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TRACKING A MOVING POINT IN THE PLANE

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Abstract. The Teichmüller theory of any hyperbolic Riemann surface R induces two closely related metrics on R in the following way. From a theorem of Bers, the fiber

$$\mathbb{K} = \Psi^{-1}([identity])$$

of the forgetful map Ψ from the Teichmüller space $Teich(R-p)$ onto the Teichmüller space $Teich(R)$ is conformal to a disc and the evaluation map $\mathbb{K} \ni [f] \mapsto f(p) \in R$ is a universal covering of R. There are two infinitesimal metrics on $\mathbb{K}$ coming from Kobayashi's construction:

(1) $Teich_{\mathbb{K}}$ is the restriction of the Teichmüller infinitesimal metric on $Teich(R-p)$ to the submanifold $\mathbb{K}$, and
(2) $Kob_{\mathbb{K}}$ is the Kobayashi metric on $\mathbb{K}$.

We show these metrics, respectively, are the lifts via the evaluation map of infinitesimal forms λ and ρ on R, where λ and ρ are the Teichmüller and Poincaré densities. λ and ρ have very different descriptions. For plane domains

$$\lambda(p) = \inf\{||\overline{\partial}V||_\infty\},$$

where the infimum is taken over all continuous functions V for which $V(p) = 1$ and V vanishes on the boundary of R, and

$$\rho(p) = \inf\{1/|f'(0)|\},$$

where the infimum is taken over all holomorphic functions f mapping the unit disc into R with $f(0) = p$. We also show

$$(1/2)Kob_{\mathbb{K}} \leq Teich_{\mathbb{K}} \leq Kob_{\mathbb{K}} \quad \text{and}$$
$$(1/2)\rho \leq \lambda \leq \rho,$$

and $\lambda/\rho = 1/2$ when R is simply connected, $\lambda/\rho = 1$ when R is a thrice punctured sphere, and in all other cases these inequalities are strict.

Introduction

A small movement of a point p to another point on any hyperbolic Riemann surface R changes the conformal structure of $R\setminus\{p\}$ but not the conformal structure of R. This idea can be expressed formally by observing that there is a natural projection between Teichmüller spaces,

$$\Psi : T(R \setminus \{p\}) \to T(R).$$

Those deformations represented by points in $T(R \setminus \{p\})$ that do not alter the corresponding points in $T(R)$ are realized by homotopy classes of motions of p on R. The map Ψ is called the forgetful map because, for any deformation in $T(R \setminus \{p\})$, it forgets the motion of p and remembers only how R was deformed. The fiber $\mathbb{K} = \Psi^{-1}([id])$ consists of the deformations caused by moving p in R while leaving

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the marked conformal structure of R fixed. This fiber is the main topic of study by Bers in [3] where he shows that when R is of finite analytic type, $\mathbb{K}$ is conformal to the universal covering of R. Then $T(R \setminus \{p\})$ factored by the natural realization of the fundamental group of R as a subgroup of the mapping class group of $R \setminus \{p\}$ is the universal curve of Riemann surfaces with a given topological type. That is, $T(R \setminus \{p\})$ factored by this group is a complex fiber space over $T(R)$ with the property that the fiber over each point τ in $T(R)$ is a Riemann surface with marked complex structure equal to the complex structure of R_τ.

This paper concerns a similar setup but with the following differences. We assume R is an open set in the extended complex plane $\overline{\mathbb{C}}$, possibly disconnected, and that its complement $E = \overline{\mathbb{C}} \setminus R$ contains three or more points. In this setting we define Teichmüller equivalence by isotopy and post composition by Möbius transformations. After normalizing at three points of E, the deformations of R in $\overline{\mathbb{C}}$ determine sets $f(E)$ complementary to $f(R)$. Thus we may also think of the points of the Teichmüller space as marked deformations of closed subsets E of $\overline{\mathbb{C}}$. This Teichmüller space is denoted by $T(E)$, so the forgetful map is

$$\Psi : T(E \cup \{p\}) \to T(E). \tag{1}$$

The fiber $\mathbb{K} = \Psi^{-1}([id])$ is a one-dimensional marked deformation space that tracks the motion of p in the component R_j of $\overline{\mathbb{C}} - E$ where p is located and for deformations in $\mathbb{K}$ the only point of $E \cup \{p\}$ that moves is p itself.

Any infinitesimal form for a metric on $T(E \cup \{p\})$ restricts to an infinitesimal form on $\mathbb{K}$. We focus attention on two special cases, namely, the Teichmüller and Kobayashi infinitesimal metrics, $Kob(\tau, V)$ and $Teich(\tau, V)$, which are defined in (6) and (7) in Section 2. These metrics are infima over the families of holomorphic functions from the unit disc into $\mathbb{K}$ and into $T(E \cup \{p\})$ with derivatives mapping to a given tangent direction. Since $\mathbb{K} \subset T(E \cup \{p\})$, the family of such functions with image in $\mathbb{K}$ is smaller and so it turns out that

$$Teich(\tau, V) \leq Kob(\tau, V). \tag{2}$$

In [8] for Riemann surfaces a reverse inequality is shown, namely,

$$Kob(\tau, V) \leq 2\, Teich(\tau, V), \tag{3}$$

where τ is a point of $\mathbb{K} \subset T(E \cup \{p\})$ and V is a tangent vector at τ.

The metric densities $Kob(\tau, V)$ and $Teich(\tau, V)$ on $\mathbb{K}$ induce metric densities on R, where the former is the Poincaré metric $\rho(z)|dz|$ on the component of R that contains p and the latter is another metric density $\lambda(z)|dz|$, which we have called the Teichmüller density in [8]. Several authors have studied the comparison between ρ and λ (see [3, 8, 10, 12, 13, 16]), but in all of these p is viewed as a variable point on a connected Riemann surface. We give two ways of constructing λ on $R = \overline{\mathbb{C}} - E$ and show that the ratio λ/ρ is a function defined on $T(R \setminus \{p\})$ bounded above and below by 1 and 1/2. In addition, we show that the ratio is strictly less than 1 unless R has only one component and that component is equal to the Riemann sphere with three points removed. We also show that $\lambda(p)/\rho(p)$ is strictly larger than 1/2 unless p belongs to a simply connected component of R.

1. Teichmüller spaces

Let E be a closed subset of the extended complex plane $\overline{\mathbb{C}}$ and consider the family $\mathcal{F}$ of all quasiconformal self-maps f of $\overline{\mathbb{C}}$. Let two elements f_0 and f_1 in $\mathcal{F}$

be equivalent if there is a Möbius transformation A that carries $f_0(E)$ to $f_1(E)$ and an isotopy through quasiconformal self-maps $g_t, 0 \leq t \leq 1$, of $\overline{\mathbb{C}}$ such that

(i) $g_0(z) = A \circ f_0(z)$ for all z in $\overline{\mathbb{C}}$,
(ii) $g_1(z) = f_1(z)$ for all z in $\overline{\mathbb{C}}$, and
(iii) $A \circ f_0(p) = g_t(p) = f_1(p)$ for all t with $0 \leq t \leq 1$ and for all p in E.

By definition the Teichmüller space $T(E)$ is the space $\mathcal{F}$ factored by this equivalence relation. If E contains zero, one, two or three points, there are no nontrivial quasiconformal deformations and the Teichmüller space $T(E)$ consists of only one point. To avoid this trivial situation we always assume E contains three or more points, say z_1, z_2 and z_3. Since the definition of $T(E)$ is independent of post composing quasiconformal self-maps of $\overline{\mathbb{C}}$ by Möbius transformations, we also assume that these quasiconformal maps fix the three points z_1, z_2 and z_3. This means that in part (iii) of the definition of equivalence of elements of $\mathcal{F}$ the Möbius transformation A is the identity. Since pullback by Möbius transformations induces isomorphisms of Teichmüller spaces, without loss of generality we may also assume that these three points are $0, 1$ and ∞.

The class of the identity map $id(z) = z$ in $T(E)$ denoted by $[id]_E$ represents a special point which we call the *base point.* If g is a quasiconformal self-map of $\overline{\mathbb{C}}$, then there is the natural isomorphism g^* from $T(g(E))$ to $T(E)$ that carries the class $[f]_{g(E)}$ to the class of $[f \circ g]_E$. In particular, it carries the base point of $T(g(E))$ to $[g]_E$ in $T(E)$, and the point $[g^{-1}]_{g(E)}$ in $T(g(E))$ to the base point of $T(E)$.

2. $\mathbb{K}$ as a holomorphic motion

If we take a point p in any component R_j of $R = \overline{\mathbb{C}} \setminus E$, then $E \cup \{p\}$ is also a closed set and there is a natural map $\Psi : T(E \cup \{p\}) \to T(E)$, called the *forgetful map,* defined by

$$\Psi([f]_{E\cup\{p\}}) = [f]_E. \tag{4}$$

It is well defined because quasiconformal maps in an isotopy joining two representatives of $[f]_{E\cup\{p\}}$ fix all of the points of $E \cup \{p\}$. These quasiconformal maps automatically extend continuously to p, and the same isotopy is an isotopy of maps that fix the points of $f(E)$.

By a theorem of Bers and Lakic, [2], [11] and [7, page 65], the cotangent space to $T(E)$ for any closed set E in $\overline{\mathbb{C}}$ containing the point ∞ is equal to the L_1-closure of the rational meromorphic functions on $\overline{\mathbb{C}}$ that are holomorphic on $\mathbb{C} - E$ and have at most simple poles on E. We shall denote this space by $A(E)$; it is the space of integrable holomorphic quadratic differentials on $\overline{\mathbb{C}} - E$. The derivative of Ψ at any point $\tau = [f]_{E\cup\{p\}}$ in $T(E \cup \{p\})$ is the linear map that is dual to the inclusion $A(f(E)) \subset A(f(E) \cup \{f(p)\})$. It follows from Theorem 3, which we prove later, that there always is a quadratic differential in $A(f(E) \cup \{f(p)\})$ and not in $A(f(E))$, so this derivative is surjective at every point of $T(E \cup \{p\})$. If furthermore $[f] \in \mathbb{K}$, then f is an orientation-preserving map that fixes the boundary of R_j, and so $f(p)$ belongs to R_j. If g is any map in the equivalence class of $[f]_{E\cup\{p\}}$ in $T(E\cup\{p\})$, then by part (iii) in the definition of equivalence for $T(E \cup \{p\})$, $g(p) = f(p)$. Therefore, there is an evaluation map $\tilde{F}$ from $T(E \cup \{p\})$ into $\overline{\mathbb{C}}$ defined by

$$\tilde{F}([f]_{E\cup\{p\}}) = f(p). \tag{5}$$

If we denote by F the restriction of $\tilde{F}$ to $\mathbb{K}$, then F maps $\mathbb{K}$ to R_j. Moreover, because R_j is open and connected one can find an isotopy of quasiconformal maps f_t that equals the identity outside of R_j and carries p to a point $f_t(p) = p_t$ in R_j such that $p_1 = f_1(p)$ is an arbitrary point of R_j. The equivalence class $[f_t]_{E\cup\{p\}}$ is a curve in $\mathbb{K}$ connecting $[id]$ to $[f_1]_{E\cup\{p\}}$ and $F([f_1]_{E\cup\{p\}}) = p_1$. This shows that F maps $\mathbb{K}$ onto R_j.

Let Π be a universal covering from the unit disc Δ onto R_j such that $\Pi(0) = p$. The map F is also a covering mapping onto R_j in the sense that for any q in R_j there is a disc D in R_j containing q such that $F^{-1}(D)$ is a disjoint union of conformal discs D_j in $\mathbb{K}$ such that F restricted to D_j is a homeomorphism of D_j onto D. Since Π is a universal covering, there is a unique lift $\tilde{\Pi}$ mapping from Δ to $\mathbb{K}$ such that $F \circ \tilde{\Pi} = \Pi$ and $\tilde{\Pi}(0)$ is the base point in $T(E \cup \{p\})$.

Theorem 1. *The lift $\tilde{\Pi}$ of Π is bijective and conformal from Δ onto $\mathbb{K}$.*

Proof. We have already observed that F maps $\mathbb{K}$ onto R_j and that it is a surjective covering of R_j and so F lifts to a covering $\tilde{\Pi}$ from Δ onto $\mathbb{K}$. If we can show $\mathbb{K}$ is simply connected, then $\tilde{\Pi}$ must be an isomorphism. By a theorem of Lieb (see [5]) the Teichmüller space $T(E)$ is isomorphic to

$$(\Pi_k T(R_k)) \times M(E),$$

where R_k runs over all the components of the complement of E in $\overline{\mathbb{C}}$ and $M(E)$ is the open ball of measurable complex-valued functions μ supported on E with $||\mu||_\infty < 1$. The isomorphism carries the Beltrami coefficients representing the points in each of the Teichmüller spaces $T(R_j)$ to Beltrami coefficients defined independently on the domains R_j. The Teichmüller metric is induced by the L_∞ metric on Beltrami coefficients. Now assume p is in a particular component R_j of the complement of E. By Bers' theorem [3] the fiber over the identity of the forgetful map from $T(R_j - \{p\})$ to $T(R_j)$ is simply connected, and by Lieb's theorem it is isomorphic to the fiber over the identity of

$$\Psi : (\Pi_{k\neq j} T(R_k)) \times M(E) \times T(R_j - \{p\}) \to (\Pi_{k\neq j} T(R_k)) \times M(E) \times T(R_j).$$

So the points of the fiber over the identity in this projection are represented precisely by the points of the fiber over the identity of $T(R_j - \{p\}) \to T(R_j)$. □

For the next result we first give some basic facts about the Kobayashi metric on a complex manifold M. Its infinitesimal form $Kob_M(\tau, V)$ is a positive number assigned to each tangent vector V at a point τ in M by the following extremal problem:

$$Kob_M(\tau, V) = \inf\{|k| \mid kg'(0) = V\}, \tag{6}$$

where the infimum is taken over all nonnegative numbers k and all holomorphic functions g mapping the unit disc Δ into M.

Because it is known that for any Teichmüller space T, $Kob_T(\tau, V) = Teich_T(\tau, V)$, [6] and [14], the infinitesimal norm $Teich_T(\tau, V)$ is given by

$$Kob_T(\tau, V) = \inf\{|k| \mid kg'(0) = V\} = Teich_T(\tau, V) = \inf\{||\overline{\partial}\tilde{V}||_\infty\}, \tag{7}$$

where the first infimum is taken over all k and all holomorphic functions g mapping Δ into T with $g(0) = \tau$ and the second infimum is over all continuous vector fields $\tilde{V}$ whose restriction to $f^\mu(E \cup \{p\})$ is equal to V and where f^μ represents the point $\tau \in T(E \cup \{p\})$. However, for complex submanifolds M of T, Kobayashi's

$$\begin{tikzcd} \Delta \arrow[r, "H"] \arrow[dr, "\Pi"'] & \mathbb{K} \arrow[d, "F"] \\ & R_j \end{tikzcd}$$

FIGURE 1

infinitesimal form for M may differ from Teichmüller's infinitesimal form for T restricted to M since the extremal problem defining $Kob_M(\tau, V)$ involves the further requirement that the holomorphic function g must have image contained in M. Thus, obviously,

$$Teich_M(\tau, V) \leq Kob_M(\tau, V). \tag{8}$$

We will see that this is usually a strict inequality when $M = \mathbb{K}$.

Schwarz's lemma implies that Kobayashi's metric for a complex disc coincides with Poincaré's metric. This observation is expressed in the following proposition.

Proposition 1. *The Kobayashi density $Kob_{\mathbb{K}}(\tau, V)$ on $\mathbb{K}$ and the Poincaré density ρ on R_j are related by the following formula:*

$$\rho(F([f]))|dF(V)| = Kob_{\mathbb{K}}([f], V). \tag{9}$$

Proof. Since Π is a universal covering of Δ onto R_j mapping 0 to p, $\frac{1}{|\Pi'(0)|}$ realizes the smallest value of $\frac{1}{|g'(0)|}$, where $g(0) = p$ and g is holomorphic mapping Δ into R_j. $\square$

The next theorem identifies the unique lifting $\tilde{\Pi}$ of the previous theorem in a different way. Define a normalized holomorphic motion $h(t, z)$ of $E \cup \{p\}$ by the following formula:

$$h(t, z) = \begin{cases} z & \text{for} \quad z \in E, \\ \Pi(t) & \text{for} \quad z = p. \end{cases} \tag{10}$$

By Slodkowski's [15] extension theorem, h extends to a normalized holomorphic motion $\tilde{h}$ of the extended complex plane. The map H that sends t to the class of $z \mapsto \tilde{h}(t, z)$ is a holomorphic map from the unit disc into $T(E \cup \{p\})$. Furthermore, since $\tilde{h}(t, z) = h(t, z) = z$ for all t and all z in E, the curve $H(t)$ represents the homotopy class that keeps all $H(t)$ in the same class in $T(E)$. Since $H(0)$ is the identity map, the image of H is in $\mathbb{K}$. Since $F(H(t)) = \tilde{h}(t, p) = h(t, p) = \Pi(t)$, the diagram in Figure 1 commutes.

Let γ be any path in $\overline{\mathbb{C}} \setminus E$ such that $\gamma(0) = p$. Take a lift $\tilde{\gamma}$ of γ under the universal covering map Π and with the property $\tilde{\gamma}(0) = 0$. Then the path $\hat{\gamma} = H(\tilde{\gamma})$ is a path in $\mathbb{K}$ that starts at the base point and whose image under F is γ. Since from the previous theorem $\mathbb{K}$ is simply connected, H is a biholomorphic surjection. Therefore, we have proved the following theorem.

Theorem 2. *The map F is a holomorphic covering from $\mathbb{K}$ onto R_j. Furthermore, the holomorphic motion h in* (10) *defines the whole fiber $\mathbb{K}$. More precisely, for every τ in $\mathbb{K}$, there exists t in Δ such that $\tau = [z \mapsto \tilde{h}(t,z)]$.*

3. Comparing metrics on $\mathbb{K}$

We now define the infinitesimal form of a metric λ naturally associated to any hyperbolic Riemann surface which turns out to be different from but equivalent to the Poincaré metric ρ. In [8] we have called it the Teichmüller density because it is the metric on R induced by $Teich(\tau, V)$ on the fiber $\mathbb{K} = \Psi^{-1}([id])$. It will turn out that the ratio $\lambda(p)/\rho(p)$ is a real-valued functional on domains $E \cup \{p\}$ that takes values always between 1/2 and 1, and takes the extreme value 1/2 only when p is in a simply connected component of R and the extreme value 1 only when R is the complement of three points in the Riemann sphere.

As a starting point we make the following definitions.

Definition 1. The infinitesimal Teichmüller norm of a vector field V defined on $E \cup \{p\}$ is

$$||V||_T = \inf\{||\overline{\partial}\tilde{V}||_\infty\}, \tag{11}$$

where the infimum is taken over all continuous, complex-valued functions $\tilde{V}$ with distributional derivatives and $\tilde{V}$ equal to V on $E \cup \{p\}$.

Definition 2. The Teichmüller density $\lambda(p)$ evaluated at a point p in R_j is equal to $||V_0||_T$, where V_0 is the function defined on $E \cup \{p\}$ which is equal to zero on E and equal to 1 at p.

Proposition 2. *The Teichmüller density $Teich_{\mathbb{K}}([f], V)$ on $\mathbb{K}$ is related to the Teichmüller density λ on R_j by the formula:*

$$\lambda(F([f]))|dF(V)| = Teich_{\mathbb{K}}([f], V), \tag{12}$$

where $F : \mathbb{K} \to R_j$ is the evaluation map $F([f]) = f(p)$.

Proof. We first prove this equality for $[f] = [id]$. If $[f] = [id]$, then $F([f]) = p$. If V is the zero vector, then $dF(V) = 0$ and (12) reduces to $0 = 0$. Suppose V is a nonzero vector field tangent to $\mathbb{K}$. Then V vanishes on E, $V(p) \neq 0$ and $\lambda(p) = ||V/V(p)||_T$. Also $dF|_{[id]}(V) = V(p)$, which follows from [1], page 104. So the left-hand side of (12) is equal to $||V||_T$. From Definition 1 and (7) this is equal to the right-hand side of (12).

Let $[f]$ be any point in $\mathbb{K}$ and $G : Teich(E \cup f(p)) \to Teich(E \cup p)$ given by $G([g]) = [g \circ f]$ is an isomorphism. Also, if $\Psi : T(E \cup \{p\}) \to T(E)$ and $\Psi_G : T(E \cup \{f(p)\}) \to T(E)$ are the forgetful maps, then $\Psi \circ G = \Psi_G$. If $\mathbb{K}'$ is the fiber over the identity of Ψ_G and $\mathbb{K}$ is the fiber over the identity of Ψ, then G restricted to $\mathbb{K}'$ is an isomorphism of $\mathbb{K}'$ onto $\mathbb{K}$. For $[g] \in \mathbb{K}'$, $(F \circ G)([g]) = F([g \circ f]) = g(f(p))$. Thus $F \circ G$ is the evaluation map for g at the point $f(p)$. By what we have just proved,

$$\lambda(FG([id]))d(FG)|_{[id]}((G^{-1})^*V) = Teich_{\mathbb{K}'}([id], (G^{-1})^*V). \tag{13}$$

But $\lambda(F \circ G([id])) = \lambda(f(p))$. Also,

$$Teich_{\mathbb{K}'}([id], (G^{-1})^*V) = Teich_{\mathbb{K}}([g], dG_{[id]}(G^{-1})^*(V)) = Teich_{\mathbb{K}}([g], V).$$

The chain rule implies

$$d(FG)|_{id}((G^{-1})^*V) = dF_{[f]}(dG_{[id]}(G^{-1})^*(V)) = dF_{[f]}(V),$$

and so (12) follows from (13). □

To get a lower bound for the infinitesimal form $\lambda(p)$ we construct a quadratic differential ψ_p canonically associated with any Riemann surface R and a point p in R. Let Π be a universal covering of a component R of the complement of E in $\overline{\mathbb{C}}$. Assume Π is normalized so that $\Pi(0) = p$ and $\Pi'(0)$ is positive. Let Γ be the universal covering group for Π. Γ is a fixed point free discrete group of Möbius transformations acting on the unit disc Δ, $\Pi \circ \gamma = \Pi$ for every $\gamma \in \Gamma$, and the factor space Δ/Γ is conformal to R.

Theorem 3. *The Poincaré theta series of* $\psi(z) = \frac{1}{\pi z}$,

$$\sum_{\gamma\in\Gamma} \psi(\gamma(z))\gamma'(z)^2 = \sum_{\gamma\in\Gamma} \frac{\gamma'(z)^2}{\pi\gamma(z)} = \psi_p(z), \tag{14}$$

defines a quadratic differential ψ_p *on* R *which is holomorphic on* $R - \{p\}$, *which has a simple pole at* p *and for which*

$$1 \le ||\psi_p|| \le 2. \tag{15}$$

Proof. The tangent vector $V = H'(0)$ to the holomorphic motion $H(t, z)$ from Section 2 vanishes on E and satisfies $V(p) = \Pi'(0)$. Put

$$V_0 = \frac{V}{\Pi'(0)}.$$

Since $1/|\Pi'(0)| = \rho(p)$ and since by Schwarz's lemma $||V||_T \le 1$, we obtain

$$||V_0||_T = \rho(p)||V||_T \le \rho(p). \tag{16}$$

On the other hand,

$$||V_0||_T \ge \frac{|V_0(\hat{\varphi})|}{||\hat{\varphi}||}$$

for every integrable quadratic differential $\hat{\varphi}$ holomorphic on $R - \{p\}$. In particular

$$\rho(p) \ge ||V_0||_T \ge \frac{|V_0(\psi_p)|}{||\psi_p||} \tag{17}$$

for the quadratic differential ψ_p given by the Poincaré theta series (14).

By the Ahlfors-Bers density theorem, ψ_p is the L_1-limit of integrable holomorphic functions ψ_p in the plane with a finite number of simple poles located at p and on the boundary of R_j. Thus to evaluate $V(\psi_p)$ we may assume it has only a finite number of poles. But if we let $\tilde{V}_0$ be a continuous extension of V_0 with bounded $\overline{\partial}$-derivative, then

$$V_0(\psi_p) = \iint \overline{\partial}\tilde{V}_0\psi_p dxdy = \iint \overline{\partial}\tilde{V}_0\psi_p \frac{d\overline{z}dz}{2i}$$

$$= \iint d\left(\tilde{V}_0\psi_p\frac{dz}{2i}\right) = -\pi res(\psi_p, p). \tag{18}$$

The only residue of $\left(\tilde{V}_0\psi_p dz\right)$ is at p and this residue is equal to $1/(\pi\Pi'(0))$, where Π is a universal covering of R_j with $\Pi(0) = p$. Since $\rho(p) = 1/|\Pi'(0)|$,

$$|V_0(\psi_p)| = \rho(p), \tag{19}$$

and from (17) we get

$$\rho(p) \geq ||V_0||_T \geq \frac{\rho(p)}{||\psi_p||},$$

which yields $||\psi_p|| \geq 1$.

On the other hand,

$$||\psi_p|| = \left|\iint_\omega \sum \frac{\gamma'(z)^2}{\pi\gamma(z)}\right| \leq \iint_\omega \sum \left|\frac{\gamma'(z)^2}{\pi\gamma(z)}\right| = \iint_{|z|<1} \frac{1}{|\pi z|} dxdy = 2,$$

where ω is a fundamental domain in Δ for the covering group. □

Theorem 4 ([8])**.** *The infinitesimal form λ on each domain R_j is equivalent to the Poincaré infinitesimal form ρ on that domain. More precisely,*

$$(1/2)\rho(p) \leq \lambda(p) \leq \rho(p). \tag{20}$$

Proof. The right-hand side of (20) follows from (16). The left-hand side follows by choosing the specific quadratic differential ψ_p on $R_j \setminus \{p\}$ given in Theorem 3. The theta series of $\psi(z) = 1/(\pi z)$ is summed over all covering transformations γ for the covering map Π of R_j. Since $||\psi_p|| \leq 2$, the definition of $\lambda(p)$ yields

$$\lambda(p) \geq \frac{|V_0(\psi_p)|}{||\psi_p||} = \frac{\rho(p)}{||\psi_p||} \geq \frac{\rho(p)}{2}. \tag{21}$$

□

Just as with the Poincaré metric, the integrated form of $\lambda(p)|dp|$ becomes a metric by the following recipe:

$$\lambda(p_1, p_2) = \inf \int_\gamma \lambda(\gamma(t))|\gamma'(t)dt|, \tag{22}$$

where the infimum is over all continuous, piecewise differentiable curves γ in R such that $\gamma(0) = p_1$ and $\gamma(1) = p_2$.

Corollary 1. *The metrics $\lambda(p_1, p_2)$ and $\rho(p_1, p_2)$, which are the integrated forms of $\lambda(p)|dp|$ and $\rho(p)|dp|$, satisfy the following inequality:*

$$(1/2)\rho(p_1, p_2) \leq \lambda(p_1, p_2) \leq \rho(p_1, p_2).$$

Proof. To define the integral in (22) it is helpful to know that the integrand is continuous, which is justified by Corollary 2 given in the next section. The inequality of this corollary then follows directly from the corresponding inequality for the infinitesimal metrics. □

4. Residues as densities

Theorem 3 shows the existence of a special quadratic differential with a simple pole at any given point p in R. We now show that the residue of a meromorphic n-differential is an $(n-1)$-differential and, consequently, the residue of a quadratic differential is a one-form. We state this observation as a lemma.

Lemma 1. *Suppose $\varphi = \varphi(z)(dz)^n$ is a meromorphic n-differential in a neighborhood of p. Then the residue of φ at p is an $(n-1)$-differential.*

Proof. Let w and z be charts such that $z = 0$ in U and $w = 0$ in V correspond to the point p. Let $\varphi_U(z)$ and $\varphi_V(w)$ be the expressions for φ in terms of the parameters z and w so that

$$\varphi_U(z) = \varphi_V(f(z))f'(z)^n,$$

where $w = f(z)$ is the transition mapping. Then

$$res(\varphi_V, p)f'(0)^{n-1} = \frac{1}{2\pi i}\int_{w(\gamma)} \varphi_V(w)f'(0)^{n-1}dw = \frac{1}{2\pi i}\int_\gamma \varphi_U(z)\frac{f'(0)^{n-1}}{f'(z)^{n-1}}dz.$$

Here γ is a simple closed curve in the z-plane winding once around $z = 0$ and oriented in the counterclockwise direction and $w(\gamma)$ has the same properties with respect to $w = 0$ in the w-plane. The integrals depend only on the homotopy class provided γ lies in a sufficiently small neighborhood of 0. Since f is univalent, the fraction $f'(0)^{n-1}/f'(z)^{n-1}$ becomes uniformly close to 1 as that neighborhood becomes sufficiently small. Therefore, the integral on the right-hand side is equal to

$$\frac{1}{2\pi i}\int_\gamma \varphi_U(z)dz = res(\varphi_U, p). \qquad \square$$

In the case that φ is a quadratic differential, $res(\varphi, p)dp$ is a holomorphic one-form and, if φ has at most simple poles, then $res(\varphi, p) = 0$ if and only if φ extends holomorphically at p. Moreover, the absolute value $|res(\varphi, p)dp|$ determines a density, that is, an infinitesimal metric. In addition to its residue, any integrable quadratic differential φ holomorphic on $E \cup \{p\}$ carries another invariant, namely, its L^1 norm. Thus we can ask for the largest residue of an integrable quadratic differential holomorphic on $R \setminus \{p\}$ subject to the condition that its L^1 norm is equal to one. Obviously, a differential that realizes this largest value will be identically equal to zero on every component of R that does not contain p.

Putting these pieces together we obtain the following theorem, which serves as an alternative definition of the Teichmüller density λ.

Theorem 5.

$$\lambda(p) = \pi \sup_{||\varphi||=1} \operatorname{Re}\, res(\varphi, p), \tag{23}$$

where the supremum is over all quadratic differentials holomorphic in $R \setminus \{p\}$ with L^1 norm one. Moreover, there is a unique quadratic differential φ_p that realizes the supremum in (23).

Proof. Let $V(p) = 1$ and V vanish on E, and let $A(E \cup \{p\})$ be the Banach space of integrable quadratic differentials φ holomorphic on the complement of $E \cup \{p\}$ with norm given by

$$||\varphi|| = \int_R |\varphi| dxdy.$$

By the Hahn-Banach and Riesz representation theorems, there exists a continuous extension $\tilde{V}$ of V with bounded $\overline{\partial}$-derivative such that

$$||\overline{\partial}\tilde{V}||_\infty = \sup\left\{\left|\int\int_R \overline{\partial}\tilde{V}\varphi\right|\right\}, \tag{24}$$

where the supremum is over all $\varphi \in A(R \setminus \{p\})$ with $||\varphi|| = 1$, and where $\tilde{V}$ is a continuous extension of V to R.

By the same residue calculation given in the proof of Theorem 3 we find that

$$\int\int_R \overline{\partial}\tilde{V}\varphi = -\pi res(\varphi, p).$$

Therefore the infimum in Definition 1 is equal to the supremum in (23).

To prove the second part of the theorem, the evaluation of the residue of φ at a point p is a linear functional $V \in A(E \cup \{p\})^*$ given by the formula

$$V(\varphi) = \frac{1}{2\pi i}\int_\gamma \varphi(z)dz, \tag{25}$$

where γ is a closed curve with winding number 1 around the point p. Now suppose φ_n is a sequence of differentials in $A(E \cup \{p\})$ with $||\varphi_n|| = 1$ such that Re $\pi res(\varphi_n, p)$ approaches the supremum in (23). From equicontinuity of bounded holomorphic functions there must be a subsequence of φ_n which converges uniformly on compact subsets of $R - \{p\}$. But it is not possible for any subsequence of φ_n to converge uniformly on compact subsets to any quadratic differential φ with $||\varphi|| < 1$, because replacing φ by $\varphi/||\varphi||$ gives a quadratic differential of norm 1 with a larger residue. Therefore, a subsequence of φ_n converges uniformly on compact subsets to φ and $||\varphi_n|| = 1 = ||\varphi||$, which implies $||\varphi_n - \varphi||$ converges to zero and so φ realizes the supremum in (23). The point φ on the unit sphere realizing the supremum is unique because the unit ball of $A(E \cup \{p\})$ is strictly convex. Thus, every convergent subsequence of φ_n converges to the same limit, which implies the original sequence itself converges to that limit. □

Corollary 2. *$\lambda(p)$ is continuous in p.*

Proof. Assume $|q - p| < \delta$. Then the Bers' $\mathcal{L}$-operator [7] yields a Banach space isomorphism $\mathcal{L}$ from $A(E \cup \{q\})$ onto $A(E \cup \{p\})$ such that

$$\max\{||\mathcal{L}||, ||\mathcal{L}^{-1}||\} < 1 + \epsilon(\delta),$$

where $\epsilon(\delta)$ is bounded by the L_∞-norm of the Beltrami coefficient of any quasiconformal self-map of R that carries q to p and is equal to the identity outside a fixed disc of radius δ_1 with center p and with $|q - p| < (1/2)\delta_1$. □

Theorem 6. *Let ρ be the Poincaré metric for R. That is, in any component R_j of R, ρ is equal to ρ_{R_j}, the Poincaré metric for the component R_j. Then for every quadratic differential φ in $A(R \setminus \{p\})$, we have*

$$|res(\varphi, p)| \leq \frac{\rho(p)}{\pi}||\varphi||.$$

Proof. From Theorem 5,

$$\lambda(p) \geq \pi|res(\varphi, p)|/||\varphi||$$

and we already know $\rho(p) \geq \lambda(p)$, so the theorem follows. □

5. Domain differentials

In the previous two sections we constructed two quadratic differentials ψ_p and φ_p in $A(E \cup \{p\})$ canonically associated to a plane domain and a point p in that domain. ψ_p is an element of $A(E \cup \{p\})$ given by a theta series of the function $q(z) = 1/(\pi z)$ summed over the elements of a Fuchsian covering group acting on the unit disc and chosen so that 0 covers p and $\Pi'(0)$ is positive. φ_p is an element of $A(E \cup \{p\})$ that maximizes the real part of $V_0(\varphi)$ with $||\varphi|| = 1$, where V_0 is a vector field vanishing on E and equal to 1 at p. We have already seen in Theorem 5 that φ_p realizing this extreme is uniquely determined. These two differentials determine the Teichmüller and Poincaré densities by the following formulae:

$$\lambda(p) = \pi \text{ Re } res(\varphi_p, p) \quad \text{and} \quad \rho(p) = \pi \text{ Re } res(\psi_p, p). \tag{26}$$

The left-hand formula comes from Theorem 5 and the right-hand comes from (18) and (19).

Before stating the next theorem we need to recall the definition of the Zygmund Banach space $Z(E)$ on an arbitrary closed set $E \subset \overline{\mathbb{C}}$. By definition, $Z(E)$ consists of all continuous complex-valued vector fields V defined on E with the property that V has a continuous extension $\tilde{V}$ to the complex plane $\overline{\mathbb{C}}$ such that $||\overline{\partial}\tilde{V}||_\infty < \infty$. The assumption that the vector field $V(z)\frac{\partial}{\partial \overline{z}}$ is continuous at $z = \infty$ implies there is a constant C such that $|\tilde{V}(z)| \leq C|z|^2$. Definition 1 defines the Teichmüller norm of V, and one easily shows that $||V||_T = 0$ if and only if V is the restriction of a quadratic polynomial to E.

There is another natural norm on $Z(E)$, which we call the cross ratio norm, and it is given by a supremum of cross ratios. To define the cross ratio norm, assume $Q = \{a, b, c, d\}$ is an arbitrary quadruple of four points contained in E and form

$$||V||_{cr} = \sup |cr(Q)\rho_{01}(cr(Q))V[Q]|,$$

where

$$V[Q] = \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(a) - V(d)}{a - d}$$

and where $\rho_{01}(p)$ is the Poincaré metric of the domain $\overline{\mathbb{C}} - \{0, 1, \infty\}$. In [7, page 67] it is shown that the two norms $||V||_T$ and $||V||_{cr}$ are equivalent by universal constants independent of the set E. Moreover, $Z(E)$ is a Banach space and the pairing between $A(E)$ and $Z(E)$,

$$(\varphi, V) = \text{ Re } \int\int \varphi \overline{\partial}\tilde{V} dxdy,$$

induces an isomorphism from $Z(E)$ onto $A(E)^*$.

In general the one-dimensional spaces in $A(E \cup \{p\})$ spanned by φ_p and ψ_p are not equal except, of course, when $A(E \cup \{p\})$ is one dimensional. The only case when $A(E \cup \{p\})$ is one dimensional occurs when E is the smallest possible set, namely, when $E = \{0, 1, \infty\}$. These comments lead to the following theorem.

Theorem 7. *If $E = \{0, 1, \infty\}$, then $\lambda(p) = \rho(p)$. Furthermore inequality* (15) *in Theorem* 3 *becomes $||\psi_p|| = 1$, $\varphi_p = \psi_p$ and*

$$\psi_p(w) = \left(\frac{1}{\pi\Pi'(0)}\right) \cdot \frac{p(p-1)}{w(w-1)(w-p)},$$

where Π is the covering map described in Theorem 3.

Proof. For every Teichmüller space $T(E \cup \{p\})$, we know that$||\varphi_p|| = 1$ and $1 \leq ||\psi_p|| \leq 2$. When $E = \{0, 1, \infty\}$, $\mathbb{K} = T(E \cup \{p\})$ and so Teichmüller's metric equals Kobayashi's metric because the Teichmüller and Kobayashi metrics coincide on any Teichmüller space. Thus, from Propositions 1 and 2, $\lambda(p) = \rho(p)$. Because of (26) this means $res(\varphi_p, p) = res(\psi_p, p)$. Note that $A(\{0, 1, p, \infty\})$ is one dimensional and generated by

$$\frac{p(p-1)}{w(w-1)(w-p)}, \tag{27}$$

which has residue at p equal to 1. From the calculation preceding (19),

$$res(\psi_p(w), p) = res(\frac{1}{\pi z}, 0) \cdot \left(\frac{dz}{dw}\Big|_{w=p} \right) = \frac{1}{\pi \Pi'(0)},$$

where $w = \Pi(z)$. Thus, after multiplication by $1/(\pi\Pi'(0))$, the formula in (27) has the same residue as ψ_p at p and so must coincide with ψ_p. Since both φ_p and ψ_p have the same residue at p and are in the same one-dimensional space, $\varphi_p = \psi_p$. □

Corollary 3 (Agard's formula)**.** *The Poincaré metric for* $\overline{\mathbb{C}} - \{0, 1, \infty\}$ *is*

$$\rho(p) = \left(\frac{1}{\pi} \int \int \left| \frac{p(p-1)}{w(w-1)(w-p)} \right| dudv \right)^{-1}. \tag{28}$$

Proof. From its definition, φ_p has norm equal to 1. Since $\psi_p = \varphi_p$, ψ_p also has norm equal to 1 and (28) is equivalent to the equation $||\psi_p|| = 1$. □

Theorem 8. *For p in any simply connected component R_j of $\overline{\mathbb{C}} - E$, $\lambda(p) = (1/2)\rho(p)$. Moreover, the pullbacks of the quadratic differentials φ_p and ψ_p to the unit disc by the Riemann map that maps p to 0 and has positive derivative at p are equal to $1/(2\pi z)$ and $1/(\pi z)$, respectively.*

Proof. Since both metrics are invariant under conformal change of coordinates, to show $\lambda(p) = (1/2)\rho(p)$ we only need to show that $\lambda_\Delta(0) = \rho_\Delta(0)/2$, where Δ is the unit disc. Since the covering group is trivial in this case, from Theorem 3 it is obvious that $\psi_0(z) = 1/(\pi z)$. We claim that $\varphi_0(z) = 1/(2\pi z)$. Since $||\varphi_0(z)|| = 1$, we know from Theorem 5 that $\lambda_\Delta(0) \geq 1/2$. The reverse inequality follows by considering the vector field $V(z) = 1 - |z|$ for $|z| \leq 1$ and $V(z) = 0$ for $|z| \geq 1$. Then $V(0) = 1$ and $\overline{\partial} V(z) = -(1/2)z^{1/2}\overline{z}^{-1/2}$, so $||\overline{\partial} V||_\infty = 1/2$. This shows that any φ for which $\pi res(\varphi, 0) > 1/2$ must have norm greater than 1. But from Theorem 5 the quadratic differential with positive residue at 0 that realizes the maximum value of $V(\varphi)$ is unique. Therefore, $\varphi_0 = 1/(2\pi z)$. □

6. Kobayashi and Teichmüller extremal domains

In this section we show that the inequalities in (20) in Theorem 4 are sharp except in two special cases. That is, we show $\lambda(p) = \rho(p)$ if and only if E consists of three points and $\lambda(p) = (1/2)\rho$ if and only if p is in a simply connected component R_j of R.

First, we make a general remark about notation. If we are given a continuous infinitesimal metric $\alpha(p)|dp|$ defined on a domain and a differentiable curve $\gamma(t), 0 \leq t \leq 1$, in that domain, we shall denote the α-arclength of γ by $\alpha(\gamma)$, that is,

$$\alpha(\gamma) = \int_0^1 \alpha(\gamma(t))|\gamma'(t)||dt|.$$

Just as in formula (22), we define the α-distance from p to q by

$$\alpha(p,q) = \inf\{\alpha(\gamma)\},$$

where the infimum is taken over all curves γ in the domain that join p to q.

Theorem 9. *If E has exactly three points, then the map H in Figure 1 is an isometry from the unit disc Δ with the hyperbolic metric onto $\mathbb{K}$ with the infinitesimal form of Teichmüller's metric. That is,*

$$Teich_{\mathbb{K}}(H(t), H(s)) = \rho_{\Delta}(t,s).$$

If E is connected, or equivalently, if every component of R is simply connected, then

$$Teich_{\mathbb{K}}(H(t), H(s)) = \frac{1}{2}\rho_{\Delta}(t,s)$$

for all t and s in Δ. In all other cases, we have

$$\frac{1}{2}\rho_{\Delta}(\gamma) < Teich_{\mathbb{K}}(H(\gamma)) < \rho_{\Delta}(\gamma),$$

for any non-constant, piecewise differentiable path γ in Δ.

Proof. Since we have already shown that $(1/2)\rho(p) \leq \lambda(p) \leq \rho(p)$, and since ρ_{Δ} is the lift to Δ of ρ and $Teich_{\mathbb{K}}$ is the lift to $\mathbb{K}$ of λ,

$$\frac{1}{2}\rho_{\Delta}(\gamma) \leq Teich_{\mathbb{K}}(H(\gamma)) \leq \rho_{\Delta}(\gamma). \tag{29}$$

Equality in (29) can hold only if the corresponding equality holds in

$$(1/2)\rho(p) \leq \lambda(p) \leq \rho(p). \tag{30}$$

We have seen that $(1/2)\rho(p) = \lambda(p)$ if the component R_j of R that contains p is simply connected. Conversely, if R_j is not simply connected, then R_j has a nontrivial function covering group Γ, and the quadratic differential

$$\psi_p(z) = \sum_{\Gamma} \frac{\gamma'(z)^2}{\pi\gamma(z)}$$

has norm strictly less than 2. From (21) this implies that $\lambda(p) > (1/2)\rho(p)$.

It remains to show that if R is not conformal to $\overline{\mathbb{C}} \setminus \{0,1,\infty\}$, then $\lambda(p) < \rho(p)$. We will prove this in the remainder of this section. □

Theorem 10. *If there is any point p in $\overline{\mathbb{C}} - E$ for which $\lambda(p) = \rho(p)$, then $A(E \cup \{p\})$ is one dimensional and $E = \{0,1,\infty\}$.*

Proof. First we show that the component of R containing p cannot be simply connected and it cannot be conformal to a punctured disc. Finally we show that if a component does not fall into one of these two cases, then it must contain a simple closed hyperbolic geodesic and whenever this is the case $\lambda(p) < \rho(p)$.

The first situation is excluded because we have already observed that if a component of R is simply connected, then $\lambda(p) = (1/2)\rho(p)$ on that component.

In the second situation, we can assume the component of R is equal to the punctured disc $\Delta' = \{z : 0 < |z| < 1\}$ and that $0 < p < 1$. Consider the vector field $V(z)\frac{\partial}{\partial z}$ defined on Δ' by

$$V(z) = \frac{1}{p\ln p}(\ln|z|)z = \frac{1}{p\ln p} \cdot \left(\frac{\ln z + \ln \overline{z}}{2}\right) z. \tag{31}$$

Notice that V vanishes at 0 and on $\{z : |z| = 1\}$, that $V(p) = 1$ and finally that

$$\mu = \overline{\partial} V = \frac{1}{2p \ln p} \cdot \frac{z}{\overline{z}}. \tag{32}$$

Thus

$$\lambda_{\Delta'}(p) \leq ||\overline{\partial} V||_\infty \leq \frac{1}{|2p \ln p|} = \rho_{\Delta'}(p).$$

But the expression in (32) is the Beltrami coefficient associated to the quadratic differential $q(z) = \frac{(dz)^2}{z^2}$ which over Δ' is a quadratic differential of infinite norm. By Theorem 5 there is a unique quadratic differential holomorphic in Δ' with the largest possible real residue at p and with L_1-norm equal to one, which we denoted by φ_p. This linear functional V is represented by the Beltrami coefficient $\lambda(p)|\varphi_p|/\varphi_p$. That is, any other Beltrami differential not of this form that represents V must have larger L_∞-norm. Since $\frac{(dz)^2}{z^2}$ is not integrable on Δ', we conclude that μ in (32) does not have the smallest possible L_∞-norm and so $\lambda(p) < \rho(p)$.

If $A(E \cup \{p\})$ is not one dimensional, then E must contain four or more points. If the component R_j of $\overline{\mathbb{C}} - E$ that contains p is neither simply connected nor conformal to a punctured disk, then R_j contains a simple closed curve homotopic to a Poincaré geodesic of positive length. Suppose that $\lambda(p) = \rho(p)$. Then by (16), $||V||_T = 1$, where $V = H'(0)$ and $H(t, z) = f^{\mu(t)}(z)$ is a holomorphic motion from Section 2. Thus, $||\mu'(0)||_\infty = 1$, and

$$V_0 = [\frac{\mu'(0)}{\Pi'(0)}].$$

Formula (25) implies that

$$V_0 = [k\lambda(p)\frac{|\varphi_p|}{\varphi_p}],$$

where $|k| = 1$ and φ_p is the differential in (26). Therefore,

$$\frac{\mu'(0)}{\Pi'(0)} = k\lambda(p)\frac{|\varphi_p|}{\varphi_p}.$$

Thus,

$$\mu'(0) = \tilde{k}\frac{|\varphi_p|}{\varphi_p},$$

where $|\tilde{k}| = 1$.

Let

$$f(t) = \iint_R \mu(t)\varphi_p dxdy.$$

Then f maps Δ to Δ and

$$f'(0) = \iint_R \mu'(0)\varphi_p dxdy = \tilde{k}.$$

Thus, by Schwarz's lemma, $f(t) = \tilde{k}t$. We have

$$\begin{aligned} |t| = |\tilde{k}t| &= |\iint_R \mu(t)\varphi_p dxdy| \\ &\leq \iint_R |\mu(t)||\varphi_p| dxdy \leq |t| \iint_R |\varphi_p| dxdy = |t|. \end{aligned}$$

Thus,

$$\mu(t)\varphi_p = \tilde{k}t|\varphi_p|$$

and $\mathbb{K}$ is a Teichmüller disc. The results in [11] or [4] now imply that E is a three-point set. □

Corollary 4. *The ratio*

$$G([f]) = \frac{\lambda(f(p))}{\rho(f(p))}$$

is a well-defined real-valued function of $[f]$ *in the Teichmüller space* $T(E \cup \{p\})$ *taking values between* $1/2$ *and* 1, *and the extreme value* $1/2$ *only when* $f(p)$ *lies in a simply connected component of* $f(R)$ *and the extreme value* 1 *only when* $f(R)$ *is the complement of three points in the Riemann sphere.*

7. Local and global Teichmüller densities

In the proof of Theorem 10 we proved that $\lambda_D(p) < \rho_D(p)$ for the case that the domain D is the punctured disc. Here we will give another method that can be applied to any domain. It involves the introduction of another density on D which we will call the *global Teichmüller density.* We have seen that the density $\lambda(p)$ is equal to the Teichmüller norm of a vector field that is equal to zero on the boundary of D and equal to 1 at p. In Section 2 we have also seen that the holomorphic motion h in (10) has an extension to the whole complex plane. Now consider all possible extensions $\tilde{h}$ of h to the whole plane and put

$$V = \frac{d}{dt}\Big|_{t=0}\tilde{h}(t,z)$$

and finally put

$$\sigma(p) = \rho(p)\inf\{||\overline{\partial}V||_\infty\}, \tag{33}$$

where the infimum is taken over all such tangent vectors V obtained as tangent vectors to holomorphic motions $\tilde{h}(t,z)$ of this type. Since σ is obtained from the infimum over a subset of the vector fields used in the definition of λ, we obviously have the following inequality:

$$\lambda(p) \le \sigma(p). \tag{34}$$

The inequality

$$\sigma(p) \le \rho(p) \tag{35}$$

is obvious because σ is given by the infimum in (33) and by Schwarz's inequality $||V||_T \le 1$. We are calling σ_D and λ_D global and local because σ_D depends on the class of vector fields that arise from global holomorphic motions defined for $|t| < 1$ that do not alter the conformal structure of D, and the tangent vectors that determine λ_D do not necessarily arise from such holomorphic motions.

There is also a corresponding metric which we call $\Sigma_{\mathbb{K}}(\tau, V)$ on the fiber $\mathbb{K}$. We can define it in the following way. Let τ be a point in $\mathbb{K}$ and consider the family $\mathcal{H}$ of all holomorphic maps H from Δ into $\mathbb{K}$ with $H(0) = \tau, H'(0) = V$ and with $F \circ H = \Pi$ in Figure 1. Let

$$\Sigma_{\mathbb{K}}(\tau, V) = \inf\{||\overline{\partial}\tilde{V}||_\infty\},$$

where the infimum is taken over all continuous extensions $\tilde{H}$ of the map h in (10) with $\tilde{V} = \tilde{H}'(0)$.

From this definition it is obvious that for any point τ in $\mathbb{K}$ and any tangent vector V at τ,

$$Teich_{\mathbb{K}}(\tau, V) \leq \Sigma_{\mathbb{K}}(\tau, V) \leq Kob_{\mathbb{K}}(\tau, V).$$

We also have a proposition for σ parallel to Propositions 1 and 2 for ρ and λ.

Proposition 3. *The Σ-density $\Sigma_{\mathbb{K}}([f], V)$ on $\mathbb{K}$ is related to the σ-density on R_j by the formula:*

$$\sigma(F([f])|dF(V)| = \Sigma_{\mathbb{K}}([f], V),$$

where $F : \mathbb{K} \to R_j$ is the evaluation map $F([f]) = f(p)$ in Figure 1.

An alternative technique for showing that $\lambda(p) < \rho(p)$ relies on two observations. First, just as in the proof of Theorem 10, a vector field V equal to 1 at p and 0 on the boundary for which $\lambda(p) = ||\overline{\partial} V||_\infty$ is unique and its $\overline{\partial}$-derivative must have Teichmüller form, that is, $\overline{\partial} V$ must be equal to

$$k\lambda(p)\frac{|\varphi_p|}{\varphi_p},$$

for some integrable holomorphic quadratic differential φ_p and some number k, with $|k| = 1$. Therefore, if we can find a vector field V with $V(p) = 1$ such that $||\overline{\partial} V||_\infty \leq \rho(p)$ and such that $\overline{\partial} V$ is not of Teichmüller form, then necessarily

$$\lambda(p) < \rho(p).$$

The second observation is that to find such a vector field V, instead of finding a formula for $V(z)$ on the whole plane as in the proof of Theorem 10, it suffices to find a holomorphic motion $H(t, z)$ that has the following properties:

(a) $H(t, z)$ extends $h(t, z)$,
(b) $H(t, z)$ is holomorphic for $|t| < 1$,
(c) $H(t, z)$ is defined and holomorphic for z in some nonempty open neighborhood U contained in D, and
(d) for any $|t| < 1$, $H(t, z)$ is injective in the variable z on its domain of definition.

If $H(t, z)$ has these properties, then by Slodkowski's theorem it extends to a holomorphic motion $\tilde{H}(t, z)$ of $\overline{\mathbb{C}}$ and putting

$$V = \frac{\partial}{\partial \overline{z}}\left(\frac{d}{dt}\Big|_{t=0}\tilde{H}(t, z)\right)$$

by Schwarz's lemma $\|\overline{\partial} V\|_\infty \leq 1$. If we can do this, then since by part (c) V is holomorphic in U, V cannot be of Teichmüller form and therefore we conclude that $\lambda(p)$ is strictly less than $\rho(p)$.

To apply this approach to the punctured disc $D = \Delta' = \{z : 0 < |z| < 1\}$ we create the partial extension $H(t, z)$ by decomposing the covering map $\Pi : \Delta \to \Delta'$ in the following way. Let $T(w)$ be the Möbius transformation

$$z = T(w) = \frac{w - i}{w + i}$$

that transforms the upper half-plane onto the unit disc and let $f = \Pi \circ T$.

By normalizing Π we may assume that f is the universal covering from the upper half-plane given by the formula

$$f(w) = \exp(2\pi i w/c),$$

where $c > 0$ and the covering group is the cyclic group generated by the translation $w \mapsto w + c$. Since $f(i) = p$ we have $c = 2\pi/\log(1/p)$. Let W be a small open set with $p \in W$. Let g be a branch of the inverse of f for which $g(p) = i$ and let U be the intersection of W with $\{w : |w| \leq p\}$. The formula for $g(z)$ in U is

$$g(z) = \frac{c}{2\pi i} \log z.$$

Now we define a partial extension of the motion $h(t, z)$ given in (10) by

$$h(t,z) = \begin{cases} f(g(z) - i + T^{-1}(t)) & \text{for } z \in U, \\ z & \text{for } |z| \geq 1 \text{ and for } z = 0. \end{cases}$$

Then $h(0, w) = f(g(w)) = w$ for all $w \in U$ and the value of $h(t, z)$ remains the same if we replace $g(z)$ by another inverse branch of f. Also h is holomorphic in t and

$$h(t,p) = f(g(p) - i + T^{-1}(t)) = f(T^{-1}(t)) = \Pi(t).$$

To check that h is injective in z assume

$$f(g(z_1) - i + T^{-1}(t)) = f(g(z_2) - i + T^{-1}(t)).$$

That implies there is an integer n such that

$$g(z_1) - i + T^{-1}(t) = g(z_2) - i + T^{-1}(t) + nc.$$

Therefore $g(z_1) = g(z_2)$ for z_1 and z_2 in U and since g is injective on U, that implies $z_1 = z_2$.

By Slodkowski's extension theorem this h can be extended to a motion of the whole plane. Any such extension will have tangent vector V whose $\overline{\partial}$-derivative vanishes on U. Thus V cannot be the extremal vector field that realizes the infimum in the definition of $\lambda(p)$ and so $\lambda(p)$ is strictly less than $\rho(p)$.

In general we can prove the following theorem.

Theorem 11. *If D is a simply connected domain, then*

$$\lambda_D(p) = \sigma_D(p) = \rho_D(p)/2,$$

if D is conformal to $\mathbb{C} \setminus \{0, 1\}$, then

$$\lambda_D(p) = \sigma_D(p) = \rho_D(p),$$

and for any other hyperbolic connected plane domain D,

$$(1/2)\rho_D(p) < \lambda_D(p) \leq \sigma_D(p) \leq \rho_D(p).$$

Proof. Since these metrics are conformally invariant, to prove the first statement we only have to show that $\sigma_\Delta(0) \leq \rho_\Delta(0)/2$, where Δ is the open unit disc. This inequality is realized by considering the holomorphic motion defined for $|t| < 1$ by

$$h(t,z) = \begin{cases} z + t(1 - |z|) & \text{for } |z| \leq 1, \\ z & \text{for } |z| \geq 1. \end{cases} \tag{36}$$

Note that $h(t, z)$ is continuous, normalized to fix three points for all $|t| < 1$ and fixes points z in the exterior of the disc. To see that $z \mapsto h(t, z)$ is injective, first observe that for $|z| \leq 1$, $|h(t, z)| \leq |z| + |t|(1 - |z|) \leq |z| + (1 - |z|) = 1$. Secondly, if $h(t, z_1) = h(t, z_2)$ and $|z_1|$ and $|z_2|$ are less than 1, then

$$z_1 - z_2 = t(|z_1| - |z_2|).$$

If z_1 is not equal to z_2 and $|t| < 1$, this implies

$$|z_1 - z_2| < ||z_1| - |z_2||,$$

which is impossible, and we conclude that $z_1 = z_2$.

To summarize, $h(t, z)$ is a holomorphic motion that fixes the boundary of the unit disc and $h(t, 0) = t$.

Finally, for z in the interior of the unit circle

$$V = \frac{d}{dt}\Big|_{t=0} h(t, z) = 1 - |z| = 1 - z^{1/2}\overline{z}^{1/2},$$

and so

$$\overline{\partial} V = -(1/2)\frac{z^{1/2}}{\overline{z}^{-1/2}}$$

and the maximum value of $|\overline{\partial} V|$ in the unit disc is $1/2$, which implies $\sigma(p) = (1/2)\rho(p)$.

The second statement of the theorem follows since we have already shown that in the case that $D = \mathbb{C} \setminus \{0, 1\}$, $\lambda(p) = \rho(p)$, and the third statement is a consequence of Theorem 9. □

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