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HOCHSCHILD (CO-)HOMOLOGY OF SCHEMES WITH TILTING OBJECT

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ABSTRACT. Given a k–scheme X that admits a tilting object T, we prove that the Hochschild (co-)homology of X is isomorphic to that of $A = \operatorname{End}_X(T)$. We treat more generally the relative case when X is flat over an affine scheme $Y = \operatorname{Spec} R$, and the tilting object satisfies an appropriate Tor-independence condition over R. Among applications, Hochschild homology of X over Y is seen to vanish in negative degrees, smoothness of X over Y is shown to be equivalent to that of A over R, and for X a smooth projective scheme we obtain that Hochschild homology is concentrated in degree zero. Using the Hodge decomposition of Hochschild homology in characteristic zero, for X smooth over Y the Hodge groups $H^q(X, \Omega^p_{X/Y})$ vanish for $p < q$, while in the absolute case they even vanish for $p \neq q$.

We illustrate the results for crepant resolutions of quotient singularities, in particular for the total space of the canonical bundle on projective space.

CONTENTS

INTRODUCTION

If A and B are derived equivalent algebras over a field k, then their Hochschild theories are isomorphic; see, for example, [38, 33]. If X and Y are smooth complex projective varieties that are derived equivalent through a Fourier–Mukai transformation, their Hochschild theories agree as well; see [18].

Here we consider the case when a scheme X is derived equivalent to an algebra A through a *classical tilting object* in its derived category of quasi-coherent sheaves and establish that the Hochschild theories of X and that of A are naturally isomorphic too.

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More precisely, for k a field, we assume that X is a k–scheme projective over an affine scheme $Z = \operatorname{Spec} K$ of finite type over k and that T is a tilting object on X. In that case, the key theorem of geometric tilting theory, recalled in Theorem 1.8 below, yields through $\mathbb{R}\operatorname{Hom}_X(T, ?)$ an exact equivalence from the derived category of quasicoherent $\mathcal{O}_X$–modules to the derived category of (right) $A = \operatorname{End}_X(T)$–modules.

Our main results, Corollary 3.5 and Theorem 4.1, are that if such a scheme X is flat over an affine k–scheme $Y = \operatorname{Spec} R$ so that the endomorphism ring A of the tilting object is as well flat over R, then the Hochschild (co-)homology of X over Y becomes naturally isomorphic to the Hochschild (co-)homology of A over R. Here, in accordance with the results in [43, 16], Hochschild theory for a flat morphism $X \to Y$ is defined to be the hyper-(co-)homology theory attached to $\Delta_*\mathcal{O}_X$, where $\Delta : X \to X \times_Y X$ is the diagonal embedding, while Hochschild theory of A over R is understood as the (co-)homology theory attached to A as a (right) module over the enveloping algebra $A^{\mathrm{op}} \otimes_R A$, which coincides with Hochschild's original definition when A is projective as an R–module, and in the flat case is the specialization of Quillen's approach in [37].

After briefly reviewing the theory and scope of tilting objects in algebra and geometry in Section 1, we investigate appropriate Tor-independence properties of tilting objects, such as flatness, in Section 2, to establish the invariance of Hochschild cohomology in Section 3 and that of Hochschild homology in Section 4. Section 5 uses the Hodge decomposition for Hochschild homology in characteristic zero to obtain the vanishing of various Hodge groups in the context of tilting, and Section 6 deals with the example of crepant resolutions of some quotient singularities.

As applications we can strengthen some earlier results. For example (see Corollary 4.2), the existence of a tilting object forces $H^i(X, \mathcal{O}_X) = 0$ for $i \neq 0$ independent of the characteristic of k, and for X flat over Y, the *negative* Hochschild homology, $\mathrm{HH}_i(X/Y)$ for $i < 0$, vanishes.

For smooth projective schemes, we even obtain $\mathrm{HH}_i(X) = 0$ for $i \neq 0$; see Theorem 4.3. Further, while smoothness of X was known to imply finite global dimension for A, here we show in Corollary 3.6 that for X and its tilting object T flat over Y, smoothness of X over Y is indeed equivalent to that of A over R, where smoothness in the non-commutative algebraic context is interpreted in the sense of van den Bergh [45].

Employing the Hodge decomposition of Hochschild cohomology in characteristic zero [43, 17], it follows for a smooth morphism $X \to Y$ over the complex numbers that the Hodge groups $H^q(X, \Omega^p_{X/Y})$ vanish for $p < q$, while in the absolute case even $H^q(X, \Omega^p_X) = 0$ for $p \neq q$.

In the final section, we illustrate the results for the total space of the canonical bundle over projective space, a crepant resolution of a quotient singularity for an algebraically closed field of characteristic zero.

We point out that in particular examples, at least in the absolute case of complex smooth projective schemes, invariance of Hochschild (co-)homology or partial consequences thereof have already been known or alluded to by some authors; see, for example, [3, 1.8.2], [19, 2.2] or the introduction of [42].

To end this introduction, we comment on the broader picture. As detailed in [30, Thm. 7.5] on any separated scheme X there exists a perfect complex E such that the derived category of quasicoherent $\mathcal{O}_X$–modules is equivalent to the derived

category of the differential graded algebra (DG algebra) $A = \mathbb{R}\operatorname{Hom}_X(E, E)$. To a category such as the latter, Toën [44, 8.1] assigns a Hochschild theory that is essentially intrinsic in his context. Thus, it should specialize both to the geometric incarnation of the Hochschild theory of X on the one hand and to the algebraic realization of the Hochschild theory of A on the other when E is a classical tilting object.

In the same vein, the flatness or Tor-independence conditions imposed here should be avoidable if one employs the theory of DG algebras and resolvents for morphisms of schemes or analytic spaces as was done in [16]. We decided, however, to present the "classical" version with a direct proof that avoids the formidable technical apparatus certainly necessary for the ultimate treatment of invariance of Hochschild theory under a larger class of exact equivalences.

1. Classical tilting objects

Let k be a field and $\mathcal{T}$ a triangulated k–linear category. With $[i]$, as usual, the ith iteration of the given (translation) auto-equivalence on $\mathcal{T}$, we set $\operatorname{Ext}^i_{\mathcal{T}}(M, N) = \operatorname{Hom}_{\mathcal{T}}(M, N[i])$. Recall that a subcategory of $\mathcal{T}$ is *thick* if it is closed under translations, exact triangles and direct summands, and *localizing* if it is further closed under all *small*, that is, set-indexed, direct sums that exist in $\mathcal{T}$. If U is any object or (full) subcategory of $\mathcal{T}$, we denote $\mathsf{Loc}\, U$ as the smallest localizing subcategory containing U in $\mathcal{T}$.

Concerning functors between triangulated k–linear categories, we only allow those that are *k–linear* and *exact*.

Tilting objects in triangulated categories. We first introduce the concept of a tilting object in a "large" triangulated category. The reason for this is that the generating condition has a formulation that is easier to verify even when one is ultimately only interested in triangulated categories whose size is bounded in some way. The relevant definition is the following.

Definition 1.1. Let $\mathcal{T}$ be a triangulated category that is closed under small direct sums. An object T in $\mathcal{T}$ is *tilting* if it is a *compact generator with only trivial self-extensions*. That is to say,

(1) (*Compactness*) The functor $\operatorname{Hom}_{\mathcal{T}}(T, ?)$ commutes with small direct sums.
(2) (*Generating*) We offer two versions that are equivalent in the presence of (1):
  (a) If N in $\mathcal{T}$ satisfies $\operatorname{Ext}^i_{\mathcal{T}}(T, N) = 0$ for each $i \in \mathbb{Z}$, then $N = 0$.
  (b) The smallest localizing subcategory that contains T is $\mathcal{T}$.
(3) (*Only trivial self-extensions*) $\operatorname{Ext}^i_{\mathcal{T}}(T, T) = 0$ for $i \neq 0$.

Remark 1.2. As to the equivalence of (a) and (b) above, note that for any object N in $\mathcal{T}$ the full subcategory ${}^{\perp}N$ consisting of those objects X from $\mathcal{T}$ with $\operatorname{Ext}^i_{\mathcal{T}}(X, N) = 0$ for each $i \in \mathbb{Z}$ is localizing. Therefore, (b) $\Rightarrow$ (a), as T in ${}^{\perp}N$, then implies $\mathsf{Loc}\, T = \mathcal{T} \subseteq {}^{\perp}N$, which means that the identity morphism on N is zero.

The converse requires T to be compact, thus (1). Namely, if T in $\mathcal{T}$ satisfies (1) and (2)(a), then the set of objects $\bigcup_{n\in\mathbb{Z}} T[n]$ *compactly generates* $\mathcal{T}$ in the sense of the definition in [36, 1.7], and Theorem 2.1.2 in that reference gives (a) $\Rightarrow$ (b).

Remark 1.3. Tilting objects as just defined are nowadays sometimes called "classical" to distinguish them from the more general notion where $\mathcal{T}$ carries a DG

enhancement, and rather than requiring no self-extensions one considers the full DG algebra of endomorphisms in the enhancement. In that case, the target category becomes that of DG modules over the DG algebra. See [30] for further details.

1.4. Despite appearances, (b) is often easier to verify than (a). If we can show that a known (set of) compact generator(s) is contained in $\mathsf{Loc}\,T$, then T is already generating by (b). For example, in practice it is often known beforehand that the triangulated category $\mathcal{T}$ in question is *compactly generated* in the sense that there is a set of compact objects G that satisfies (a) or, equivalently, (b). To then test that a given object T is compact and generates, it suffices to check

(c) The smallest thick subcategory in $\mathcal{T}$ that contains T equals the subcategory $\mathcal{T}^c$ of all compact objects.

Another way to employ the equivalence of the generating conditions is as follows. Assume T in $\mathcal{T}$ satisfies (1) and (2) and let $L : \mathcal{T} \to \mathcal{T}'$ be an exact functor into a triangulated category also closed under small direct sums. Also assume that L commutes with small direct sums, for example, if L admits a right adjoint. It then follows that in $\mathcal{T}'$ the full subcategory $L(\mathcal{T})$ is contained in $\mathsf{Loc}\,L(T)$. Thus, if $L(\mathcal{T})$ contains a generating set of compact objects for $\mathcal{T}'$, then $L(T)$ satisfies (1) and (2) in $\mathcal{T}'$ as soon as it is again compact, and only (3) remains to be verified to establish $L(T)$ as a tilting object in $\mathcal{T}'$.

Example 1.5. (See [32, 33] for a more general account and further references.) For a ring B, denote $D(B)$ as the full derived category of right B–modules.

Assume it is known that the triangulated category $\mathcal{T}$ is equivalent to the derived category of some k–algebra B. In that case $\mathcal{T}$ must contain a tilting object T, as any ring, when considered as a module over itself, is a tilting object in its own derived category.

By Rickard's fundamental result [38], for any tilting object T the category $\mathcal{T}$ is equivalent to $D(\mathrm{End}_{\mathcal{T}}(T))$. To be more precise, assume there is an equivalence $F\colon \mathcal{T} \to D(B)$. As any equivalence, the functor F will preserve compactness, whence $F(T)$ is a *perfect* complex of B–modules, as those complexes are precisely the compact objects in $D(B)$; see [35] or [20, Prop. 9.6]. As the generating property and lack of self-extensions are as well preserved by the equivalence F, Rickard tells us that $? \otimes^{\mathbb{L}}_B F(T)$ provides an equivalence from $D(B)$ onto $D(A')$, where $A' = \mathrm{End}_B(F(T))$. Thus, $F(?) \otimes^{\mathbb{L}}_B F(T)$ provides an equivalence from $\mathcal{T}$ to $D(A')$. Finally note that F induces an isomorphism of algebras $A = \mathrm{End}_{\mathcal{T}}(T) \cong A' = \mathrm{End}_B(F(T))$, whence in summary, $\mathcal{T} \cong D(\mathrm{End}_{\mathcal{T}}(T))$, as claimed.

It follows that tilting objects *detect all k–algebras B* that satisfy $D(B) \cong \mathcal{T}$ as triangulated categories in the sense that the assignment $T \mapsto \mathrm{End}_{\mathcal{T}}(T)$ is a surjection onto the isomorphism classes of those rings B. It would be interesting to understand the fibres of this assignment. Note, for example, that the *Picard group* of $\mathcal{T}$, that is, the group of all auto-equivalences of $\mathcal{T}$, operates on those fibres.

Derived equivalent algebras have isomorphic Hochschild (co-)homology, which thus becomes an invariant of the derived category $\mathcal{T}$, retrievable from the endomorphism algebra of any tilting object.

Tilting objects in geometry. Here we are mainly interested in the situation where $\mathcal{T}$ is a "*geometric*" triangulated category, in that $\mathcal{T} \cong D(X) = D(\mathsf{QCoh}(X))$, the derived category of quasi-coherent sheaves on some scheme X over k. For any noetherian quasi-projective scheme, the triangulated category $D(X)$ is closed under

small direct sums; see e.g. [36, Example 1.3] and the references therein. Moreover, that category is compactly generated, and the compact objects are exactly the perfect complexes, as soon as X is quasi-compact and separated; see [36, Prop. 2.5].

1.6. The category of schemes we will consider consists of those k–schemes X such that the structure morphism $X \to \operatorname{Spec} k$ can be factored as $X \xrightarrow{p} Z \xrightarrow{q} \operatorname{Spec} k$ with p projective and Z an affine scheme of finite type over k. The morphisms between such schemes are the morphisms over $\operatorname{Spec} k$.

For such a scheme X, the triangulated category $D(X)$ is thus in particular k–linear, closed under small direct sums, and compactly generated.

Remark 1.7. Let us point out that the affine scheme Z appearing above plays only an auxiliary role. Indeed, assume Y is any affine scheme over k of finite type and suppose the structure morphism $X \to \operatorname{Spec} k$ factors through a morphism $f : X \to Y$ as well as through a projective morphism $p : X \to Z$ as before. Then $Z' = Y \times_k Z$ is again affine of finite type over k, and the induced morphism $X \to Z'$ factors p and thus, is in turn projective; see [25, Prop. 5.5.5].

This flexibility implies, for example, that the category of schemes under consideration is closed under fibre products over affine schemes of finite type, so that for X, X' in that category and $f : X \to Y, f' : X' \to Y$ morphisms to an affine scheme Y of finite type over k, the fibre product $X \times_Y X'$ again belongs to the category.

In particular, the category is closed under base change by morphisms $Y' \to Y$ of affine schemes of finite type over k. That is, with $f : X \to Y$ a morphism, $X' = X \times_Y Y'$ is again in that category if X is.

The reason to restrict ourselves to the category of schemes in Section 1.6 is the following structural result that has its origin in Beilinson's seminal paper [4] and was developed further through [2], [8, Thm. 6.2] and [11]. The form given here is [30, Thm. 7.6]. To abbreviate, we call T from $D(X)$ a *tilting object on* X if it is one for that triangulated category.

Theorem 1.8. *If X as in Section* 1.6 *admits a tilting object T, then with $A = \operatorname{End}_X(T)$ the following hold:*

(1) *The functor $T_* = \mathbb{R}\operatorname{Hom}_{\mathcal{O}_X}(T, \ )$ induces an equivalence from $D(X)$ to $D(A)$. Its left adjoint $T^* = (\ ) \otimes^{\mathbb{L}}_A T$ provides the inverse equivalence.*
(2) *The equivalence T_* carries $D^b(\mathsf{Coh}(X))$, the bounded derived category of coherent $\mathcal{O}_X$–modules, to $D^b(\mathsf{mod}\, A)$, the bounded derived category of finitely generated right A–modules.*
(3) *If X is smooth, then A has finite global dimension.*

Remark 1.9. If X maps to some affine k–scheme $Y = \operatorname{Spec} R$, then the equivalence T_* is R–linear; thus, A is naturally an R–algebra. If $X \xrightarrow{p} Z = \operatorname{Spec} K \xrightarrow{q} \operatorname{Spec} k$ is a factorization with p projective, as is supposed to exist, then, in view of [26, Thm. 2.4.1(i)], the ring A is a *finite* K–algebra. In particular, as a ring, A is noetherian on either side and (module–)finite over its centre.

We now turn to some basic examples.

Tilting in the absolute case.

1.10. If X is already projective over the field k and T is a tilting object on it, then $A = \operatorname{End}_X(T)$ is a finite-dimensional k–algebra, and so its Grothendieck group

$K_0(A)$ of finitely generated modules is free abelian of finite rank. In view of the equivalence, this is then also isomorphic to the Grothendieck group $K_0(X)$ of coherent $\mathcal{O}_X$–modules, and so the type of projective varieties that can carry a tilting object is severely restricted by the requirement that $K_0(X)$ be free abelian of finite rank.

1.11. If the field k is algebraically closed, then (see, for example, [1, p. 35f]) any finite-dimensional k–algebra A is Morita–equivalent to a *basic* algebra, an algebra A' with a complete set $\{e_i\}_{i=1,\dots,N}$ of primitive orthogonal idempotents such that $e_iA' \cong e_jA'$ as right A–modules only if $i = j$. The modules e_iA' are then, up to an A'–module isomorphism, the unique indecomposable (right) projective A'–modules, and, with $\operatorname{rad} A'$ the radical of A', the modules $S_i = e_iA'/e_i \operatorname{rad} A'$ represent precisely the different isomorphism classes of simple A'–modules. Their respective classes form an integral basis of the Grothendieck group $K_0(A')$, isomorphic to $\mathbb{Z}^N$.

Further information is encoded in the *quiver* attached to A', with vertices labeled by the indices $i = 1, \dots, N$, with the number of arrows from the vertex j to the vertex i equal to the (finite) dimension over k of $\operatorname{Ext}^1_{A'}(S_j, S_i)$.

For T a tilting object on X and $A = \operatorname{End}_X(T)$, combining the inverse of the Morita–equivalence $D(A) \xrightarrow{\cong} D(A')$ with the inverse T^* to T_* maps each e_iA' to an indecomposable direct summand E_i of the initial tilting object T, and the direct sum $T' = \bigoplus_{i=1}^N E_i$ is again a tilting object on X. The difference between T and T' is just that T may contain several copies of the same object E_i, that is, $T \cong \bigoplus_{i=1}^N E_i^{n_i}$ for suitable integers $n_i > 0$. As the number N of the pairwise non-isomorphic indecomposable summands equals the rank of the free abelian group $K_0(X)$, it is an invariant of the scheme.

1.12. In the literature, instead of the tilting object, often the set $\mathbb{E} = \{E_1, \dots, E_N\}$ of its indecomposable, pairwise non-isomorphic direct summands is considered. Many authors have studied the special case, when this set forms further what is also known as a *full, strongly exceptional collection* on X, in that in addition to $T = \bigoplus_i E_i$ being a tilting object, it is asked that the indices be (partially) ordered so that $\operatorname{Hom}_X(E_j, E_i) = 0$ for $j > i$, and the summands E_i are furthermore required to be *simple*, meaning $\mathbb{R}\operatorname{Hom}_X(E_i, E_i) \cong k[0]$ for each i; see, for example, [39, 8]. Finally, let us also mention the weaker notion of full exceptional collections, where it is only required that $\operatorname{Ext}^n_X(E_j, E_i) = 0$ for any n when $j > i$. In this situation (see [8]) the sequence $\mathbb{E}$ induces on the triangulated category of coherent sheaves on X an *admissible filtration* with layers that are semi-simple triangulated categories, but the whole derived category is not guaranteed to be of the form $D(A)$ for some algebra A.

Example 1.13. Smooth projective varieties, at least over the complex numbers, that admit a tilting object include projective spaces, quadrics, Fano surfaces, various toric varieties [29] and a sample of rational homogeneous varieties [7, 40, 41], as well as products of such varieties [7], and (iterated) projective bundles over any of these [21].

Dubrovin [22, 4.2.2] predicts in the context of complex varieties the existence of a full, strongly exceptional collection for (smooth, projective) Fano varieties exactly when their *quantum cohomology is semi-simple*; see [3] for further comments.

For rational homogeneous manifolds $X = G/P$, with G a connected complex semi-simple Lie group and $P \subseteq G$ a parabolic subgroup, Catanese conjectures (see

[7]) there should exist a tilting object, namely even a full strongly exceptional poset indexed by the Bruhat-Chevalley partial order of Schubert varieties in X.

Remark 1.14. If $\mathcal{T}$ is equivalent to the derived category of a k–algebra as in Example 1.5, there are usually many non-isomorphic, even Morita non-equivalent such algebras; see [32] for a more detailed discussion. However, in the situation of Theorem 1.8, the scheme X will often be unique up to isomorphism in view of the reconstruction theorem by Bondal and Orlov [10].

Local or open Calabi-Yau varieties.

1.15. Other intriguing examples where Theorem 1.8 applies are provided by some *local* or *open Calabi-Yau varieties*. These include crepant resolutions of quotient singularities $\mathbb{C}^n/G$, for G a finite subgroup of $SL_n(\mathbb{C})$; see [30, 7.2 ff.] for a detailed discussion of what is known or conjectured. The twisted group algebra $\mathbb{C}[z_1, ..., z_n]*G$ then appears as the (suspected) endomorphism algebra of a tilting object.

A second class of such examples arises as the total space of the canonical line bundle on those smooth projective Fano varieties that themselves carry a tilting object as the foundation of a *geometric helix*. The endomorphism ring is then a "*rolled-up helix algebra*", a term coined by Bridgeland; see [13, Thm. 3.6] and [12] for further details.

The canonical bundle over projective space falls into both the classes just mentioned, and we use it as the running example to illustrate our results below. Thus, we spend a few lines to review this case, referring to the indicated references for details.

1.16. Let $\mathbb{P} = \mathbb{P}^{n-1} = \mathbb{P}_k(V)$ be the projective space defined by an n–dimensional vector space V over the field k, with $n \geqslant 2$.

In Beilinson's paper [4] that started it all, the author exhibited two tilting objects on such a projective space $\mathbb{P}$, namely

$$T_0 = \bigoplus_{i=0}^{n-1} \mathcal{O}_{\mathbb{P}}(i-n+1) \quad \text{and} \quad T_1 = \bigoplus_{i=0}^{n-1} \Omega^i_{\mathbb{P}}(i),$$

with $\Omega^i_{\mathbb{P}}$ the $\mathcal{O}_{\mathbb{P}}$-module of differential forms of degree i. The associated endomorphism algebras are

$$A_0 = \operatorname{End}_{\mathbb{P}}(T_0) \cong \bigoplus_{i,j=0}^{n-1} \operatorname{Hom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(j-n+1), \mathcal{O}_{\mathbb{P}}(i-n+1)) \cong \bigoplus_{i,j=0}^{n-1} \mathbb{S}\mathrm{ym}_{i-j}(V),$$

$$A_1 = \operatorname{End}_{\mathbb{P}}(T_1) \cong \bigoplus_{i,j=0}^{n-1} \operatorname{Hom}_{\mathbb{P}}(\Omega^j_{\mathbb{P}}(j), \Omega^i_{\mathbb{P}}(i)) \cong \bigoplus_{i,j=0}^{n-1} \Lambda^{j-i}(V^*),$$

where V^* denotes the k–dual vector space. Either algebra can be viewed as a quiver algebra on n vertices labeled, say, $0, ..., n-1$, with arrows from i to $i+1$ corresponding to (a basis of) V, respectively V^*, and with quadratic relations given respectively by the kernels of the natural maps $V \otimes V \to \mathbb{S}\mathrm{ym}_2 V$ and $V^* \otimes V^* \to \Lambda^2 V^*$.

There are many more tilting objects, even full strongly exceptional sequences on $\mathbb{P}$; see [12] for a recent discussion.

1.17. Bondal [8, 9] showed that the algebras A_0 and A_1 above are Koszul-duals of each other, in the sense that $A_{1-i} \cong A_i^! = \operatorname{Ext}^\bullet_{A_i}(A_i/\operatorname{rad} A_i, A_i/\operatorname{rad} A_i)$ for $i = 0, 1$, where $\operatorname{rad} A$ again denotes the radical of the algebra A. Note that in either case, the semi-simple k–algebra $A_i/\operatorname{rad} A_i$ is just a product of n copies of the base field k with componentwise operations. In particular, either algebra A_i is artinian, Koszul and of finite global dimension equal to $n-1$.

1.18. Now let $\pi : X \to \mathbb{P}$ be the (affine) canonical bundle, the total space of the line bundle $\omega_{\mathbb{P}} = \Omega^{n-1}_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(-n)$ over $\mathbb{P}$. Note that this means by convention $\pi_*\mathcal{O}_X \cong \mathbb{S}\mathrm{ym}_{\mathbb{P}}(\omega_{\mathbb{P}}^{-1})$; thus, $X = \mathbb{V}_{\mathbb{P}}(\omega_{\mathbb{P}}^{-1})$ in the notation of [25]. The smooth variety X is a *local*, also called *open Calabi-Yau variety* in that its canonical bundle in turn is trivial, $\omega_X = \Omega^{n+1}_X \cong \mathcal{O}_X$.

As noted by Bridgeland [12, 13], any tilting object T given by a full strongly exceptional sequence on $\mathbb{P}$ pulls back to a tilting object π^*T on X. While π is an affine map, contracting the zero section in the affine bundle X yields a projective map $p : X \to Z = \operatorname{Spec} K$, where

$$K = \bigoplus_{m \geqslant 0} H^0(\mathbb{P}, (\omega_{\mathbb{P}}^{-1})^{\otimes m}) \cong \bigoplus_{m \geqslant 0} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(mn)) \cong k[x_1, ..., x_n]^{(n)}$$

is the nth Veronese subring of the polynomial ring $S = \mathbb{S}\mathrm{ym}(V) \cong k[x_1, ..., x_n]$, spanned by all polynomials homogeneous of degree a multiple of n. If the characteristic of k does not divide n and if k further contains the nth roots of unity, one may identify K as well as the invariant ring under the action of the cyclic group μ_n, generated by the corresponding roots of unity, acting diagonally on the variables x_i. That is to say, $K \cong S^{\mu_n}$. The ring K is evidently of finite type over k, and so Theorem 1.8 applies. Note that X is the natural, and crepant, desingularisation of the isolated singularity of Z, whence it also fits into the first class of examples mentioned in Section 1.15.

1.19. The endomorphism ring of π^*T_0 on the canonical bundle X is easy to describe:

$$B_0 = \operatorname{End}_X(\pi^*T_0) \cong \bigoplus_{i,j=0}^{n-1} S(i-j)^{(n)}.$$

It can be viewed as the algebra of $(n \times n)$–matrices, with the entries at position (i, j) as sums of polynomials homogeneous of degree $i-j+mn$ for $m \geqslant 0$.

If we again assume that the characteristic of k does not divide n and that k contains the corresponding roots of unity, then this algebra is isomorphic via the usual discrete Fourier transform to the twisted group algebra defined by the diagonal action of the cyclic group μ_n on the polynomial ring S, that is, $B_0 \cong S*\mu_n$. This identification also exhibits B_0 clearly as a positively graded k–algebra, with the subalgebra in degree zero the semi-simple group algebra $k\mu_n \cong A_0/\operatorname{rad} A_0$.

1.20. It is easily established directly that B_0 is indeed of finite global dimension equal to n and homologically homogeneous, which means that all simple modules have the same projective dimension. Furthermore, it is a Calabi-Yau algebra in that it is (graded) Gorenstein with its a–invariant equal to 0, that is to say $\mathbb{R}\operatorname{Hom}_{B_0}(k, B_0) \cong k[-n]$.

The algebra B_0 is again Koszul, its Koszul-dual being the trivial extension algebra of A_1 by its k–dual $\mathbb{D}(A_1) = \mathrm{Hom}_k(A_1, k)$, that is, $B_0^! \cong A_1 \ltimes \mathbb{D}(A_1)$, an artinian symmetric algebra. In terms of representation by a quiver, that for B_0 is obtained from the one for A_0 by adding a copy of (a basis of) V as additional arrows from vertex $n-1$ to vertex 0, but keeping the same relations. The quiver for $B_0^!$ can be obtained from that of A_1 by again adding a copy of (a basis of) V^* as additional arrows from vertex $n-1$ to 0 and keeping the same relations.

As recently established by Bocklandt, Schedler and Wemyss [6], these facts imply that the relations in the algebra B_0 can then be described through the derivatives of a single quiver (super-)potential, the simple loop in the underlying quiver of B_0 that corresponds to the socle element in the exterior algebra.

Bridgeland's quoted work, of which the preceding paragraph is essentially a synopsis, shows that the same properties are inherited by all endomorphism algebras of tilting objects on X that come from a helix on $\mathbb{P}$. In particular, the reader may want to make explicit the structure of $B_1 = \mathrm{End}_X(\pi^* T_1)$.

2. Flatness and Tor-independence of tilting objects

Tor–independence conditions. To simplify the investigation of the behaviour of Hochschild (co-)homology under tilting, we will impose some flatness, or at least some Tor–independence assumptions on tilting objects. To this end, we make the following definition, the notion of pseudo-flatness generalised from [5, Defn. (80)]; see also [15].

Definition 2.1. Let $\mathcal{T}$ be a triangulated category and assume furthermore that it is R–linear over some commutative k–algebra R.

We call a tilting object T in $\mathcal{T}$ *flat* over R if its endomorphism ring A is flat as an R–algebra. The object is *pseudo-flat* if only $\mathrm{Tor}_i^R(A, A) = 0$ for each $i \neq 0$.

If $\mathcal{T}, \mathcal{T}'$ is a pair of R–linear triangulated categories, T a tilting object in $\mathcal{T}$ and T' a tilting object in $\mathcal{T}'$, with endomorphism R–algebras A, A', respectively, then these tilting objects are *Tor–independent* over R if $\mathrm{Tor}_i^R(A, A') = 0$ for $i \neq 0$.

Clearly, flatness implies pseudo-flatness, which in turn means that T is Tor–independent of itself, and any of these properties will automatically hold if R is semi-simple, for example, a field. As concerns (pseudo-)flatness, we offer the following fact that covers most known cases.

Lemma 2.2. *Let $f : X \to Y = \mathrm{Spec}\, R$ be a morphism from a scheme X as in Section* 1.6 *to an affine scheme Y of finite type over k and assume T is a tilting object on X. If $\mathcal{H}_i(\mathcal{E}nd_X(T) \otimes^{\mathbb{L}}_R N) = 0$ for all $i > 0$ and every R–module N, then T is a* flat *tilting object over R.*

The hypthesis is satisfied in particular if X is flat *over* $\mathrm{Spec}\, R$ *and the endomorphism $\mathcal{O}_X$–algebra $\mathcal{E}nd_X(T)$ is quasi-isomorphic to a locally free $\mathcal{O}_X$–module, necessarily concentrated in degree* 0.

Proof. The exact functors $\mathbb{R}f_*(\mathcal{E}nd_X(T) \otimes^{\mathbb{L}}_R ?)$ and $A \otimes^{\mathbb{L}}_R ?$ from $D(R)$ to itself are isomorphic by the projection formula. The assumption ensures that when applied to an R–module the first functor has only cohomology in non-negative degrees, while the second one always has only cohomology in non-positive (cohomological) degrees. Thus, $A \otimes_R ?$ is exact on R–modules, equivalently, A is flat over R, whence T is flat over R by definition. $\square$

Example 2.3. If the tilting object T is flat over Z for some *projective* morphism $p : X \to Z = \operatorname{Spec} K$ to an affine scheme, then its endomorphism ring A is a finite projective K–module, as it is already known to be (module-)finite over K by Remark 1.9.

Example 2.4. In our running example, the tilting object $\pi^* T_0$ on the anticanonical bundle X over $\mathbb{P}^n_k$ is not flat over the affine scheme obtained from collapsing the zero section in X. However, the corresponding ring $K = S^{(n)}$ is Cohen–Macaulay and admits a Noether normalization, a finite morphism $\operatorname{Spec} K \to \operatorname{Spec} R$, with R smooth over k. One may take, for example, $R = k[x_0^n, ..., x_n^n]$ as the subring of the polynomial ring S generated by the indicated powers of the variables. This ring R is itself a polynomial ring, and the explicit description of B_0 in Section 1.19 shows that B_0 is a maximal Cohen-Macaulay module over K, thus, projective as an R–module. It follows, by definition, that $\pi^* T_0$ is flat over R.

Duals and products of tilting objects. Next we note that the class of tilting objects is closed under taking duals and "transversal products". At least in the absolute case over an algebraically closed field, this is certainly folklore, but here we include the details in the relative situation for completeness.

2.5. If X, X' are schemes over some common scheme Y, denote $p_X, p_{X'}$ as the canonical projections from the fibre product $X \times_Y X'$ onto its factors and $\mathbb{L}p_X^*, \mathbb{L}p_{X'}^*$ as the respective derived inverse image functors. Given complexes M, M' of quasi-coherent sheaves on X, X' respectively, we set $M \boxtimes M' = \mathbb{L}p_X^* M \otimes^{\mathbb{L}}_{X \times_Z X'} \mathbb{L}p_{X'}^* M'$.

As in Remark 1.7 we will always assume that X, X' are schemes as in Section 1.6 and that Y is affine of finite type over k so that the fibre product $X \times_Y X'$ is still in the category of schemes fixed in Section 1.6.

Proposition 2.6. *Let T be a tilting object on X and T' a tilting object on X', respectively. Set $A = \operatorname{End}_X(T)$, as before, and $A' = \operatorname{End}_{X'}(T')$.*

(1) *The $\mathcal{O}_X$–dual $T^\vee = \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(T, \mathcal{O}_X)$ of the perfect complex T is again a tilting object with $\operatorname{End}_X(T^\vee) \cong \operatorname{End}_X(T)^{\mathrm{op}}$ as K–algebras. In particular, $T^\vee$ is (pseudo-)flat along with T.*
(2) *Assume X, X' are flat over the affine scheme $Y = \operatorname{Spec} R$. If T, T' are Tor–independent over R, then $T \boxtimes T'$ is a tilting object on $X \times_Y X'$ with $\operatorname{End}_{X \times_Y X'}(T \boxtimes T') \cong A \otimes_R A'$.*

Proof. When restricted to the thick subcategory of perfect complexes, the functor $\mathbb{R}\mathcal{H}om_X(?, \mathcal{O}_X)$ becomes an exact duality, whence $T^\vee$ is perfect and without self-extensions along with T. Now T generates all of $\mathcal{T} = D(X)$, so, in particular, by Definition 1.1(2b) the thick subcategory of perfect complexes is contained in $\mathsf{Loc}\, T$, and then, due to the duality and the observation in Section 1.4, that category is as well contained in $\mathsf{Loc}(T^\vee)$. As $D(X)$ is generated by its perfect complexes, $T^\vee$ is also a generator. Finally, the duality $\mathbb{R}\mathcal{H}om_X(?, \mathcal{O}_X)$ induces an R–algebra anti-isomorphism from $\operatorname{End}_X(T)$ onto $\operatorname{End}_X(T^\vee)$, whence $\operatorname{End}_X(T^\vee) \cong \operatorname{End}_X(T)^{\mathrm{op}}$.

As concerns (2), $T \boxtimes T'$ is a perfect complex on $X \times_Y X'$ as X, X' are flat over Y. To confirm that this object generates, note first that due to projectivity over some affine scheme, X carries an ample invertible sheaf, say, $\mathcal{L}$, and the powers $\mathcal{L}^n$ for $n \in \mathbb{Z}$ are contained in $\mathsf{Loc}\, T$, as that category is, after all, of $D(X)$. The functor $? \boxtimes T'$ from $D(X)$ to $D(X \times_Y X')$ commutes with small direct sums, and so again

employing the observation in Section 1.4, we get that $\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^n \boxtimes T'$ is contained in $\mathsf{Loc}(T \boxtimes T') \subseteq D(X \times_Y X')$.

Applying the same argument to the functor $\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^n \boxtimes ?$ from $D(X')$ to $D(X \times_Y X')$ and an ample invertible sheaf $\mathcal{L}'$ on X', it follows that $\bigoplus_{n,m\in\mathbb{Z}} \mathcal{L}^n \boxtimes \mathcal{L}'^m$ is in turn contained in $\mathsf{Loc} \bigoplus_{n\in\mathbb{Z}} \mathcal{L}^n \boxtimes T'$, which we just saw to be contained in $\mathsf{Loc}\, T \boxtimes T'$. Now the invertible sheaf $\mathcal{L} \boxtimes \mathcal{L}'$ is ample on $X \times_Y X'$, whence its powers and translates generate all of $D(X \times_Y X')$; see [36, Example 1.10]. It follows that $T \boxtimes T'$ already generates, as claimed.

It remains to verify the vanishing conditions. By the projection formula and flat base change, $\mathbb{R}\operatorname{Hom}_{X\times_Y X'}(T \boxtimes T', T \boxtimes T') \cong A \otimes^{\mathbb{L}}_R A'$, whence the result follows from the Tor-independence of T, T'. □

Remark 2.7. The fact that $T^\vee$ is a tilting object along with T restricts the class of algebras A that occur as endomorphism rings of tilting objects on a given scheme X in that such algebras are then derived equivalent to their own opposite algebras, $D(A) \cong D(A^{\mathrm{op}})$, in view of Proposition 2.6(1).

Corollary 2.8. *If X is flat over Y, and T a tilting object on X that is pseudo-flat over Y, then $T^{\mathrm{ev}} = T^\vee \boxtimes T$ is tilting on $X \times_Y X$ with endormorphism algebra $A^{\mathrm{ev}} = A^{\mathrm{op}} \otimes_R A$.* □

To end this section, we note the following permanence property with respect to base change.

Proposition 2.9. *Let X be flat over the affine scheme Y and $u : Y' \to Y$ an affine morphism, with Y' as well of finite type over k. With $u' : X' = X \times_Y Y' \to X$ the induced morphism, if T is a tilting object on X that is flat over Y, then $T' = u'^*T$ is a tilting object on X' that is flat over Y'.*

Proof. Pulling back along u' preserves perfection of complexes as X is flat over Y. Thus, T' is perfect in $D(X')$. The powers of any ample invertible $\mathcal{O}_X$–module $\mathcal{L}$ pull back to powers of an ample invertible $\mathcal{O}_{X'}$–module, and those are contained in $u'^*(\mathsf{Loc}\, T) \subseteq \mathsf{Loc}\, u'^*T$. As those powers generate $D(X')$, it follows that $T' = u'^*T$ in turn generates too. Note that this is, of course, the same argument as in the proof of Proposition 2.6(2), applied to $T' \cong T \boxtimes \mathcal{O}_{Y'}$.

With regard to vanishing, $p'_*\mathcal{E}nd_{X'}(T') \cong p_*u'^*\mathcal{E}nd_X(T) \cong u^*p_*\mathcal{E}nd_X(T)$ by flat base change. Now $A = p_*\mathcal{E}nd_X(T)$ is flat over Y by assumption, whence $A' = u^*A$ is concentrated in degree zero and flat over Y'. □

3. Hochschild cohomology under tilting

Hochschild (co-)homology of morphisms of schemes. If $X \to Y$ is a flat morphism of schemes or analytic spaces, then the reasonable analogue of Hochschild theory for algebras is given by the hyper-(co-)homology of the structure sheaf of the diagonal in $X \times_Y X$. To be more precise, we recall the definition and refer the reader to [43, 16] for the general picture.

Definition 3.1. Let $f : X \to Y$ be a flat morphism, $\Delta : X \hookrightarrow X \times_Y X$ the embedding of the diagonal, and denote $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X$ as the structure sheaf of the diagonal. The *Hochschild cohomology* of X over Y with values in a complex $\mathcal{M}$ from $D(X \times_Y X)$ is then

$$\mathrm{HH}^\bullet(X/Y, \mathcal{M}) = \operatorname{Ext}^\bullet_{X\times_Y X}(\mathcal{O}_\Delta, \mathcal{M}),$$

while the *Hochschild homology* of X over Y with values in $\mathcal{M}$ is defined as the hypercohomology

$$\mathrm{HH}_\bullet(X/Y,\mathcal{M}) = H^{-\bullet}(X,\mathbb{L}\Delta^*\mathcal{M}).$$

We simply write $\mathrm{HH}^\bullet(X/Y)$, respectively $\mathrm{HH}_\bullet(X/Y)$, if $\mathcal{M} = \mathcal{O}_\Delta$ is the structure sheaf of the diagonal.

In the absolute case, when X is projective over the field k, we abbreviate further, $\mathrm{HH}^\bullet(X,\mathcal{M}) = \mathrm{HH}^\bullet(X/\operatorname{Spec} k,\mathcal{M})$ and $\mathrm{HH}^\bullet(X) = \mathrm{HH}^\bullet(X,\mathcal{O}_\Delta)$.

Remark 3.2. If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_{X\times_Y X}$–module, then the definition implies that $\mathrm{HH}^i(X/Y,\mathcal{M}) = 0$ for $i < 0$, while $\mathrm{HH}_j(X/Y,\mathcal{M}) = 0$ for $j < -\dim X$. In general, there is no upper bound for the non-vanishing.

However, if X is locally Cohen–Macaulay and smooth over Y, then $\mathcal{O}_\Delta$ is perfect, thus, isomorphic to a finite complex of locally free sheaves, and locally the length of such a locally free resolution can be bounded by the relative dimension $\dim X - \dim Y$ according to the Auslander–Buchsbaum formula. With $\mathcal{P}$ such a locally free resolution, $\operatorname{Ext}^i_{X\times_Y X}(\mathcal{O}_\Delta,\mathcal{M}) \cong H^i(X\times_Y X, \mathcal{M}\otimes_{X\times_Y X}\mathcal{P}^\vee)$, where $\mathcal{P}^\vee$ denotes the $\mathcal{O}_{X\times_Y X}$–dual of $\mathcal{P}$. The hypercohomology groups on the right-hand side vanish for $i > 2\dim X$, the sum of the length of $\mathcal{P}$ and the dimension of $X\times_Y X$. Therefore, in this case the Hochschild cohomology $\mathrm{HH}^i(X/Y,\mathcal{M})$ is concentrated in the range $0 \leqslant i \leqslant 2\dim X$ for any (quasi-)coherent $\mathcal{O}_{X\times_Y X}$–module $\mathcal{M}$. Similarly, $\mathrm{HH}_j(X/Y,\mathcal{M})$ will be concentrated in the range $-\dim X \leqslant j \leqslant \dim X - \dim Y$.

The following observation is somewhat pedantic but allows us to clearly exhibit the action of Hochschild cohomology on homology.

Remark 3.3. If $Y = \operatorname{Spec} R$ is affine, with R some commutative ring, then the Hochschild (co-)homology is naturally a graded R–module. To make this structure explicit, note that

$$\mathrm{HH}_\bullet(X/Y,\mathcal{M}) = H^{-\bullet}(X,\mathbb{L}\Delta^*\mathcal{M}) \cong H^0(Y, H^{-\bullet}(\mathbb{R}f_*\mathbb{L}\Delta^*\mathcal{M}))$$

and call the complex $\mathbb{R}f_*\mathbb{L}\Delta^*\mathcal{M}$ in $D(R) \cong D(Y)$ the *Hochschild complex* on Y with coefficients in $\mathcal{M}$.

Factoring $f : X \to Y$ through the diagonal embedding as $f : X \xrightarrow{\Delta} X\times_Y X \xrightarrow{f\cdot p} Y$, where $p : X\times_Y X \to X$ denotes any of the natural projections, and using the projection formula and exactness of Δ_*, this complex can also be displayed as

$$(\dagger)\qquad \mathbb{R}f_*\mathbb{L}\Delta^*\mathcal{M} \cong \mathbb{R}(fp)_*\Delta_*\mathbb{L}\Delta^*\mathcal{M} \cong \mathbb{R}(fp)_*(\mathcal{M}\otimes^{\mathbb{L}}_{X\times_Y X}\mathcal{O}_\Delta).$$

A class $\xi \in \mathrm{HH}^i(X/Y)$ in Hochschild cohomology is represented by a morphism $\xi : \mathcal{O}_\Delta \to \mathcal{O}_\Delta[i]$ in $D(X\times_Y X)$, and the induced morphism of complexes of R–modules

$$\mathbb{R}(fp)_*(\mathcal{M}\otimes^{\mathbb{L}}_{X\times_Y X}\xi) : \mathbb{R}(fp)_*(\mathcal{M}\otimes^{\mathbb{L}}_{X\times_Y X}\mathcal{O}_\Delta) \to \mathbb{R}(fp)_*(\mathcal{M}\otimes^{\mathbb{L}}_{X\times_Y X}\mathcal{O}_\Delta)[i]$$

represents the R–linear action $\xi\star? : \mathrm{HH}_\bullet(X/Y,\mathcal{M}) \to \mathrm{HH}_{\bullet-i}(X/Y,\mathcal{M})$ of Hochschild cohomology on homology.

While the preceding definition and remark apply for any flat morphism, we now return to the situation where X is a scheme in the category described in Section 1.6 and Y is an affine scheme of finite type over a field k.

Preservation of the diagonal. With notation as in the previous section, the key result of this section is that, for a pseudo-flat tilting object T, the diagonal is preserved under tilting by $T^\vee \boxtimes T$, in that the structure sheaf of the diagonal in $X \times_Y X$ is transformed into A with its canonical bimodule structure.

To abbreviate, we henceforth set $T^{\mathrm{ev}} = T^\vee \boxtimes T$ and accordingly write the equivalence induced by this tilting object as

$$T^{\mathrm{ev}}_* = \mathbb{R}\operatorname{Hom}_{X\times_Y X}(T^\vee \boxtimes T,\ ) : D(X \times_Y X) \xrightarrow{\cong} D(A^{\mathrm{ev}}).$$

Theorem 3.4. *If X is flat over Y, and T a tilting object on X that is pseudo-flat over Y, then $T^{\mathrm{ev}}_*(\mathcal{O}_\Delta) \cong A$ in $D(A^{\mathrm{ev}})$, where A on the right is endowed with its canonical (right) A^{ev}–module structure.*

Proof. Consider the following chain of equalities and isomorphisms, where the first line simply replaces $\mathcal{O}_\Delta$ by its definition, and the subsequent isomorphisms result, in turn, from the adjunction $(\mathbb{L}\Delta^*, \mathbb{R}\Delta_* = \Delta_*)$, Serre's "diagonal trick"; that is, the identification of functors $\mathbb{L}\Delta^*(-\boxtimes -) \cong - \otimes^{\mathbb{L}}_X -$. Then we have the adjunction $(\otimes^{\mathbb{L}}_X, \mathbb{R}\mathcal{H}om_X)$ and finally the natural identification of T with $T^{\vee\vee}$:

$$\begin{aligned}
\mathbb{R}\operatorname{Hom}_{X\times_Y X}(T^\vee \boxtimes T, \mathcal{O}_\Delta) &= \mathbb{R}\operatorname{Hom}_{X\times_Y X}(T^\vee \boxtimes T, \Delta_*\mathcal{O}_X)\\
&\cong \mathbb{R}\operatorname{Hom}_X(\mathbb{L}\Delta^*(T^\vee \boxtimes T), \mathcal{O}_X)\\
&\cong \mathbb{R}\operatorname{Hom}_X(T^\vee \otimes^{\mathbb{L}}_X T, \mathcal{O}_X)\\
&\cong \mathbb{R}\operatorname{Hom}_X(T, \mathbb{R}\mathcal{H}om_X(T^\vee, \mathcal{O}_X))\\
&\cong \mathbb{R}\operatorname{Hom}_X(T, T)\\
&\cong A.
\end{aligned}$$

That the identification is one of bimodules follows easily from the fact that A acts (from the left) through endomorphisms on the second factor in $T^\vee \boxtimes T$, while A^{op} acts (from the left) on the first one. □

As immediate consequences, we obtain the following results.

Corollary 3.5. *Assume X is flat over $Y = \operatorname{Spec} R$, and T is a pseudo-flat tilting object on it. The functor T^{ev}_* then induces an isomorphism of graded R–algebras*

$$(*) \qquad \operatorname{Ext}^\bullet_{X\otimes_Y X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \operatorname{Ext}^\bullet_{A^{\mathrm{ev}}}(A, A).$$

That is, the Hochschild cohomology ring $\mathrm{HH}^\bullet(X/Y)$ of X over Y is isomorphic to the Hochschild cohomology ring $\mathrm{HH}^\bullet(A/R)$ of A over R.

Moreover, for any complex $\mathcal{M}$ in $D(X \times_Y X)$, the same functor induces an isomorphism

$$\operatorname{Ext}^\bullet_{X\otimes_Y X}(\mathcal{O}_\Delta, \mathcal{M}) \cong \operatorname{Ext}^\bullet_{A^{\mathrm{ev}}}(A, T^{\mathrm{ev}}_*\mathcal{M})$$

of graded right modules over that ring isomorphism. □

Preservation of smoothness. In [45], van den Bergh introduced *smoothness* for an algebra over a field to mean that its Hochschild dimension, equal to the projective dimension of the algebra as a module over its enveloping algebra, is finite. Extending this definition to algebras over an arbitrary commutative ring, we additionally require that the algebra is *flat* over that ring. With this definition, we can now formulate the following improvement over Theorem 1.8(3).

Corollary 3.6. *If X and its tilting object T are flat over Y, then X is* smooth *over Y if, and only if, A is* smooth *over R.*

Proof. The flat morphism $X \to Y = \operatorname{Spec} R$ is smooth if, and only if, $\mathcal{O}_\Delta$ is a compact object in $D(X \times_Y X)$, while A is smooth over R if, and only if, A is flat over R and its projective dimension over A^{ev} is finite; equivalently, A is a compact object in $D(A^{\mathrm{ev}})$. Now A is flat over R as T is a flat tilting object, and $\mathcal{O}_\Delta$ is compact if, and only if, its image $T_*^{\mathrm{ev}}\mathcal{O}_\Delta = A$ under the equivalence T_*^{ev} is compact. □

Remark 3.7. When algebras A and B of finite type over an algebraically closed field are derived equivalent, then finite global dimension of one implies finite global dimension of the other, but even in the artinian case these dimensions need not be the same. In the artinian case, finite global dimension is equivalent to smoothness as the global dimension equals the projective dimension of the algebra as a module over its enveloping algebra; see [28]. We do not know whether algebras that appear as endomorphism rings of tilting objects on the same smooth projective scheme X may have different global dimensions. However, using the equality of global dimension and of projective dimension over the enveloping algebra, we get immediately from Corollary 3.5 that the global dimension of $A = \operatorname{End}_X(T)$ is bounded from below by $\max\{n \mid \operatorname{HH}^n(X) \neq 0\}$, and in all the examples we are aware of even equality holds. Note that in view of Remark 3.2 we have $\max\{i \mid \operatorname{HH}^i(X) \neq 0\} \leq 2 \dim X$, and this inequality will usually be strict.

For a concrete example, if $X = \mathbb{P}^{n-1}$ as in Section 1.16, then $\max\{i \mid \operatorname{HH}^i(X) \neq 0\} = n - 1 = \dim X$ equals the global dimension of either endomorphism algebra A_i, for $i = 0, 1$, of the respective tilting object T_i there.

3.8. Given that the Hochschild cohomology rings of X and A are isomorphic, it seems reasonable to suspect that X and A indeed have isomorphic deformation theories as well. At least, the given tilting object lifts to any flat deformation of X as it has no higher self-extensions, and such a lifting might conceivably still serve as a tilting object on the deformation, with the endomorphism ring a deformation of the original algebra A. However, we will not pursue this problem further. Instead, we now turn to Hochschild homology.

4. Hochschild homology and tilting

Invariance of the Hochschild complex.

Theorem 4.1. *Assume $f : X \to Y = \operatorname{Spec} R$ is flat, and T is a tilting object on X that is pseudo-flat over Y. One then has a natural isomorphism of functors*

$$\mathbb{R}f_*\mathbb{L}\Delta^*(?) \cong T_*^{\mathrm{ev}}(?) \otimes^{\mathbb{L}}_{A^{\mathrm{ev}}} A : D(X \times_Y X) \longrightarrow D(R),$$

and for any complex $\mathcal{M}$ in $D(X \times_Y X)$, the functor T_*^{ev} induces an isomorphism of graded R–modules $\operatorname{HH}_\bullet(X/Y, \mathcal{M}) \cong \operatorname{HH}_\bullet(A/R, T_*^{\mathrm{ev}}\mathcal{M})$, linear over the isomorphism $(*)$ in Hochschild cohomology. In particular, $\operatorname{HH}_\bullet(X/Y) \cong \operatorname{HH}_\bullet(A/R)$.

Proof. This proof essentially uses the fact that $\mathbb{R}f_* : D(X) \to D(R)$ admits a *right adjoint* $f^!$; see [36, Example 4.2]. Indeed, one then has the following chain of isomorphisms of R–modules, natural both in $\mathcal{M}$ from $D(X \times_Y X)$ and N from $D(R)$. The first one arises from the adjunctions $(\mathbb{R}f_*, f^!)$ and $(\mathbb{L}\Delta^*, \Delta_*)$,

$$\operatorname{Hom}_R(\mathbb{R}f_*\mathbb{L}\Delta^*\mathcal{M}, N) \cong \operatorname{Hom}_{X \times_Y X}(\mathcal{M}, \Delta_* f^! N),$$

the next two from applying the equivalence T_*^{ev} and then expanding its definition in the second argument,

$$\begin{aligned}&\cong \operatorname{Hom}_{A^{\mathrm{ev}}}(T_*^{\mathrm{ev}}\mathcal{M}, T_*^{\mathrm{ev}}\Delta_* f^! N))\\ &= \operatorname{Hom}_{A^{\mathrm{ev}}}(T_*^{\mathrm{ev}}\mathcal{M}, \mathbb{R}\operatorname{Hom}_{X\times_Y X}(T^\vee \boxtimes T, \Delta_* f^! N)),\end{aligned}$$

while again using the adjunction $(\mathbb{L}\Delta^*, \Delta_*)$, this time in the second argument, and then employing the *diagonal trick* $\mathbb{L}\Delta^*(-\boxtimes -) \cong -\otimes^{\mathbb{L}}_X -$ as above, transforms this expression isomorphically into

$$\cong \operatorname{Hom}_{A^{\mathrm{ev}}}(T_*^{\mathrm{ev}}\mathcal{M}, \mathbb{R}\operatorname{Hom}_X(T^\vee \otimes^{\mathbb{L}}_X T, f^! N)).$$

Perfection of T yields the identification $T^\vee \otimes^{\mathbb{L}}_X T \cong \mathbb{R}\mathcal{H}om_X(T,T)$, which in turn provides the isomorphism

$$\cong \operatorname{Hom}_{A^{\mathrm{ev}}}(T_*^{\mathrm{ev}}\mathcal{M}, \mathbb{R}\operatorname{Hom}_X(\mathbb{R}\mathcal{H}om_X(T,T), f^! N)).$$

Once again applying the adjunction $(\mathbb{R}f_*, f^!)$, together with the quasi-isomorphisms $\mathbb{R}f_*\mathbb{R}\mathcal{H}om_X(T,T) \cong \mathbb{R}\operatorname{Hom}_X(T,T) \cong A$ in $D(R)$, results in the isomorphisms

$$\begin{aligned}&\cong \operatorname{Hom}_{A^{\mathrm{ev}}}(T_*^{\mathrm{ev}}\mathcal{M}, \mathbb{R}\operatorname{Hom}_R(\mathbb{R}f_*\mathbb{R}\mathcal{H}om_X(T,T), N),\\ &\cong \operatorname{Hom}_{A^{\mathrm{ev}}}(T_*^{\mathrm{ev}}\mathcal{M}, \mathbb{R}\operatorname{Hom}_R(A, N)),\end{aligned}$$

and finally the adjunction $\operatorname{Hom}_{A^{\mathrm{ev}}}(U, \mathbb{R}\operatorname{Hom}_R(V,W)) \cong \operatorname{Hom}_R(U \otimes^{\mathbb{L}}_{A^{\mathrm{ev}}} V, W)$ for complexes of (right) A^{ev}–modules U, V and (symmetric) R–modules V, W establishes the isomorphism

$$\cong \operatorname{Hom}_R((T_*^{\mathrm{ev}}\mathcal{M}) \otimes^{\mathbb{L}}_{A^{\mathrm{ev}}} A, N).$$

In summary, the bi-functors $\operatorname{Hom}_R(\mathbb{R}f_*\mathbb{L}\Delta^*(-), ?)$ and $\operatorname{Hom}_R((T_*^{\mathrm{ev}}(-) \otimes^{\mathbb{L}}_{A^{\mathrm{ev}}} A, ?)$ on $D(X\times_Y X)^{\mathrm{op}} \times D(R)$ are isomorphic, from which the first claim follows.

To see that this isomorphism of functors is linear over the isomorphism in Hochschild cohomology, one may rewrite the above chain of isomorphisms, beginning from the description

$$\mathbb{R}f_*\mathbb{L}\Delta^*\mathcal{M} \cong \mathbb{R}(fp)_*(\mathcal{M} \otimes^{\mathbb{L}}_{X\times_Y X} \mathcal{O}_\Delta)$$

in (†) in Remark 3.3 above. The justification of each individual step in the following chain of isomorphisms is the same as before, except that this time we use the right adjoint $(fp)^!$ to $\mathbb{R}(fp)_*$:

$$\begin{aligned}&\operatorname{Hom}_R(\mathbb{R}(fp)_*(\mathcal{M} \otimes^{\mathbb{L}}_{X\times_Y X} \mathcal{O}_\Delta), N)\\ &\cong \operatorname{Hom}_{X\times_Y X}(\mathcal{M}, \mathbb{R}\mathcal{H}om_{X\times_Y X}(\mathcal{O}_\Delta, (fp)^! N))\\ &\cong \operatorname{Hom}_{A^{\mathrm{ev}}}(T_*^{\mathrm{ev}}\mathcal{M}, T_*^{\mathrm{ev}}\mathbb{R}\mathcal{H}om_{X\times_Y X}(\mathcal{O}_\Delta, (fp)^! N))\\ &= \operatorname{Hom}_{A^{\mathrm{ev}}}(T_*^{\mathrm{ev}}\mathcal{M}, \mathbb{R}\operatorname{Hom}_{X\times_Y X}(T^\vee \boxtimes T, \mathbb{R}\mathcal{H}om_{X\times_Y X}(\mathcal{O}_\Delta, (fp)^! N)))\\ &\cong \operatorname{Hom}_{A^{\mathrm{ev}}}(T_*^{\mathrm{ev}}\mathcal{M}, \mathbb{R}\operatorname{Hom}_R(\mathbb{R}(fp)_*((T^\vee \boxtimes T) \otimes^{\mathbb{L}}_{X\times_Y X} \mathcal{O}_\Delta), N)),\end{aligned}$$

and, again using (†) in Remark 3.3, but now on the term $\mathbb{R}\operatorname{Hom}_R(\mathbb{R}(fp)_*((T^\vee \boxtimes T) \otimes^{\mathbb{L}}_{X\times_Y X} \mathcal{O}_\Delta))$, we find

$$\begin{aligned}\mathbb{R}(fp)_*((T^\vee \boxtimes T) \otimes^{\mathbb{L}}_{X\times_Y X} \mathcal{O}_\Delta) &\cong \mathbb{R}f_*\mathbb{L}\Delta^*(T^{\mathrm{ev}})\\ &\cong T^{\mathrm{ev}}_*(T^{\mathrm{ev}}) \otimes^{\mathbb{L}}_{A^{\mathrm{ev}}} A\\ &\cong A^{\mathrm{ev}} \otimes^{\mathbb{L}}_{A^{\mathrm{ev}}} A\\ &\cong A,\end{aligned}$$

whence the action of the Hochschild cohomology of X through the argument $\mathcal{O}_\Delta$ on the geometric side is seen to be transported into the action of the Hochschild cohomology of A on the second factor in $(T^{\mathrm{ev}}_*\mathcal{M}) \otimes^{\mathbb{L}}_{A^{\mathrm{ev}}} A$ on the algebraic side. This proves that the isomorphism of functors is indeed linear over the corresponding isomorphism in cohomology, as desired. □

Vanishing of negative Hochschild homology. For algebras, Hochschild homology of modules is necessarily concentrated in non-negative (homological) degrees, that is, $\mathrm{HH}_i(A/R, M) = 0$ for $i < 0$ and M an A^{ev}–module. On the geometric side, however, as we pointed out in Remark 3.2, a priori, $\mathrm{HH}_i(X/Y, \mathcal{M})$ for a $\mathcal{O}_{X\times_Y X}$–module $\mathcal{M}$ is only guaranteed to vanish for $i < -\dim X$, due to the appearance of hypercohomology in the definition of the latter. In the presence of a tilting object, we get better vanishing behaviour.

Corollary 4.2. *If $f : X \to Y = \operatorname{Spec} R$ is flat, and X admits a tilting object that is pseudo-flat over Y, then*

(1) $\mathrm{HH}_i(X/Y) = 0$ *for* $i < 0$*;*
(2) $H^i(X, \mathcal{O}_X) = 0$ *for* $i \neq 0$*.*

Proof. Statement (1) follows immediately from Theorem 4.1 as for any algebra $\mathrm{HH}_i(A/R) \cong \mathrm{HH}_i(X/Y)$ vanishes for $i < 0$. For (2), note that the co-unit of the adjunction $(\mathbb{L}\Delta^*, \Delta_*)$ provides for a natural morphism $\mathbb{L}\Delta^*(\mathcal{O}_\Delta) \cong \mathbb{L}\Delta^*\Delta_*(\mathcal{O}_X) \to \mathcal{O}_X$ whose image under Δ_* becomes a split epimorphism as for any adjunction. In view of (†) in Remark 3.3, this shows that $H^i(\mathbb{R}f_*\mathcal{O}_X) \cong H^i(X, \mathcal{O}_X)$ splits off as a direct summand of $\mathrm{HH}_{-i}(X/Y)$, whence the first claim implies the second. □

Vanishing of Hochschild homology in the absolute case. For a scheme that is projective and smooth over a field, we get an even stronger result.

Theorem 4.3. *If a smooth projective scheme X over a field k carries a tilting object, then* $\mathrm{HH}_i(X) = 0$ *for* $i \neq 0$*.*

Proof. Let $A = \operatorname{End}_X(T)$ be the endomorphism algebra of a tilting object on X so that $\mathrm{HH}_\bullet(A) \cong \mathrm{HH}_\bullet(X)$. As recalled before (see Theorem 1.8(3) and Corollary 3.6), A is then necessarily of finite global dimension, even smooth. A result essentially due to Keller [34, Prop. 2.5] (see [27, Prop. 6] for further details) then says that $\mathrm{HH}_i(A) = 0$ for $i \neq 0$. □

5. The Hodge decomposition in characteristic zero

We now restrict ourselves to smooth morphisms over a field of characteristic zero that we may assume to be the complex numbers by a suitable base change to an algebraically closed field and application of the Lefschetz principle. In this situation,

Hochschild (co-)homology admits a *Hodge decomposition* (see again [43, 17] for details) in that

$$\mathrm{HH}_i(X/Y) \cong \bigoplus_{p-q=i} H^q(X, \Omega^p_{X/Y}),$$
$$\mathrm{HH}^i(X/Y) \cong \bigoplus_{p+q=i} H^q(X, \mathcal{T}^p_{X/Y}) \cong \bigoplus_{p+q=i} \mathrm{Ext}^q_X(\Omega^p_{X/Y}, \mathcal{O}_X),$$

where $\Omega^p_{X/Y}$ denotes the $\mathcal{O}_X$–module of relative differential forms of order p and $\mathcal{T}^p_{X/Y}$ the indicated exterior power of the tangent sheaf $\mathcal{T}_{X/Y} = \mathcal{H}om_X(\Omega^1_{X/Y}, \mathcal{O}_X)$.

In the absolute case, when $Y = \operatorname{Spec} \mathbb{C}$, Kodaira–Serre duality yields the isomorphisms $H^q(X, \Omega^p_X) \cong H^{n-q}(X, \Omega^{n-p}_X)^*$, and so vanishing of Hochschild homology in negative degrees in Corollary 4.2(1) implies already vanishing in positive degrees, reproving Theorem 4.3 in this case.

Theorem 5.1. *If X is a smooth complex projective variety that carries a tilting object, then the Hodge groups satisfy $H^q(X, \Omega^p_X) = 0$ for $p \neq q$, that is, $\mathrm{HH}_i(X) = 0$ for $i \neq 0$, and $\mathrm{HH}_0(X) \cong \bigoplus_p H^p(X, \Omega^p_X)$.* □

Question 5.2. How is the Hodge decomposition reflected on the algebraic side? In other words, knowing $A = \mathrm{End}_X(T)$ for some complex projective variety X with tilting object T, can one read the Hodge groups or Hodge filtration algebraically on $\mathrm{HH}_0(A)$?

In the relative situation, when $f : X \to Y$ is a smooth morphism over a field of characteristic zero, we get at least the following result on the Hodge modules from Corollary 4.2.

Theorem 5.3. *If $f : X \to Y = \operatorname{Spec} R$ is smooth, and X carries a tilting object that is pseudo-flat over Y, then $H^q(X, \Omega^p_{X/Y}) = 0$ for $q > p$.* □

6. Hochschild (co-)homology for some open Calabi–Yau varieties

6.1. Assume that the finite group $G \subset SL(V)$, for some finite-dimensional complex vector space V, *satisfies the crepant resolution conjecture* in that there is a resolution of singularities $X \to V/G$ with X carrying a tilting object whose endomorphism ring is isomorphic to the twisted group ring $\mathcal{O}(V)*G$; see [30, Ch. 7] for details and known cases and [24] for far-reaching consequences.

6.2. The Hochschild (co-)homology of X is then easily obtained, as it is known, including the multiplicative structure on cohomology, on the algebraic side for twisted group rings; see e.g. [23, 24, 42].

To recall the algebraic result succinctly, we use the *fixed-point scheme*

$$Z = \{(v, g) \in V \times G \mid g(v) = v\} \subseteq V \times G.$$

It is just the disjoint union $Z \cong \coprod_{g \in G} V^g$ of the linear spaces $V^g = \mathrm{Ker}(\mathrm{id} - g) \subseteq V$ of fixed points of the various elements g in G. It is in particular naturally an affine scheme, on which the group G still acts through $h(z, g) = (h(z), hgh^{-1})$. The resulting quotient Z/G can be identified, non-canonically, with $\coprod_{[g] \in G/\sim} V^g/C_g$, where the disjoint union runs over the conjugacy classes $G/\sim$ of G, and $[g] \in G/\sim$ denotes the (arbitrary) choice of one element from each conjugacy class with C_g

its centralizer in G. This scheme is thus the disjoint union of quotient singularities resulting from the action of the stabilizers of the conjugacy classes in G on the respective fixed linear subspace. It contains the original quotient singularity V/G as the connected component corresponding to the identity element in G.

In these terms, one has the following result, a straightforward reinterpretation of [23, Thm. 26 & Thm. 31]. It is presented in this form in [24] for cohomology.

Theorem 6.3. *Given a finite-dimensional complex vector space V with coordinate ring $S = \mathcal{O}(V)$, and a finite subgroup $G \subset SL(V)$, the Hochschild (co-)homology of the twisted group ring $S{*}G$ is concentrated in (co-)homological degrees $0 \leqslant i \leqslant \dim_{\mathbb{C}} V$ and given, as modules over S^G, through*

$$\mathrm{HH}_i(S{*}G) \cong (\Omega^i_Z)^G \cong \bigoplus_{[g]\in G/\sim} (\Omega^i_{V^g})^{C_g}$$

and

$$\begin{aligned}\mathrm{HH}^i(S{*}G) &\cong \mathrm{HH}_{\dim_{\mathbb{C}} V - i}(S{*}G) \otimes_{\mathbb{C}} (\det V)^{-1}\\ &\cong \left(\Omega_Z^{\dim_{\mathbb{C}} V - i}\right)^G \otimes_{\mathbb{C}} (\det V)^{-1}\\ &\cong \bigoplus_{[g]\in G/\sim} \left(\mathcal{T}_{V^g}^{i-\dim_{\mathbb{C}} V/V^g} \otimes \det(V/V^g)^{-1}\right)^{C_g},\end{aligned}$$

where $\mathcal{T}_{V^g}$ denotes as before the tangent sheaf and the superscript indicates the appropriate exterior power of it. □

On the geometric side, this then yields the following information.

Corollary 6.4. *If $X \to V/G$ is a crepant resolution for a finite subgroup $G \subset SL(V)$ so that $D(X) \simeq D(\mathcal{O}(V){*}G)$, then*

$$\mathrm{HH}_i(X) \cong \bigoplus_p H^p(X, \Omega_X^{p+i}) \cong (\Omega^i_Z)^G \cong \bigoplus_{[g]\in G/\sim} (\Omega^i_{V^g})^{C_g}.$$

In particular,

$$\mathrm{HH}_0(X) \cong \bigoplus_p H^p(X, \Omega_X^p) \cong \mathcal{O}(Z)^G \cong \prod_{[g]\in G/\sim} \mathcal{O}(V^g)^{C_g}.$$

Moreover,

$$\mathrm{HH}^i(X) \cong \mathrm{HH}_{\dim X - i}(X) \otimes (\det V)^{-1}.$$

In particular,

$$\mathrm{HH}^0(X) \cong H^0(X, \mathcal{O}_X) \cong \mathcal{O}(V)^G.$$

□

Question 6.5. There again remains the question whether there is a direct interpretation of the direct sum decomposition of Hochschild homology into the Hodge modules in terms of the group representation data that appear on the right-hand side, or what information relating those direct sum decompositions might reveal. Such interpretations abound for Kleinian surface singularities, the case of $G \subset SL_2(\mathbb{C})$, where one has various dictionaries provided through the McKay correspondence and its dual; see [14] for a rather comprehensive geometric account.

On the cohomological side, in general, one can interpret part of the group representation data as orbifold cohomology of the assumed crepant resolution if the group acts further symplectically; see [24, Theorem 1.2] and the related discussion therein.

Remark 6.6. Theorem 6.3 extends to finite subgroups $G \subset GL(V)$, with the only modification being that the factor $(\det V)^{-1}$ in the description of Hochschild cohomology has to move inside, $\mathrm{HH}^i(S{*}G) \cong \left(\Omega_Z^{\dim_{\mathbb{C}} V - i} \otimes_{\mathbb{C}} (\det V)^{-1}\right)^G$, and that in the direct sum description that follows only those conjugacy classes that lie in $G \cap SL(V)$ contribute; see [23, Example 35] for details.

The canonical bundle on projective space revisited.

6.7. As our final example we return to the example of $X = \mathbb{V}_{\mathbb{P}}(\omega_{\mathbb{P}}^{-1})$, the canonical bundle over $\mathbb{P} = \mathbb{P}(V)$. As before, we assume $\dim V = n \geqslant 2$, and, for simplicity, we work over $k = \mathbb{C}$. In this situation (see Section 1.19) the result $\mathrm{HH}(X) \cong \mathrm{HH}(\mathcal{O}(V){*}\mu_n)$, with μ_n acting diagonally through multiplication by roots of unity, tells us in view of Theorem 6.3 that

$$\begin{aligned} \mathrm{HH}_0(X) &\cong \mathcal{O}(V)^{\mu_n} \oplus \bigoplus_{1 \neq g \in \mu_n} \mathbb{C} \cong \mathcal{O}(V)^{\mu_n} \oplus \mathbb{C}^n, \\ \mathrm{HH}_i(X) &\cong (\Omega_V^i)^{\mu_n} \quad \text{for } i \neq 0 \end{aligned}$$

as any element $g \neq 1$ in μ_n leaves only the origin fixed, whence $Fix(g) = \{0\}$ and μ_n acts trivially on $\mathbb{C} = \mathcal{O}(Fix(g))$.

6.8. Using the fact that $\mathrm{HH}_i(X) = \bigoplus_{q \geqslant 0} H^q(X, \Omega_X^{i+q})$, this result can easily be reconfirmed geometrically. In fact, the Zariski–Jacobi sequence for the smooth affine structural morphism $\pi : X \to \mathbb{P}$ is

$$0 \to \pi^*\Omega_{\mathbb{P}}^1 \to \Omega_X^1 \to \Omega_{X/\mathbb{P}}^1 \to 0$$

and, as $\pi_*\mathcal{O}_X = \mathbb{S}\mathrm{ym}_{\mathbb{P}}(\omega_{\mathbb{P}}^{-1})$, the $\mathcal{O}_X$–module of relative differentials $\Omega_{X/\mathbb{P}}^1$ can be identified with $\pi^*(\omega_{\mathbb{P}}^{-1})$. Taking exterior powers yields for any $p \geqslant 0$ a short exact sequence of $\mathcal{O}_X$–modules

$$0 \to \pi^*\Omega_{\mathbb{P}}^p \to \Omega_X^p \to \pi^*(\Omega^{p-1} \otimes_{\mathbb{P}} \omega_{\mathbb{P}}^{-1}) \to 0.$$

Taking into account that $H^\bullet(X, \ ) \cong H^\bullet(\mathbb{P}, \pi_*(\ ))$ and that $H^q(X, \Omega_X^p) = 0$ for $q > p$ by Theorem 5.3, the resulting long exact cohomology sequence becomes

$$\begin{aligned} 0 &\longrightarrow H^0(\mathbb{P}, \pi_*\pi^*\Omega_{\mathbb{P}}^p) \longrightarrow H^0(X, \Omega_X^p) \longrightarrow H^0(\mathbb{P}, \pi^*\pi_*\Omega_{\mathbb{P}}^{p-1} \otimes \omega_{\mathbb{P}}^{-1}) \\ \longrightarrow \cdots &\longrightarrow H^i(\mathbb{P}, \pi_*\pi^*\Omega_{\mathbb{P}}^p) \longrightarrow H^i(X, \Omega_X^p) \longrightarrow H^i(\mathbb{P}, \pi^*\pi_*\Omega_{\mathbb{P}}^{p-1} \otimes \omega_{\mathbb{P}}^{-1}) \\ \longrightarrow \cdots &\longrightarrow H^{p+1}(\mathbb{P}, \pi_*\pi^*\Omega_{\mathbb{P}}^p) \longrightarrow 0 \end{aligned}$$

where the occurring tensor products are taken over $\mathcal{O}_{\mathbb{P}}$. The first three terms form a short exact sequence, isomorphic to

$$0 \to \bigoplus_{m \geqslant 0} H^0(\mathbb{P}, \Omega_{\mathbb{P}}^p(mn)) \to H^0(X, \Omega_X^p) \to \bigoplus_{m \geqslant 1} H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(mn)) \to 0.$$

It can be identified with the corresponding short exact sequence that results from restricting the Koszul complex over V to degrees that are multiples of m,

$$0 \to \bigoplus_{m \geqslant 0} H^0(\mathbb{P}, \Omega^p_{\mathbb{P}}(mn)) \to (\Omega^p_V)^{(n)} \to \bigoplus_{m \geqslant 1} H^0(\mathbb{P}, \Omega^{p-1}_{\mathbb{P}}(mn)) \to 0,$$

and it thereby yields the isomorphism

$$(\Omega^p_V)^{(n)} \cong H^0(X, \Omega^p_X) \subseteq \mathrm{HH}_p(X)$$

from the invariant differential forms on V onto the indicated direct summand of the Hochschild homology of X. Using the fact that for $q \geqslant 1, m \geqslant 0$, the cohomology groups $H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(m))$ vanish except for $p = q$ and $m = 0$, in which case $H^p(\mathbb{P}, \Omega^p_{\mathbb{P}}) = \mathbb{C}$, we reaffirm the result stated above. In particular, the $\mathcal{O}(V)^G$–torsion submodule of $\mathrm{HH}_0(X)$ appears first as $\bigoplus_{1 \neq g \in \mu_n} \mathbb{C}$ and second as $\bigoplus_{p=1}^{n-1} H^p(X, \Omega^p_X)$.

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