Theorem A. Moreover, every smooth arc of diffeomorphisms in $\mathcal{U}$ can be C^r-approached by an analytic arc of analytic diffeomorphisms in $\mathcal{U}$. From the proof of Theorem A, we will deduce the following corollary:

Corollary 1.1. *For every three-dimensional manifold M^3 endowed with a real analytic structure, there is a non-empty C^1-open subset $\mathcal{U} \subset \mathrm{Diff}^1(M^3)$ such that the subset of analytic diffeomorphisms $f \in \mathcal{U}$ such that f has no attractors is C^r-dense in $\mathcal{U}$ for every $r \geq 1$.*

In the statement of Theorem A, the C^r-generic diffeomorphisms f in the open set $\mathcal{U}$ have no attractors but infinitely many repellers. This motivates the following problem:

Problem 2. For a three-dimensional manifold M^3 is there a non-empty open set $\mathcal{O} \subset \mathrm{Diff}^r(M^3)$ and a dense (open and dense, residual) set $\mathcal{D} \subset \mathcal{O}$ such that any $f \in \mathcal{D}$ has neither attractors nor repellers?

[4] Chain recurrent topological attractors of C^1-generic diffeomorphisms are homoclinic classes, hence are transitive.

[5] Some authors use the terminology "weak attractor" instead of quasi-attractor.

Theorem B. *For every compact manifold M with $\dim M \geq 4$, there is a non-empty open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$ such that:*

- *there are hyperbolic periodic saddles $p_{1,f}, \ldots, p_{k,f}$ varying continuously with $f \in \mathcal{U}$, whose chain recurrence classes $\Lambda_{1,f}, \ldots, \Lambda_{k,f}$ are the unique quasi-attractors of f;*
- *there are hyperbolic periodic saddles $q_{1,f}, \ldots, q_{\ell,f}$ varying continuously with $f \in \mathcal{U}$, whose chain recurrence classes $\Sigma_{1,f}, \ldots, \Sigma_{\ell,f}$ are the unique quasi-repellers of f;*
- *the set $\{f \in \mathcal{U}, f$ has neither attractors nor repellers$\}$ is C^r-residual in $\mathcal{U} \cap \mathrm{Diff}^r(M)$ for every $r \geq 1$.*

Our results can be easily adapted for vector fields, building locally generic vector fields having finitely many (non-singular) quasi-attractors but no attractors. However, one of the main differences between diffeomorphisms and flows is the existence of singularities, in particular when these singularities are not isolated from the regular part of the limit set of the flow.

This phenomenon was first suspected experimentally by Lorenz [Lo] and then proved rigourously in [Gu, ABS, GuW], where the authors exhibited, in dimension 3, a C^1-open set of vector fields having a robust attractor containing infinitely many periodic orbits accumulating on a saddle singularity. Their construction (known as the *geometric model of a Lorenz attractor*) leads to the notion of singular attractors, which have been studied in extends on 3-manifolds: for instance, if the presence of a singularity inside the attractor prevents the usual definition of hyperbolicity, robust singular attractors in dimension 3 always present a kind of weak hyperbolicity called a *singular hyperbolicity*; see [MPP1, MPP2]. In particular, they satisfy the *star condition*: C^1-robustly all the periodic orbits are hyperbolic. [LGW, GWZ, MM] show that in any dimension, robust singular attractors satisfying the star condition are singular hyperbolic. Recent examples [BKR] [BLY] show that robust singular attractors may satisfy neither the star condition nor the singular hyperbolicity. However, even these new examples admit a strong stable direction, invariant by the flow and dominated by a center-unstable bundle.

Hence it is natural to ask:

Problem 3. Does every C^1-robustly transitive singular attractor of a vector field admit a strong stable bundle?

Indeed, this question was our first motivation for this work. Before presenting our results, let us make a comment on this question. First consider the non-singular case:

- Examples of a (non-singular) robustly transitive attractor whose tangent flow does not admit any dominated splitting are already known (just consider the suspension flows of robustly transitive diffeomorphisms without invariant hyperbolic bundles in [BV]).
- On the other hand, [BDP, BGV] implies that the linear Poincaré flow on the normal bundle over a robustly transitive attractor always admits a dominated splitting.

Now consider the case of a robustly transitive singular attractor. The linear Poincaré flow is not defined on the singularity: for this reason, it is not clear which kind of hyperbolicity (or dominated splitting) will satisfy the singular attractors. This difficulty has been solved in dimension 3 by Morales-Pacifico-Pujals [MPP1, MPP2] by defining the notion of singular hyperbolicity.

Now we state our result for flows. Our construction can be adapted in order to build a *robust singular quasi-attractor* whose tangent bundle doesn't have any dominated splitting with respect to the tangent flow.

Theorem C. *There is a non-empty open set $\mathcal{U}$ of the space $\mathcal{X}^r(B^4)$ of C^r-vector fields on the 4-ball such that:*

- *any $X \in \mathcal{U}$ is transverse to the boundary and entering inside the ball;*
- *any $X \in \mathcal{U}$ has a unique zero 0_X in B^4; one denotes by Λ_X the chain recurrence class of 0_X;*
- *any $X \in \mathcal{U}$ has a unique quasi-attractor in B^4 which is Λ_X;*
- *the subset $\{X \in \mathcal{U},\ \Lambda_X$ is not an attractor$\}$ is C^r-residual in $\mathcal{U}$;*
- *for $X \in \mathcal{U}$, there is no dominated splitting for the tangent flow of X on Λ_X.*

1.1. **Organization of the paper.** Our main result is the construction in Section 3 of an example of locally generic diffeomorphisms of the solid torus $S^1 \times \mathbb{D}^2$ without attractors.

Putting the solid torus in a ball B^3, we get a model of an attracting ball without attractors, which allows us, in Section 4, to replace the sinks of a gradient-like diffeomorphism by these attracting balls without attractors, thus proving Theorem A.

Multiplying this ball B^3 by a normal contraction, one gets in Section 4.4 an attracting ball B^n, for $n > 3$, without attractors and repellers. This section completes the proof of Theorem B.

Section 5 considers the case of vector fields and shows that our construction in Section 3 leads to locally generic vector fields X on 4-manifolds having a unique quasi-attractor Λ_X and no attractors. Furthermore, Λ_X is the chain recurrence class of a singularity of X.

Section 6 concludes this paper by discussing alternative notions of attractors which could be better adapted to generic dynamical systems.

2. Notation, definitions and preliminaries

2.1. **Disks and balls.** For every $d \in \mathbb{N}$ and $r \in \mathbb{R}$, we denote by $\mathbb{D}^d(r)$ the closed ball in $\mathbb{R}^d$ centered at 0 and with radius r, i.e., $\mathbb{D}^d(r) = \{x \in \mathbb{R}^d : \|x\| \leq r\}$. For simplicity, we denote $\mathbb{D}^d = \mathbb{D}^d(1)$. Given a compact Riemannian manifold M, a point $x \in M$, and a real number $\delta > 0$, we denote $B_\delta(x) = \{y \in M : \ d(x,y) \leq \delta\}$, the compact ball centered at x and with radius r.

Recall that every orientation preserving diffeomorphism of $\mathbb{D}^2$ is smoothly isotopic to the identity.

An *essential disk in* $S^1 \times \mathbb{D}^2$ is an embedding $D \colon \mathbb{D}^2 \hookrightarrow S^1 \times \mathbb{D}^2$ whose boundary $\partial D = D(\partial \mathbb{D}^2)$ is contained in $\partial \left(S^1 \times \mathbb{D}^2\right) = S^1 \times S^1$ and is not homotopic to a point in $S^1 \times S^1$.

2.2. **Hyperbolicity, partial hyperbolicity, dominated splitting.** Let f be a diffeomorphism of a manifold M of dimension d, x a periodic point of f, and π its period. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ be the moduli of the eigenvalues of the differential $Df^\pi(x)$. The point x is *hyperbolic* if $\lambda_i \neq 1$ for all $i \in \{1, \ldots, d\}$. The point x is *sectionally area expanding* (or *sectionally expanding*) if

$$\lambda_i\lambda_j > 1, \text{ for all } i,j \in \{1,\dots,d\}, i \neq j.$$

We say a compact invariant set Λ of f is *hyperbolic* if there are a Df-invariant splitting

$$TM|_\Lambda = E^s \oplus E^u$$

and constants $C > 0$, $\lambda \in (0,1)$ such that for any $x \in \Lambda$ and $n \in \mathbb{N}$,

$$\|Df^n|_{E^s(x)}\| \leq C\lambda^n, \ \|Df^{-n}|_{E^u(x)}\| \leq C\lambda^n.$$

The bundles E^s and E^u are called the *stable* and *unstable bundle* of Λ, respectively. They are always continuous bundles so that the dimensions $\dim E^s(x)$ and $\dim E^u(x)$ are locally constant. If $\dim E^s(x)$ is independent on $x \in \Lambda$, then we call $\dim E^s$ the index of the hyperbolic set Λ.

We say Λ is a *basic set* if Λ is an isolated hyperbolic transitive set: there is an open neighborhood U of Λ such that $\Lambda = \bigcap_{i\in\mathbb{Z}} f^i(U)$.

Given a compact invariant set Λ, a Df-invariant splitting $TM|_\Lambda = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is *dominated*, and we denote $E_1 \oplus_< E_2 \oplus_< \cdots \oplus_< E_k$ if the dimensions $\dim(E_i)$ are constant over Λ and if there are constants $C > 0$, $\lambda \in (0,1)$ such that for any $x \in \Lambda$, $n \in \mathbb{N}$ and $i \in \{1,\dots,k-1\}$, we have

$$\|Df^n|_{E_i(x)}\|\|Df^{-n}|_{E_{i+1}(f^n(x))}\| \leq C\lambda^n.$$

A Df-invariant bundle E is called *(uniformly) contracting* if there are constants $C > 0$, $\lambda \in (0,1)$ such that for any $x \in \Lambda$ and $n \in \mathbb{N}$, we have $\|Df^n|_{E(x)}\| \leq C\lambda^n$. It is called *(uniformly) expanding* if it is contracting for f^{-1}.

A dominated splitting $E_1 \oplus_< E_2 \oplus_< \cdots \oplus_< E_k$ is *partially hyperbolic* if E_1 is uniformly contracting or E_k is uniformly expanding.

2.3. **Cone fields associated to a dominated splitting.** Given a compact set $V \subset M$, a continuous (not necessarily invariant) bundle $F \subset T_V M$, and a positive number $\alpha > 0$, *the cone field on V associated to F of size $\alpha > 0$* is

$$C^F_\alpha(x) = \{v \in T_xM : \ \exists v_F \in F, v_{F^\perp} \in F^\perp, \text{ s.t. } v = v_F + v_{F^\perp}, |v_{F^\perp}| \leq \alpha|v_F|\}$$

for $x \in V$, where $F^\perp$ is the orthogonal subbundle of F.

We say a *cone field C^F_α is strictly Df-invariant* if there is $\beta \in (0,\alpha)$ such that, for any $x \in V$ such that $f(x) \in V$, we have

$$Df(C^F_\alpha(x)) \subset C^F_\beta(f(x)).$$

If an invariant compact set Λ has a dominated splitting $T_\Lambda M = E \oplus_< F$, then there is $\alpha_0 > 0$ such that for any $\alpha \in (0,\alpha_0)$, there is $N \in \mathbb{N}$ such that the cone field C^E_α is strictly Df^{-N}-invariant and the cone field C^F_α is strictly Df^N-invariant.

2.4. **Conley theory and quasi-attractors.** Let (X,d) be a compact metric space and $f\colon X \to X$ a homeomorphism.

For any $x,y \in X$, we denote $x \dashv y$ if for every $\epsilon > 0$ there is an ϵ-pseudo orbit joining x to y. That is, there are $n > 0$ and a sequence of points $\{x = x_0, x_1, \dots, x_n = y\}$ verifying $d(f(x_i), x_{i+1}) < \epsilon$ for $0 \leq i \leq n-1$.

We say that x is chain recurrent if $x \dashv x$, and we denote by $\mathcal{R}(f)$ the set of chain recurrent points of f, called the *chain recurrent set* of f.

An invariant compact set K of X is *chain recurrent* (or *chain transitive*) if every point $x \in K$ is chain recurrent for the restriction $f|_K$. In other words, $K = \mathcal{R}(f|_K)$.

We say x and y are *chain equivalent* if $x \dashv y$ and $y \dashv x$. The chain equivalence is an equivalence relation on $\mathcal{R}(f)$. For any $x \in \mathcal{R}(f)$, the equivalence class of x is called *the chain recurrence class of* x and is denoted by $C(x)$.

A *quasi-attractor* Λ is a chain transitive set which admits a base of neighborhoods which are attracting regions (this implies that Λ is a chain recurrence class).

2.5. **Plykin attractor.** Let $\mathrm{Diff}^1_0(\mathbb{D}^2, \mathrm{Int}(\mathbb{D}^2))$ denote the space of orientation preserving C^1-embeddings

$$\phi\colon \mathbb{D}^2 \to \mathrm{Int}(\mathbb{D}^2).$$

Notice that the elements of $\mathrm{Diff}^1_0(\mathbb{D}^2, \mathrm{Int}(\mathbb{D}^2))$ are all isotopic, in particular are isotopic to any linear contraction of $\mathbb{D}^2$.

In [Pl] Plykin built a non-empty open subset $\mathcal{P} \subset \mathrm{Diff}^1_0(\mathbb{D}^2, \mathrm{Int}(\mathbb{D}^2))$ such that for any $\phi \in \mathcal{P}$ the chain recurrent set of ϕ consists in the union of a non-trivial hyperbolic attractor A_ϕ and a finite set of periodic sources.

We denote by $\mathcal{P}_0 \subset \mathcal{P}$ the non-empty open subset of diffeomorphisms such that the hyperbolic attractor A_ϕ contains a fixed point x_ϕ which is an area expanding saddle point:[6]

$$Det(D\phi(x_\phi)) > 1.$$

2.6. **Solenoid maps associated to a braid in $S^1 \times \mathbb{D}^2$.** A *connected braid* γ *of* $S^1 \times \mathbb{D}^2$ is (the isotopy class of) an embedding of the circle S^1 in $S^1 \times \mathbb{D}^2$, transverse to the fibers $\{\theta\} \times \mathbb{D}^2$, for $\theta \in S^1$. The projection $S^1 \times \mathbb{D}^2 \to S^1$ induces on γ a finite covering of the circle; we denote by $n_\gamma \neq 0$ the order of this finite cover.

For any braid γ, we denote by $\mathcal{U}_\gamma$ the (non-empty) open subset of diffeomorphisms $f\colon S^1 \times \mathbb{D}^2 \hookrightarrow \mathrm{Int}(S^1 \times \mathbb{D}^2)$ such that $f(S^1 \times \{0\})$ is isotopic to the braid γ.

We call *canonical solenoid maps* associated to a braid γ the maps built as follows: denote $n = n_\gamma$; we choose a representative $\gamma\colon S^1 \to S^1 \times \mathbb{D}^2$ of the braid having the form $\gamma(t) = (nt, z(t))$. We fix $\delta > 0$ such that

$$\begin{gathered}\text{for all } t \in S^1, \quad d(z(t), \{nt\} \times \partial\mathbb{D}^2) > 2\delta;\\ \text{for any } t_1, t_2 \in S^1, \quad \big(t_1 \neq t_2 \text{ and } nt_1 = nt_2 \in S^1\big) \Rightarrow d(z(t_1), z(t_2)) > 2\delta.\end{gathered}$$

Now the map $f_{\gamma,\delta}$ defined on $S^1 \times \mathbb{D}^2$ by $f_{\gamma,\delta}(t,z) = (nt, \delta z + z(t))$ belongs to $\mathcal{U}_\gamma$ and is called a *canonical solenoid map* associated to a braid γ.

2.7. **Partially hyperbolic solenoid maps.** For every $\alpha > 0$ we denote by $\mathcal{C}_\alpha$ the cone field on $S^1 \times \mathbb{D}^2$ defined by

$$\mathcal{C}_\alpha(x) = \{u = (u_1, u_2) \in T_x(S^1 \times \mathbb{D}^2) = \mathbb{R} \times \mathbb{R}^2 \text{ such that } |u_2| \le \alpha |u_1|\}.$$

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[6] The fact that $\mathcal{P}_0$ is non-empty is because one starts with any Plykin attractor and a periodic point p of it. We can perform a local one-parameter family of deformation supported in a neighborhood of p, keeping the hyperbolic structure, so that for the parameter 1 the continuation of p has jacobian larger than 1. As the hyperbolic structure is kept along the deformation, the structural stability implies that for parameter 1 one has a Plykin attractor with one area expanding saddle.

We denote by $\mathcal{U}_\gamma^{part.hyp}$ the set of diffeomorphisms $f\in\mathcal{U}_\gamma$ such that there are $\alpha>0$ and $\ell\in\mathbb{N}\setminus\{0\}$ such that:

- the cone field $\mathcal{C}_\alpha$ is strictly invariant under Df^ℓ;
- there is $\lambda>1$ such that for every $x\in S^1\times\mathbb{D}^2$ and every vector $u=(u_1,u_2)\in\mathcal{C}_\alpha(x)$ one has
$$|v_1|\geq\lambda|u_1|, \text{ where } Df^\ell(u)=(v_1,v_2)\in T_{f^\ell(x)}(S^1\times\mathbb{D}^2).$$

The set $\mathcal{U}_\gamma^{part.hyp}$ is a C^1-open subset of $\mathcal{U}_\gamma$. Moreover, one easily verifies:

Lemma 2.1. *Let $f\in\mathcal{U}_\gamma$ be of the form $(t,z)\mapsto(\alpha(t),\varphi_t(z))$, where $\alpha(t)$ is in the isotopic class of γ. Assume that:*

- *for every $t\in S^1$ one has*
$$\left|\frac{d}{dt}\alpha(t)\right|>1;$$
- *for every $(t,z)\in S^1\times\mathbb{D}^2$ one has*
$$\|D_z(\varphi_t)\|<\left|\frac{d}{dt}\alpha(t)\right|.$$

Then f is partially hyperbolic; more precisely, $f\in\mathcal{U}_\gamma^{part.hyp}$.

As a direct consequence one gets:

Corollary 2.2. *For every braid γ with $|n_\gamma|\geq 2$ every canonical solenoid map f associated to γ belongs to $\mathcal{U}_\gamma^{part.hyp}$.*

In particular, the open set $\mathcal{U}_\gamma^{part.hyp}$ is non-empty.

Corollary 2.3. *Let $f\in\mathcal{U}_\gamma$ satisfy the hypotheses of Lemma 2.1 and $\Phi\colon S^1\times\mathbb{D}^2\to S^1\times\mathbb{D}^2$ be a diffeomorphism of the form $(t,z)\mapsto(t,h_t(z))$, where Φ_t is an orientation preserving diffeomorphism of $\mathbb{D}^2$.*

Then the map $g=\Phi^{-1}f\Phi$ belongs to $\mathcal{U}_\gamma^{part.hyp}$.

2.8. **Realization of a map $\varphi\in\mathrm{Diff}_0^1(\mathbb{D}^2,\mathrm{Int}(\mathbb{D}^2))$ by a solenoid map $f\in\mathcal{U}_\gamma^{part.hyp}$.** The aim of this section is to prove:

Proposition 2.4. *Given any $\varphi\in\mathrm{Diff}_0^1(\mathbb{D}^2,\mathrm{Int}(\mathbb{D}^2))$ and any braid γ with $|n_\gamma|\geq 2$, there is a diffeomorphism $f\in\mathcal{U}_\gamma^{part.hyp}$ such that the disk $\{0\}\times\mathbb{D}^2$ is positively invariant (and normally hyperbolic) and the restriction of f to $\{0\}\times\mathbb{D}^2$ is φ.*

Proof. We denote $n=n_\gamma$. We choose a representative $\gamma\colon S^1\to S^1\times\mathbb{D}^2$, $\gamma(t)=(nt,z(t))$. Consider a canonical solenoid map $f_{\gamma,\delta}$, associated to the braid γ, for some $0<\delta<1$. Recall that $f_{\gamma,\delta}(t,z)=(nt,z(t)+h_\delta(z))$, where $h_\delta\colon\mathbb{D}^2\to\mathrm{Int}(\mathbb{D}^2)$ is the homothety of ration δ.

Consider $\varphi\in\mathrm{Diff}_0^1(\mathbb{D}^2,\mathrm{Int}(\mathbb{D}^2))$. By using Corollary 2.3, one just needs to prove Proposition 2.4 for a conjugate of φ by an orientation preserving diffeomorphism of $\mathbb{D}^2$.

This allows us to assume that $\varphi(\mathbb{D}^2)$ is contained in the disk $\mathbb{D}^2(\delta)$ of radius δ and that there is a differentiable isotopy from φ to the homothety h_δ, whose image remains contained in $\mathbb{D}^2(\delta)$. More precisely, there is a C^1-map $\Phi\colon\mathbb{D}^2\times[-1,1]\to\mathrm{Int}(\mathbb{D}^2)$ of the form $\Phi(x,t)=\varphi_t(x)$ where:

- for every $t\in[-1,1]$ one has $\varphi_t\in\mathrm{Diff}_0^1(\mathbb{D}^2,\mathrm{Int}(\mathbb{D}^2))$,
- for every $t\in[-1,1]$ one has $\varphi_t(\mathbb{D}^2)\subset\mathbb{D}^2(\delta)$,

- $\varphi_0 = \varphi$, and
- $\varphi_t = h_\delta$ if $|t| \geq \frac{1}{2}$.

We denote $C = \max_{t\in[-1,1], z\in\mathbb{D}^2} \|D_z(\varphi_t)\|$.

Let $\psi\colon [-\frac{1}{|n|}, \frac{1}{|n|}] \to [-1,1]$ be a diffeomorphism such that there is $0 < \varepsilon < \frac{1}{2|n|}$ with the following properties:

- $\psi(\frac{1}{n}) = 1$ and $\psi(-\frac{1}{n}) = -1$;
- for every $t \in [-\frac{1}{|n|}, \frac{1}{|n|}]$ one has $\left|\frac{d}{dt}\psi(t)\right| > 1$;
- $\left|\frac{d}{dt}\psi(t)\right| > 2C$ for every $t \in [-\varepsilon, \varepsilon]$;
- $\left|\frac{d}{dt}\psi(t)\right| = |n|$ for $|t| \geq \frac{1}{|n|} - \varepsilon$.

We define $f\colon S^1 \times \mathbb{D}^2 \to \mathrm{Int}(S^1 \times \mathbb{D}^2)$ as follows:

- $f(t,z) = (\psi(t), z(\frac{\psi(t)}{|n|}) + \varphi_{\frac{t}{\varepsilon}}(z))$ if $t \in [-\varepsilon, \varepsilon]$,
- $f(t,z) = (\psi(t), z(\frac{\psi(t)}{|n|}) + h_\delta(z))$ if $|t| \in [\varepsilon, \frac{1}{|n|}]$,
- $f(t,z) = (nt, z(t) + h_\delta(z))$ if $t \notin [-\frac{1}{|n|}, \frac{1}{|n|}]$.

Notice that $f(\{0\} \times \mathbb{D}^2) \subset \{0\} \times \mathbb{D}^2$; the disk is normally hyperbolic and the restriction of f to that disk induces φ. One concludes the proof of Proposition 2.4 by proving:

Claim 1. The map f defined above belongs to $\mathcal{U}_\gamma^{part.hyp}$.

Proof. We first notice that the image $f(t,0)$ belongs to the curve $\gamma(S^1)$. In other words $f(t,0) = \gamma(\tau_t)$, where $t \mapsto \tau_t$ is a diffeomorphism of the circle. So the image of $\{t\} \times \mathbb{D}^2$ is contained in a disc of radius δ in $\{n.\tau_t\} \times \mathbb{D}^2$ centered at $\gamma(\tau_t)$. As a consequence, if $r \neq s$, then $f(\{r\} \times \mathbb{D}^2) \cap f(\{s\} \times \mathbb{D}^2) = \emptyset$. One deduces that f is injective, hence is a diffeomorphism from $S^1 \times \mathbb{D}^2$ onto its image contained in $f_{\gamma,\delta}(S^1 \times \mathbb{D}^2)$. One deduces that f belongs to $\mathcal{U}_\gamma$.

In order to get the partial hyperbolicity, we will verify that f satisfies the hypotheses of Lemma 2.1 in each of the possible expressions. We first notice that f keeps invariant the trivial foliation of $S^1 \times \mathbb{D}^2$ by the disks $\{t\} \times \mathbb{D}^2$. It remains to get the control of the derivative of f.

The map f coincides with $f_{\gamma,\delta}$ out of $[-\frac{1}{|n|}, \frac{1}{|n|}] \times \mathbb{D}^2$, giving the condition in this region. On $\left([-\frac{1}{|n|}, -\varepsilon] \cup [\varepsilon, \frac{1}{|n|}]\right) \times \mathbb{D}^2$, one notices that the derivative of the restriction of f to each disk $\{t\} \times \mathbb{D}^2$ is the homothety of ratio $0 < \delta < 1$; hence the conclusion holds because $\left|\frac{d}{dt}\psi(t)\right| > 1$. Finally, for $t \in [\varepsilon, \varepsilon]$ the derivative of the restriction of f to the disk $\{t\} \times \mathbb{D}^2$ is bounded by the constant C, and $\left|\frac{d}{dt}\psi(t)\right| > C$, by assumption. □

□

3. An attracting solid torus $S^1 \times \mathbb{D}^2$ without attractors

Our main results are consequences of a construction in the solid torus $S^1 \times \mathbb{D}^2$, that we explain in this section.

3.1. **Plykin attractors on normally hyperbolic disks, for solenoid maps.**

Recall that $\mathcal{P}_0$ is the open set of structurally stable diffeomorphisms in $\mathrm{Diff}^1_0(\mathbb{D}^2, \mathrm{Int}(\mathbb{D}^2))$, defined in Section 2.5, whose non-wandering set consists exactly in the union of a non-trivial hyperbolic attractor (a Plykin attractor) and a finite set of periodic sources.

Given any braid γ with $|n_\gamma| \geq 2$, let $\mathcal{U}_\gamma^{Ply}$ denote the set of diffeomorphisms $f \in \mathcal{U}_\gamma^{part.hyp}$ such that f leaves positively invariant a normally hyperbolic essential disk (which is defined in Subsection 2.1) D_f and such that the restriction ϕ_f of f to D_f is C^1-conjugate to an element $\phi \in \mathcal{P}_0$.

As a corollary of Proposition 2.4 one gets:

Corollary 3.1. *Given any braid γ with $|n_\gamma| \geq 2$, the set $\mathcal{U}_\gamma^{Ply}$ is a non-empty C^1-open subset of $\mathcal{U}_\gamma^{part.hyp}$.*

Proof. Proposition 2.4 implies that $\mathcal{U}_\gamma^{Ply}$ is non-empty. It is open because the disk D_f is normally hyperbolic, hence persists by perturbation and varies C^1-continuously with f. Hence the restriction ϕ_f varies C^1-continuously with f. One concludes by recalling that $\mathcal{P}_0$ is an open subset of $\mathrm{Diff}_0^1(\mathbb{D}^2, \mathrm{Int}(\mathbb{D}^2))$. $\square$

Let $\gamma \subset S^1 \times \mathbb{D}^2$ be a braid with $|n_\gamma| \geq 2$. Consider $f \in \mathcal{U}_\gamma^{Ply}$. By definition of $\mathcal{U}_\gamma^{Ply}$ and $\mathcal{P}_0$, one has the following properties:

- there are $\alpha_f > 0$ and $\ell > 0$ such that the cone field $\mathcal{C}_{\alpha_f}$ is strictly invariant by Df^ℓ;
- the disk D_f is positively invariant and normally hyperbolic; hence the disk D_f is transverse to the cone field $\mathcal{C}_{\alpha_f}$;
- the restriction ϕ_f of f to D_f belongs to $\mathcal{P}_0$; hence, the disk D_f contains a Plykin attractor A_f of ϕ_f; as the disk D_f is normally hyperbolic, A_f is a hyperbolic basic set for f;
- the Plykin attractor A_f contains a hyperbolic fixed point $p_f = x_{\phi_f}$ such that $Det(D\phi_f(p_f)) > 1$; as a consequence, the product of any two eigenvalues of $Df(p_f)$ has modulus larger than 1; hence, the point p_f is a sectionally expanding fixed point of f;
- we denote by Λ_f the chain recurrence class of A_f; in an equivalent way, Λ_f is the chain recurrence class of the fixed point p_f.

3.2. **Statement of our main result.**

Theorem 1. *Given any braid γ with $|n_\gamma| \geq 2$:*

(1) *For every $f \in \mathcal{U}_\gamma^{Ply}$, the chain recurrence class Λ_f is the unique quasi-attractor of f.*
(2) *We denote*

$$\mathcal{U}_{wild,\gamma} = \{f \in \mathcal{U}_\gamma^{Ply}, \Lambda_f \cap \overline{\{\textit{sources of } f\}} \neq \emptyset\}.$$

In particular, Λ_f is not an attractor for $f \in \mathcal{U}_{wild,\gamma}$. Then, for every $r \geq 1$, the subset $\mathcal{U}_{wild,\gamma}$ is residual in $\mathcal{U}_\gamma^{Ply}$ for the C^r-topology.

According to [BC], for C^1-generic diffeomorphisms, the ω-limit set $\omega(x)$ of any generic point x of the manifold is a quasi-attractor. Hence item (1) of Theorem 1 implies:

Corollary 3.2. *There is a C^1-residual subset of $\mathcal{U}_\gamma^{Ply}$ of diffeomorphisms f for which the basin of Λ_f is residual in $S^1 \times \mathbb{D}^2$.*

We don't know if Corollary 3.2 holds for C^r-topology, $r > 1$. However, we think that it is possible to prove:

Problem 4. There is a C^2-open subset of $\mathcal{U}_\gamma^{Ply}$ of diffeomorphisms for which Λ_f carries an SRB-measure whose basin has total Lebesgue measure in $S^1 \times \mathbb{D}^2$.

3.3. **First step of the proof of Theorem 1: Uniqueness of the quasi-attractor.**

Proof. Let $U \subset S^1 \times \mathbb{D}^2$ be an open attracting region of f: $f(\overline{U}) \subset U$. Consider a segment $\sigma \subset U$ which is tangent to $\mathcal{C}_{\alpha_f}$. As Df^ℓ leaves strictly invariant the cone field $\mathcal{C}_{\alpha_f}$ and expands the vectors in that cone field, the forward iterates $f^{n\ell}(\sigma)$, $n > 0$, remain tangent to $\mathcal{C}_{\alpha_f}$ and their length tends to ∞. One deduces that there is $n > 0$ such that $f^n(\sigma) \cap D_f \neq \emptyset$. Hence $f^n(U) \cap D_f$ contains a non-empty open set. By definition of $\mathcal{P}_0$ the basin of the Plykin attractor A_f of ϕ_f is a dense open subset of D_f. As a consequence, $f^n(U)$ contains a point x in this basin. So $\omega(x, f) \subset A_f$. However, $\omega(x) \subset U$ because U is by definition an attracting region. So $U \cap A_f \neq \emptyset$. As A_f is transitive and U is an attracting region, this implies $A_f \subset U$. In the same way, the chain recurrence class Λ_f of A_f is contained in U.

Recall that a quasi-attractor is a chain recurrence class which is the intersection of a decreasing sequence of attracting regions. This implies that every quasi-attractor of f contains Λ_f, and hence is equal to Λ_f.

On the other hand, as $S^1 \times \mathbb{D}^2$ is an attracting region, it contains at least one quasi-attractor. This concludes the proof. $\square$

3.4. **Robust homoclinic tangencies.** Now our construction consists in proving:

Proposition 3.3. *For every $f \in \mathcal{U}_\gamma^{Ply}$ and every point x of the hyperbolic basic set A_f, there is $y \in A_f$ such that $W^u(x)$ and $W^s(y)$ meet tangentially at one point.*

Idea of the proof of Proposition 3.3. Proposition 3.3 is completely analogous to [As, Proposition 3.1]. We just recall the ideas of the proof for completeness.

The hyperbolic set A_f is a hyperbolic attractor of ϕ_f. Furthermore, by hypothesis, the basin $W^s(A_f)$ contains the whole disk D_f, punctured by the finite set R_f of repelling periodic points (contained in $f(D_f)$). Hence $D_f \setminus R_f$ is foliated by the stable manifolds of the point in A_f. Let us denote $\mathcal{F}^s$ as this foliation.

On the other hand, as D_f is an essential disk, transverse to the cone field $\mathcal{C}_{\alpha_f}$, for every circle $S^1 \times \{z\}$ the image $f(S^1 \times \{z\})$ transversely cuts D_f in exactly $|n_\gamma| > 1$ points (always with the same orientation). In particular, $f(S^1 \times \mathbb{D}^2)$ cuts D_f in exactly $|n_\gamma|$ connected components, and one of them is $f(D_f)$. Let $\Delta(f)$ be another component. Notice that $\mathcal{F}^s$ induces a foliation of the disk $\Delta(f)$, already denoted by $\mathcal{F}^s$.

Recall that A_f is a lamination whose leaves are the unstable leaves for ϕ_f of the points $z \in A_f$. These leaves are tangent to the center-unstable direction of A_f considered as a basic set for f, and we denote them by $L^c(z)$.

The unstable leaves of the points of A_f are C^1-surfaces. More precisely, for every point $z \in A_f$, the unstable manifold $W^u(z)$ for f is the union of all the strong unstable leaves $L^{uu}(x)$ for $x \in L^c(z)$.

Each strong unstable leaf is a curve tangent to the cone field $\mathcal{C}_{\alpha_f}$, contained in $f(S^1 \times \mathbb{D}^2)$ and of infinite length. In particular, every sufficiently large segment of a strong unstable leaf cuts the disk $\Delta(f)$. We endow the strong unstable leaves with the orientation induced by the orientation of the circle S^1. Hence, for any point $z \in A_f$ one has a well-defined point $h(z) \in \Delta(f)$ which is the first intersection point of $L^{uu}(z)$ with $\Delta(f)$. Notice that the map h is continuous.

Now $\mathcal{L}_f = h(A_f)$ is a regular 1-dimensional compact lamination contained in $\Delta(f)$. Moreover, the leaves are C^1-curves varying C^1-continuously because they are obtained as the (transverse) intersection of $\Delta(f)$ with the unstable manifolds of the points $z \in A_f$.

Given any compact 1-dimensional lamination by uniformly C^1-curves of a 2-disk endowed with a non-singular foliation, every leaf of the lamination admits tangency points with the foliation. So every leaf of the lamination $\mathcal{L}_f$ admits tangency points with $\mathcal{F}^s$, ending the proof. □

3.5. **Proof of Theorem 1.** In proving Theorem 1 we will show:

Proposition 3.4. *Given any braid γ with $|n_\gamma| \geq 2$, and any $\varepsilon > 0$, the set*

$$\mathcal{U}_{n,\gamma} = \{f \in \mathcal{U}_\gamma^{Ply}, \exists\ q_{n,f} \text{ hyperbolic periodic source, } d(p_f, q_{n,f}) < \frac{1}{n}\}$$

is open for the C^1-topology and is dense in $\mathcal{U}_\gamma^{Ply} \cap \operatorname{Diff}^r(S^1 \times \mathbb{D}^2, \operatorname{Int}(S^1 \times \mathbb{D}^2))$ for the C^r-topology, for every $r \geq 1$.

Notice that $\bigcap_{n \in \mathbb{N}^*} \mathcal{U}_{n,\gamma} \subset \mathcal{U}_{wild,\gamma}$. Hence Proposition 3.4 implies that $\mathcal{U}_{wild,\gamma}$ is residual in $\mathcal{U}_\gamma^{Ply}$ for the C^r-topology, for any $r \geq 1$, ending the proof of Theorem 1.

The fact that $\mathcal{U}_{n,\gamma}$ is C^1-open is a simple consequence of the continuous dependence of hyperbolic periodic points, for the C^1-topology. The difficulty is to prove the C^r-density. As the set of C^{r+1}-diffeomorphisms is dense in the set of C^r-diffeomorphisms for the C^r-topology, the C^r-density of $\mathcal{U}_{n,\gamma}$ is implied by the C^{r+1}-density. Hence it is enough to prove the C^r-density of $\mathcal{U}_{n,\gamma}$ for r large enough.

However, the C^1-density can be proved by an argument of a different nature involving specific C^1-perturbations lemmas. We present both arguments in the next sections.

3.6. **C^1-density of $\mathcal{U}_{n,\gamma}$.** Recall that $\mathcal{U}_\gamma^{Ply}$ is contained in $\mathcal{U}_\gamma^{part.hyp}$. Hence every $f \in \mathcal{U}_\gamma^{Ply}$ is partially hyperbolic on $S^1 \times \mathbb{D}^2$: there is a dominated splitting

$$T_x(S^1 \times \mathbb{D}^2) = E^{cs}(x) \oplus_{<} E^u(x),$$

for $x \in \bigcap_{n \in \mathbb{Z}} f^n(S^1 \times \mathbb{D}^2)$, where $\dim E^{cs} = 2$, $\dim E^u = 1$, and the vectors in E^u are uniformly expanded.

For every $f \in \mathcal{U}_\gamma^{Ply}$, consider the set Σ_f of hyperbolic periodic saddle points which are homoclinically related with the point p_f (i.e. whose stable and unstable manifolds transversally cut the unstable and stable manifold of p_f, respectively). Let $\Sigma_{f,0} \subset \Sigma_f$ be the set of saddle points $p \in \Sigma_f$ which are sectionally expanding. In other words, $p \in \Sigma_f$ belongs to $\Sigma_{f,0}$ if

$$\left| Det\left(Df^{\pi(p)}|_{E^{cs}(p)} \right) \right| > 1,$$

where $\pi(p)$ is the period of p and $Df^{\pi(p)}|_{E^{cs}(p)}$ is the restriction of the derivative at the period to the center stable bundle at p.

Recall that the point p_f is sectionally expanding (i.e. $p_f \in \Sigma_{f,0}$). A classical argument, formalized by using the notion of transitions in [BDP] and used by many authors, implies that for every f the set $\Sigma_{f,0}$ is dense in the homoclinic class $H(p_f, f)$ of p_f (i.e. the closure of Σ_f). More precisely, there is a sequence $p_{f,i} \in \Sigma_{f,0}$, $i \in \mathbb{N}$, such that, for every $\delta > 0$ and for every i large enough, the orbit of $p_{f,i}$ is δ-dense in $H(p_f, f)$. We denote by π_i the period of $p_{f,i}$.

Now, according to [BC], for C^1-generic $f \in \mathcal{U}_\gamma^{Ply}$ the homoclinic class $H(p_f, f)$ coincides with the chain recurrence class Λ_f. As a consequence one gets:

Lemma 3.5. *For C^1-generic $f \in \mathcal{U}_\gamma^{Ply}$, the closure of $\Sigma_{f,0}$ contains Λ_f.*

According to Proposition 3.3, for any $f \in \mathcal{U}_\gamma^{Ply}$ the chain recurrence class Λ_f contains a tangency point q_f of $W^u(p_f)$ with $W^s(A_f)$. One deduces:

Lemma 3.6. *For every $f \in \mathcal{U}_\gamma^{Ply}$ the 2-dimensional bundle E^{cs} does not admit any dominated splitting along Λ_f.*

Proof. We argue by contradiction, assuming that there is a dominated splitting $E^{cs} = E_1 \oplus_< E_2$ on Λ_f: this splitting defines a dominated splitting $T_{\Lambda_f}(S^1 \times \mathbb{D}^2) = E_1 \oplus_< E_2 \oplus_< E^u$ on Λ_f. Then the stable manifold $W^u(p_f)$ is tangent to $E_2 \oplus E^u$. Furthermore, for every $x \in A_f$ and every $y \in W^s(x) \cap \Lambda_f$, the stable manifold $W^s(x)$ is tangent to $E_1(y)$ at y. This prevents $W^u(p_f)$ from having a tangency point with $W^s(x)$ for $x \in A_f$, hence contradicts Proposition 3.3. $\square$

As a direct corollary one gets:

Corollary 3.7. *For every C^1-generic $f \in \mathcal{U}_\gamma^{Ply}$, the 2-dimensional bundle E^{cs} does not admit any dominated splitting along $\Sigma_{f,0}$.*

Now, an argument of Mañé in [Ma] (see also [BDP]) shows that, for every $\varepsilon > 0$ and every i large enough, there is an ε-C^1-perturbation $g_i \in \mathcal{U}_\gamma^{Ply}$ of f which coincides with f on the orbit of $p_{f,i}$ and out an arbitrarily small neighborhood of this orbit, and such that the (real or complex) eigenvalues of $Dg_i^{\pi_i}(p_{f,i})$ corresponding to the center-stable bundle E^{cs} have the same modulus. Furthermore, as $p_{f,i}$ was sectionally expanding for f, this modulus can be taken larger than 1. As the eigenvalue corresponding to the unstable bundle is also larger than 1, one gets that the orbit of $p_{f,i}$ is a hyperbolic source for g_i. Hence, choosing $\varepsilon > 0$ small enough (so that the continuation p_{g_i} of p_f remains arbitrarily close to p_f) and i large enough (so that the orbit of $p_{f,i}$ is passing arbitrarily close to p_f), one gets $g_i \in \mathcal{U}_{n,\gamma}$, ending the proof of the density of $\mathcal{U}_{n,\gamma}$ in $\mathcal{U}_\gamma^{Ply}$ for the C^1-topology.

3.7. **C^r-density of $\mathcal{U}_{n,\gamma}$ for $r \geq 2$.** We now consider

$$\mathcal{U}_\gamma^{r,Ply} = \mathcal{U}_\gamma^{Ply} \cap \mathrm{Diff}^r(S^1 \times \mathbb{D}^2, \mathrm{Int}(S^1 \times \mathbb{D}^2))$$

endowed with the C^r-topology, for $r \geq 2$.

According to Proposition 3.3, for every $f \in \mathcal{U}_\gamma^{r,Ply}$ the unstable manifold $W^u(p_f)$ presents a tangency point q_f with the stable manifold of a point $z_f \in A_f$. Notice that $W^u(p_f)$ and $W^s(z_f)$ are C^r-immersed submanifold, and $r \geq 2$. By performing an arbitrarily small C^r perturbation of f, one may assume that the tangency point q_f is a quadratic tangency point.

Then, for every g in a small C^2-neighborhood $\mathcal{V}$ of f the tangency point q_f of $W^u(p_f)$ with the stable foliation of A_f has a unique continuation q_g, the quadratic tangency point of $W^s(p_g)$ with the stable foliation of A_g. This tangency point varies continuously with g.

Notice that the positive orbit of q_f is contained in the invariant normally hyperbolic disk D_f containing A_f. The negative orbit of q_f is not contained in D_f: by construction, q_f belongs to the lamination $\mathcal{L}_f = h(A_f)$, hence, is the first return map on D_f of the strong unstable leaf of a point $y_f \in A_f$ (i.e. $q_f = h(y_f)$); so for $n > 0$ large $f^{-n}(q_f)$ is a point contained in the local strong unstable leaf of

$f^{-n}(y_f) \in A_f \subset D_f$. So, one can perform a small C^r-perturbation of f in a neighborhood of $f^{-n}(q_f)$ without modifying the restriction of f to the disk D_f, hence without modifying the stable foliation $\mathcal{F}_f$ of A_f in D_f. So we get:

Lemma 3.8. *There is a C^r arc $\{f_t\}, t \in [0,1]$, of C^r-diffeomorphisms $f_t \in \mathcal{V}$ such that:*

- *$f_0 = f$;*
- *for every $t \in [0,1]$, f_t coincides with f on the disk D_f (in particular, the stable foliation $\mathcal{F}_{f_t}$ of A_{f_t} is $\mathcal{F}_f$);*
- *the tangency point $q_t = q_{f_t}$ defines an arc transverse to the stable foliation $\mathcal{F}_f$.*

Recall that A_f is a (transitive) hyperbolic attractor for the restriction on f to D_f and that the fixed point p_f belongs to A_f. Hence the stable manifold of p_f is a dense leaf of the foliation A_f. As a consequence one gets:

Corollary 3.9. *There is a sequence $t_n > 0$ tending to 0 such that, for every $n \in \mathbb{N}$, the point q_{t_n} is a quadratic tangency point of the stable manifold of p_f with the unstable manifold of p_f, and p_f is a hyperbolic sectionally expanding point of f_{t_n}.*

Hence $g = f_{t_n}$ is an arbitrarily small C^r-perturbation of f having a quadratic homoclinic tangency point associated to a sectionally expanding fixed point $p_g = p_f$. This situation has been studied in [PV]:

Theorem 2 ([PV])**.** *If $\{g_s\}_{s\in[0,1]}$ is a generic arc of C^r-diffeomorphisms ($r \geq 2$) and there is a periodic hyperbolic point p of g_0 which is sectionally expanding and such that $W^s(p, g_0) \cap W^u(p, g_0)$ contains a quadratic tangency point q, then there are a sequence s_i converging to 0 and periodic sources q_i of g_{s_i} converging to q.*

Notice that, for large i, the orbits of the periodic sources q_i are passing arbitrarily close to the point p. As a consequence, for i large the diffeomorphism g_{s_i} belongs to $\mathcal{U}_{n,\gamma}$ and is an arbitrarily C^r-small perturbation of g which is an arbitrarily C^r-small perturbation of f. This proves the C^r-density of $\mathcal{U}_{n,\gamma}$ in $\mathcal{U}_\gamma^{Ply}$, ending the proof of Proposition 3.4.

3.7.1. *One-parameter version of Proposition* 3.4 *for proving Corollary* 1.1. Let us cite precisely [PV] because the statements are precise but informally formulated, and we will now use their full strength:

Theorem 3 ([PV, Main Theorem])**.** *Near any smooth diffeomorphisms exhibiting a homoclinic tangency associated to a sectionally dissipative saddle, there is a residual subset of an open set of diffeomorphisms such that each of its elements displays infinitely many coexisting sinks.*

We point out that our methods also imply a one-parameter version of this theorem: A generic unfolding of such a (quadratic) homoclinic tangency yields to residual subsets of intervals in the parameter line whose corresponding diffeomorphisms exhibit infinitely many sinks. (...) (...) "Smooth" in the statement above means of class $\mathcal{C}^2$ (...).

Let us add to these statements that it is clear from the proof in [PV] that the announced sinks contain a subsequence of points accumulating on the (continuation of) the initial saddle.

Noticing that a sectionally expanding saddle is a sectionally dissipative saddle for the inverse diffeomorphism, we can now formulate a stronger version of Proposition 3.4:

Corollary 3.10. *Consider any braid γ with $|n_\gamma| \geq 2$, any $r > 1$ and any $\mathcal{C}^r$-open subset $\mathcal{V}$ of $\mathcal{U}_\gamma^{Ply} \cap \mathrm{Diff}^r(S^1 \times \mathbb{D}^2, \mathrm{Int}(S^1 \times \mathbb{D}^2))$.*

Then there is a non-empty $\mathcal{C}^r$-open set $\mathcal{V}_{arc}$ of arcs $[0,1] \to \mathcal{V}$ such that any arc $\{f_t\}_{t\in[0,1]} \in \mathcal{V}_{arc}$ intersects $\mathcal{U}_{wild,\gamma}$ along diffeomorphisms f_t for t in a residual subset of a non-empty open subset of $[0,1]$.

Proof. Just define $\mathcal{V}_{arc}$ as the $\mathcal{C}^r$ open set of arcs $[0,1] \to \mathcal{V}$ displaying a generic quadratic tangency associated to the sectionally expanding dissipative saddle p_{f_t} for some $t \in]0,1[$. Such a set is always open, and Lemma 3.8 implies that it is non-empty. Then [PV] implies that each of these arcs contains a residual subset of a (non-empty) interval corresponding to diffeomorphisms displaying infinitely many periodic hyperbolic sinks accumulating on p_{f_t}, ending the proof. ☐

4. Non-existence of attractors for diffeomorphisms

4.1. An attracting ball B^3 without attractors.

Theorem A is obtained from Theorem 1 by building locally generic diffeomorphisms of an attracting ball B^3 without topological attractors and with a unique quasi-attractor:

Theorem 4. *There is a non-empty C^1-open subset $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{D}^3, \mathrm{Int}(\mathbb{D}^3))$ and, for $f \in \mathcal{U}$, a hyperbolic periodic point p_f varying continuously with f such that:*

1. *the diffeomorphism f is C^1-conjugated with the homothety $z \mapsto \frac{1}{2}z$ in a neighborhood of the sphere $\partial\mathbb{D}^3$;*
2. *for every $f \in \mathcal{U}$, the chain recurrence class $\Lambda_f = C(p_f)$ is the unique quasi-attractor of f;*
3. *for every $r \geq 1$, the subset*

$$\mathcal{U}_{wild} = \{f \in \mathcal{U}, \Lambda_f \cap \overline{\{\textit{sources of } f\}} \neq \emptyset\}$$

is residual for the C^r-topology.

The next lemma can be easily proved by using the same kind of perturbations used for the *derived from Anosov* diffeomorphisms in [Sm]. We leave the details of the construction to the reader.

Lemma 4.1. *Let $f\colon S^1 \times \mathbb{D}^2 \to \mathrm{Int}(S^1 \times \mathbb{D}^2)$ be a solenoid map such that $\bigcap_{n\in\mathbb{N}} f^n(S^1 \times \mathbb{D}^2)$ is a hyperbolic attractor. Then, there is g isotopic to f, which coincides with f in a neighborhood of the boundary $\partial(S^1 \times \mathbb{D}^2)$ and such that the chain recurrent set in $S^1 \times \mathbb{D}^2$ consists in exactly one fixed hyperbolic sink ω and a hyperbolic basic set of saddle type (i.e. neither attracting nor repelling). Moreover, if f is orientation preserving, one may require that the derivative $Dg(\omega)$ is the homothety of ratio $\frac{1}{2}$.*

Proof of Theorem 4. According to [Gi] there is a diffeomorphism f_0 of the 3-sphere S^3 admitting a torus T with the following properties:

- the torus T bounds two solid tori Δ_1 and Δ_2;
- $f_0(\Delta_1)$ is contained in the interior of Δ_1, and the restriction $f_0|_{\Delta_1}$ is a hyperbolic Smale-solenoid attractor corresponding to a 2-braid γ;

- $f_0^{-1}(\Delta_2)$ is contained in the interior of Δ_2, and the restriction $f_0^{-1}|_{\Delta_2}$ is a hyperbolic Smale-solenoid attractor corresponding to a 2-braid γ.

We now modify f_0 by surgery in both solid tori Δ_1 and Δ_2 in order to get a diffeomorphism f_1 with the following properties:

- f_1 coincides with f_0 in the neighborhood of the torus T; as a consequence $f_1(\Delta_1) \subset \mathrm{Int}(\Delta_1)$ and $f_1^{-1}(\Delta_2) \subset \mathrm{Int}(\Delta_2)$;
- the restriction of f_1 to the solid torus Δ_1 belongs to the C^1-open set $f \in \mathcal{U}_\gamma^{Ply}$;
- the intersection of the chain recurrent set $\mathcal{R}(f_1)$ with Δ_2 consists exactly in a hyperbolic fix source α_1 and a non-trivial hyperbolic set K_1 of saddle type (this is obtained by applying Lemma 4.1 to the restriction of f^{-1} to the solid torus $f(\Delta_2)$).

Now one removes from S^3 the interior of a small ball B centered at α_1. Then $\mathcal{B} = S^3 \setminus \mathrm{Int}(B)$ is a compact ball diffeomorphic to $\mathbb{D}^3$. Furthermore $f_1(\mathcal{B})$ is contained in the interior of $\mathcal{B}$. Now there is a C^1-neighborhood $\mathcal{U}$ of f_1 such that every $f \in \mathcal{U}$ satisfies the following properties:

- there is a diffeomorphism $\varphi\colon \mathcal{B} \to \mathbb{D}^3$ such that $\varphi f \varphi^{-1}\colon \mathbb{D}^3 \to \mathbb{D}^3$ coincides with the homothety $z \mapsto \frac{1}{2}z$ in a neighborhood of $S^2 = \partial\mathbb{D}^3$;
- the image of the solid torus $\Delta_1 \subset \mathcal{B}$ is contained in its interior, and the restriction $f|_{\Delta_1}$ belongs to $\mathcal{U}_\gamma^{Ply}$; one denotes by Λ_f the unique quasi-attractor of f contained in Δ_1 and by p_f the hyperbolic sectionally expanding saddle point in Λ_f associated to $f|_{\Delta_1} \in \mathcal{U}_\gamma^{Ply}$;
- the intersection of the chain recurrent set $\mathcal{R}(f)$ with $\mathcal{B} \setminus \mathrm{Int}(\Delta_1)$ is a hyperbolic basic set of saddle type.

One concludes by noticing that C^r-generic diffeomorphisms $f \in \mathcal{U}$ induce by restriction on Δ_1 C^r-generic diffeomorphisms in $\mathcal{U}_\gamma^{Ply}$. As a consequence, there is a sequence of hyperbolic sources converging to a point in Λ_f, preventing Λ_f from being an attractor. □

4.2. **End of the proof of Theorem A.** In proving Theorem A one considers the time one map of the flow of a gradient vector field of a Morse function on M. Then one replaces the diffeomorphism in a neighborhood of each sink by a diffeomorphism in the open set $\mathcal{U}$ built in Theorem 4.

Remark 4.2. Let M be a compact orientable 3-manifold. Using the fact that M admits a Heegaard splitting in two handlebodies, one easily verifies that M admits a gradient-like diffeomorphism having a unique sink. As a consequence, we can assume that $k = 1$ in the statement of Theorem A.

4.3. **End of the proof of Corollary 1.1.** Consider a C^1-open set $\mathcal{O}$ of diffeomorphisms of M given by Theorem A and Remark 4.2 above. So every $f \in \mathcal{O}$ admits a unique quasi-attractor contained in a ball B, and the restriction of f to that ball belongs to the open set $\mathcal{U}$ built in Theorem 4. Moreover, the quasi-attractor is contained in a solid torus T contained in the ball B, and the restriction of f to T belongs to $\mathcal{U}_\gamma^{Ply}$ for some 2-braid γ.

Then given any $r > 1$ and any $\mathcal{C}^r$-open subset $\mathcal{O}_0$ of $\mathcal{O}$ there is a $\mathcal{C}^r$-open set $\mathcal{O}_{0,arc}$ of arcs $[0,1] \to \mathcal{O}_0$ such that for any $\{f_t\}_{t\in[0,1]} \in \mathcal{O}_{0,arc}$ that restriction to T belongs to the $\mathcal{C}^r$ open set $\mathcal{V}_{arc}$ built in Corollary 3.10. In particular, f_t has infinitely many repellers converging to the unique quasi-attractor, for t in a residual subset of a non-empty open set of parameters.

To complete the proof of Corollary 1.1 it remains to recall that the set of analytic arcs is $\mathcal{C}^r$ dense in the set of $\mathcal{C}^r$ arcs. In particular, $\mathcal{O}_{0,arc}$ contains an analytic arc, proving that $\mathcal{O}_0$ contains analytic diffeomorphisms without attractors.

4.4. **Non-existence of attractors and repellers in higher dimensions: Proof of Theorem B.** Multiplying our construction in B^3 by a transverse contraction allows us to get

Lemma 4.3. *Given any $d > 3$, there is a non-empty C^1-open subset*

$$\mathcal{U}_d \subset \mathrm{Diff}^1(\mathbb{D}^d, \mathrm{Int}(\mathbb{D}^d))$$

such that every $f \in \mathcal{U}_d$ satisfies the following properties:

(1) *the diffeomorphism f is C^1 conjugated with the homothety $z \mapsto \frac{1}{2}z$ in a neighborhood of the sphere $\partial\mathbb{D}^d$;*
(2) *the chain recurrent set of f is contained in a normally hyperbolic 3-disc D_f;*
(3) *the restriction $f|_{D_f}$ belongs to the open subset $\mathcal{U}$ given by Theorem 4; in particular, for every $f \in \mathcal{U}_d$ the chain recurrence class Λ_f of the fixed point p_f is the unique quasi-attractor of f.*

As a consequence, for every $r \geq 1$, the subset

$$\mathcal{U}_{wild} = \{f \in \mathcal{U}_d, \Lambda_f \cap \overline{\{\text{ sources of the restriction } f|_{D_f}\}} \neq \emptyset\}$$

is residual for the C^r-topology. Then $f \in \mathcal{U}_{wild}$ has neither attractors nor repellers in $\mathbb{D}^d$.

Given any manifold M with $\dim(M) > 3$, one considers a diffeomorphism f_0 which is the time-one map of a Morse function. Now, one builds a diffeomorphism f_1 obtained from f_0 as follows:

- one replaces f_0, in a small ball centered to each sink, by a diffeomorphism in the open set $\mathcal{U}_d$ built in Lemma 4.3;
- one replaces f_0^{-1}, in a small ball centered to each source, by the inverse of a diffeomorphism in the open set $\mathcal{U}_d$ built in Lemma 4.3.

Now the open set announced in Theorem B is obtained by considering a small neighborhood of the diffeomorphism f_1 above.

5. Singular flows: Proof of Theorem C

Our example for flow is very similar to the examples built for Theorem 1, so we will just sketch the construction.

We consider an open set $\mathcal{U}$ of vector fields on $\mathbb{R}^4$ such that every $X \in \mathcal{U}$ satisfies the following properties:

- the vector field X admits a transverse cross section Σ diffeomorphic to a solid torus $S^1 \times \mathbb{D}^2$;
- the vector field X has a unique singular point 0_X which is a saddle with $\dim(W^s(0_X)) = 3$; the eigenvalues of the derivative $D_{0_X}X$ are

$$\lambda_1 < \lambda_2 < \lambda_3 < 0 < \lambda_4,$$

  with $\lambda_4 + \lambda_1 > 0$;

- there is an essential disc $D_0 \subset \Sigma$, transverse to all the circles $S^1 \times \{z\}, z \in \mathbb{D}^2$, and contained in the local stable manifold of the saddle point 0_X;
- the first return map on Σ is well defined on $\Sigma \setminus D_0$, and the image is contained in the interior of Σ; we denote it by $P\colon \Sigma \setminus D_0 \to \mathrm{Int}(\Sigma)$;
- the first return map P leaves invariant a splitting $T\Sigma = E^{cs} \oplus E^u$ which is a dominated splitting with $\dim E^{cs} = 2$ and $\dim E^u = 1$; moreover, E^u is transverse to the discs $\{t\} \times \mathbb{D}^2$, $t \in S^1$;
- the bundle E^u is uniformly expanding by a factor larger than 3; more precisely, given any non-zero vector u tangent to $E^u(x)$, $x \in \Sigma$, denote $u = u_h + u_v$, where u_h is tangent to the S^1 fiber through x and u_v is tangent to the $\mathbb{D}^2$ fiber through x;
  assume $x \in \Sigma \setminus D_0$ and let $w = D_x P(u) = w_h + w_v$; then we require
  $$|w_h| > 3|u_h|;$$
- there is an essential disc $D_1 \subset \Sigma \setminus D_0$, invariant by P (i.e. $P(D_1) \subset \mathrm{Int}(D_1)$), normally hyperbolic, and such that the restriction $P|_{D_1}$ is smoothly conjugated to an element of the open set $\mathcal{P}_0$ of structurally stable diffeomorphisms in $\mathrm{Diff}^1_0(\mathbb{D}^2, \mathrm{Int}(\mathbb{D}^2))$, defined in Section 2.5; in particular, the chain recurrent set of $P|_{D_1}$ consists of a Plykin attractor A_X and finitely many repelling points, and the Plykin attractor A_X contains a fixed point p_X which is sectionally expanding;
- there is an essential disc $D_2 \subset \Sigma \setminus D_0$, invariant by P, normally hyperbolic, and such that the restriction $P|_{D_2}$ has a unique fixed point q_X; the disc D_2 is contained in the stable manifold of q_X (for the map P); finally, the derivative $D_{q_X}(P)$ of P at q_X has a complex (non-real) eigenvalue, corresponding to the tangent space $T_{q_X} D_2$.

It is not hard to build a non-empty open set $\mathcal{U}$ of vector fields satisfying all the properties above (see also [BLY], which contains the details of an analogous construction).

As in the proof of Theorem 1, one verifies that, for any open subset $O \subset \Sigma$ there is $n > 0$ such that $f^n(O)$ meets D_0, D_1 and D_2. This implies that every attracting region for X which meets Σ contains the singular point 0_X, the Plykin attractor A_X (and hence its orbits by the flow of X) and the orbit γ_X of the point q_X. Hence there is a unique quasi-attractor Λ_X for the orbits of X through Σ, and this quasi-attractor contains 0_X, A_X and γ_X. An analogous argument shows that, for every $X \in \mathcal{U}$, the invariant manifolds of A_X for P present a tangency point. This implies that C^r-generic paths in $\mathcal{U}$ unfold generic homoclinic bifurcations associated to p_X, implying that, for C^r-generic $X \in \mathcal{U}$ the quasi-attractor Λ_X is accumulated by periodic sources, which prevents Λ_X from being an attractor.

One concludes the proof of Theorem C by proving

Lemma 5.1. *For any $X \in \mathcal{U}$, the tangent flow of X on Λ_X does not admit any dominated splitting.*

Proof. Assume that there is a dominated splitting $TM|_{\Lambda_X} = E \oplus_< F$ for the tangent flow of X. This dominated splitting induces on $\Sigma \cap \Lambda_X$ a dominated splitting $T\Sigma|_{\Sigma \cap \Lambda_X} = E_\Sigma \oplus F_\Sigma$ invariant by P (just consider $E_\Sigma = (E + \mathbb{R}X) \cap T\Sigma$ and $F_\Sigma = (F + \mathbb{R}X) \cap T\Sigma$).

The fact that q_X belongs to $\Lambda_X \cap \Sigma$ implies that $\dim E_\Sigma = 2$. One deduces that $E_\Sigma = E^{cs}$ and $F_\Sigma = E^u$. As a consequence one gets two possibilities for the splitting $T_xM = E(x) \oplus F(x)$ at $x \in \Lambda_X \cap \Sigma$:

- either $E = E^{cs} \oplus \mathbb{R}X$ and $F \subsetneq E^u \oplus \mathbb{R}X$
- or $E \subsetneq E^{cs} \oplus \mathbb{R}X$ and $F = E^u \oplus \mathbb{R}X$.

In the first case, X is tangent to E along Λ_X. However, Λ_X contains the unstable manifold of 0_X (a quasi-attractor always contains its unstable manifold). This manifold consists in 0_X and 2 orbits of X. Hence $W^u(0_X)$ is tangent to E. This implies that $E(0_X)$ contains the eigenspace corresponding to λ_4, which contradicts the fact that E is dominated by F.

In the second case, X is tangent to F along Λ_X. However, for $x \in A_X$, the space $E^{cs}(x)$ contains vectors tangent to the hyperbolic attractor $A_X \subset D_1$ and hence contains vectors which are exponentially expanded by the derivative DP^n, for $n \to +\infty$. This implies that the space $E(x)$ contains vectors $u \in E(x)$ and a sequence of times $t_n \to +\infty$ such that $X_{t_n}(x) = P^n(x) \in \Sigma$ and

$$\lim_{n\to+\infty} |(X_{t_n})_*(u)| = +\infty,$$

where $(X_t)_*$ denotes the derivative of the time t of the flow of X.

On the other hand, $X(x) \in F(x)$ but $|(X_{t_n})_*(X(x))|$ remains bounded, contradicting the fact that F dominates E.

Hence both cases lead to a contradiction, ending the proof. $\square$

6. Changing the definition of attractors

With a better understanding of the complexity of generic dynamics, people tried the definition of attractors in order to ensure their existence.

6.1. **Palis' approach from the point of view of ergodic theory.**

From the ergodic viewpoint, an attractor Λ of f should satisfy the following:

- "*indecomposable property*": there is an ergodic invariant probability measure μ such that $\mathrm{supp}(\mu) = \Lambda$;
- "*attracting property*": its basin $B(\Lambda)$ has positive Lebesgue measure, where

$$x \in B(\Lambda) \Longleftrightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} = \mu$$

(here δ_z stands for the Dirac measure at the point z).

Conjecture (Palis [P1, P2, P3]). *There is a dense set $\mathcal{D} \subset \mathrm{Diff}^r(M)$ such that for any $f \in \mathcal{D}$, f has only finitely many (ergodic) attractors, and the union of the basins of attractors forms a full Lebesgue measure set in M.*

Palis completed his conjecture by continuity properties of the basins of the attractors with respect to the diffeomorphism.

6.2. **A topological approach.**

In [H], Hurley proved that, for generic homeomorphisms of a compact manifold, the ω-limit set of every generic point is a quasi-attractor, and he stated the following conjecture.

Conjecture (Hurley). *For C^r-generic diffeomorphisms the ω-limit sets of generic points in M are quasi-attractors.*

This conjecture has been proved in [MP, BC] for the C^1-topology and remains open in more regular topologies.

However, the information given by Hurley's conjecture is very weak: every C^0-generic homeomorphism h has uncountably many quasi-attractors. Furthermore, the first author recently announced that (in dimension larger than 2) the closure of the basin of each quasi-attractor has empty interior.[7] In the setting of the C^1-topology, [BD] shows that there are locally generic diffeomorphisms having an uncountable family of quasi-attractors which are at the same time quasi-repellers. In particular, the basin of each of them is reduced to the quasi-attractor itself (which is a Cantor set).

Let us define a new notion of attractor which will allow us to propose a new conjecture.

Definition 6.1.
- A *residual attractor* of a diffeomorphism f is a chain recurrence class admitting a neighborhood U which is an attracting region and such that the ω-limit set of the generic points in U is Λ.
- A *locally residual attractor* of a diffeomorphism f is a chain recurrence class admitting an open set U such that the ω-limit set of the generic points in U is Λ. Notice here U may not be a neighborhood of Λ.

Remark 6.2.
- For C^1-generic diffeomorphisms, one can deduce from [BC] that the residual attractors are exactly the quasi-attractors which are isolated in the set of quasi-attractors: they admit a neighborhood disjoint from any other quasi-attractor.
- For C^1-generic diffeomorphisms, our notion of *locally residual attractor* coincides with the notion of *generic attractor* introduced by Milnor in [Mi]. More precisely, Milnor first defines a *minimal attractor* for the ergodic point of view: the basin has positive Lebesgue measure and every proper subset's basin only has zero Lebesgue measure. Then, on the topological generic setting he writes: "There is an analogous concept of *generic-attractor*. The definition will be left to the reader." Hence, a *generic attractor* is an invariant set whose basin is a locally residual set and such that every proper subset's basin is meager. As Hurley's conjecture is proved for C^1-generic diffeomorphisms, Milnor's generic attractors of a C^1-generic diffeomorphism are its locally residual attractors.

The locally generic examples built in Theorem A have finitely many residual attractors, and the union of their basin is a residual subset of the whole manifold M. This motivates the following problem:

Problem 5.
(1) Is it true that C^r-generic diffeomorphisms have at least one (locally) residual attractor?
(2) For any C^r-generic diffeomorphism, is it true that the ω-limit set of every generic point is a (locally) residual attractor?

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[7] The proof of this last fact (the closure of each basin has empty interior) was found by the first author of this present paper, writing this conclusion. The idea of the proof is the following. Consider a homeomorphism h and any open set U. Then, one can perform a C^0 small perturbation h_0 of h such that U meets the basin of two attracting regions of h_0. Now a standard argument of genericity leads to the announced statement. Hurley kindly wrote us that he did not notice this fact.

(A positive answer to these questions is known for locally residual attractors of C^1-generic diffeomorphisms. See the next section, devoted to the C^1-topology.)

We would like to better understand these residual attractors, in particular to understand if they are associated to periodic orbits. Recall that the *homoclinic class* of a periodic orbit is the closure of the transverse intersection of its invariant manifolds. It is an invariant compact set canonically associated to the periodic orbit. [BC] shows that, for C^1-generic diffeomorphisms, the chain recurrence class of a periodic orbit is its homoclinic class. As a consequence, isolated chain recurrence classes of C^1-generic diffeomorphisms are homoclinic classes (in particular, this holds for topological attractors). As we noticed above, the residual attractors are the quasi-attractors which are isolated in the set of quasi-attractors. It seems natural to ask:

Problem 6. Let Λ be a residual attractor of a generic diffeomorphism. Is Λ the homoclinic class of a periodic orbit?

6.3. **Remarks on the C^1 topology.** For C^1-generic non-critical (i.e. far from homoclinic tangencies) diffeomorphisms, [Y] gave a positive answer to Problems 5 and 6 proving that every quasi-attractor is a homoclinic class. Since for C^1-generic diffeomorphisms we can have only countably many homoclinic classes, together with the results in [MP, BC] there is at least one locally residual attractor. Furthermore, the (countable) union of the basins of the locally residual attractors is a residual subset of the manifold.

In a forthcoming work, we will obtain more precise results for the C^1-topology.

- In contrast to Theorem A, we will prove that, for a closed surface M^2, every diffeomorphism in a C^1-dense and open subset of $\mathrm{Diff}^1(M^2)$ has a hyperbolic attractor.[8] Notice that it's still unknown whether C^r-generic surface diffeomorphisms must have an attractor or not for $r > 1$.
- As a complement of Theorem A, for any compact three-dimensional manifold M^3 without boundary, we will construct a C^1 open set $\mathcal{U} \subset \mathrm{Diff}^1(M^3)$ such that C^1-generic $f \in \mathcal{U}$ have neither attractors nor repellers.[9]
- Together with S. Gan, we will give a positive answer to Problems 5 and 6 in the setting of a partially hyperbolic splitting with a 1-dimensional center bundle. In this setting, we prove that for C^1-generic diffeomorphisms every quasi-attractor is a residual attractor.

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[8] After ending this preprint, we were told that this result is already known and contained in the non-published Araujo's thesis [Ar]. However, from [PS, page 964], Araujo's proof is unclear.

[9] For the examples built in Theorem A, there are infinitely many repellers.

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