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LARGE AFFINE SPACES OF NON-SINGULAR MATRICES

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Abstract. Let $\mathbb{K}$ be an arbitrary (commutative) field with at least three elements. It was recently proven that an affine subspace of $\mathrm{M}_n(\mathbb{K})$ consisting only of non-singular matrices must have a dimension less than or equal to $\binom{n}{2}$. Here, we classify, up to equivalence, the subspaces whose dimension equals $\binom{n}{2}$. This is done by classifying, up to similarity, all the $\binom{n}{2}$-dimensional linear subspaces of $\mathrm{M}_n(\mathbb{K})$ consisting of matrices with no non-zero invariant vector, reinforcing a classical theorem of Gerstenhaber. Both classifications only involve the quadratic structure of the field $\mathbb{K}$.

1. Introduction

1.1. **Introduction and basic definitions.** In this article, we let $\mathbb{K}$ be an arbitrary (commutative) field. We denote by $\mathrm{M}_n(\mathbb{K})$ the algebra of square matrices with n rows and entries in $\mathbb{K}$, and by $\mathrm{GL}_n(\mathbb{K})$ its group of invertible elements. We also denote by $\mathrm{M}_{n,p}(\mathbb{K})$ the vector space of matrices with n rows, p columns and entries in $\mathbb{K}$. The transpose of a matrix M is denoted by M^T.

An **affine** subspace $\mathcal{V}$ of $\mathrm{M}_n(\mathbb{K})$ is the translate of a linear subspace V of $\mathrm{M}_n(\mathbb{K})$: then V is uniquely determined by $\mathcal{V}$ (it is the set of all matrices M such that $M + \mathcal{V} = \mathcal{V}$) and is called the **translation vector space** of $\mathcal{V}$.

Given two linear (or affine) subspaces V and W of $\mathrm{M}_n(\mathbb{K})$, we say that V and W are equivalent, and we write $V \sim W$, if $W = PVQ$ for some $(P, Q) \in \mathrm{GL}_n(\mathbb{K})^2$; we say that V and W are similar, and we write $V \simeq W$, if $W = PVP^{-1}$ for some $P \in \mathrm{GL}_n(\mathbb{K})$.

Two matrices A and B of $\mathrm{M}_n(\mathbb{K})$ are called **congruent**, and we write $A \approx B$, if $A = PBP^T$ for some $P \in \mathrm{GL}_n(\mathbb{K})$. Finally, two quadratic forms q and q' on vector spaces over $\mathbb{K}$ are called **similar** if q' is equivalent to $\lambda\, q$ for some $\lambda \in \mathbb{K} \smallsetminus \{0\}$.

Here, we are concerned with the geometry of $\mathrm{GL}_n(\mathbb{K}) \cup \{0\}$ as a cone in the vector space $\mathrm{M}_n(\mathbb{K})$.

From the linear algebraist's viewpoint, the natural questions that one may ask are the following ones:

- What is the minimal linear (resp. affine) subspace of $\mathrm{M}_n(\mathbb{K})$ containing $\mathrm{GL}_n(\mathbb{K})$?
- What is the minimal linear subspace of $\mathrm{M}_n(\mathbb{K})$ containing $\mathrm{M}_n(\mathbb{K}) \smallsetminus \mathrm{GL}_n(\mathbb{K})$?
- What are the maximal linear (resp. affine) subspaces included in $\mathrm{GL}_n(\mathbb{K}) \cup \{0\}$?

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- What are the maximal linear (resp. affine) subspaces included in $\mathrm{M}_n(\mathbb{K}) \smallsetminus \mathrm{GL}_n(\mathbb{K})$?

The first two problems have easy answers: $\mathrm{GL}_n(\mathbb{K})$ always spans $\mathrm{M}_n(\mathbb{K})$, the affine subspace it generates is $\mathrm{M}_n(\mathbb{K})$ unless $n = 1$ and $\#\,\mathbb{K} = 2$, and $\mathrm{M}_n(\mathbb{K}) \smallsetminus \mathrm{GL}_n(\mathbb{K})$ spans $\mathrm{M}_n(\mathbb{K})$ unless $n = 1$.

The last two questions have no clear answer however and depend widely on the field $\mathbb{K}$. For example, $\mathrm{GL}_n(\mathbb{C}) \cup \{0\}$ contains no 2-dimensional linear subspace, while $\mathrm{GL}_{2n}(\mathbb{R}) \cup \{0\}$ always does. As for singular linear subspaces (i.e. linear subspaces included in $\mathrm{M}_n(\mathbb{K}) \smallsetminus \mathrm{GL}_n(\mathbb{K})$), a classification of them is generally considered to be out of reach, even for an algebraically closed field, although much progress was made in the 1980s in understanding their structure (see the works of Atkinson, Lloyd and Stephens [1, 2, 3, 4] and our recent [12]).

Rather than try to classify all the linear (resp. affine) subspaces contained in $\mathrm{GL}_n(\mathbb{K})$ or $\mathrm{M}_n(\mathbb{K}) \smallsetminus \mathrm{GL}_n(\mathbb{K})$, a more modest approach is to find the maximal dimension for such a subspace and to classify the linear (resp. affine) subspaces with a maximal *dimension*. To this day, this problem has been almost entirely solved:

- A linear subspace included in $\mathrm{GL}_n(\mathbb{K}) \cup \{0\}$ has dimension at most n; linear subspaces in $\mathrm{GL}_n(\mathbb{K}) \cup \{0\}$ with dimension n correspond to the structures of (possibly non-associative and non-unital) division algebras on $\mathbb{K}^n$ that are compatible with its vector space structure (see e.g. the last section of [13]). Note that no such subspace exists when $n \geq 2$ and $\mathbb{K}$ is algebraically closed.
- An affine subspace included in $\mathrm{M}_n(\mathbb{K}) \smallsetminus \mathrm{GL}_n(\mathbb{K})$ has dimension at most $n(n-1)$. If its dimension is $n(n-1)$, then it is equivalent to the space of matrices with zero as last column or to its transpose (unless $n = 2$ and $\#\,\mathbb{K} = 2$, in which case there is an additional equivalence class). This is a classical result of Dieudonné [5] (see also [11] for a simplified proof) which may be used to classify the endomorphisms of the vector space $\mathrm{M}_n(\mathbb{K})$ that stabilize $\mathrm{GL}_n(\mathbb{K})$ (see [13]).

Here, we will focus on the affine subspaces of $\mathrm{M}_n(\mathbb{K})$ that are included in $\mathrm{GL}_n(\mathbb{K})$. Let $\mathcal{V}$ be such a subspace, and choose $P \in \mathcal{V}$. Then $P^{-1}\mathcal{V}$ is also included in $\mathrm{GL}_n(\mathbb{K})$, contains the identity matrix I_n and has the same dimension as $\mathcal{V}$. Denoting by H its translation vector space, we see that $I_n - \lambda M \in \mathrm{GL}_n(\mathbb{K})$ for every $\lambda \in \mathbb{K}$ and $M \in H$; hence the linear subspace H has the following two equivalent properties:

(i) For every $M \in H$, one has $\mathrm{Sp}(M) \subset \{0\}$, where $\mathrm{Sp}(M)$ denotes the set of eigenvalues of M *in the field* $\mathbb{K}$.
(ii) No matrix of H possesses a non-zero invariant vector in $\mathbb{K}^n$.

Definition 1. A linear subspace H of $\mathrm{M}_n(\mathbb{K})$ is said to have a **trivial spectrum** if no matrix of H possesses a non-zero invariant vector in $\mathbb{K}^n$.

Note that for such a linear subspace H with a trivial spectrum, the affine subspace $I_n + H$ is included in $\mathrm{GL}_n(\mathbb{K})$, and so is any subspace equivalent to it. For example, if we denote by $\mathrm{NT}_n(\mathbb{K})$ the space of strictly upper triangular matrices of $\mathrm{M}_n(\mathbb{K})$, then $I_n + \mathrm{NT}_n(\mathbb{K})$ is an affine subspace of non-singular matrices with dimension $\binom{n}{2}$.

It follows that classifying up to equivalence the affine subspaces of non-singular matrices essentially amounts to classifying up to similarity the linear subspaces of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum. When $\mathbb{K}$ is algebraically closed, the linear subspaces with a trivial spectrum are the linear subspaces of nilpotent matrices: a famous theorem of Gerstenhaber [6] states that the dimension of such a subspace is bounded above by $\binom{n}{2}$ and that equality occurs only for subspaces similar to $\mathrm{NT}_n(\mathbb{K})$. It is only very recently that the upper bound $\binom{n}{2}$ has been shown to apply to linear subspaces with a trivial spectrum for an arbitrary field (see the works of Quinlan [8] and our own [10]):

Theorem 1. *Let V be a linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum. Then $\dim V \leq \binom{n}{2}$.*

Definition 2. A linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum is called **maximal**[1] if its dimension is $\binom{n}{2}$.

Our aim is to classify the maximal linear subspaces of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum. Unlike the case of nilpotent linear subspaces, the structure of the ground field $\mathbb{K}$ plays a large part in this classification. For example, if there exists a polynomial $t^2 - at - b \in \mathbb{K}[t]$ with degree two and no root in $\mathbb{K}$, then the line spanned by the companion matrix $\begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}$ is obviously a maximal linear subspace of $\mathrm{M}_2(\mathbb{K})$ with a trivial spectrum and it is not similar to $\mathrm{NT}_2(\mathbb{K})$. Another example is given by the space $\mathrm{A}_n(\mathbb{R})$ of skew-symmetric real matrices, which has a trivial spectrum and dimension $\binom{n}{2}$, although it is not similar to $\mathrm{NT}_n(\mathbb{R})$ if $n \geq 2$.

1.2. **Reducibility.**

Notation 3. Let V and W be respective subsets of $\mathrm{M}_n(\mathbb{K})$ and $\mathrm{M}_p(\mathbb{K})$. Set

$$V \vee W := \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \mid (A, B, C) \in V \times \mathrm{M}_{n,p}(\mathbb{K}) \times W \right\} \subset \mathrm{M}_{n+p}(\mathbb{K}).$$

Note that if V and W are maximal linear subspaces with a trivial spectrum, then $V \vee W$ is a linear subspace with a trivial spectrum and dimension $\binom{n}{2} + \binom{p}{2} + pn = \binom{n+p}{2}$; hence it is maximal. Notice also that the composition law $\vee$ is associative.

Definition 4. A maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum is called **irreducible** if the only linear subspaces of $\mathbb{K}^n$ it stabilizes are $\{0\}$ and $\mathbb{K}^n$ (and we call it reducible otherwise).

Conversely, let H be a maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum. Assume that there is a $p \in [\![1, n-1]\!]$ such that $F := \mathbb{K}^p \times \{0\}$ is stabilized by every matrix of H. Then we may write every matrix of H as

$$M = \begin{bmatrix} f(M) & g(M) \\ 0 & h(M) \end{bmatrix} \quad \text{for some } (f(M), g(M), h(M)) \in \mathrm{M}_p(\mathbb{K}) \times \mathrm{M}_{p,n-p}(\mathbb{K}) \times \mathrm{M}_{n-p}(\mathbb{K}).$$

Therefore $V := f(H)$ and $W := h(H)$ are linear subspaces respectively of $\mathrm{M}_p(\mathbb{K})$ and $\mathrm{M}_{n-p}(\mathbb{K})$, each with a trivial spectrum, and since

$$\binom{n}{2} = \dim H \leq \dim V + \dim W + \dim g(H) \leq \binom{p}{2} + \binom{n-p}{2} + p(n-p) = \binom{n}{2},$$

[1]This should not be confused with the concept of maximality in the set of linear subspaces with a trivial spectrum ordered by the inclusion of subsets.

we find that both V and W are maximal. Hence $H \subset V \vee W$, and since the dimensions are equal, we deduce that $H = V \vee W$.

Conjugating H with an appropriate invertible matrix, this generalizes as follows: if H is not irreducible, then $H \simeq V \vee W$ for some maximal linear subspaces V and W with trivial spectra. This yields:

Proposition 2. *Let H be a maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum. Then there are irreducible maximal linear subspaces $V_1, \dots, V_p$ with trivial spectra such that*

$$H \simeq V_1 \vee V_2 \vee \cdots \vee V_p.$$

This suggests that we focus our attention on the irreducible maximal subspaces.

1.3. **Main theorems.** Denote by $\mathrm{A}_n(\mathbb{K})$ the set of *alternate* matrices of $\mathrm{M}_n(\mathbb{K})$, i.e. the skew-symmetric ones with a zero diagonal, i.e. the ones for which $\forall X \in \mathbb{K}^n,\ X^T A X = 0$.

Definition 5. A matrix $P \in \mathrm{M}_n(\mathbb{K})$ is called **non-isotropic** if the quadratic form $X \mapsto X^T P X$ is non-isotropic, i.e. $\forall X \in \mathbb{K}^n \smallsetminus \{0\},\ X^T P X \neq 0$.

Notice, in that case, that P is non-singular and that P^{-1} is non-isotropic. The subspace $P\,\mathrm{A}_n(\mathbb{K})$ then has dimension $\binom{n}{2}$ and has a trivial spectrum: indeed, given $A \in \mathrm{A}_n(\mathbb{K})$ and $X \in \mathbb{K}^n$,

$$PAX = X \Rightarrow P^{-1}X = AX \Rightarrow X^T P^{-1} X = 0 \Rightarrow X = 0.$$

We may now state our main results.

Theorem 3. *Assume that $\#\mathbb{K} \geq 3$. Let n be a positive integer. Then the irreducible maximal linear subspaces of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum are the subspaces of the form $P\,\mathrm{A}_n(\mathbb{K})$ for a non-isotropic matrix $P \in \mathrm{GL}_n(\mathbb{K})$.*

Theorem 4 (Classification theorem for maximal linear subspaces with a trivial spectrum)**.** *Assume that $\#\mathbb{K} \geq 3$. Let V be a maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum. Then there is a list $(P_1, \dots, P_p) \in \mathrm{GL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{n_p}(\mathbb{K})$ of non-isotropic matrices such that*

$$V \simeq P_1\,\mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee P_p\,\mathrm{A}_{n_p}(\mathbb{K}).$$

The integer p is uniquely determined by V and, for every $k \in [\![1, p]\!]$, the matrix P_k is uniquely determined by V up to congruence and multiplication by a non-zero scalar. Moreover, given another list $(Q_1, \dots, Q_p) \in \mathrm{GL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{n_p}(\mathbb{K})$, if Q_k is congruent to a scalar multiple of P_k for each $k \in [\![1, p]\!]$, then

$$V \simeq Q_1\,\mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee Q_p\,\mathrm{A}_{n_p}(\mathbb{K}).$$

If $\mathbb{K}$ is quadratically closed, it follows that there is no irreducible maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum for $n \geq 2$. If $\mathbb{K}$ is finite (with at least three elements), then every 3-dimensional quadratic form over $\mathbb{K}$ is isotropic; hence $\mathrm{M}_n(\mathbb{K})$ contains no irreducible maximal linear subspace with a trivial spectrum for $n \geq 3$. We deduce the following corollaries:

Corollary 5. *Let $\mathbb{K}$ be a quadratically closed field. Then $\mathrm{NT}_n(\mathbb{K})$ is, up to similarity, the sole maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum.*

Corollary 6. *Let $\mathbb{K}$ be a finite field with at least three elements. Let V be a maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum.*

Then there are matrices $M_1, \dots, M_p$, either equal to $0 \in \mathrm{M}_1(\mathbb{K})$ or belonging to $\mathrm{M}_2(\mathbb{K})$ with no eigenvalue in $\mathbb{K}$, such that

$$V \simeq \mathbb{K}\, M_1 \vee \cdots \vee \mathbb{K}\, M_p.$$

Each M_k is then uniquely determined by V up to similarity and multiplication by a non-zero scalar.

We may finally state the structure theorem for affine subspaces of non-singular matrices.

Theorem 7 (Classification theorem for large affine subspaces of non-singular matrices)**.** *Assume that $\#\mathbb{K} \geq 3$. Let $\mathcal{V}$ be an $\binom{n}{2}$-dimensional affine subspace of $\mathrm{M}_n(\mathbb{K})$ included in $\mathrm{GL}_n(\mathbb{K})$. Then there is a list $(P_1, \dots, P_p) \in \mathrm{GL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{n_p}(\mathbb{K})$ of non-isotropic matrices such that $n = n_1 + \cdots + n_p$ and*

$$\mathcal{V} \sim I_n + \big(P_1\, \mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee P_p\, \mathrm{A}_{n_p}(\mathbb{K})\big).$$

The integer p is uniquely determined by $\mathcal{V}$ and, for $1 \leq k \leq p$, the similarity class of the non-isotropic quadratic form $X \mapsto X^T P_k X$ is uniquely determined by $\mathcal{V}$. Moreover, given another list $(Q_1, \dots, Q_p) \in \mathrm{GL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{n_p}(\mathbb{K})$, if $X \mapsto X^T Q_k X$ is similar to $X \mapsto X^T P_k X$ for each $k \in [\![1, p]\!]$, then

$$\mathcal{V} \sim I_n + \big(Q_1\, \mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee Q_p\, \mathrm{A}_{n_p}(\mathbb{K})\big).$$

Note that the existence of $(P_1, \dots, P_p)$ is a trivial consequence of Theorem 4 using the considerations of Paragraph 1.1.

As a consequence, $\binom{n}{2}$-dimensional affine subspaces of $\mathrm{M}_n(\mathbb{K})$ included in $\mathrm{GL}_n(\mathbb{K})$ are classified, up to equivalence, by the lists of the form $([\varphi_1], \dots, [\varphi_p])$, where the φ_k's are finite-dimensional non-isotropic quadratic forms over $\mathbb{K}$, the $[\varphi_k]$'s are their similarity classes, and $\sum_{k=1}^{p} \dim \varphi_k = n$. For the field of real numbers, this has the following striking corollary:

Corollary 8. *Let $\mathcal{V}$ be an affine subspace of $\mathrm{M}_n(\mathbb{R})$ included in $\mathrm{GL}_n(\mathbb{R})$ with dimension $\binom{n}{2}$. Then there is a unique list $(n_1, \dots, n_p)$ of positive integers such that $n = n_1 + \cdots + n_p$ and*

$$\mathcal{V} \sim I_n + \big(\mathrm{A}_{n_1}(\mathbb{R}) \vee \cdots \vee \mathrm{A}_{n_p}(\mathbb{R})\big).$$

1.4. **Totally intransitive action of a space of matrices.** Proving the previous theorems will require an extensive use of the following concept and of the subsequent remark:

Definition 6. Let V be a linear subspace of $\mathrm{M}_n(\mathbb{K})$. For $X \in \mathbb{K}^n$, set

$$VX := \big\{MX \mid X \in V\big\}.$$

Note that VX is always a linear subspace of $\mathbb{K}^n$.

We say that V acts **totally intransitively** on $\mathbb{K}^n$ if $VX \neq \mathbb{K}^n$ for every $X \in \mathbb{K}^n$, which is equivalent to having $\dim(VX) < n$ for every $X \in \mathbb{K}^n$.

Remark 1. If V has a trivial spectrum, then $X \notin VX$ for every $X \in \mathbb{K}^n \smallsetminus \{0\}$; hence V acts totally intransitively on $\mathbb{K}^n$.

Moreover $V^T := \big\{M^T \mid M \in V\big\}$ also has a trivial spectrum; hence

$$\forall X \in \mathbb{K}^n, \quad \dim(VX) < n \quad \text{and} \quad \dim(V^T X) < n.$$

1.5. **Structure of the paper.** We will start (Section 2) with general considerations on the spaces of the type $P\,\mathrm{A}_n(\mathbb{K})$ with $P \in \mathrm{GL}_n(\mathbb{K})$. Using some of the obtained results, we will then prove the uniqueness statements in Theorems 4 and 7 (Section 3). The proof of Theorem 3 will be carried out in Section 4 by induction on n, starting from $n = 2$ and using a recent lemma that was proved in [10]: this is, by far, the most technical part of the paper. In Section 5, we will easily derive Gerstenhaber's theorem from Theorem 4 in the case $\#\mathbb{K} \geq 3$. In Section 6, we will show that Theorem 3 fails for $n = 3$ and $\mathbb{K} \simeq \mathbb{F}_2$. The case $\#\mathbb{K} = 2$ remains a very exciting challenge that we will not undertake here.

2. Basic properties of the spaces $P\,\mathrm{A}_n(\mathbb{K})$

First we consider $P\,\mathrm{A}_n(\mathbb{K})$ for an arbitrary $P \in \mathrm{GL}_n(\mathbb{K})$. To start with, note that, for every $Q \in \mathrm{GL}_n(\mathbb{K})$, one has

$$P\,\mathrm{A}_n(\mathbb{K})\,Q = P(Q^T)^{-1}Q^T\,\mathrm{A}_n(\mathbb{K})\,Q = \big(P(Q^T)^{-1}\big)\,\mathrm{A}_n(\mathbb{K}),$$

which immediately shows that $\{P\,\mathrm{A}_n(\mathbb{K}) \mid P \in \mathrm{GL}_n(\mathbb{K})\}$ is an equivalence class (for the equivalence of spaces of matrices).

In order to move forward, we need some basic properties of $\mathrm{A}_n(\mathbb{K})$: for this, we equip $\mathbb{K}^n$ with the non-degenerate symmetric bilinear form $(X, Y) \mapsto X^TY$.

Lemma 9. *For any $X \in \mathbb{K}^n \smallsetminus \{0\}$, one has*

$$\mathrm{A}_n(\mathbb{K})X = \{X\}^\perp$$

and in particular $\dim(\mathrm{A}_n(\mathbb{K})X) = n - 1$.

Proof. This is obvious if X is the first vector e_1 of the canonical basis of $\mathbb{K}^n$. In the general case, we may find some $P \in \mathrm{GL}_n(\mathbb{K})$ such that $Pe_1 = X$, and note that

$$\begin{aligned}\mathrm{A}_n(\mathbb{K})X = (P^T)^{-1}P^T\,\mathrm{A}_n(\mathbb{K})Pe_1 &= (P^T)^{-1}\,\mathrm{A}_n(\mathbb{K})e_1\\ &= (P^T)^{-1}\{e_1\}^\perp = \{Pe_1\}^\perp = \{X\}^\perp.\end{aligned}$$

□

We may now determine, amongst the spaces of the above form, those with a trivial spectrum:

Lemma 10. *Let $P \in \mathrm{GL}_n(\mathbb{K})$. Then $P\,\mathrm{A}_n(\mathbb{K})$ has a trivial spectrum if and only if P is non-isotropic.*

Proof. The "if" part has already been dealt with in the beginning of Section 1.3. Assume that P is isotropic. Then obviously $(P^T)^{-1}$ is also isotropic; hence we find a non-zero vector $X \in \mathbb{K}^n$ such that $X^T(P^T)^{-1}X = 0$, i.e. $P^{-1}X \in \{X\}^\perp$. Then Lemma 9 shows that $P^{-1}X = AX$ for some $A \in \mathrm{A}_n(\mathbb{K})$; hence $(PA)X = X$, which shows that $P\,\mathrm{A}_n(\mathbb{K})$ does not have a trivial spectrum. □

Proposition 11. *Let $P \in \mathrm{GL}_n(\mathbb{K})$ be a non-isotropic matrix. Then $P\,\mathrm{A}_n(\mathbb{K})$ is an irreducible maximal subspace with a trivial spectrum.*

Proof. It only remains to show that $P\,\mathrm{A}_n(\mathbb{K})$ is irreducible. We use a *reductio ad absurdum* by assuming that it has a non-trivial stable subspace $F \subset \mathbb{K}^n$ with dimension $p \in [\![1, n-1]\!]$. Then $F^\perp$ is stabilized by $(P\,\mathrm{A}_n(\mathbb{K}))^T = \mathrm{A}_n(\mathbb{K})P^T$. Choosing an arbitrary non-zero vector $X \in F$, we have $\dim(P\,\mathrm{A}_n(\mathbb{K})X) = \dim\{X\}^\perp = n-1$; hence $p = n - 1$.

However, choosing a non-zero vector $Y \in F^{\perp}$ yields $\dim(\mathrm{A}_n(\mathbb{K})P^T Y) = n-1$; hence $n-p = n-1$. This yields $n = 2$ and $p = 1$, in which case every matrix of $P\,\mathrm{A}_n(\mathbb{K})$ must be nilpotent (since it has an eigenvector and 0 is the sole possible eigenvalue in $\mathbb{K}$), contradicting the fact that every non-zero matrix of $P\,\mathrm{A}_2(\mathbb{K})$ is non-singular. □

We now investigate when two spaces of the form $P\,\mathrm{A}_n(\mathbb{K})$ are similar. Here is our basic result:

Lemma 12. *Let $P \in \mathrm{GL}_n(\mathbb{K})$. Then $P\,\mathrm{A}_n(\mathbb{K}) = \mathrm{A}_n(\mathbb{K})$ if and only if P is a scalar multiple of the identity.*

Proof. The "if" part is trivial. Assume conversely that $P\,\mathrm{A}_n(\mathbb{K}) = \mathrm{A}_n(\mathbb{K})$. Let $X \in \mathbb{K}^n \smallsetminus \{0\}$. Then $P\,\mathrm{A}_n(\mathbb{K})X = \mathrm{A}_n(\mathbb{K})X$ yields that P stabilizes the hyperplane $\{X\}^{\perp}$; hence P^T stabilizes $\mathrm{span}(X)$. Since this holds for every non-zero $X \in \mathbb{K}^n$, this shows that P^T is a scalar multiple of the identity; hence P is also. □

The following corollary will be our starting point for the uniqueness statement in Theorem 4:

Proposition 13. *Let $(P, Q) \in \mathrm{GL}_n(\mathbb{K})^2$. Then $P\,\mathrm{A}_n(\mathbb{K}) \simeq Q\,\mathrm{A}_n(\mathbb{K})$ if and only if $P \approx \lambda Q$ for some $\lambda \in \mathbb{K} \smallsetminus \{0\}$.*

Proof. If $P = \lambda\, RQR^T$ for some $R \in \mathrm{GL}_n(\mathbb{K})$ and some $\lambda \in \mathbb{K} \smallsetminus \{0\}$, then

$$P\,\mathrm{A}_n(\mathbb{K}) = RQR^T\,\mathrm{A}_n(\mathbb{K}) = R(QR^T\,\mathrm{A}_n(\mathbb{K})R)R^{-1} = R(Q\,\mathrm{A}_n(\mathbb{K}))R^{-1}.$$

Conversely, assume that $P\,\mathrm{A}_n(\mathbb{K}) = R(Q\,\mathrm{A}_n(\mathbb{K}))R^{-1}$ for some $R \in \mathrm{GL}_n(\mathbb{K})$. Then the above computation yields $(RQR^T)^{-1}P\,\mathrm{A}_n(\mathbb{K}) = \mathrm{A}_n(\mathbb{K})$; hence Lemma 12 yields a non-zero scalar λ such that $(RQR^T)^{-1}P = \lambda I_n$. Therefore $P = R(\lambda\, Q)R^T$. □

Remark 2 (A crucial remark). Let E be a finite-dimensional vector space and b a (possibly non-symmetric) bilinear form on E such that $\forall x \in E \smallsetminus \{0\},\ b(x,x) \neq 0$. Given a non-zero vector $x \in E$, the hyperplane $H := \{y \in E :\ b(x,y) = 0\}$ is then a complementary subspace of $\mathrm{span}(x)$ in E. By induction on the dimension of spaces, it follows that there exists a basis $(f_1, \ldots, f_n)$ of E which is *right-orthogonal* for b, i.e. $b(f_i, f_j) = 0$ for every $(i,j) \in [\![1,n]\!]^2$ satisfying $i < j$.

For a non-isotropic matrix $P \in \mathrm{GL}_n(\mathbb{K})$, this may be interpreted as follows: P is congruent to a lower-triangular matrix T, and hence $P\,\mathrm{A}_n(\mathbb{K})$ is similar to $T\,\mathrm{A}_n(\mathbb{K})$. This remark will play a major part in our proof of Theorem 3.

Now, given non-isotropic matrices P and Q of $\mathrm{GL}_n(\mathbb{K})$, we may examine when the two affine subspaces $I_n + P\,\mathrm{A}_n(\mathbb{K})$ and $I_n + Q\,\mathrm{A}_n(\mathbb{K})$ are equivalent.

Proposition 14. *Let P and Q be non-isotropic matrices of $\mathrm{GL}_n(\mathbb{K})$.*

Then $I_n + P\,\mathrm{A}_n(\mathbb{K}) \sim I_n + Q\,\mathrm{A}_n(\mathbb{K})$ if and only if the quadratic forms $X \mapsto X^T PX$ and $X \mapsto X^T QX$ are similar.

Proof.

- Assume first that $I_n + P\,\mathrm{A}_n(\mathbb{K}) \sim I_n + Q\,\mathrm{A}_n(\mathbb{K})$, and choose a pair $(R,S) \in \mathrm{GL}_n(\mathbb{K})^2$ such that $R(I_n + P\,\mathrm{A}_n(\mathbb{K})) = (I_n + Q\,\mathrm{A}_n(\mathbb{K}))S$. Obviously S belongs to $(I_n + Q\,\mathrm{A}_n(\mathbb{K}))S$; hence $S = R(I_n + PA)$ for some $A \in \mathrm{A}_n(\mathbb{K})$. By comparing the translation vector spaces of $R(I_n + P\,\mathrm{A}_n(\mathbb{K}))$ and $(I_n + Q\,\mathrm{A}_n(\mathbb{K}))S$, we also find that $RP\,\mathrm{A}_n(\mathbb{K}) = Q\,\mathrm{A}_n(\mathbb{K})S = Q(S^T)^{-1}\,\mathrm{A}_n(\mathbb{K})$. Therefore Proposition 13 yields a non-zero

scalar λ such that $RP = \lambda\, Q(S^T)^{-1}$. It follows that $S^T = (I_n - AP^T)R^T$ and

$$\lambda\, Q = RPS^T = RP(I_n - AP^T)R^T = RPR^T - (RP)A(RP)^T.$$

Since A is alternate, we find $\lambda\, X^TQX = X^T(RPR^T)X = (R^TX)^TP(R^TX)$ for every $X \in \mathbb{K}^n$, and the quadratic forms $X \mapsto X^TQX$ and $X \mapsto X^TPX$ are similar because R^T is non-singular.

- Conversely, assume that $X \mapsto X^TQX$ and $X \mapsto X^TPX$ are similar. Then there is a non-singular matrix $R \in \mathrm{GL}_n(\mathbb{K})$, a non-zero scalar λ and an alternate matrix A' such that $\lambda Q = RPR^T + A'$. The matrix $A := -(RP)^{-1}A'((RP)^T)^{-1}$ is congruent to $-A'$ and is therefore alternate. We set $S := R(I_n + PA)$. Note that $S = RP(P^{-1} + A)$ is non-singular: indeed, $\forall X \in \mathbb{K}^n \smallsetminus \{0\},\ X^T(P^{-1} + A)X = X^TP^{-1}X \neq 0$ since P^{-1} is non-isotropic; hence $P^{-1} + A$ is non-singular.

  However, $S^T = (I_n - AP^T)R^T$; therefore

  $$RPS^T = RPR^T - (RP)A(RP)^T = RPR^T + A' = \lambda\, Q.$$

  We deduce that

  $$R\big(P\,\mathrm{A}_n(\mathbb{K})\big) = \lambda\, Q(S^T)^{-1}\,\mathrm{A}_n(\mathbb{K}) = \big(Q\,\mathrm{A}_n(\mathbb{K})\big)S.$$

  We have just proven that the affine subspaces $R(I_n + P\,\mathrm{A}_n(\mathbb{K}))$ and $(I_n + Q\,\mathrm{A}_n(\mathbb{K}))S$ have S as a common point and have the same translation vector space; hence they are equal. This yields $I_n + P\,\mathrm{A}_n(\mathbb{K}) \sim I_n + Q\,\mathrm{A}_n(\mathbb{K})$.

□

Finally, the following lemma will be a major key to unlock our proof of Theorem 3:

Lemma 15. *Let $n \geq 3$. Assume $\#\mathbb{K} \geq 3$. Let V be an $\binom{n}{2}$-dimensional linear subspace of $\mathrm{M}_n(\mathbb{K})$ which acts totally intransitively on $\mathbb{K}^n$.*

Assume that there exists a linear hyperplane H of V such that $H \subset \mathrm{A}_n(\mathbb{K})$. Then $V = \mathrm{A}_n(\mathbb{K})$.

Proof. Let $A \in V$. We prove that A is alternate, i.e. that the quadratic form $q : X \mapsto X^TAX$ is zero. We denote by $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{K}^n$.

Let $X \in \mathbb{K}^n \smallsetminus \{0\}$. If $\dim(HX) = n-1$, then $AX \in HX$ since $HX \subset VX \subsetneq \mathbb{K}^n$, and hence $q(X) = 0$.

If $\dim(HX) = n - 1$ for every $X \in \mathbb{K}^n \smallsetminus \{0\}$, then we readily have $q = 0$.

Assume now that $\dim(HX_1) < n - 1$ for some $X_1 \in \mathbb{K}^n \smallsetminus \{0\}$.

This shows that there exists $X_2 \in \mathbb{K}^n \smallsetminus \mathrm{span}(X_1)$ such that $X_2^TMX_1 = 0$ for every $M \in H$. Let $X_3 \in \mathbb{K}^n \smallsetminus \mathrm{span}(X_1, X_2)$. We may choose a non-singular matrix $P \in \mathrm{GL}_n(\mathbb{K})$ such that $Pe_i = X_i$ for every $i \in [\![1, 3]\!]$.

Then $V' := P^TVP$ acts totally intransitively on $\mathbb{K}^n$ and contains the hyperplane $H' := P^THP \subset \mathrm{A}_n(\mathbb{K})$. We now have $e_2^TMe_1 = 0$ for every $M \in H'$; hence H' is included in the space V_1 of all alternate matrices $A = (a_{i,j})$ of $\mathrm{M}_n(\mathbb{K})$ such that $a_{2,1} = 0$. The dimension of this space is obviously $\binom{n}{2} - 1$, and therefore $H' = V_1$. Then it is obvious that $\dim(H'e_3) = n - 1$ and hence $\dim(HX_3) = n - 1$.

We have therefore proven that

$$\forall X \in \mathbb{K}^n \smallsetminus \mathrm{span}(X_1, X_2),\ q(X) = 0.$$

It now suffices to show that q vanishes everywhere on $\mathrm{span}(X_1, X_2)$.

Let $X \in \operatorname{span}(X_1, X_2) \smallsetminus \{0\}$. We choose an arbitrary vector $X_3 \in \mathbb{K}^n \smallsetminus \operatorname{span}(X_1, X_2)$. The plane $\operatorname{span}(X, X_3)$ satisfies $\operatorname{span}(X, X_3) \cap \operatorname{span}(X_1, X_2) = \operatorname{span}(X)$. Since $\#\mathbb{K} > 2$, this plane has at least four distinct 1-dimensional subspaces, three of which are different from $\operatorname{span}(X)$. We deduce that the quadratic form q vanishes on at least three 1-dimensional subspaces of $\operatorname{span}(X, X_3)$. Classically, this shows that q vanishes everywhere on $\operatorname{span}(X, X_3)$ (indeed, a non-zero homogeneous polynomial of degree 2 on $\mathbb{K}^2$ has at most 2 zeroes in the projective line $\mathbb{P}(\mathbb{K}^2)$). In particular $q(X) = 0$. We deduce that $q = 0$, which completes our proof. $\square$

3. The uniqueness statement in the two classification theorems

The uniqueness statement in Theorem 4 is equivalent to the following result, which we prove right away:

Proposition 16. *Let $(P_1, \dots, P_p)$ and $(Q_1, \dots, Q_q)$ be two families of non-isotropic matrices, respectively of $\mathrm{GL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{n_p}(\mathbb{K})$ and $\mathrm{GL}_{m_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{m_q}(\mathbb{K})$. In order that*

$$P_1 \,\mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee P_p \,\mathrm{A}_{n_p}(\mathbb{K}) \simeq Q_1 \,\mathrm{A}_{m_1}(\mathbb{K}) \vee \cdots \vee Q_q \,\mathrm{A}_{m_q}(\mathbb{K}),$$

it is necessary and sufficient that $q = p$ and that P_k be congruent to a scalar multiple of Q_k for every $k \in [\![1, p]\!]$.

Proof. The "sufficient condition" statement follows immediately from Proposition 13. For the converse statement, set $V := P_1 \,\mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee P_p \,\mathrm{A}_{n_p}(\mathbb{K})$ and $W := Q_1 \,\mathrm{A}_{m_1}(\mathbb{K}) \vee \cdots \vee Q_q \,\mathrm{A}_{m_q}(\mathbb{K})$.

For $k \in [\![1, p]\!]$, set $F_k := \mathbb{K}^{n_1 + \cdots + n_k} \times \{0\} \subset \mathbb{K}^n$, where $n := n_1 + \cdots + n_p$. Also set $F_0 = \{0\}$ and denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{K}^n$. Set $k \in [\![1, p]\!]$. Our key statement is the set of equalities:

$$\forall X \in F_k \smallsetminus F_{k-1}, \quad \dim(VX) = n_1 + \cdots + n_k - 1.$$

Note first that the case $X = e_{n_1 + \cdots + n_{k-1} + 1}$ follows trivially from Lemma 9.

Now consider an arbitrary vector $X \in F_k \smallsetminus F_{k-1}$. Then $e_1, \dots, e_{n_1 + \cdots + n_{k-1}}, X$ are linearly independent, and may therefore be completed as a basis $(e_1, \dots, e_{n_1 + \cdots + n_{k-1}}, X, f_2, \dots, f_{n_k})$ of F_k. Therefore

$$\mathcal{B} := (e_1, \dots, e_{n_1 + \cdots + n_{k-1}}, X, f_2, \dots, f_{n_k}, e_{n_1 + \cdots + n_k + 1}, \dots, e_n)$$

is a basis of $\mathbb{K}^n$ and the matrix of coordinates R of $\mathcal{B}$ in the canonical basis of $\mathbb{K}^n$ belongs to $\mathrm{GL}_{n_1}(\mathbb{K}) \vee \cdots \vee \mathrm{GL}_{n_p}(\mathbb{K})$ and satisfies $Re_{n_1 + \cdots + n_{k-1} + 1} = X$. This yields a list of non-isotropic matrices $(P'_1, \dots, P'_p) \in \mathrm{GL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{n_p}(\mathbb{K})$ for which

$$R^{-1} V R \subset P'_1 \,\mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee P'_p \,\mathrm{A}_{n_p}(\mathbb{K})$$

and therefore $R^{-1} V R = P'_1 \,\mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee P'_p \,\mathrm{A}_{n_p}(\mathbb{K})$ as the dimensions equal $\binom{n}{2}$ on both sides. Applying the special case of $e_{n_1 + \cdots + n_{k-1} + 1}$ to $R^{-1} V R$ then yields $\dim(VX) = \dim(R^{-1} V X) = \dim(R^{-1} V R)(R^{-1} X) = n_1 + \cdots + n_k - 1$.

It follows that

$$\left\{\dim(VX) \mid X \in \mathbb{K}^n\right\} = \{0, n_1 - 1, n_1 + n_2 - 1, \dots, n_1 + \cdots + n_p - 1\}$$

has cardinality $p + 1$. The same holds for W instead of V with the m_j's in place of the n_k's. Since V is similar to W, one has $\left\{\dim(VX) \mid X \in \mathbb{K}^n\right\} = \left\{\dim(WX) \mid\right.$

$X \in \mathbb{K}^n\big\}$ and we deduce successively that $q = p$ and $(n_1, \dots, n_p) = (m_1, \dots, m_q)$. Now, set $P \in \mathrm{GL}_n(\mathbb{K})$ such that $W = P^{-1}VP$. For every $k \in [\![1, p]\!]$, observe that

$$\begin{aligned}\big\{X \in \mathbb{K}^n :\ \dim VX \leq n_1 + \cdots + n_k - 1\big\} &= F_k \\ &= \big\{X \in \mathbb{K}^n :\ \dim WX \leq n_1 + \cdots + n_k - 1\big\};\end{aligned}$$

hence P stabilizes F_k. This shows that $P \in \mathrm{GL}_{n_1}(\mathbb{K}) \vee \cdots \vee \mathrm{GL}_{n_p}(\mathbb{K})$, which in turn proves that $P_k \,\mathrm{A}_{n_k}(\mathbb{K})$ is similar to $Q_k \,\mathrm{A}_{n_k}(\mathbb{K})$ for every $k \in [\![1, p]\!]$. Proposition 13 finally yields that P_k is congruent to a scalar multiple of Q_k, for every $k \in [\![1, p]\!]$. $\square$

Proposition 17. *Let $(P_1, \dots, P_p)$ and $(Q_1, \dots, Q_q)$ be two families of non-isotropic matrices, respectively in $\mathrm{GL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{n_p}(\mathbb{K})$ and $\mathrm{GL}_{m_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{m_q}(\mathbb{K})$. In order that*

$$\begin{aligned}&\big(I_{n_1} + P_1 \,\mathrm{A}_{n_1}(\mathbb{K})\big) \vee \cdots \vee \big(I_{n_p} + P_p \,\mathrm{A}_{n_p}(\mathbb{K})\big) \\ &\qquad\qquad \sim \big(I_{m_1} + Q_1 \,\mathrm{A}_{m_1}(\mathbb{K})\big) \vee \cdots \vee \big(I_{m_q} + Q_q \,\mathrm{A}_{m_q}(\mathbb{K})\big),\end{aligned}$$

it is necessary and sufficient that $q = p$ and that the (non-isotropic) quadratic form $X \mapsto X^T P_k X$ be similar to $X \mapsto X^T Q_k X$ for every $k \in [\![1, p]\!]$.

Proof. The "sufficient condition" statement follows trivially from Proposition 14. For the converse statement, let us set $\mathcal{V} := (I_{n_1} + P_1 \,\mathrm{A}_{n_1}(\mathbb{K})) \vee \cdots \vee (I_{n_p} + P_p \,\mathrm{A}_{n_p}(\mathbb{K}))$ and $\mathcal{W} := (I_{m_1} + Q_1 \,\mathrm{A}_{m_1}(\mathbb{K})) \vee \cdots \vee (I_{m_q} + Q_q \,\mathrm{A}_{m_q}(\mathbb{K}))$, and assume that $\mathcal{V} \sim \mathcal{W}$. Choose two non-singular matrices R and S such that $\mathcal{W} = R\mathcal{V}S$. Denote by V (resp. by W) the translation vector space of $\mathcal{V}$ (resp. of $\mathcal{W}$), and set $n := \sum_{k=1}^{p} n_k$. Then

$$\mathcal{W} = (RS)S^{-1}(I_n + V)S = (RS)(I_n + S^{-1}VS).$$

In particular $RS \in \mathcal{W}$ and the comparison of translation vector spaces yields $S^{-1}VS = (RS)^{-1}W$. The first result yields that RS is upper block-triangular with diagonal blocks $R_1, \dots, R_q$, where $R_k \in \mathrm{GL}_{m_k}(\mathbb{K})$ for every $k \in [\![1, q]\!]$. Thus

$$S^{-1}VS = (RS)^{-1}W = (R_1^{-1}Q_1) \,\mathrm{A}_{m_1}(\mathbb{K}) \vee \cdots \vee (R_q^{-1}Q_q) \,\mathrm{A}_{m_q}(\mathbb{K})$$

and the $R_k^{-1}Q_k$'s are necessarily non-isotropic since $S^{-1}VS$ has a trivial spectrum. We deduce from Proposition 16 that $(n_1, \dots, n_p) = (m_1, \dots, m_q)$. With the line of reasoning from the proof of Proposition 16, we also find that $S \in \mathrm{GL}_{n_1}(\mathbb{K}) \vee \cdots \vee \mathrm{GL}_{n_p}(\mathbb{K})$. However we already know that RS belongs to $\mathrm{GL}_{n_1}(\mathbb{K}) \vee \cdots \vee \mathrm{GL}_{n_p}(\mathbb{K})$ and hence $R = (RS)S^{-1} \in \mathrm{GL}_{n_1}(\mathbb{K}) \vee \cdots \vee \mathrm{GL}_{n_p}(\mathbb{K})$.

Returning to $R\mathcal{V}S = \mathcal{W}$ finally entails that $I_{n_k} + Q_k \,\mathrm{A}_{n_k}(\mathbb{K})$ is equivalent to $I_{n_k} + P_k \,\mathrm{A}_{n_k}(\mathbb{K})$ for each $k \in [\![1, p]\!]$, and Proposition 14 then yields that $X \mapsto X^T P_k X$ is similar to $X \mapsto X^T Q_k X$ for each $k \in [\![1, p]\!]$. $\square$

4. Structure of the irreducible maximal spaces with a trivial spectrum

In the entire section, we assume $\#\mathbb{K} \geq 3$. We will prove Theorem 3 by induction. The case $n = 1$ needs no explanation.

4.1. **The case $n = 2$.** Let V be an irreducible maximal linear subspace of $\mathrm{M}_2(\mathbb{K})$ with a trivial spectrum. Then $V = \mathrm{span}(M)$ for some $M \in \mathrm{M}_2(\mathbb{K}) \smallsetminus \{0\}$ with no non-zero eigenvalue. If 0 is an eigenvalue of M, then M is triangularizable and V is not irreducible.

Hence M is non-singular. Setting $K := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $P := MK^{-1}$, we readily have $P\,\mathrm{A}_2(\mathbb{K}) = \mathrm{span}(M) = V$ and Lemma 10 shows that P is non-isotropic.

4.2. **Setting things up.** Let $n \geq 2$ and assume that the result of Theorem 3 holds for any positive integer $k \leq n$. Let $V \subset \mathrm{M}_{n+1}(\mathbb{K})$ be a maximal subspace with a trivial spectrum. Denote by $(e_1, \ldots, e_{n+1})$ the canonical basis of $\mathbb{K}^{n+1}$. We wish to show that V is reducible or similar to $P\,\mathrm{A}_{n+1}(\mathbb{K})$ for some $P \in \mathrm{GL}_{n+1}(\mathbb{K})$, in which case Lemma 10 guarantees that P must be non-isotropic.

Of course, this amounts to finding a basis of $\mathbb{K}^{n+1}$ in which all the endomorphisms $X \mapsto MX$ of $\mathbb{K}^{n+1}$, for $M \in V$, have a "reduced" shape that is essentially the one described in Theorem 4. The first problem is how to select the last vector f_{n+1} of such a basis. Since the rank of an alternate matrix is even, an obvious necessary condition is that V should not contain any matrix with $\mathrm{span}(f_{n+1})$ as column space. Our starting point is that such a vector exists (and may even be chosen amongst the canonical basis of $\mathbb{K}^{n+1}$). This has already been proven in [10, Proposition 10]: we reproduce a proof since it is short and the result is crucial to our study.

Lemma 18. *Let W be a linear subspace of $\mathrm{M}_p(\mathbb{K})$ with a trivial spectrum. Then there exists a non-zero vector $X \in \mathbb{K}^p$ such that W contains no matrix M with $\mathrm{span}(X)$ as column space.*

Proof. Denote by $(e_1, \ldots, e_p)$ the canonical basis of $\mathbb{K}^p$. For $X \in \mathbb{K}^p \smallsetminus \{0\}$, set $W_X := \big\{M \in W : \mathrm{Im}(M) \subset \mathrm{span}(X)\big\}$. For $(i,j) \in [\![1,p]\!]^2$, denote by $E_{i,j}$ the matrix of $\mathrm{M}_p(\mathbb{K})$ with zero entries everywhere except at the (i,j)-spot where the entry is 1.

We prove, by induction on p, that there exists an index $i \in [\![1,p]\!]$ such that $W_{e_i} = \{0\}$. The case $p = 1$ is trivial.

Assume that $W_{e_i} \neq \{0\}$ for every $i \in [\![1,p]\!]$, denote by W' the linear subspace of W consisting of its matrices with zero as last row, and write every $M \in W'$ as

$$M = \begin{bmatrix} J(M) & [?]_{(p-1)\times 1} \\ [0]_{1\times(p-1)} & 0 \end{bmatrix} \quad \text{with } J(M) \in \mathrm{M}_{p-1}(\mathbb{K}).$$

Then $J(W')$ is a linear subspace of $\mathrm{M}_{p-1}(\mathbb{K})$ with a trivial spectrum. The induction hypothesis yields an index $i \in [\![1,p-1]\!]$ such that $J(W')_{e_i} = \{0\}$.

Since $W_{e_i} \neq \{0\}$, we find a matrix $M \in W$ such that $\mathrm{Im}(M) = \mathrm{span}(e_i)$. Then $M \in W'$ and it follows from $J(W')_{e_i} = \{0\}$ that M is a non-zero scalar multiple of $E_{i,p}$. Therefore $E_{i,p} \in W$.

Now, taking an arbitrary permutation matrix $P \in \mathrm{GL}_n(\mathbb{K})$ and applying the previous step to PWP^{-1} yields the following generalization: for every $j \in [\![1,p]\!]$, there exists an integer $f(j) \in [\![1,p]\!] \smallsetminus \{j\}$ such that $E_{f(j),j} \in W$.

We choose a *cycle* for the map $f : [\![1,p]\!] \to [\![1,p]\!]$, i.e. a list $(j_1, \ldots, j_r)$ of distinct elements of $[\![1,p]\!]$ such that $f(j_1) = j_2, \ldots, f(j_{r-1}) = j_r$ and $f(j_r) = j_1$. The matrix $A := \sum_{k=1}^{r} E_{f(j_k),j_k}$ then belongs to W although 1 is an eigenvalue of it (a

corresponding eigenvector being $\sum_{k=1}^{r} e_{j_k}$). This is a contradiction, which shows that $W_{e_i} = \{0\}$ for some $i \in [\![1, p]\!]$. $\square$

By conjugating V with an appropriate invertible matrix, we then lose no generality assuming that no matrix of V has $\operatorname{span}(e_{n+1})$ as column space and that $Ve_{n+1} \subset \operatorname{span}(e_1, \dots, e_n)$ (since $e_{n+1} \notin Ve_{n+1}$). This means that every matrix of V has a 0 entry at the $(n+1, n+1)$-spot.

In order to complete the choice of a "good" basis for V, we now turn to the first n vectors $f_1, \dots, f_n$. The basic idea is to find the projections of $f_1, \dots, f_n$ onto $\operatorname{span}(e_1, \dots, e_n)$ and alongside $\operatorname{span}(e_{n+1})$ by applying the induction hypothesis to a subspace of $\mathrm{M}_n(\mathbb{K})$ that is deduced from V (the space V_{ul} defined below), and then apply the induction hypothesis once more to find the projections of $f_1, \dots, f_n$ onto $\operatorname{span}(e_{n+1})$ alongside $\operatorname{span}(e_1, \dots, e_n)$.

Consider the subspace W of V consisting of its matrices with zero as last column. For $M \in W$, write

$$M = \begin{bmatrix} K(M) & [0]_{n\times 1} \\ L(M) & 0 \end{bmatrix} \quad \text{with } K(M) \in \mathrm{M}_n(\mathbb{K}) \text{ and } L(M) \in \mathrm{M}_{1,n}(\mathbb{K}),$$

and set

$$V_{ul} := K(W)$$

(the subscript "*ul*" stands for "upper left"). The rank theorem shows that

$$\dim V = \dim W + \dim(Ve_{n+1}) \quad \text{and} \quad \dim W = \dim \operatorname{Ker} K + \dim V_{ul}.$$

However, our assumptions mean that $\operatorname{Ker} K = \{0\}$; hence

$$\dim V = \dim V_{ul} + \dim(Ve_{n+1}).$$

Obviously, V_{ul} is a linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum; hence $\dim V_{ul} \leq \binom{n}{2}$. Moreover $\dim(Ve_{n+1}) \leq n$ since V acts totally intransitively on $\mathbb{K}^{n+1}$. We deduce that

$$\binom{n+1}{2} = \dim V = \dim V_{ul} + \dim(Ve_{n+1}) \leq \binom{n}{2} + n = \binom{n+1}{2};$$

hence

$$\dim V_{ul} = \binom{n}{2} \quad \text{and} \quad \dim(Ve_{n+1}) = n.$$

In this reduced situation, we conclude that:

1. V_{ul} is a maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum.
2. $Ve_{n+1} = \operatorname{span}(e_1, \dots, e_n)$.

Applying the induction hypothesis to V_{ul} together with Remark 2 shows that we may find non-isotropic *lower-triangular* matrices $P_1, \dots, P_r$ such that

$$V_{ul} \simeq P_1 \, \mathrm{A}_{n_1}(\mathbb{K}) \vee \dots \vee P_r \, \mathrm{A}_{n_r}(\mathbb{K}).$$

This shows that, by conjugating V with a well-chosen matrix of the form

$$\begin{bmatrix} R & [0]_{n\times 1} \\ [0]_{1\times n} & 1 \end{bmatrix}$$

for some $R \in \mathrm{GL}_n(\mathbb{K})$, we lose no generality assuming that

$$V_{ul} = P_1 \, \mathrm{A}_{n_1}(\mathbb{K}) \vee \dots \vee P_r \, \mathrm{A}_{n_r}(\mathbb{K}) \quad \text{and} \quad P_1 = \begin{bmatrix} 1 & [0]_{1\times(n_1-1)} \\ C_1' & P_1' \end{bmatrix}$$

for some lower-triangular matrix $P_1' \in \mathrm{M}_{n_1-1}(\mathbb{K})$ (possibly of size 0) and some column matrix $C_1' \in \mathrm{M}_{n_1-1,1}(\mathbb{K})$.

Remark 3 (An important remark on block-diagrams). *From now on, and unless specified otherwise, every matrix M of V will be systematically seen with the following 3×3 block decomposition:*

$$M = \begin{bmatrix} ? & [?]_{1\times(n-1)} & ? \\ [?]_{(n-1)\times 1} & [?]_{n-1} & [?]_{(n-1)\times 1} \\ ? & [?]_{1\times(n-1)} & ? \end{bmatrix};$$

i.e. the four question marks represent single entries, while the others represent submatrices with sizes as indicated by the subscript (where the central subscript $n-1$ denotes an $(n-1)\times(n-1)$ block).

If $n_1 > 1$, we set $s := r$, $(i_1, \dots, i_s) := (n_1 - 1, n_2, \dots, n_r)$ and $(R_1, \dots, R_s) := (P_1', P_2, \dots, P_r)$.

If $n_1 = 1$, we set $s := r - 1$, $(i_1, \dots, i_s) := (n_2, \dots, n_r)$ and $(R_1, \dots, R_s) := (P_2, \dots, P_r)$.

In any case, we set

$$V_m := R_1 \, \mathrm{A}_{i_1}(\mathbb{K}) \vee \cdots \vee R_s \, \mathrm{A}_{i_s}(\mathbb{K})$$

(the subscript "m" stands for "middle").

Here are two consequences of the above reductions (with the block decompositions laid out in Remark 3):

(i) For every $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$, the subspace V contains a matrix of the form

$$\begin{bmatrix} ? & L & 0 \\ ? & ? & 0 \\ ? & ? & 0 \end{bmatrix}.$$

(ii) For every $U \in V_m$, the subspace V contains a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ ? & ? & 0 \end{bmatrix}.$$

Proof of statement (i). Let $L_1 \in \mathrm{M}_{1,n_1-1}(\mathbb{K})$. Then

$$P_1 \times \begin{bmatrix} 0 & L_1 \\ -L_1^T & [0]_{n_1-1} \end{bmatrix} = \begin{bmatrix} ? & L_1 \\ [?]_{(n_1-1)\times 1} & [?]_{n_1-1} \end{bmatrix}$$

and $\begin{bmatrix} 0 & L_1 \\ -L_1^T & [0]_{n_1-1} \end{bmatrix}$ is alternate. Since $V_{ul} = P_1 \, \mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee P_r \, \mathrm{A}_{n_r}(\mathbb{K})$, we deduce that, for every $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$, the subspace V_{ul} contains a matrix of the form $\begin{bmatrix} ? & L \\ [?]_{(n-1)\times 1} & [?]_{n-1} \end{bmatrix}$, and the conclusion follows from the definition of V_{ul}. $\square$

Proof of statement (ii). We will only tackle the case $n_1 > 1$, the case $n_1 = 1$ being essentially similar (and even simpler). For every $M \in P_2 \, \mathrm{A}_{n_2}(\mathbb{K}) \vee \cdots \vee P_r \, \mathrm{A}_{n_r}(\mathbb{K})$ and every $N \in \mathrm{M}_{n_1-1,n-n_1}(\mathbb{K})$, we know that V_{ul} contains the matrix

$$\begin{bmatrix} 0 & [0]_{1\times(n_1-1)} & [0]_{1\times(n-n_1)} \\ [0]_{(n_1-1)\times 1} & [0]_{n_1-1} & N \\ [0]_{(n-n_1)\times 1} & [0]_{(n-n_1)\times(n_1-1)} & M \end{bmatrix}.$$

Let $A \in \mathrm{A}_{n_1-1}(\mathbb{K})$. Then

$$P_1 \times \begin{bmatrix} 0 & [0]_{1\times(n_1-1)} \\ [0]_{(n_1-1)\times 1} & A \end{bmatrix} = \begin{bmatrix} 0 & [0]_{1\times(n_1-1)} \\ [0]_{(n_1-1)\times 1} & P_1'A \end{bmatrix}$$

and it follows that V_{ul} contains a matrix of the form

$$\begin{bmatrix} 0 & [0]_{1\times(n_1-1)} & [0]_{1\times(n-n_1)} \\ [0]_{(n_1-1)\times 1} & P_1'A & [0]_{(n_1-1)\times(n-n_1)} \\ [0]_{(n-n_1)\times 1} & [0]_{(n-n_1)\times(n_1-1)} & [0]_{(n-n_1)\times(n-n_1)} \end{bmatrix}.$$

With the respective definitions of V_m and V_{ul}, point (ii) follows easily. □

Now let $C \in \mathrm{M}_{n-1,1}(\mathbb{K})$. Since $Ve_{n+1} = \mathrm{span}(e_1, \ldots, e_n)$, we know that V contains a matrix of the form

$$\begin{bmatrix} ? & ? & 0 \\ ? & ? & C \\ ? & ? & 0 \end{bmatrix}.$$

Adding an appropriate matrix given by statement (i), and remembering that 0 is the only possible eigenvalue for a matrix in V, we deduce:

(iii) V contains a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ ? & ? & C \\ ? & ? & 0 \end{bmatrix}.$$

Now denote by V' the subspace of V consisting of its matrices with zero as first row. For $M \in V'$, write

$$M = \begin{bmatrix} 0 & [0]_{1\times n} \\ [?]_{n\times 1} & \mathcal{J}(M) \end{bmatrix} \quad \text{with } \mathcal{J}(M) \in \mathrm{M}_n(\mathbb{K}),$$

and set

$$V_{lr} := \mathcal{J}(V')$$

(the subscript "lr" stands for "lower right"). Note that the subspace V_{lr} of $\mathrm{M}_n(\mathbb{K})$ has a trivial spectrum and that it contains:

(a) A matrix of the form $\begin{bmatrix} U & [0]_{(n-1)\times 1} \\ [?]_{1\times(n-1)} & 0 \end{bmatrix}$ for every $U \in V_m$ (by statement (ii));

(b) A matrix of the form $\begin{bmatrix} [?]_{n-1} & C \\ [?]_{1\times(n-1)} & 0 \end{bmatrix}$ for every $C \in \mathrm{M}_{n-1,1}(\mathbb{K})$ (by statement (iii)).

Since $\dim V_m = \binom{n-1}{2}$, we deduce that $\dim V_{lr} \geq \binom{n-1}{2} + (n-1) = \binom{n}{2}$. However $\dim V_{lr} \leq \binom{n}{2}$ since V_{lr} has a trivial spectrum. It thus follows from statements (a) and (b) that:

(c) V_{lr} contains, for every $U \in V_m$, a *unique* matrix of the form $\begin{bmatrix} U & [0]_{(n-1)\times 1} \\ ? & 0 \end{bmatrix}$;

(d) Every matrix of V_{lr} with zero as last column has the form $\begin{bmatrix} U & [0]_{(n-1)\times 1} \\ ? & 0 \end{bmatrix}$ for some $U \in V_m$.

A key point now is that V_{lr} is a maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum. One may thus be tempted to apply the induction hypothesis to V_{lr}. However, the problem is that using a new change of basis blindingly risks destroying the previous reduced form of V_{ul}! As we shall now see, the fact that V_m is already reduced forces V_{lr} to be in the reduced form of Theorem 4 (i.e. no further change of basis is necessary at this point).

Claim 1. The subspace V_{lr} has a "roughly reduced" shape; i.e., there exists an integer $q \geq 1$, a non-isotropic matrix $Q \in \mathrm{GL}_q(\mathbb{K})$ and a maximal subspace W of $\mathrm{M}_{n-q}(\mathbb{K})$ with a trivial spectrum such that

$$V_{lr} = W \vee Q\,\mathrm{A}_q(\mathbb{K}).$$

Proof. Applying the induction hypothesis to V_{lr}, we recover a matrix $P \in \mathrm{GL}_n(\mathbb{K})$, a non-isotropic matrix $Q' \in \mathrm{GL}_q(\mathbb{K})$ (possibly with $q = n$) and a maximal subspace W' of $\mathrm{M}_{n-q}(\mathbb{K})$ with a trivial spectrum such that

$$PV_{lr}P^{-1} = W' \vee Q'\,\mathrm{A}_q(\mathbb{K}).$$

Note, using statement (b), that $\dim(V_{lr}e_n) = n-1$, whereas $\dim(PV_{lr}P^{-1}x) < n-1$ for every $x \in \mathrm{span}(e_1, \ldots, e_{n-q})$ (since W' acts totally intransitively on $\mathbb{K}^{n-q}$). Hence $Pe_n \notin \mathrm{span}(e_1, \ldots, e_{n-q})$. Multiplying P with a well-chosen matrix of $\mathrm{GL}_{n-q}(\mathbb{K}) \vee \mathrm{GL}_q(\mathbb{K})$, we lose no generality assuming that $Pe_n = e_n$.

Assume first that $q = 1$. Then $V_{lr}e_n = \mathrm{span}(e_1, \ldots, e_{n-1}) = (PV_{lr}P^{-1})e_n$ while $PV_{lr}P^{-1}e_n = P(V_{lr}e_n)$, which shows that P stabilizes $\mathrm{span}(e_1, \ldots, e_{n-1})$. Therefore $P \in \mathrm{GL}_{n-1}(\mathbb{K}) \vee \{1\}$ and $V_{lr} = W \vee \mathrm{A}_1(\mathbb{K})$ for some maximal linear subspace W of $\mathrm{M}_{n-1}(\mathbb{K})$ with a trivial spectrum.

Assume, for the rest of the proof, that $q > 1$. Our aim is to prove that $P \in \mathrm{GL}_{n-q}(\mathbb{K}) \vee \mathrm{GL}_q(\mathbb{K})$, and it will follow that $V_{lr} = W \vee Q\,\mathrm{A}_q(\mathbb{K})$ for some maximal linear subspace W of $\mathrm{M}_{n-q}(\mathbb{K})$ with a trivial spectrum and some non-isotropic matrix $Q \in \mathrm{GL}_q(\mathbb{K})$.

Set

$$H := \big\{M \in V_{lr} : \; Me_n = 0\big\};$$

i.e., H is the set of all matrices of V_{lr} with 0 as last column. Notice that $PHP^{-1} = \big\{M \in PV_{lr}P^{-1} : \; Me_n = 0\big\}$ since $Pe_n = e_n$. Notice also that

$$\mathrm{span}(e_1, \ldots, e_{n-1-i_s}) \subset \mathrm{span}(e_1, \ldots, e_{n-1}) = V_{lr}e_n$$

(this uses statement (b) and the fact that $V_{lr}e_n \neq \mathbb{K}^n$) and that

$$\mathrm{span}(e_1, \ldots, e_{n-q}) \subset (PV_{lr}P^{-1})e_n\ .$$

- *Case 1:* $i_s > 1$.
  - We first claim that

$$\forall x \in V_{lr}e_n, \quad \dim Hx < n-2 \;\Leftrightarrow\; x \in \mathrm{span}(e_1, \ldots, e_{n-1-i_s}). \tag{1}$$

Indeed, let $x \in \mathrm{span}(e_1, \ldots, e_{n-1})$ seen as a vector of $\mathbb{K}^{n-1}$ with the canonical identification $\mathbb{K}^{n-1} \simeq \mathbb{K}^{n-1} \times \{0\} \subset \mathbb{K}^n$. By statements (c) and (d), one has

$$\dim V_m x \leq \dim Hx \leq 1 + \dim V_m x.$$

If $x \in \mathrm{span}(e_1, \ldots, e_{n-1-i_s})$, then the line of reasoning from the proof of Proposition 16 yields $\dim V_m x \leq n - i_s - 2$ and hence $\dim Hx \leq n - i_s - 1 < n-2$; otherwise $\dim V_m x = n-2$ and hence $\dim Hx \geq n-2$.

– Moreover, we claim that

$$\forall x \in (PV_{lr}P^{-1})e_n,\ \dim(PHP^{-1}x) < n-2 \ \Leftrightarrow\ x \in \operatorname{span}(e_1,\dots,e_{n-q}). \tag{2}$$

The implication $\Leftarrow$ follows from $PV_{lr}P^{-1} = W' \vee Q'\,\mathrm{A}_q(\mathbb{K})$ since W' acts totally intransitively on $\mathbb{K}^{n-q}$ and $q > 1$.
For the converse implication, notice first that the equality $PV_{lr}P^{-1} = W' \vee Q'\,\mathrm{A}_q(\mathbb{K})$ yields $(PV_{lr}P^{-1})e_n = \operatorname{span}(e_1,\dots,e_{n-q}) \oplus G$ for some $(q-1)$-dimensional subspace G of $\operatorname{span}(e_{n-q+1},\dots,e_n)$ which does not contain e_n (note that $(PV_{lr}P^{-1})e_n$ cannot contain e_n since $PV_{lr}P^{-1}$ has a trivial spectrum). Consider a vector $x \in G \smallsetminus \{0\}$. The subspace PHP^{-1} contains, for every $A \in \mathrm{A}_{q-1}(\mathbb{K})$, and every $B \in \mathrm{M}_{n-q,q}(\mathbb{K})$ with zero as last column, the matrix

$$\begin{bmatrix} [0]_{n-q} & B \\ [0]_{q\times(n-q)} & C \end{bmatrix}, \quad \text{where } C = Q' \times \begin{bmatrix} A & [0]_{(q-1)\times 1} \\ [0]_{1\times(q-1)} & 0 \end{bmatrix}.$$

Since x belongs to $\operatorname{span}(e_{n-q+1},\dots,e_n)$ and is linearly independent from e_n, it easily follows that $\dim(PHP^{-1})x \geq n-2$.
Now let $x \in (PV_{lr}P^{-1})e_n \smallsetminus \operatorname{span}(e_1,\dots,e_{n-q})$. Then we have a decomposition $x = z + y$ with $z \in \operatorname{span}(e_1,\dots,e_{n-q})$ and $y \in G \smallsetminus \{0\}$. Obviously, there exists a non-singular matrix $R \in \{I_{n-q}\} \vee \{I_q\}$ such that $Rx = y$. Replacing P with RP, we thus reduce the situation to the one where $x \in G \smallsetminus \{0\}$, which we have treated before. Implication $\Rightarrow$ in statement (2) follows.

Since $X \mapsto PX$ is linear, $Pe_n = e_n$, $\operatorname{span}(e_1,\dots,e_{n-q}) \subset (PV_{lr}P^{-1})e_n$, and $\operatorname{span}(e_1,\dots,e_{n-1-i_s}) \subset V_{lr}e_n$, we deduce from statements (1) and (2) that $X \mapsto PX$ induces an isomorphism from $\operatorname{span}(e_1,\dots,e_{n-1-i_s})$ to $\operatorname{span}(e_1,\dots,e_{n-q})$; hence $i_s = q-1$ and $P \in \mathrm{GL}_{n-q}(\mathbb{K}) \vee \mathrm{GL}_q(\mathbb{K})$.

- *Case 2:* $i_s = 1$.
  – Notice first that $\operatorname{span}(e_1,\dots,e_{n-1}) = V_{lr}e_n$ and

$$\forall x \in V_{lr}e_n,\ \dim(Hx \cap V_{lr}e_n) < n-2 \quad \text{if } x \in \operatorname{span}(e_1,\dots,e_{n-2}). \tag{3}$$

Indeed, for every $x \in \operatorname{span}(e_1,\dots,e_{n-2})$, statements (c) and (d) show that $\dim(Hx \cap V_{lr}e_n) \leq \dim(V_m x)$ (where x is naturally seen as a vector of $\mathbb{K}^{n-1}$), and the definition of V_m shows, since $i_s = 1$, that $\dim(V_m x) < n-2$.

– On the other hand, we claim that

$$\begin{aligned} &\forall x \in (PV_{lr}P^{-1})e_n,\ \dim\big((PHP^{-1})x \cap (PV_{lr}P^{-1})e_n\big) < n-2 \\ &\qquad\qquad\qquad\qquad \Leftrightarrow\ x \in \operatorname{span}(e_1,\dots,e_{n-q}). \end{aligned} \tag{4}$$

Indeed, for any $x \in \operatorname{span}(e_1,\dots,e_{n-q})$, one has

$$\dim\big((PHP^{-1})x \cap (PV_{lr}P^{-1})e_n\big) \leq \dim(PV_{lr}P^{-1})x \leq n-q-1 < n-2.$$

Conversely, let $x \in (PV_{lr}P^{-1})e_n \smallsetminus \operatorname{span}(e_1,\dots,e_{n-q})$. Note first that $(PHP^{-1})x \subset (PV_{lr}P^{-1})e_n$. In order to see this, we naturally identify $\mathbb{K}^n$ with $\mathbb{K}^{n-q} \oplus \mathbb{K}^q$: the identity $PV_{lr}P^{-1} = W' \vee Q'\,\mathrm{A}_q(\mathbb{K})$ yields $(PV_{lr}P^{-1})e_n = \mathbb{K}^{n-q} \times \big[Q'(\mathbb{K}^{q-1} \times \{0\})\big]$ while, for every $M \in PHP^{-1}$, the column space of M is included in $\mathbb{K}^{n-q} \times Q'\operatorname{Im}(N)$, where

$N = \begin{bmatrix} A_M & [0]_{(q-1)\times 1} \\ [0]_{1\times(q-1)} & 0 \end{bmatrix}$ for some $A_M \in \mathrm{A}_{q-1}(\mathbb{K})$; this shows that $\operatorname{Im} M \subset (PV_{lr}P^{-1})e_n$ for every $M \in PHP^{-1}$.

With the same arguments as in the proof of statement (2), one may prove that $\dim(PHP^{-1})x = n-2$, and hence $\dim\big((PHP^{-1})x \cap (PV_{lr}P^{-1})e_n\big) = \dim(PHP^{-1})x = n-2$. Therefore statement (4) is established.

From statements (3) and (4), we deduce that the linear injection $X \mapsto PX$ maps $\operatorname{span}(e_1,\dots,e_{n-2})$ into $\operatorname{span}(e_1,\dots,e_{n-q})$, which shows that $q=2$, $i_s = 1 = q-1$ and $P \in \mathrm{GL}_{n-q}(\mathbb{K}) \vee \mathrm{GL}_q(\mathbb{K})$. This finishes our proof. □

Now that we know that V_{lr} is "roughly reduced", we may use the shape of V_m to better grasp that of V_{lr}.

Take W, q and Q as in Claim 1. If $q=1$, then obviously $W = V_m$.

Assume now that $q>1$ and split

$$Q = \begin{bmatrix} Q_1 & [?]_{(q-1)\times 1} \\ [?]_{1\times(q-1)} & ? \end{bmatrix}$$

with $Q_1 \in \mathrm{M}_{q-1}(\mathbb{K})$. Then Q_1 is still non-isotropic and statement (d) shows that V_m contains $W \vee Q_1\,\mathrm{A}_{q-1}(\mathbb{K})$, and hence $V_m = W \vee Q_1\,\mathrm{A}_{q-1}(\mathbb{K})$ since the dimensions are equal on both sides. By applying the induction hypothesis to W and by using the same arguments as in the proof of Proposition 16, we deduce that $W = R_1\,\mathrm{A}_{i_1}(\mathbb{K}) \vee \cdots \vee R_{s-1}\,\mathrm{A}_{i_{s-1}}(\mathbb{K})$ and $Q_1\,\mathrm{A}_{q-1}(\mathbb{K}) = R_s\,\mathrm{A}_{i_s}(\mathbb{K})$.

Therefore

$$V_{lr} = \begin{cases} R_1\,\mathrm{A}_{i_1}(\mathbb{K}) \vee \cdots \vee R_s\,\mathrm{A}_{i_s}(\mathbb{K}) \vee \mathrm{A}_1(\mathbb{K}) & \text{if } q=1, \\ R_1\,\mathrm{A}_{i_1}(\mathbb{K}) \vee \cdots \vee R_{s-1}\,\mathrm{A}_{i_{s-1}}(\mathbb{K}) \vee Q\,\mathrm{A}_q(\mathbb{K}) & \text{if } q>1. \end{cases}$$

Assume again that $q>1$. Then Q need not be lower-triangular, so we have to reduce the situation a little further.

Since $Ve_{n+1} = \operatorname{span}(e_1,\dots,e_n)$, we find that $Q\,\mathrm{A}_q(\mathbb{K})e_q = \operatorname{span}(e_1,\dots,e_{q-1})$ which shows that Q stabilizes $\operatorname{span}(e_1,\dots,e_{q-1})$, i.e.

$$Q = \begin{bmatrix} T_0 & C \\ [0]_{1\times(q-1)} & \alpha \end{bmatrix}$$

for some $T_0 \in \mathrm{GL}_{q-1}(\mathbb{K})$ and some $\alpha \in \mathbb{K} \smallsetminus \{0\}$.

Note, since Q is non-singular, that a matrix of the form $M = QA$, with $A \in \mathrm{A}_q(\mathbb{K})$, has zero as last column if and only if A has zero as last column. It then follows from the shape of V_m that $T_0\,\mathrm{A}_{q-1}(\mathbb{K}) = R_s\,\mathrm{A}_{i_s}(\mathbb{K})$. Therefore $(R_s^{-1}T_0)\,\mathrm{A}_{q-1}(\mathbb{K}) = \mathrm{A}_{q-1}(\mathbb{K})$, and we deduce from Lemma 12 that T_0 is a scalar multiple of R_s. Since we may replace Q with a scalar multiple of itself, we lose no generality assuming that $T_0 = R_s$.

Finally we define

$$T_1 := \begin{bmatrix} I_{q-1} & C' \\ [0]_{1\times(q-1)} & 1 \end{bmatrix} \in \mathrm{GL}_q(\mathbb{K}),$$

where $C' := -T_0^{-1}C$, so that $Q' := (T_1)^T Q T_1$ is lower-triangular; we replace V with RVR^{-1} for

$$R := \begin{bmatrix} I_{n+1-q} & [0]_{(n+1-q)\times q} \\ [0]_{q\times(n+1-q)} & T_1^T \end{bmatrix}.$$

For the sake of convenience (and symmetry), we now set

$$P := P_1 \quad \text{and} \quad p := n_1.$$

Let us see how the situation looks after all those reductions:

(i) We still have

$$V_{ul} = P\,\mathrm{A}_p(\mathbb{K}) \vee P_2\,\mathrm{A}_{n_2}(\mathbb{K}) \vee \cdots \vee P_r\,\mathrm{A}_{n_r}(\mathbb{K})$$

and

$$V_m = R_1\,\mathrm{A}_{i_1}(\mathbb{K}) \vee \cdots \vee R_s\,\mathrm{A}_{i_s}(\mathbb{K})$$

with the above notation (nothing has changed there).

Recall that $(i_1, \dots, i_s) = (n_2, \dots, n_r)$ if $p = 1$; otherwise $(i_1, \dots, i_s) = (n_1 - 1, n_2, \dots, n_r)$.

(ii) Either $q = 1$ and then

$$V_{lr} = R_1\,\mathrm{A}_{i_1}(\mathbb{K}) \vee \cdots \vee R_s\,\mathrm{A}_{i_s}(\mathbb{K}) \vee Q\,\mathrm{A}_1(\mathbb{K}) \quad \text{with } Q = 1,$$

or $q > 1$ and then

$$V_{lr} = R_1\,\mathrm{A}_{i_1}(\mathbb{K}) \vee \cdots \vee R_{s-1}\,\mathrm{A}_{i_{s-1}}(\mathbb{K}) \vee Q\,\mathrm{A}_q(\mathbb{K})$$

and

$$Q = \begin{bmatrix} R_s & [0]_{i_s \times 1} \\ L_1 & \alpha \end{bmatrix} \quad \text{with } \alpha \in \mathbb{K} \smallsetminus \{0\} \text{ and } L_1 \in \mathrm{M}_{1,q-1}(\mathbb{K}).$$

We set $\alpha := 1$ if $q = 1$.

(iii) Recall finally that if $p > 1$, then $P = \begin{bmatrix} 1 & [0]_{1\times(p-1)} \\ C_1 & R_1 \end{bmatrix}$ for some $C_1 \in \mathrm{M}_{p-1,1}(\mathbb{K})$.

(iv) No matrix of V has $\mathrm{span}(e_{n+1})$ as column space (no change there).

However, one important thing has changed: if $q > 1$, we no longer have $Ve_{n+1} = \mathrm{span}(e_1, \dots, e_n)$; rather $Ve_{n+1} = \mathrm{span}(e_1, \dots, e_{n+1-q}) \oplus H$ for some linear hyperplane H of $\mathrm{span}(e_{n+2-q}, \dots, e_{n+1})$ which does not contain e_{n+1}. We still have $e_1 \in Ve_{n+1}$, nevertheless. Set finally

$$Z := \begin{bmatrix} R_1 & & (0) \\ & \ddots & \\ (0) & & R_s \end{bmatrix} \in \mathrm{GL}_{n-1}(\mathbb{K}).$$

From there, V will remain essentially fixed. We will prove separately:

- That the case $p = n = q$ (i.e. V_{ul} and V_{lr} are **glued**) leads to the equivalence of V with $\mathrm{A}_{n+1}(\mathbb{K})$.
- That the case $p \neq n$ or $q \neq n$ (i.e. V_{ul} and V_{lr} are **unglued**) leads to the reducibility of V.

Prior to studying the two cases separately, we continue with general considerations that apply to both of them.

4.3. **Special types of matrices in V.** With the matrices L_1 and C_1 from the previous paragraph[2], set

$$\widetilde{L_1} := \begin{bmatrix}[0]_{1\times(n-q)} & L_1\end{bmatrix} \in \mathrm{M}_{1,n-1}(\mathbb{K}) \quad \text{and} \quad \widetilde{C_1} := \begin{bmatrix} C_1 \\ [0]_{(n-p)\times 1}\end{bmatrix} \in \mathrm{M}_{n-1,1}(\mathbb{K}).$$

Notation 7. For an arbitrary $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$, we define $\overline{L}$ as the matrix of $\mathrm{M}_{1,n-1}(\mathbb{K})$ with the same first $p-1$ entries as L and all the other ones equal to zero.

For an arbitrary $C \in \mathrm{M}_{n-1,1}(\mathbb{K})$, we define $\overline{C}$ as the matrix of $\mathrm{M}_{n-1,1}(\mathbb{K})$ with the same last $q-1$ entries as C and all the other ones equal to zero.

Using the respective shapes of V_{ul}, V_m and V_{lr}, we now find important classes of matrices in V, together with an isolated matrix. First of all, taking arbitrary row matrices $L_0 \in \mathrm{M}_{1,p-1}(\mathbb{K})$ and $L_0' \in \mathrm{M}_{1,n-p}(\mathbb{K})$, we know that V_{ul} contains a matrix of the form $\begin{bmatrix} PA & N \\ [0]_{(n-p)\times p} & [0]_{n-p}\end{bmatrix}$ with $A = \begin{bmatrix} 0 & L_0 \\ -L_0^T & [0]_{p-1}\end{bmatrix}$ and $N = \begin{bmatrix} L_0' \\ [0]_{(p-1)\times(n-p)}\end{bmatrix}$. Therefore, using the block decomposition of matrices of V explained in Remark 3, we find that:

- For every $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$, there is a unique[3] $A_L \in V$ of the form

$$A_L = \begin{bmatrix} 0 & L & 0 \\ -Z\overline{L}^T & \widetilde{C_1}L & 0 \\ f(L) & \varphi(L) & 0\end{bmatrix},$$

  and $f : \mathrm{M}_{1,n-1}(\mathbb{K}) \to \mathbb{K}$ and $\varphi : \mathrm{M}_{1,n-1}(\mathbb{K}) \to \mathrm{M}_{1,n-1}(\mathbb{K})$ are linear maps.

Let $U \in V_m$, which we write as a block-triangular matrix

$$U = \begin{bmatrix} [?]_{n-1-i_s} & [?]_{n-1-i_s,i_s} \\ [0]_{i_s,n-1-i_s} & R_sA\end{bmatrix}$$

with $A \in \mathrm{A}_s(\mathbb{K})$. With the respective structures of V_{ul} and V_m and the fact that V contains no matrix with column space $\mathrm{span}(e_{n+1})$, we know that V contains a unique matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ ? & ? & 0\end{bmatrix}.$$

Since

$$Q \times \begin{bmatrix} A & [0]_{(i_s-1)\times 1} \\ [0]_{1\times(i_s-1)} & 0\end{bmatrix} = \begin{bmatrix} R_sA & [0]_{(i_s-1)\times 1} \\ L_1R_s^{-1}(R_sA) & 0\end{bmatrix},$$

the structure of V_{lr} yields that the above matrix of V has $\begin{bmatrix} ? & \widetilde{L_1}Z^{-1}U & 0\end{bmatrix}$ as last row. Therefore:

- For every $U \in V_m$, there is a unique $E_U \in V$ of the form

$$E_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ h(U) & \widetilde{L_1}Z^{-1}U & 0\end{bmatrix}.$$

[2]Setting $L_1 := 0$ if $q = 1$, and $C_1 := 0$ if $p = 1$.

[3]As the map K from the beginning of Section 4.2 is one-to-one.

We know that some matrix of V has $\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^T$ as last column. Summing it with a well-chosen matrix of type A_L, we deduce:

- The subspace V contains a matrix

$$J = \begin{bmatrix} a & 0 & 1 \\ C_1' & ? & 0 \\ b & L_1' & 0 \end{bmatrix} \quad \text{with } (a,b) \in \mathbb{K}^2 \text{ and } (L_1', C_1') \in \mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K}).$$

With the above matrices A_L and J, we find that $\dim(e_1^T V) \geq n$. We already knew that $\dim V = \binom{n+1}{2}$ and $\dim V_{lr} = \binom{n}{2}$; hence the rank theorem shows that the map $\mathcal{J}$ from Section 4.2 yields an isomorphism from the subspace of all matrices of V with zero as first row to V_{lr}. Using the structure of V_{lr} with the same method as in the definition of the A_L matrices, we thus find one last important class of matrices in V:

- For every $C \in \mathrm{M}_{n-1,1}(\mathbb{K})$, there is a unique $B_C \in V$ of the form

$$B_C = \begin{bmatrix} 0 & 0 & 0 \\ \psi(C) & 0 & C \\ g(C) & -\alpha\,\overline{C}^T (Z^{-1})^T & \widetilde{L_1} Z^{-1} C \end{bmatrix}$$

  and $g : \mathrm{M}_{n-1,1}(\mathbb{K}) \to \mathbb{K}$ and $\psi : \mathrm{M}_{n-1,1}(\mathbb{K}) \to \mathrm{M}_{n-1,1}(\mathbb{K})$ are linear maps.

Remark 4. The above matrices span V: a straightforward computation shows indeed that the linear subspaces $\{A_L \mid L \in \mathrm{M}_{1,n-1}(\mathbb{K})\}$, $\{B_C \mid C \in \mathrm{M}_{n-1,1}(\mathbb{K})\}$, $\{E_U \mid U \in V_m\}$ and $\mathrm{span}(J)$ are independent, and the sum of their dimensions is $(n-1) + (n-1) + \binom{n-1}{2} + 1 = \binom{n+1}{2} = \dim V$.

From now on, our main task is to refine our understanding of the matrices of the types A_L, B_C, E_U and J: the basic strategy is to form well-chosen linear combinations of those special matrices and use the fact that none of them may have a non-zero eigenvalue. Most of the time, we will simply apply the fact that both V and V^T act totally intransitively on $\mathbb{K}^{n+1}$. Let us start by considering the maps φ and ψ in the A_L and B_C matrices.

Claim 2. The maps φ and ψ are scalar multiples of the identity.

Proof. Let $C \in \mathrm{M}_{n-1,1}(\mathbb{K})$ and $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$. Denote by x (resp. y) the vector of $\mathrm{span}(e_2, \ldots, e_n)$ with coordinate matrix C (resp. L^T) in the basis $(e_2, \ldots, e_n)$. We prove that

$$LC = 0 \Rightarrow \big(\varphi(L)\,C = 0 \text{ and } L\,\psi(C) = 0\big). \tag{5}$$

Assume that $LC = 0$. Notice then that both A_L and B_C stabilize the plane $\mathrm{span}(x, e_{n+1})$ and that the respective matrices of their induced endomorphisms in the basis (x, e_{n+1}) are $\begin{bmatrix} 0 & 0 \\ \varphi(L)C & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ t_1 & t_2 \end{bmatrix}$ for some $(t_1, t_2) \in \mathbb{K}^2$. Since V has a trivial spectrum, we deduce that

$$\forall \lambda \in \mathbb{K}, \quad \begin{vmatrix} 1 & 1 \\ t_1 + \lambda\,\varphi(L)C & 1 + t_2 \end{vmatrix} \neq 0;$$

hence $\varphi(L)C = 0$.

Similarly, notice that A_L^T and B_C^T both stabilize $\mathrm{span}(e_1, y)$ and the respective matrices of their induced endomorphisms in the basis (e_1, y) are $\begin{bmatrix} 0 & s_1 \\ 1 & s_2 \end{bmatrix}$ and

$\begin{bmatrix} 0 & L\psi(C) \\ 0 & 0 \end{bmatrix}$ for some $(s_1, s_2) \in \mathbb{K}^2$. With the above line of reasoning, we deduce that $L\psi(C) = 0$.

We may now conclude the proof. For the non-degenerate bilinear mapping $(L, C) \mapsto LC$ on $\mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K})$, we deduce from (5) that φ stabilizes the orthogonal subspace of every linear hyperplane of $\mathrm{M}_{1,n-1}(\mathbb{K})$; hence φ stabilizes every 1-dimensional linear subspace of $\mathrm{M}_{1,n-1}(\mathbb{K})$, which shows that φ is a scalar multiple of the identity. With the same line of reasoning, we see that ψ is also a scalar multiple of the identity. $\square$

We now have two scalars λ and μ such that:

$$\forall L \in \mathrm{M}_{1,n-1}(\mathbb{K}), \quad A_L = \begin{bmatrix} 0 & L & 0 \\ -Z\overline{L}^T & \widetilde{C_1}L & 0 \\ f(L) & \lambda L & 0 \end{bmatrix}$$

and

$$\forall C \in \mathrm{M}_{n-1,1}(\mathbb{K}), \quad B_C = \begin{bmatrix} 0 & 0 & 0 \\ \mu C & 0 & C \\ g(C) & -\alpha\,\overline{C}^T(Z^{-1})^T & \widetilde{L_1}Z^{-1}C \end{bmatrix}.$$

Claim 3. The map h vanishes everywhere on V_m.

Proof. Choose $t \in \mathbb{K}$ such that $\mu + t \neq 0$ and $a + t \neq 0$ (this is feasible since $\#\mathbb{K} \geq 3$). Observe then that

$$\begin{cases} \forall U \in V_m, & E_U(e_1 + t\,e_{n+1}) = \begin{bmatrix} 0 \\ [0]_{(n-1)\times 1} \\ h(U) \end{bmatrix}, \\ \forall C \in \mathrm{M}_{n-1,1}(\mathbb{K}), & B_C(e_1 + t\,e_{n+1}) = \begin{bmatrix} 0 \\ (\mu + t)\,C \\ ? \end{bmatrix}, \\ & J(e_1 + t\,e_{n+1}) = \begin{bmatrix} a + t \\ ? \\ ? \end{bmatrix}. \end{cases}$$

However $V(e_1 + te_{n+1})$ is a strict linear subspace of $\mathbb{K}^{n+1}$. Judging from the vectors $B_C(e_1 + t\,e_{n+1})$ and the vector $J(e_1 + t\,e_{n+1})$, we deduce that $V(e_1 + te_{n+1})$ cannot contain e_{n+1}. This shows that $h(U) = 0$ for every $U \in V_m$. $\square$

It follows that

$$\forall U \in V_m, \quad E_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & \widetilde{L_1}Z^{-1}U & 0 \end{bmatrix}.$$

From there, we need to study the glued and unglued cases separately.

4.4. **The case where V_{ul} and V_{lr} are glued.** In this section, we assume $p = q = n$. In this case, we simply have $\widetilde{L_1} = L_1$, $\widetilde{C_1} = C_1$, $Z = R_1 = R_s$, $V_m = Z\,\mathrm{A}_{n-1}(\mathbb{K})$ and $\forall (L, C) \in \mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K})$, $\overline{L} = L$ and $\overline{C} = C$. Our aim is to prove that V is equivalent to $\mathrm{A}_{n+1}(\mathbb{K})$.

Claim 4. One has

$$\forall (L,C) \in \mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K}), \quad f(L) = -L_1 L^T \quad \text{and} \quad g(C) = \mu L_1 Z^{-1} C.$$

Proof. Let $t \in \mathbb{K} \smallsetminus \{-a\}$. Note that $J(e_1 + te_{n+1})$ has $a+t$ as first entry, whereas

$$\begin{cases} \forall L \in \mathrm{M}_{1,n-1}(\mathbb{K}), \quad A_L(e_1 + te_{n+1}) = \begin{bmatrix} 0 \\ -ZL^T \\ f(L) \end{bmatrix}, \\ \forall C \in \mathrm{M}_{n-1,1}(\mathbb{K}), \quad B_C(e_1 + te_{n+1}) = \begin{bmatrix} 0 \\ (\mu+t)C \\ g(C) + tL_1 Z^{-1} C \end{bmatrix}. \end{cases}$$

Judging from $J(e_1 + te_{n+1})$, the vector space $V(e_1 + te_{n+1})$ cannot contain $\operatorname{span}(e_2, \dots, e_{n+1})$. Thus $V(e_1 + te_{n+1}) \cap \operatorname{span}(e_2, \dots, e_{n+1}) = \big\{ A_L(e_1 + te_{n+1}) \mid L \in \mathrm{M}_{1,n-1}(\mathbb{K}) \big\}$ (since the first space has a dimension less than n and obviously contains the second one). Using the B_C matrices, it follows that

$$\forall C \in \mathrm{M}_{n-1,1}(\mathbb{K}), \quad g(C) + tL_1 Z^{-1} C = (\mu + t)\, f\big(-C^T (Z^{-1})^T\big).$$

Since this holds for several values of t, we deduce that

$$\forall C \in \mathrm{M}_{n-1,1}(\mathbb{K}), \quad g(C) = -\mu f(C^T (Z^{-1})^T) \quad \text{and} \quad L_1 Z^{-1} C = -f\big(C^T (Z^{-1})^T\big),$$

which obviously yields the claimed results. □

Therefore, for any $(L, C, U) \in \mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K}) \times Z\, \mathrm{A}_{n-1}(\mathbb{K})$, we have

$$A_L = \begin{bmatrix} 0 & L & 0 \\ -ZL^T & C_1 L & 0 \\ -L_1 L^T & \lambda L & 0 \end{bmatrix}; \quad B_C = \begin{bmatrix} 0 & 0 & 0 \\ \mu C & 0 & C \\ \mu L_1 Z^{-1} C & -\alpha C^T (Z^{-1})^T & L_1 Z^{-1} C \end{bmatrix}$$

and

$$E_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & L_1 Z^{-1} U & 0 \end{bmatrix}.$$

Now set

$$T := \begin{bmatrix} 1 & 0 & 0 \\ C_1 & Z & 0 \\ \lambda & L_1 & \alpha \end{bmatrix} \in \mathrm{GL}_{n+1}(\mathbb{K}) \quad \text{and} \quad T' := \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{n-1} & 0 \\ -\mu & 0 & 1 \end{bmatrix} \in \mathrm{GL}_{n+1}(\mathbb{K}).$$

A straightforward computation shows that, for every $(L, C, U) \in \mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K}) \times (Z\, \mathrm{A}_{n-1}(\mathbb{K}))$:

$$T^{-1} A_L\, T' = \begin{bmatrix} 0 & L & 0 \\ -L^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad T^{-1} B_C\, T' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & Z^{-1} C \\ 0 & -(Z^{-1} C)^T & 0 \end{bmatrix}$$

and

$$T^{-1} E_U\, T' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Z^{-1} U & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $T^{-1} V T'$ contains a linear hyperplane of $\mathrm{A}_{n+1}(\mathbb{K})$. Since V acts totally intransitively on $\mathbb{K}^{n+1}$, this is also the case of $T^{-1} V T'$; hence Lemma 15 shows that $T^{-1} V T' = \mathrm{A}_{n+1}(\mathbb{K})$. We deduce that V is equivalent to $\mathrm{A}_{n+1}(\mathbb{K})$ and may

thus be written as $Y\,\mathrm{A}_{n+1}(\mathbb{K})$ for some $Y \in \mathrm{GL}_{n+1}(\mathbb{K})$, and Lemma 10 yields that Y is non-isotropic. This completes the case where V_{ul} and V_{lr} are glued.

4.5. **The case where V_{ul} and V_{lr} are unglued.** Here, we assume that $p < n$ or $q < n$. Note that this means that $p = 1$ or $q = 1$ or there are several diagonal blocks $R_1\,\mathrm{A}_{i_1}(\mathbb{K}), \dots, R_s\,\mathrm{A}_{i_s}(\mathbb{K})$ in the block decomposition of V_m discussed earlier. Note in particular that $p + q \leq n + 1$.

Our aim is to prove that V is reducible. Since the matrices A_L, B_C, E_U and J span V, it suffices to find a non-trivial linear subspace of $\mathbb{K}^{n+1}$ which is stabilized by all of them. In that prospect, we start by analyzing f and g.

Claim 5. One has $f = 0$, and $g(C) = 0$ for every $C \in \mathrm{M}_{n-1,1}(\mathbb{K})$ such that $\overline{C} = 0$.

Proof. We start by proving that

$$\forall L \in \mathrm{M}_{1,n-1}(\mathbb{K}), \quad \overline{L} = 0 \Rightarrow f(L) = 0$$

and

$$\forall C \in \mathrm{M}_{n-1,1}(\mathbb{K}), \quad \overline{C} = 0 \Rightarrow g(C) = 0.$$

We choose $t \in \mathbb{K}$ such that $\mu + t \neq 0$ and $a + t \neq 0$. Then, for every $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$ such that $\overline{L} = 0$, one has

$$A_L(e_1 + te_{n+1}) = \begin{bmatrix} 0 \\ [0]_{(n-1)\times 1} \\ f(L) \end{bmatrix};$$

hence $f(L) = 0$ with the same argument as in the proof of Claim 3.

Now choose $x \in \mathbb{K}$ such that $\lambda + x \neq 0$ and $x \neq 0$. Then

$$\begin{cases} \forall L \in \mathrm{M}_{1,n-1}(\mathbb{K}), & (xe_1 + e_{n+1})^T A_L = \begin{bmatrix} f(L) & (\lambda + x)L & 0 \end{bmatrix}, \\ & (xe_1 + e_{n+1})^T J = \begin{bmatrix} ? & [?]_{1\times(n-1)} & x \end{bmatrix}. \end{cases}$$

Since $V^T(xe_1 + e_{n+1})$ is a strict linear subspace of $\mathbb{K}^{n+1}$, those matrices show that e_1 cannot belong to $V^T(xe_1 + e_{n+1})$. However

$$\forall C \in \mathrm{M}_{n-1,1}(\mathbb{K}),\ (xe_1 + e_{n+1})^T B_C = \begin{bmatrix} g(C) & -\alpha\,\overline{C}^T(Z^{-1})^T & \widetilde{L_1}Z^{-1}C \end{bmatrix}.$$

Therefore, if $\overline{C} = 0$, then $\widetilde{L_1}Z^{-1}C = 0$ and hence $g(C) = 0$.

Let $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$. The column matrix $C := Z\overline{L}^T$ has null entries starting from the p-th, and since $p + q \leq n + 1$, this yields $\overline{C} = 0$. Therefore $g(C) = 0$ and

$$\big((\mu + t)A_L + B_C\big)(e_1 + te_{n+1}) = \begin{bmatrix} 0 \\ [0]_{(n-1)\times 1} \\ (\mu + t)f(L) \end{bmatrix}.$$

The above argument then shows that $f(L) = 0$. □

In particular, we have

$$\forall L \in \mathrm{M}_{1,n-1}(\mathbb{K}),\ A_L = \begin{bmatrix} 0 & L & 0 \\ -Z\overline{L}^T & \widetilde{C_1}L & 0 \\ 0 & \lambda L & 0 \end{bmatrix}.$$

We now distinguish between two cases, whether $p < n$ or $p = n$.

Claim 6. If $p < n$, then $Ve_1 \subset \mathrm{span}(e_1, \dots, e_p)$.

Proof. Assume that $p < n$.

Write

$$C_1' = \begin{bmatrix} c_1' \\ \vdots \\ c_{n-1}' \end{bmatrix}.$$

Let $i \in [\![p, n-1]\!]$ (note that such an integer exists).

Let $(x, y, z) \in \mathbb{K}^3$ be such that $x + \lambda z \neq 0$. Denote by $C_i'' \in \mathrm{M}_{n-1,1}(\mathbb{K})$ the column matrix with all entries 0 except the i-th which equals 1. Note that, for every $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$, both column matrices $\widetilde{C_1}$ and $Z\overline{L}^T$ have zero entries starting from the p-th: for $\widetilde{C_1}$, this comes from its very definition; for $Z\overline{L}^T$, this is obvious if $p = 1$ because then $\overline{L} = 0$; otherwise this comes from the fact that Z stabilizes $\mathbb{K}^{p-1} \times \{0\} \subset \mathbb{K}^{n-1}$ (as $i_1 = p - 1$) and that $\overline{L} \in \mathbb{K}^{p-1} \times \{0\}$. It follows that the $(i+1)$-th row of every A_L matrix is zero. Setting $\gamma := \widetilde{L_1} Z^{-1} C_i''$, we therefore have:

$$\begin{cases} \forall L \in \mathrm{M}_{1,n-1}(\mathbb{K}), & (xe_1 + ye_{i+1} + ze_{n+1})^T A_L = \begin{bmatrix} 0 & (x + \lambda z)\, L & 0 \end{bmatrix}, \\ & (xe_1 + ye_{i+1} + ze_{n+1})^T J = \begin{bmatrix} ax + c_i' y + bz & [?]_{1\times(n-1)} & x \end{bmatrix}, \\ & (xe_1 + ye_{i+1} + ze_{n+1})^T B_{C_i''} \\ & \qquad = \begin{bmatrix} \mu y + g(C_i'')z & [?]_{1\times(n-1)} & y + \gamma z \end{bmatrix}. \end{cases}$$

Since $V^T(xe_1 + ye_{i+1} + ze_{n+1}) \neq \mathbb{K}^{n+1}$, we deduce that

$$\begin{vmatrix} ax + c_i' y + bz & x \\ \mu y + g(C_i'')z & y + \gamma z \end{vmatrix} = 0.$$

Notice that, with an arbitrary $(y, z) \in \mathbb{K}^2$ being fixed, the above equation is linear in x and has several solutions; hence

$$(c_i' y + bz)(y + \gamma z) = 0 \quad \text{and} \quad a(y + \gamma z) - (\mu y + g(C_i'')z) = 0.$$

Both equations have a degree less than or equal to 2 in both variables. Since $\#\,\mathbb{K} > 2$, we deduce that

$$c_i' = 0; \quad c_i'\gamma + b = 0; \quad a = \mu \quad \text{and} \quad a\gamma = g(C_i'').$$

Therefore $a = \mu$, $b = 0$ and $c_p' = \cdots = c_{n-1}' = 0$. Since Z is non-singular and stabilizes $\mathbb{K}^{p-1} \times \{0\}$, we may thus find $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$ such that $C_1' = Z\overline{L}^T$. The first column of $A_L + J$ is

$$\begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix};$$

therefore $a = 0$ (because $A_L + J$ has no non-zero eigenvalue). It follows that $\mu = 0$ and $g(C_i'') = 0$ for every $i \in [\![p, n-1]\!]$. Since g is linear, $g(C) = 0$ whenever $\overline{C} = 0$ (by Claim 5), and $p - 1 < n - q + 1$, we deduce that $g = 0$.

For any matrix of type A_L, B_C, E_U or J, we have therefore found that its first column has null entries starting from the $(p+1)$-th. This yields our claim since these matrices span V. □

Claim 7. Assume that $p = n$ (and therefore $q = 1$). Then $\lambda = b = 0$ and $L_1' = 0$.

This shows that all the matrices A_L, B_C, E_U and J have zero as last row in the case $p = n$.

Proof. Since $q = 1$, one has $\widetilde{L_1} = 0$, while $\overline{C} = 0$ for every $C \in \mathrm{M}_{1,n-1}(\mathbb{K})$. This leads to $f = 0$ and $g = 0$ by Claim 5.

Therefore $\forall (L, C) \in \mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K})$,

$$A_L = \begin{bmatrix} 0 & L & 0 \\ -ZL^T & ? & 0 \\ 0 & \lambda L & 0 \end{bmatrix} \quad \text{and} \quad B_C = \begin{bmatrix} 0 & 0 & 0 \\ \mu C & 0 & C \\ 0 & 0 & 0 \end{bmatrix}.$$

Write $L_1' = \begin{bmatrix} l_1' & \cdots & l_{n-1}' \end{bmatrix}$. Let $i \in [\![1, n-1]\!]$. Denote by $L_i'' \in \mathrm{M}_{1,n-1}(\mathbb{K})$ the row matrix with all entries zero except the i-th, which equals one.

Let $(x, z) \in \mathbb{K}^2$ such that $\mu x + z \neq 0$. Then

$$\begin{cases} \forall C \in \mathrm{M}_{n-1,1}(\mathbb{K}), \quad B_C(xe_1 + e_{i+1} + ze_{n+1}) = \begin{bmatrix} 0 \\ (\mu x + z)C \\ 0 \end{bmatrix}, \\ \qquad\qquad\qquad\qquad A_{L_i''}(xe_1 + e_{i+1} + ze_{n+1}) = \begin{bmatrix} 1 \\ [?]_{(n-1)\times 1} \\ \lambda \end{bmatrix}, \\ \qquad\qquad\qquad\qquad J(xe_1 + e_{i+1} + ze_{n+1}) = \begin{bmatrix} ax + z \\ [?]_{(n-1)\times 1} \\ bx + l_i' \end{bmatrix}. \end{cases}$$

We deduce that $\begin{vmatrix} 1 & ax + z \\ \lambda & bx + l_i' \end{vmatrix} = 0$. Since, for a given $x \in \mathbb{K}$, this holds for several values of z, we successively deduce that $\lambda = 0$ and $\forall x \in \mathbb{K},\ b\,x + l_i' = 0$, which yields $\lambda = b = l_i' = 0$. Therefore $L_1' = 0$. $\square$

In two special cases, we may now conclude that V is reducible: if $p = 1$, then Claim 6 shows that $\mathrm{span}(e_1)$ is stabilized by V; if $p = n$, then Claims 5 and 7 show that $\mathrm{span}(e_1, \ldots, e_n)$ is stabilized by V (indeed, in that case, $q = 1$ and hence $\widetilde{L_1} = 0$ and $\overline{C} = 0$ for every $C \in \mathrm{M}_{n-1,1}(\mathbb{K})$).

Assume finally that $1 < p < n$. Then $Ve_1 \subset \mathrm{span}(e_1, \ldots, e_p)$ by Claim 6. Note that the change of basis matrix $R = \begin{bmatrix} I_{n+1-q} & 0 \\ 0 & T_1^T \end{bmatrix}$ from Section 4.2 leaves $\mathrm{span}(e_1, \ldots, e_p)$ invariant as $p \leq n + 1 - q$. Therefore we also have $(R^{-1}VR)e_1 \subset \mathrm{span}(e_1, \ldots, e_p)$, and some of our recent findings may be summed up as follows:

Proposition 19. *Let V be a maximal subspace of $\mathrm{M}_{n+1}(\mathbb{K})$ with a trivial spectrum such that:*

(i) $Ve_{n+1} = \mathrm{span}(e_1, \ldots, e_n)$;

(ii) there are lower-triangular non-isotropic matrices $P \in \mathrm{GL}_p(\mathbb{K})$, $P_2 \in \mathrm{GL}_{n_2}(\mathbb{K})$, $\ldots, P_r \in \mathrm{GL}_{n_r}(\mathbb{K})$, with $1 < p < n$, such that $V_{ul} = P\,\mathrm{A}_p(\mathbb{K}) \vee P_2\,\mathrm{A}_{n_2}(\mathbb{K}) \vee \cdots \vee P_r\,\mathrm{A}_{n_r}(\mathbb{K})$.

Then $Ve_1 \subset \mathrm{span}(e_1, \ldots, e_p)$.

Note that the fact that V contains no matrix with column space $\mathrm{span}(e_{n+1})$, our starting point in Section 4.2, is a consequence of assumptions (i) and (ii) of

Proposition 19 (using the rank theorem to compute the dimension of V from that of V_{ul}, as in the beginning of Section 4.2).

Now, all we need to complete the unglued case is to show that any V satisfying the assumptions of Proposition 19 is reducible. Let V be such a subspace, with the above notation. Let $x \in \operatorname{span}(e_1, \dots, e_p) \smallsetminus \{0\}$. Recall that the bilinear form $b : (X, Y) \in (\mathbb{K}^p)^2 \mapsto X^T P Y$ is non-isotropic, and hence non-degenerate. Denote by X_0 the matrix of coordinates of x in $(e_1, \dots, e_p)$. In the hyperplane $H := \{Y \in \mathbb{K}^p : X_0^T Y = 0\}$, we may therefore find a "right-sided orthogonal basis" $(f_2, \dots, f_p)$, i.e. $b(f_i, f_j) = 0$ for every $(i, j) \in [\![2, p]\!]^2$ with $i < j$. We then choose a non-zero vector f_1 such that $b(f_1, f_j) = 0$ for every $j \in [\![2, p]\!]$. It follows that $(f_1, \dots, f_p)$ is a basis of $\mathbb{K}^p$. Denoting by S the matrix of coordinates of $(f_1, f_2, \dots, f_p)$ in $(e_1, \dots, e_p)$, the matrix $P' := S^T P S$ is lower-triangular and

$$S^T \big(P \, \mathrm{A}_p(\mathbb{K})\big)(S^T)^{-1} = P' \, \mathrm{A}_p(\mathbb{K}).$$

Then set $T_2 := \begin{bmatrix} S^T & 0 \\ 0 & I_{n+1-p} \end{bmatrix} \in \mathrm{GL}_{n+1}(\mathbb{K})$ and

$$V' := T_2 \, V \, T_2^{-1}.$$

Notice finally that T_2 stabilizes $\operatorname{span}(e_1, \dots, e_n)$, fixes e_{n+1}, and obviously

$$V'_{ul} = P' \, \mathrm{A}_p(\mathbb{K}) \vee P_2 \, \mathrm{A}_{n_2}(\mathbb{K}) \vee \cdots \vee P_r \, \mathrm{A}_{n_r}(\mathbb{K}).$$

Thus Proposition 19 applied to V' shows that $V' e_1 \subset \operatorname{span}(e_1, \dots, e_p)$. However S maps $\operatorname{span}(e_2, \dots, e_p)$ to $\operatorname{span}(f_2, \dots, f_p)$; hence $S^T X_0 \in \operatorname{span}(e_1) \smallsetminus \{0\}$. This yields

$$V x \subset \operatorname{span}(e_1, \dots, e_p).$$

We conclude that $\operatorname{span}(e_1, \dots, e_p)$ is a non-trivial invariant subspace for V; hence V is reducible. This completes our proof of Theorem 3.

5. On large spaces of nilpotent matrices

In this short section, we show that the following famous theorem of Gerstenhaber on linear subspaces of nilpotent matrices is an easy consequence of Theorem 4:

Theorem 20 (Gerstenhaber's theorem)**.** *Let $\mathbb{K}$ be a field with at least three elements, and V be a linear subspace of $\mathrm{M}_n(\mathbb{K})$ such that $\dim V = \binom{n}{2}$ and every matrix of V is nilpotent. Then V is similar to $\mathrm{NT}_n(\mathbb{K})$.*

See [6] for the original proof under the more restrictive assumption $\#\mathbb{K} \geq n$, [7] for a very elegant proof using trace maps and a theorem of Jacobson, and [14] for a proof with no restriction on the cardinality of $\mathbb{K}$.

Proof. The assumptions show that V is a maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with a trivial spectrum. Then $V \simeq P_1 \, \mathrm{A}_{n_1}(\mathbb{K}) \vee \cdots \vee P_p \, \mathrm{A}_{n_p}(\mathbb{K})$ for non-isotropic matrices $P_1, \dots, P_p$. Since every matrix of V is nilpotent, every matrix of $P_k \, \mathrm{A}_{n_k}(\mathbb{K})$ is nilpotent for every $k \in [\![1, p]\!]$.

Let $q \geq 2$ be a positive integer and $P \in \mathrm{GL}_q(\mathbb{K})$, and assume that P is non-isotropic and every element of $P \, \mathrm{A}_q(\mathbb{K})$ is nilpotent. Note that q is odd since $\mathrm{A}_q(\mathbb{K})$ contains non-singular matrices when q is even. Then $\operatorname{tr}(PA) = 0$ for every $A \in \mathrm{A}_q(\mathbb{K})$, which shows that P is symmetric. Since q is odd and P is non-singular, P is not alternate; hence it is congruent to a non-singular diagonal matrix D (even

if $\mathbb{K}$ has characteristic 2; see [9, Chapter 35]). Thus $D\,\mathrm{A}_q(\mathbb{K})$ is similar to $P\,\mathrm{A}_q(\mathbb{K})$ and must therefore have a trivial spectrum. Finally, set $K := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and

$$A := \begin{bmatrix} K & [0]_{2\times(q-2)} \\ [0]_{(q-2)\times 2} & [0]_{q-2} \end{bmatrix} \in \mathrm{A}_q(\mathbb{K}),$$

and note that DA is obviously non-nilpotent, a contradiction.

Returning to V, we deduce that $n_1 = \cdots = n_p = 1$; hence $V \simeq \mathrm{NT}_n(\mathbb{K})$. □

6. On the exceptional case of $\mathbb{F}_2$

In the proof of Theorem 4, we have repeatedly used the assumption that the field $\mathbb{K}$ had at least 3 elements. The reader will therefore not be surprised by the following counterexample, which shows that Theorem 4 fails for the field $\mathbb{F}_2$. Observe first that there is no non-isotropic matrix in $\mathrm{GL}_3(\mathbb{F}_2)$ (since every 3-dimensional quadratic form over a finite field is isotropic); hence no maximal linear subspace of $\mathrm{M}_3(\mathbb{F}_2)$ with a trivial spectrum has the form $P\,\mathrm{A}_3(\mathbb{F}_2)$.

Consider the following matrices of $\mathrm{M}_3(\mathbb{F}_2)$:

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Using the identities $\forall x \in \mathbb{F}_2,\ x + x = 0$ and $x^2 = x$, a straightforward computation yields

$$\forall (x, y, z) \in \mathbb{F}_2^3, \quad \det(I_3 + x\,A + y\,B + z\,C) = 1.$$

Therefore the 3-dimensional subspace $V := \mathrm{span}(A, B, C)$ has a trivial spectrum. The fact that $A + B$ is non-singular shows however that V is irreducible. If V were indeed reducible, then there would exist a 1-dimensional subspace W of $\mathrm{M}_2(\mathbb{F}_2)$ such that $V \simeq \{0\} \vee W$ or $V \simeq W \vee \{0\}$, and in both cases every matrix of V would be singular.

The classification of the irreducible maximal subspaces of $\mathrm{M}_n(\mathbb{F}_2)$ with a trivial spectrum thus remains an unresolved issue.

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