

SYMPLECTIC COVARIANCE PROPERTIES FOR SHUBIN AND BORN–JORDAN PSEUDO-DIFFERENTIAL OPERATORS

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Dedicated to H.G. Feichtinger on his 60th birthday

ABSTRACT. Among all classes of pseudo-differential operators only the Weyl operators enjoy the property of symplectic covariance with respect to conjugation by elements of the metaplectic group. In this paper we show that there is, however, a weaker form of symplectic covariance for Shubin’s τ -dependent operators, in which the intertwiners are no longer metaplectic, but are still invertible non-unitary operators. We also study the case of Born–Jordan operators, which are obtained by averaging the τ -operators over the interval $[0, 1]$ (such operators have recently been studied by Boggiatto and his collaborators, and by Toft). We show that covariance still holds for these operators with respect to a subgroup of the metaplectic group.

1. INTRODUCTION

In the early years of quantum mechanics physicists were confronted with an ordering problem: assume that some quantization process associates to the real variables x_j (position) and p_j (momentum) two operators \widehat{X}_j and \widehat{P}_j satisfying the canonical commutation rule $\widehat{X}_j\widehat{P}_j - \widehat{P}_j\widehat{X}_j = i\hbar$. Then what should the operator associated to the monomial $x_j^m p_j^\ell$ be? The first to give a mathematically motivated answer was Weyl [27]; he was developing his ideas on a group theoretical approach to quantization which led to the prescription

$$(1.1) \quad x_j^m p_j^\ell \xrightarrow{\text{Weyl}} \frac{1}{2^\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} \widehat{P}_j^{\ell-k} \widehat{X}_j^m \widehat{P}_j^k.$$

It turns out that the Weyl ordering is a particular case of the more general “ τ -ordering” : for any real number τ one defines

$$(1.2) \quad x_j^m p_j^\ell \xrightarrow{\tau} \sum_{k=0}^{\ell} \binom{\ell}{k} (1-\tau)^k \tau^{\ell-k} \widehat{P}_j^{\ell-k} \widehat{X}_j^m \widehat{P}_j^k,$$

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which reduces to Weyl’s prescription when $\tau = \frac{1}{2}$. From now on we will assume that $\widehat{X}_j f = x_j f$ and $\widehat{P}_j f = (1/2\pi i)\partial_{x_j} f$. The τ -ordering (1.2) is itself a particular case of the Shubin pseudo-differential calculus (Shubin [23]): given a symbol a the τ -pseudo-differential operator $A_\tau = \text{Op}_\tau(a)$ is formally defined by

$$(1.3) \quad A_\tau f(x) = \iint e^{2\pi i p(x-y)} a(\tau x + (1 - \tau)y, p) f(y) dp dy.$$

For $\tau = \frac{1}{2}$ we recover the well-known Weyl correspondence: $A_{1/2} = A_{\text{Weyl}} = \widehat{A}$, where

$$(1.4) \quad \widehat{A}f(x) = \iint e^{2\pi i p(x-y)} a(\frac{1}{2}(x + y), p) f(y) dp dy.$$

Using Schwartz’s kernel theorem it is not difficult to show that for every continuous linear operator $A : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ and for every $\tau \in \mathbb{R}$ there exists $a \in \mathcal{S}'(\mathbb{R}^{2n})$ such that $A = \text{Op}_\tau(a)$; the τ -operators are thus of a very general nature. Now, it is (reasonably) well known (Stein [24], Wong [28]) that among all τ -operators only the Weyl operators enjoy a symmetry property known as “symplectic covariance with respect to the metaplectic group”:

$$\text{If } \text{Op}_\tau(a \circ S) = \widehat{S} \text{Op}_\tau(a) \widehat{S}^{-1} \text{ for every } \widehat{S} \in \text{Mp}(2n, \mathbb{R}), \text{ then } \tau = \frac{1}{2}.$$

Here $\text{Mp}(2n, \mathbb{R})$ is the metaplectic group and S the projection of $\widehat{S} \in \text{Mp}(2n, \mathbb{R})$ on the symplectic group $\text{Sp}(2n, \mathbb{R})$. This property is very important in Hörmander’s approach to Weyl’s calculus in Chapter 18 (Theorem 18.5.9) of [19].

Symplectic covariance in the sense above is thus really a *characteristic property* of Weyl pseudo-differential calculus. In fact, one shows more generally that:

Let $a \mapsto \text{Op}(a)$ be a linear mapping from $\mathcal{S}'(\mathbb{R}^{2n})$ to the space of linear operators that is continuous in the topology of $\mathcal{S}'(\mathbb{R}^{2n})$. Assume that: (i) if $a = a(x)$, $a \in L^\infty(\mathbb{R}^n)$, then $\text{Op}(a)$ is multiplication by $a(x)$; (ii) if $S \in \text{Sp}(2n, \mathbb{R})$, then $\text{Op}(a \circ S) = \widehat{S} \text{Op}(a) \widehat{S}^{-1}$. Then $a \mapsto \text{Op}(a)$ is the Weyl correspondence

(Stein [24], §12.7, Wong, Chapter 30), so the property of symplectic covariance really singles out Weyl operators among all possible “quantization schemes”.

The principal aim of this paper is to report on the fact that there exists a weaker form of symplectic covariance for τ -operators which reduces to the case above when $\tau = \frac{1}{2}$. In fact, we will show in Proposition 3 that to each $S \in \text{Sp}(2n, \mathbb{R})$ with $\det(S - I) \neq 0$ we can associate an invertible operator $R_\tau(S) : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ such that

$$(1.5) \quad R_\tau(S) \text{Op}_\tau(a) = \text{Op}_\tau(a \circ S) R_\tau(S).$$

The operator $R_\tau(S)$ is explicitly given by the simple looking expression

$$(1.6) \quad R_\tau(S) = \sqrt{|\det(S - I)|} \int \widehat{T}_\tau(Sz) \widehat{T}_\tau(-z) dz,$$

where $\widehat{T}_\tau(z)$ is up to a factor depending on τ and z to the Heisenberg operator $\widehat{T}(z)$ familiar from harmonic analysis. The twisted symbol of $R_\tau(S)$, viewed as

τ -pseudo-differential operator, is given by the formula

$$(1.7) \quad r_\sigma(z) = \frac{1}{\sqrt{|\det(S-I)|}} e^{i\pi M(S)z^2},$$

where $M(S)$ is the symplectic Cayley transform of S .

The operator $R_\tau(S)$ is generally not unitary, and thus does not generate a copy of $\text{Mp}(2n, \mathbb{R})$. However it does generate a group $\text{Mp}_\tau(2n, \mathbb{R})$ of invertible operators $R'_\tau(S)$ (S an arbitrary element of $\text{Sp}(2n, \mathbb{R})$) which is, as we will see, algebraically isomorphic to the metaplectic group $\text{Mp}(2n, \mathbb{R})$. This group is actually a group of intertwiners for τ -operators, in the sense that we have the property

$$(1.8) \quad R'_\tau(S) \text{Op}_\tau(a) = \text{Op}_\tau(a \circ S) R'_\tau(S)$$

which truly generalizes the usual symplectic covariance with respect to the metaplectic group which only holds for Weyl operators.

We will also study symplectic covariance properties for Born–Jordan pseudo-differential operators. These operators were very recently introduced in de Gosson and Luef [17], where it was remarked that the constructions of Boggiatto and his collaborators [1, 2, 3] are related to a quantization procedure going back to Born and Jordan [4] and historically anterior to the work of Weyl [27]. Born and Jordan's quantization is based on the prescription

$$(1.9) \quad x^m p^\ell \xrightarrow{\text{BJ}} \frac{1}{\ell+1} \sum_{k=0}^{\ell} \widehat{P}^{\ell-k} \widehat{X}^m \widehat{P}^k,$$

an elementary calculation shows that this correspondence is obtained by averaging the τ -ordering (1.2) over the interval $[0, 1]$. This suggests that one define more generally the Born–Jordan pseudo-differential operator with symbol a by the formula

$$(1.10) \quad A_{\text{BJ}} = \int_0^1 A_\tau d\tau.$$

We will see that while one cannot expect any formula of the type (1.5), Born–Jordan operators are nevertheless symplectically covariant with respect to a subgroup of $\text{Mp}(2n, \mathbb{R})$, containing the Fourier transform as well as unitary rescalings.

A topic we have not included in this paper is the study of the properties of these operators on Feichtinger's [6, 7] modulation spaces. It turns out that there is a close relationship between Shubin operators and Shubin classes, which are in turn part of the family of modulation spaces (L^2 -spaces with radial symmetric weights). We will come back to these properties in future work.

Notation 1. The Euclidean scalar product of two vectors u and v on \mathbb{R}^m is denoted indifferently by $u \cdot v$ or by uv . When X is a symmetric matrix we will often write Xu^2 for $Xu \cdot u$. The standard symplectic form σ on $\mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{R}^{2n}$ is defined by $\sigma(z, z') = px' - p'x$. If $z = (x, p)$, $z' = (x', p')$, the corresponding symplectic group is $\text{Sp}(2n, \mathbb{R})$. We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on \mathbb{R}^n and by $\mathcal{S}'(\mathbb{R}^n)$ its dual (the tempered distributions). The normalizations we use correspond to those familiar with the theory of pseudo-differential operators and harmonic analysis. For instance, the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is

$$Ff(x) = \int e^{-2\pi i x x'} f(x') dx'$$

(it corresponds to the choice $\hbar = 1/2\pi$ in the quantum-mechanical literature).

2. THE SHUBIN CALCULUS

2.1. Definitions and main properties.

2.1.1. *The pseudo-differential operators A_τ .* The τ -pseudo-differential operator $A_\tau = \text{Op}_\tau(a)$ with symbol $a \in \mathcal{S}'(\mathbb{R}^n)$ is, by definition, the operator with distributional kernel

$$(2.1) \quad K_\tau(x, y) = F_2^{-1} [a(\tau x + (1 - \tau)y, \cdot)](x - y),$$

where F_2^{-1} is the inverse Fourier transform in the second set of variables. We can thus formally write (Shubin [23])

$$(2.2) \quad A_\tau f(x) = \iint e^{2\pi i p(x-y)} a(\tau x + (1 - \tau)y, p) f(y) dp dy.$$

One easily verifies by using this expression that the (formal) adjoint of $A_\tau = \text{Op}_\tau(a)$ is given by

$$(2.3) \quad \text{Op}_\tau(a)^* = \text{Op}_{1-\tau}(\bar{a}).$$

2.1.2. *The operators $\widehat{T}_\tau(z)$.* Let $\widehat{T}(z_0)$ be the Heisenberg operator: it is defined for $f \in \mathcal{S}'(\mathbb{R}^n)$ by

$$(2.4) \quad \widehat{T}(z_0)f(x) = e^{2\pi i(p_0 x - \frac{1}{2}p_0 x_0)} f(x - x_0),$$

where $z_0 = (x_0, p_0)$. Let τ be a real parameter and set, more generally,

$$(2.5) \quad \widehat{T}_\tau(z_0)f(x) = e^{2\pi i(p_0 x - (1-\tau)p_0 x_0)} f(x - x_0);$$

that is, equivalently,

$$(2.6) \quad \widehat{T}_\tau(z_0) = e^{i\pi(2\tau-1)p_0 x_0} \widehat{T}(z_0).$$

We have $\widehat{T}_{1/2}(z_0) = \widehat{T}(z_0)$, and

$$(2.7) \quad \widehat{T}_\tau(z_0)^{-1} = \widehat{T}_{1-\tau}(-z_0).$$

It is immediate to check the following relations:

$$(2.8) \quad \widehat{T}_\tau(z_0)\widehat{T}_\tau(z_1) = e^{2\pi i\sigma_\tau(z_0, z_1)} \widehat{T}_\tau(z_1)\widehat{T}_\tau(z_0),$$

$$(2.9) \quad \widehat{T}_\tau(z_0 + z_1) = e^{-i\pi\sigma_\tau(z_0, z_1)} \widehat{T}_\tau(z_0)\widehat{T}_\tau(z_1),$$

where

$$\sigma_\tau(z_0, z_1) = 2(1 - \tau)p_0 x_1 - 2\tau p_1 x_0.$$

Note that σ_τ is a bilinear form, but not in general a symplectic form (it fails to be antisymmetric if $\tau \neq \frac{1}{2}$).

For many purposes it is useful to write formula (2.2) in terms of the operators $\widehat{T}_\tau(z)$:

$$(2.10) \quad A_\tau f = \text{Op}_\tau(a)f = \int a_\sigma(z)\widehat{T}_\tau(z)f dz,$$

where a_σ is the symplectic Fourier transform of a ; that is,

$$a_\sigma(z) = \int e^{-2\pi i\sigma(z, z')} a(z') dz'.$$

Following the usage in the theory of Weyl operators, we will call a_σ the “twisted symbol of A_τ ”. The distributional kernel of A_τ can then be written

$$(2.11) \quad K_\tau(x, y) = F_2^{-1} [a_\sigma(x - y, \cdot)] (\tau x + (1 - \tau)y);$$

it is often more suitable for calculations than formula (2.1) above.

2.1.3. *A composition formula.* The τ -operators can be composed in exactly the same way as usual Weyl operators:

Proposition 1. *Let $A_\tau = \text{Op}_\tau(a)$ and $B_\tau = \text{Op}_\tau(b)$ with $a, b \in \mathcal{S}(\mathbb{R}^{2n})$ (or, more generally, $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $b \in \mathcal{S}(\mathbb{R}^{2n})$). We have $A_\tau B_\tau = C_\tau = \text{Op}_\tau(c)$ with*

$$(2.12) \quad c_\sigma(z) = \int e^{i\pi\sigma_\tau(z-z', z')} a_\sigma(z - z') b_\sigma(z') dz',$$

where the integral should be viewed as a distributional bracket when $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $b \in \mathcal{S}(\mathbb{R}^{2n})$.

Proof. Writing

$$(2.13) \quad A_\tau = \int a_\sigma(z) \widehat{T}_\tau(z) dz \quad \text{and} \quad B_\tau = \int b_\sigma(z) \widehat{T}_\tau(z) dz.$$

we have

$$A_\tau B_\tau = \iint a_\sigma(z_0) b_\sigma(z_1) \widehat{T}_\tau(z_0) \widehat{T}_\tau(z_1) dz_0 dz_1,$$

and hence, using formula (2.9),

$$A_\tau B_\tau = \iint e^{i\pi\sigma_\tau(z_0, z_1)} a_\sigma(z_0) b_\sigma(z_1) \widehat{T}_\tau(z_0 + z_1) dz_0 dz_1.$$

The composition formula (2.12) follows by making the change of variables $z = z_0 + z_1, z' = z_1$. □

2.2. The τ -Wigner transform. Boggiatto and his collaborators [1, 2, 3] have recently studied a τ -dependent Wigner transform $W_\tau(f, g)$ (which has independently been considered by Toft [25]). This transform is related to the Shubin τ -pseudodifferential calculus. Averaging W_τ over τ in the interval $[0, 1]$ leads to an element of the Cohen class, i.e. to a transform of the type $Q(f, g) = W_\tau(f, g) * \theta$ where $\theta \in \mathcal{S}'(\mathbb{R}^{2n})$.

The following result relates the operator A_τ to the τ -Wigner transform:

Proposition 2. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. We have the formula*

$$(2.14) \quad (A_\tau f | g)_{L^2} = \langle a, W_\tau(f, g) \rangle,$$

where $W_\tau(f, g)$ is the τ -dependent cross-Wigner transform of (f, g) defined by

$$(2.15) \quad W_\tau(f, g)(z) = \int e^{-2\pi i y p} f(x + \tau y) \overline{g(x - (1 - \tau)y)} dy.$$

Proof. We have

$$\langle a, W_\tau(f, g) \rangle = \int e^{-2\pi i y p} a(z) f(x + \tau y) \overline{g(x - (1 - \tau)y)} dy dp dx.$$

Setting $y' = x + \tau y =, x' = x - (1 - \tau)y$ we get

$$\langle a, W_\tau(f, g) \rangle = \int e^{-2\pi i(x' - y')p} a((1 - \tau)x' + \tau y', p) f(y') \overline{g(x')} dy' dp dx',$$

and hence the equality (2.14) in view of (2.2). □

Formula (2.14) yields an alternative definition of the operator $A_\tau f$ for an arbitrary symbol $a \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$: choose $g \in \mathcal{S}(\mathbb{R}^n)$; then $W_\tau(f, g) \in \mathcal{S}(\mathbb{R}^{2n})$ and the distributional bracket $\langle a, W_\tau(f, g) \rangle$ is thus defined; by definition A_τ is the continuous operator $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ defined by the right-hand side of (2.14).

Remark 2.1. The result above is closely related to (1.4) (and to some extent to Proposition 4.11) in Toft [25].

We notice that the τ -dependent Wigner transform $W_\tau \psi = W_\tau(f, g)$ satisfies the same marginal properties as the ordinary Wigner transform: for every $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have

$$(2.16) \quad \int W_\tau f(x, p) dp = |f(x)|^2, \quad \int W_\tau f(x, p) dx = |\widehat{f}(p)|^2$$

(see Boggiatto et al. [1]).

In the case $\tau = 1$ the transform W_τ reduces to the Rihaczek distribution

$$(2.17) \quad \mathcal{R}(f, g)(z) = e^{2\pi i p \cdot x} \widehat{f}(x) \overline{\widehat{g}(x)}$$

for $f, g \in L^2(\mathbb{R}^n)$. When $\tau = 0$ we get the so-called dual Rihaczek distribution

$$(2.18) \quad \mathcal{R}^*(f, g)(z) = e^{-2\pi i p \cdot x} f(x) \overline{\widehat{g}(x)}.$$

3. SYMPLECTIC COVARIANCE IN SHUBIN CALCULUS

3.1. A class of intertwining operators.

3.1.1. *The symplectic Cayley transform.* We will use the following notation:

$$\begin{aligned} \text{Sp}_{(0)}(2n, \mathbb{R}) &= \{S \in \text{Sp}(2n, \mathbb{R}) : \det(S - I) \neq 0\}, \\ \text{Sym}_{(0)}(2n, \mathbb{R}) &= \{M \in \text{Sym}(2n, \mathbb{R}) : \det(M - \frac{1}{2}J) \neq 0\}. \end{aligned}$$

Let $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$. By definition the symplectic Cayley transform (introduced in de Gosson [12, 13, 14, 15, 16]) of S is the symmetric matrix given by

$$(3.1) \quad M(S) = \frac{1}{2}J(S + I)(S - I)^{-1}$$

(the symmetry of $M(S)$ is readily verified using the relation $S^T J S = S J S^T = J$, which is equivalent to $S \in \text{Sp}(2n, \mathbb{R})$). It is clear that the mapping $M(\cdot)$ is a bijection $\text{Sp}_{(0)}(2n, \mathbb{R}) \rightarrow \text{Sym}_{(0)}(2n, \mathbb{R})$ and that the inverse of that bijection is given by

$$(3.2) \quad S = (M - \frac{1}{2}J)^{-1}(M + \frac{1}{2}J).$$

We have the properties

$$(3.3) \quad M(S^{-1}) = -M(S)$$

and, when in addition $S', SS' \in \text{Sp}_{(0)}(2n, \mathbb{R})$,

$$(3.4) \quad M(SS') = M(S) + (S^T - I)^{-1}J(M(S) + M(S'))^{-1}J(S - I)^{-1}.$$

3.1.2. *The intertwining operators* $R_\tau(S)$. We will need the following well-known generalization of the Fresnel formula (see e.g. Folland [8], Appendix A): let X be a real invertible matrix of dimension m ; then

$$(3.5) \quad \int e^{-2\pi iuv} e^{i\pi Xv^2} dv = |\det X|^{-1/2} e^{\frac{i\pi}{4} \text{sign } X} e^{-i\pi X^{-1}u^2},$$

where $\text{sign } X$ is the difference between the number of > 0 and < 0 eigenvalues of X . Using this formula and the two propositions above we set out to study the operators

$$(3.6) \quad R_\tau(S) = \sqrt{|\det(S - I)|} \int \widehat{T}_\tau(Sz) \widehat{T}_\tau(-z) dz$$

defined for $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$.

Proposition 3. (i) *Let* $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$. *The operator* $R_\tau(S)$ *is a continuous mapping* $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ *satisfying*

$$(3.7) \quad R_\tau(S) \widehat{T}_\tau(z) = \widehat{T}_\tau(Sz) R_\tau(S),$$

and we have, for $a \in \mathcal{S}'(\mathbb{R}^{2n})$,

$$(3.8) \quad R_\tau(S) \text{Op}_\tau(a) = \text{Op}_\tau(a \circ S) R_\tau(S).$$

(ii) *Let* $S, S' \in \text{Sp}_{(0)}(2n, \mathbb{R})$ *and assume that* $SS' \in \text{Sp}_{(0)}(2n, \mathbb{R})$ *as well. We have*

$$(3.9) \quad R_\tau(SS') = e^{i\frac{\pi}{4} \text{sign } M(SS')} R_\tau(S) R_\tau(S').$$

(iii) *We also have*

$$(3.10) \quad R_\tau(S^{-1}) = R_\tau(S)^{-1} = R_{1-\tau}(S)^*.$$

Proof. (i) It is equivalent to show that the operators

$$\Gamma_\tau(S) = \int \widehat{T}_\tau(Sz) \widehat{T}_\tau(-z) dz$$

are such that $\Gamma_\tau(S) \widehat{T}_\tau(z) = \widehat{T}_\tau(Sz) \Gamma_\tau(S)$. Let $f \in \mathcal{S}(\mathbb{R}^n)$; in view of formula (2.9) we have

$$\Gamma_\tau(S) f = \int e^{i\pi\sigma(Sz, z)} \widehat{T}_\tau((S - I)z) f dz.$$

Since $S - I$ is a linear automorphism, $\widehat{T}_\tau((S - I)z) : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$; hence $\Gamma_\tau(S) f \in \mathcal{S}(\mathbb{R}^n)$. The continuity of $\Gamma_\tau(S)$ is straightforward to verify. Set

$$F(z, z_0) = \widehat{T}_\tau(Sz) \widehat{T}_\tau(-z) \widehat{T}_\tau(z_0),$$

$$G(z, z_0) = \widehat{T}_\tau(Sz_0) \widehat{T}_\tau(Sz) \widehat{T}_\tau(-z).$$

By repeated use of formula (2.9) we obtain

$$F(z, z_0) = e^{-i\pi\sigma(Sz - z_0, z - z_0)} \widehat{T}_\tau((S - I)z + z_0),$$

$$G(z, z_0) = e^{-i\pi\sigma((S - I)z_0 + Sz_0, z)} \widehat{T}_\tau((S - I)z + Sz_0);$$

hence $G(z - z_0, z_0) = F(z, z_0)$. It follows that $\int F(z, z_0) dz = \int G(z, z_0) dz$, and hence the equality (3.7). That the operators $R_\tau(S)$ satisfy the intertwining relation (3.8) follows by using definition (2.10) of $\text{Op}_\tau(a)$.

(ii) (Cf. the proof of Proposition 4.2 in de Gosson [14].) For brevity we write $M = M(S)$, $M' = M(S')$. In view of the composition formula (2.12) the twisted symbol c_σ of $R_\tau(S)R_\tau(S')$ is

$$c_\sigma(z) = K \int e^{i\pi[\sigma(z,z')+\Phi(z,z')]} dz',$$

where the constant K and the phase Φ are given by

$$K = |\det(S - I)(S' - I)|^{-1/2},$$

$$\Phi(z, z') = Mz^2 - 2Mz \cdot z' + (M + M')z'^2.$$

A straightforward calculation shows that

$$\sigma(z, z') - 2Mz \cdot z' = -2J(S - I)^{-1}z \cdot z';$$

hence

$$\sigma(z, z') + \Phi(z, z') = -2J(S - I)^{-1}z \cdot z' + Mz^2 + (M + M')z'^2.$$

It follows that

$$(3.11) \quad c_\sigma(z) = K e^{i\pi Mz^2} \int e^{-2\pi i J(S-I)^{-1}z \cdot z'} e^{i\pi(M+M')z'^2} dz'.$$

Applying the Fresnel formula (3.5) with $X = M + M'$ to (3.12) and replacing K with its value we get

$$(3.12) \quad c_\sigma(z) = |\det[(M + M')(S - I)(S' - I)]|^{-1/2} e^{\frac{i\pi}{4} \text{sign}(M+M')} e^{2\pi i \Theta(z)},$$

where the phase Θ is given by

$$\Theta(z) = [M + (S^T - I)^{-1}J(M + M')^{-1}J(S - I)^{-1}] z^2$$

$$= M(SS')z^2$$

(the second equality in view of formula (3.4)). Noting that by definition of the symplectic Cayley transform we have

$$M + M' = J(I + (S - I)^{-1} + (S' - I)^{-1}),$$

it follows, again using property (3.4), that

$$\det[(M + M')(S - I)(S' - I)] = \det[(S - I)(M + M')(S' - I)]$$

$$= \det(SS' - I),$$

which concludes the proof of the product formula (3.9).

(iii) Let us first show that $R_\tau(S^{-1}) = R_\tau(S)^{-1}$. Let c be the symbol of $C = R_\tau(S)R_\tau(S^{-1})$; we claim that $c_\sigma(z) = \delta(z)$, and hence $C = I$. Noting that $\det(S^{-1} - I) = \det(S - I) \neq 0$, formula (3.11) in the proof of part (ii) shows that we have

$$c_\sigma(z) = L e^{i\pi Mz^2} \int e^{-2\pi i J(S-I)^{-1}z \cdot z'} e^{i\pi(M+M(S^{-1}))z'^2} dz',$$

where $L = |\det(S - I)|^{-1}$. Since $M(S^{-1}) = -M$ we have, setting $z'' = (S^T - I)^{-1}Jz'$,

$$c_\sigma(z) = \frac{e^{i\pi Mz^2}}{|\det(S - I)|} \int e^{-2\pi i J(S-I)^{-1}z \cdot z'} dz'$$

$$= e^{i\pi Mz^2} \int e^{2\pi i z z''} dz''.$$

Hence $c_\sigma(z) = \delta(z)$ by the Fourier inversion formula, which proves that $C = I$. Let us finally show that $R_\tau(S^{-1}) = R_{1-\tau}(S)^*$. We have

$$\begin{aligned} R_\tau(S^{-1}) &= \frac{1}{\sqrt{|\det(S^{-1} - I)|}} \int e^{i\pi M(S^{-1})z^2} \widehat{T}_\tau(z) dz \\ &= \frac{1}{\sqrt{|\det(S - I)|}} \int e^{-i\pi M(S)z^2} \widehat{T}_\tau(z) dz. \end{aligned}$$

Hence, using formula (2.3) for the adjoint of a τ -operator,

$$R_\tau(S^{-1})^* = \frac{1}{\sqrt{|\det(S - I)|}} \int e^{i\pi M(S)z^2} \widehat{T}_{1-\tau}(z) dz = R_{1-\tau}(S),$$

which is the same thing as $R_\tau(S^{-1}) = R_{1-\tau}(S)^*$. □

Notice that formula (3.10) shows that the operators $R_\tau(S)$ are unitary if and only if $\tau = \frac{1}{2}$ (the Weyl case (1.4)). See de Gosson [12, 13, 14, 15, 16] for a detailed discussion of this and related properties.

3.1.3. Application to the τ -Wigner function. The usual cross-Wigner function $W(f, g)$ has the following well-known (and very useful) property of symplectic covariance: for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $S \in \text{Sp}(2n, \mathbb{R})$ we have

$$(3.13) \quad W(\widehat{S}f, \widehat{S}g)(z) = W(f, g)(S^{-1}z),$$

where $\widehat{S} \in \text{Mp}(2n, \mathbb{R})$ is any of the two metaplectic operators which cover S . In the τ -dependent case this result must be modified as follows:

Proposition 4. *Let $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. We have*

$$(3.14) \quad W_\tau(R_\tau(S)f, R_{1-\tau}(S)g)(z) = W_\tau(f, g)(S^{-1}z).$$

Proof. Let $A_\tau = \text{Op}_\tau(a)$. Recall that $(\text{Op}_\tau(a)f|g)_{L^2} = \langle a|W_\tau(f, g) \rangle$ (formula (2.14)). In view of the second equality (3.10) we have

$$\begin{aligned} (R_\tau(S) \text{Op}_\tau(a)f|R_{1-\tau}(S)g)_{L^2} &= (R_{1-\tau}(S)^* R_\tau(S) \text{Op}_\tau(a)f|g)_{L^2} \\ &= (\text{Op}_\tau(a)f|g)_{L^2} \\ &= \langle a|W_\tau(f, g) \rangle. \end{aligned}$$

On the other hand, using the intertwining property (3.8), we have

$$\begin{aligned} (R_\tau(S) \text{Op}_\tau(a)f|R_{1-\tau}(S)g)_{L^2} &= (\text{Op}_\tau(a \circ S)R_\tau(S)f|R_{1-\tau}(S)g)_{L^2} \\ &= \langle a \circ S, W_\tau(R_\tau(S)f, R_{1-\tau}(S)g) \rangle \\ &= \langle a, W_\tau(R_\tau(S)f, R_{1-\tau}(S)g) \circ S^{-1} \rangle \end{aligned}$$

(the last identity by using the change of variables $z \mapsto S^{-1}z$ and the fact that $\det S = 1$). Formula (3.14) follows. □

Notice that we recover the usual symplectic covariance formula (3.13) for the cross-Wigner transform when $\tau = \frac{1}{2}$.

3.2. The operators $R_\tau(S)$ as pseudo-differential operators.

3.2.1. *The twisted symbol of $R_\tau(S)$.* The following result identifies $R_\tau(S)$ as a τ -pseudo-differential operator:

Proposition 5. *Let $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$. We have*

$$(3.15) \quad R_\tau(S) = \int r_\sigma(z) \widehat{T}_\tau(z) dz,$$

where the twisted symbol r_σ is given by the formula

$$(3.16) \quad r_\sigma(z) = \frac{1}{\sqrt{|\det(S - I)|}} e^{i\pi M(S)z^2}.$$

Proof. We have (see the proof of Proposition 3)

$$R_\tau(S) = \int e^{i\pi\sigma(Sz,z)} \widehat{T}_\tau((S - I)z) dz.$$

The change of variables $z' = (S - I)z$ yields

$$\Gamma_\tau(S) = |\det(S - I)|^{-1} \int e^{i\pi\sigma(S(S-I)^{-1}z',z')} \widehat{T}_\tau(z') dz'.$$

Since $S(S - I)^{-1} = I + (S - I)^{-1}$ we have

$$\begin{aligned} \sigma(S(S - I)^{-1}z', z') &= \sigma((S - I)^{-1}z', z') \\ &= J(S - I)^{-1}z' \cdot z' \\ &= \left(\frac{1}{2}J + J(S - I)^{-1}\right) z' \cdot z' \\ &= M(S)z' \cdot z', \end{aligned}$$

and hence the expression (3.16) for $r_\sigma(z)$. □

3.2.2. *Example: The Fourier transform.* For practical calculations the kernel formula (2.11) is useful. It immediately yields

$$(3.17) \quad K_\tau(S)(x + y, y) = \frac{1}{\sqrt{|\det(S - I)|}} \int e^{2\pi i(\tau x + y)p} e^{i\pi M(S)z^2} dp.$$

This formula can be used for the calculation of explicit expressions for the operators $R_\tau(S)$. Let us give an example. Choosing $S = J$ we have $M(S) = \frac{1}{2}I$; a straightforward computation yields

$$(3.18) \quad K_\tau(J)(x, y) = e^{i\frac{2\pi}{4}} e^{\frac{i\pi}{2}(x-y)^2} e^{-2\pi i(\tau x + (1-\tau)y)^2}.$$

Notice that when $\tau = \frac{1}{2}$ we get

$$(3.19) \quad K_{1/2}(J)(x, y) = e^{i\frac{2\pi}{4}} e^{-2\pi xy}.$$

Hence $R_{1/2}(J)$ is, up to a factor, the usual Fourier transform. In fact we have

$$R_{1/2}(J) \in \text{Mp}(2n, \mathbb{R});$$

$R_{1/2}(J)$ is the metaplectic operator \widehat{J} with projection J on the symplectic group (see e.g. de Gosson [12, 16], Folland [8]). This is not pure coincidence, as we will see in a moment when we discuss the metaplectic group.

3.2.3. *Application to Kohn–Nirenberg operators.* Kohn–Nirenberg operators are widely used in time-frequency analysis. They are called normally ordered operators in quantum mechanics, and standard pseudo-differential operators in microlocal analysis, where they have been an important tool since the early 1970s. They correspond to Shubin operators with the choice $\tau = 1$, leading via formula (1.2) of the Introduction to the ordering

$$(3.20) \quad x_j^m p_j^\ell \xrightarrow{\text{KN}} \widehat{X}_j^m \widehat{P}_j^\ell$$

for monomial symbols. More generally, in view of (2.2), Kohn–Nirenberg operators $A_1 = a(x, D)$ are formally given by the expression

$$(3.21) \quad a(x, D)f(x) = \int e^{-2\pi i p x} a(x, p) \widehat{f}(p) dp$$

where $\widehat{f} = Ff$ is the Fourier transform of f ; the function a is usually referred to as the Kohn–Nirenberg symbol of $a(x, D)$. The formula above actually makes sense as soon as $f \in \mathcal{S}(\mathbb{R}^n)$ and $a \in L^\infty(\mathbb{R}^{2n})$ (this latter condition will be satisfied in what follows), but one can give a meaning to (3.21) for much larger classes of symbols (e.g. the Hörmander classes [19]). Let us identify the intertwining operator $R_1(S)$.

Proposition 6. *Assume that $\det(S \pm I) \neq 0$. The Kohn–Nirenberg symbol of the operator $R_1(S)$ is*

$$(3.22) \quad r(z) = 2^{-n/2} e^{\frac{i\pi}{4} \text{sign } M(S)} \frac{1}{\sqrt{|\det(S + I)|}} e^{4i\pi M(-S)z^2},$$

where $M(-S) = JM(S)J$. Conversely, every function $r(z) = Ce^{4i\pi Mz^2}$ with M real symmetric and $\det(M - \frac{1}{2}J) \neq 0$ is such that

$$(3.23) \quad r(x, D)a(x, D) = (a \circ S)(x, D)r(x, D),$$

where $S = (M - \frac{1}{2}J)^{-1}(M + \frac{1}{2}J)$.

Proof. Since the symplectic Fourier transform F_σ is involutive, we have $r(z) = F_\sigma r_\sigma(z) = F r_\sigma(Jz)$. Using the Fresnel formula (3.5) we obtain after a few straightforward calculations

$$r(z) = 2^{-n/2} e^{\frac{i\pi}{4} \text{sign } M(S)} \frac{1}{\sqrt{|\det(S + I)|}} e^{i\pi JM(S)^{-1}Jz \cdot z}.$$

Using definition (3.1) of the symplectic Cayley transform we see that $JM(S)^{-1}J = 4M(-S)$, hence formula (3.22). The converse property is obvious, taking property (3.2) of the symplectic Cayley transform into account. \square

As an illustration of (3.22) we see that the Kohn–Nirenberg symbol $j(z)$ of the Fourier transform leads, in the case of odd $n = 2k + 1$, to the simple formula

$$(3.24) \quad j(z) = 2^{-(k+1)} (-i)^k e^{-2\pi i z^2}.$$

In the case of even n we have $\det(J + I) = 0$, so formula (3.22) does not apply.

3.2.4. *Discussion of the non-unitarity of $R_\tau(S)$.* The relations (2.8)–(2.9) satisfied by the operators $\widehat{T}_\tau(z)$ are the same as those satisfied by the usual Heisenberg operators $\widehat{T}(z)$. Now recall how the existence of the “metaplectic representation” is often justified in the literature (see, e.g., Folland [8], §4.2, Reiter [22], Stein [24], §12.7). We first remark that the formula $\rho(z, t) = e^{2\pi it}\widehat{T}(z)$ defines a unitary and irreducible representation (the “Schrödinger representation”) of the Heisenberg group $H(2n)$, which is $\mathbb{R}^{2n} \times \mathbb{R}$ equipped with the group law

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2}\sigma(z, z')).$$

For $S \in \text{Sp}(2n, \mathbb{R})$ define $\rho^S(z, t) = \rho(Sz, t)$; ρ^S is also a representation of $H(2n)$, and since ρ^S and ρ are the same on the center $\{(0, t) : t \in \mathbb{R}\}$ of $H(2n)$ they are equivalent. Hence there exists a unitary operator U_S on $L^2(\mathbb{R}^n)$ such that $U_S\widehat{T}(z) = \widehat{T}(Sz)U_S$. Since the representation ρ is irreducible, it follows that this formula defines the operator U_S uniquely up to a constant multiple with modulus one (Stone–von Neumann’s theorem). The operators U_S do not lead to a representation of $\text{Sp}(2n, \mathbb{R})$ because $U_{SS'} = c(S, S')U_SU_{S'}$, where c is a non-trivial cocycle. They do thus only lead to a projective representation of the symplectic group. Using standard group cohomology one then shows however that there exists a function $\mu(S) : \text{Sp}(2n, \mathbb{R}) \rightarrow \mathbb{C}$ such that the operators $\tilde{S} = \mu(S)U_S$ form a projective representation of the universal covering $\pi : \widetilde{\text{Sp}}(2n, \mathbb{R}) \rightarrow \text{Sp}(2n, \mathbb{R})$, which descends to a unitary representation of the double covering $\pi : \text{Sp}_2(2n, \mathbb{R}) \rightarrow \text{Sp}(2n, \mathbb{R})$. This is the metaplectic group $\text{Mp}(2n, \mathbb{R})$. (The index $\mu(S)$ is related to the Maslov index; it has been studied in detail in de Gosson [9, 10, 12, 16], following earlier work of Leray [21].) The reason for which the operators $R_\tau(S)$ are not unitarily equivalent to the elements of $\text{Mp}(2n, \mathbb{R})$ comes precisely from the fact that the $R_\tau(S)$ are not unitary (unless $\tau = \frac{1}{2}$; recall from Proposition 3 that $R_\tau(S)^* = R_{1-\tau}(S)^{-1}$). Therefore, these operators can, at best, generate a projective representation of the symplectic group. We will show in the next section that it is however possible to construct a group of non-unitary operators having properties similar to those of the metaplectic group $\text{Mp}(2n, \mathbb{R})$. However, this will need a careful study of the generators of the latter, following some of our previous work.

3.3. **The issue of L^2 -boundedness: A partial result.** We will prove that each operator $R_\tau(S)$ is bounded on $L^2(\mathbb{R}^n)$ provided that the parameter τ is sufficiently close to the “Weyl value” $\frac{1}{2}$. Intuitively speaking, it means that such operators are small deformations of the corresponding metaplectic operator \widehat{S} (see the description of $\text{Mp}(2n, \mathbb{R})$ in the next section) and therefore bounded on the square integrable functions. This partial result is in accordance with the theory of the “ ε -Wigner transform”

$$(3.25) \quad W^{(\varepsilon)}(f, g) = \frac{1}{2\varepsilon} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} W_\tau(f, g) d\tau,$$

developed by Boggiatto et al. in [2], where they show that $W^{(\varepsilon)}(f, g)$ has properties similar to those of $W(f, g)$ as $\varepsilon \rightarrow 0^+$.

Proposition 7. *Let $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$. There exists $\varepsilon > 0$ such that $R_\tau(S) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ for $\frac{1}{2} - \varepsilon \leq \tau \leq \frac{1}{2} + \varepsilon$, and there exists a constant $C(\varepsilon, S)$ such*

that

$$(3.26) \quad \|R_\tau(S)f\|_{L^2} \leq C(\varepsilon, S)\|f\|_{L^2}$$

for all $f \in L^2(\mathbb{R}^n)$.

Proof. We begin by noting that formulae (3.15), (3.16) in Proposition 5 can be rewritten, using (2.6), as

$$\begin{aligned} R_\tau(S) &= \frac{1}{\sqrt{|\det(S - I)|}} \int e^{i\pi M(S)z^2} e^{i\pi(2\tau-1)px} \widehat{T}(z) dz \\ &= \frac{1}{\sqrt{|\det(S - I)|}} \int e^{i\pi M'z^2} \widehat{T}(z) dz, \end{aligned}$$

where the symmetric matrix M' is given by

$$M' = M + (\tau - \frac{1}{2})N \quad , \quad N = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

By definition of the symplectic Cayley transform we have

$$(3.27) \quad M(S) = \frac{1}{2}J + J(S - I)^{-1}$$

(formula (3.2)); hence $\det(M(S) - \frac{1}{2}J) \neq 0$. It follows from the continuity of the determinant function that we will have $\det(M' - \frac{1}{2}J) \neq 0$ as well if $\tau - \frac{1}{2}$ is chosen small enough, and in this case $M' = M(S')$ for some $S' \in \text{Sp}_{(0)}(2n, \mathbb{R})$. We can thus write

$$R_\tau(S) = \frac{1}{\sqrt{|\det(S - I)|}} \int e^{i\pi M(S')z^2} \widehat{T}(z) dz,$$

and hence

$$(3.28) \quad R_\tau(S) = \sqrt{|\det(S' - I)|} R_{1/2}(S'),$$

so $R_\tau(S)$ is a multiple of a metaplectic operator. Formula (3.26) follows. □

It would certainly be interesting to see whether the result above can be improved. We note that formula (3.28) shows explicitly why the operators $R_\tau(S)$ are generally not unitary.

4. THE GROUP $\text{Mp}_\tau(2n, \mathbb{R})$

4.1. The metaplectic group $\text{Mp}(2n, \mathbb{R})$.

4.1.1. *Definition in terms of quadratic Fourier transforms.* We recall that the metaplectic group $\text{Mp}(2n, \mathbb{R})$ is a faithful unitary representation in $L^2(\mathbb{R}^n)$ of the double cover $\pi : \text{Sp}_2(2n, \mathbb{R}) \rightarrow \text{Sp}(2n, \mathbb{R})$ of the symplectic group. One of the possible definitions of $\text{Mp}(2n, \mathbb{R})$ is the following (Leray [21], de Gosson [10, 12, 16]): it is the group generated by the operators $\hat{S}_{W,m} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ with distributional kernel

$$(4.1) \quad K_{W,m}(x, x') = i^{m-n/2} \sqrt{|\det B^{-1}|} e^{2\pi i W(x, x')},$$

where $W(x, x')$ (the “generating quadratic form”) and B are defined as follows. If $S_W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is symplectic with $\det B \neq 0$, then

$$(4.2) \quad W(x, x') = \frac{1}{2}DB^{-1}x^2 - B^{-1}x \cdot x' + \frac{1}{2}B^{-1}Ax'^2;$$

the integer m (the ‘‘Maslov index’’) is defined by

$$(4.3) \quad m\pi \equiv \arg \det B \pmod{2\pi}.$$

The operator $\widehat{S}_{W,m}$ is invertible with inverse

$$(4.4) \quad \widehat{S}_{W,m}^{-1} = \widehat{S}_{W^*,m^*}, \quad W^*(x, x') = -W(x', x), \quad m^* = n - m.$$

Notice that the condition $\det B \neq 0$ means that S_W is a free symplectic matrix; equivalently $S(0 \times \mathbb{R}^n) \cap (0 \times \mathbb{R}^n) = 0$. The quadratic form W is a generating function for S ; that is, $(x, p) = S(x', p')$ is equivalent to $p = \partial_x W(x, x')$ and $p' = -\partial_{x'} W(x, x')$. We notice, for further use, that the operator $\widehat{S}_{W,m}$ can be written as a product of elementary unitary operators:

$$(4.5) \quad \widehat{S}_{W,m} = \widehat{V}_{-DB^{-1}} \widehat{M}_{B^{-1},m} \widehat{J} \widehat{V}_{-B^{-1}A},$$

where $\widehat{J} = e^{-i\frac{\pi}{4}} F$ and

$$(4.6) \quad \widehat{V}_{-DB^{-1}} f(x) = e^{i\pi DB^{-1}x^2} f(x), \quad \widehat{M}_{B^{-1},m} f(x) = i^m \sqrt{|\det B^{-1}|} f(B^{-1}x).$$

Observe that DB^{-1} and $B^{-1}A$ are symmetric matrices because S is symplectic. Projecting on $\text{Sp}(2n, \mathbb{R})$, formula (4.5) yields the matrix factorization

$$(4.7) \quad S_W = V_{-DB^{-1}} M_{B^{-1}} J V_{-B^{-1}A}$$

with

$$(4.8) \quad V_{-DB^{-1}} = \begin{pmatrix} I & 0 \\ DB^{-1} & I \end{pmatrix}, \quad M_{B^{-1}} = \begin{pmatrix} B & 0 \\ 0 & (B^T)^{-1} \end{pmatrix}.$$

The matrices V_{-P} , M_L generate, together with J , the symplectic group, and the corresponding operators \widehat{V}_{-P} , $\widehat{M}_{L,m}$, \widehat{J} generate the group $\text{Mp}(2n, \mathbb{R})$.

4.1.2. A factorization result. An essential (and not so well-known) fact (Leray [21]) is that every $\widehat{S} \in \text{Mp}(2n, \mathbb{R})$ can be written as the product $\widehat{S}_{W,m} \widehat{S}_{W',m'}$ of two metaplectic operators with kernel of the type (4.1). It is related to the fact that every $S \in \text{Sp}(2n, \mathbb{R})$ can be written as a product $S_W S_{W'}$ of two free symplectic matrices (de Gosson [10, 12, 16]). These factorizations are of course non-unique. We will need the following refinement of this property:

Lemma 4.1. *The generating quadratic forms W and W' in the factorizations $\widehat{S} = \widehat{S}_{W,m} \widehat{S}_{W',m'}$, $S = S_W S_{W'}$ can be chosen in such a way that $\det [(S_W - I)(S_{W'} - I)] \neq 0$, i.e. $S_W, S_{W'} \in \text{Sp}_{(0)}(2n, \mathbb{R})$.*

Proof. We first remark that

$$(4.9) \quad \det(S_W - I) = \det(-B) \det [B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}]$$

(Lemma 4 in de Gosson [11]). The next step consists in remarking that in view of the factorization (4.5) we have

$$(4.10) \quad \widehat{S}_{W,m} \widehat{S}_{W',m'} = \widehat{V}_{-P} \widehat{M}_{B^{-1},m} \widehat{J} \widehat{V}_{-(P'+Q)} \widehat{M}_{(B')^{-1},m'} \widehat{J} \widehat{V}_{-Q'}$$

with $P = DB^{-1}$, $P' = D'(B')^{-1}$, $Q = B^{-1}A$, $Q' = (B')^{-1}A'$. We next remark that the right-hand side of (4.10) remains unchanged if we replace P' with $P' + \lambda I$ and Q with $Q' - \lambda I$. Choosing λ so that

$$\begin{aligned} \det [P + Q - \lambda I + B^{-1} - (B^T)^{-1}] &\neq 0, \\ \det [P' + Q' + \lambda I - (B')^{-1} - (B'^T)^{-1}] &\neq 0, \end{aligned}$$

we have $\det(S_W - I) \neq 0$ and $\det(S_{W'} - I) \neq 0$ in view of the equality (4.9). The lemma follows by replacing, if necessary, $W(x, x')$ with $W(x, x') - \frac{1}{2}\lambda x^2$ and $W'(x, x')$ with $W'(x, x') + \frac{1}{2}\lambda x'^2$. \square

We remark that the symmetric matrix

$$(4.11) \quad W'' = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}$$

is the Hessian matrix of the generating quadratic form W .

4.2. The group $\text{Mp}_\tau(2n, \mathbb{R})$. In de Gosson [11] (also see [12, 13, 14, 16]) we proved the following result:

Proposition 8. *Let $\widehat{S} \in \text{Mp}(2n, \mathbb{R})$ have the projection $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$. Then*

$$(4.12) \quad \widehat{S} = i^{\nu(\widehat{S})} \sqrt{|\det(S - I)|} \int \widehat{T}(Sz)\widehat{T}(-z)dz,$$

where the integer $\nu(\widehat{S})$ corresponds to the choice (modulo 2) of an argument of $\det(S - I)$:

$$(4.13) \quad \nu(\widehat{S})\pi \equiv n\pi + \arg \det(S - I) \pmod{2\pi}.$$

When $S = S_W$ is a free symplectic matrix we have

$$(4.14) \quad \nu(\widehat{S}_{W,m}) \equiv m - \text{Inert } W'' \pmod{2},$$

where $\text{Inert } W''$ is the number of negative eigenvalues of the Hessian matrix (4.11).

It follows that our operators $R_\tau(S)$ coincide, up to a unimodular factor, with metaplectic operators when $\tau = \frac{1}{2}$:

Corollary 1. *For $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$ the operators $R(S) = R_{1/2}(S)$ are, up to a unimodular factor $i^{\nu(S)}$, elements of the metaplectic group $\text{Mp}(2n, \mathbb{R})$.*

Proof. In [14, 15] we have shown that

$$(4.15) \quad R^\nu(S) = \frac{i^{\nu(S)}}{\sqrt{|\det(S - I)|}} \int e^{i\pi M(S)z^2} \widehat{T}(z)dz,$$

where $\nu(S)$ is the Conley–Zehnder [5] index (which we describe below). The result follows since $\widehat{T}_{1/2}(z) = \widehat{T}(z)$. \square

The Conley–Zehnder index $i_{\text{CZ}}(\widetilde{S})$ is, loosely speaking, a Maslov-type intersection index measuring the number of intersections of a symplectic path $\widetilde{S} : t \mapsto S_t$, $0 \leq t \leq 1$, joining the identity to $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$ with the variety $\{S : \det(S - I) = 0\}$, which plays the role of a “caustic”. More precisely, $i_{\text{CZ}}(\widetilde{S})$ associates to the path $t \mapsto S_t$ an integer which only depends on the homotopy class (with fixed endpoints) of that path, and such that

$$(4.16) \quad \text{sign } \det(S - I) = (-1)^{n - i_{\text{CZ}}(\widetilde{S})}.$$

Equivalently,

$$(4.17) \quad \frac{1}{\pi} \arg \det(S - I) \equiv n - i_{\text{CZ}}(\widetilde{S}) \pmod{2}.$$

In [11] we have proven that when S is a free symplectic matrix S_W , then

$$(4.18) \quad i_{\text{CZ}}(\widetilde{S}_W) \equiv m + n - \text{Inert } W'' \pmod{2}.$$

For a detailed discussion of that index (including an extension to the case $\det(S - I) = 0$ and its expression in terms of the Leray–Maslov index [21]) see our paper [15].

In analogy with the definition of $\text{Mp}(2n, \mathbb{R})$ in terms of the basic metaplectic operators $\widehat{S}_{W,m}$ we define a group $\text{Mp}_\tau(2n, \mathbb{R})$ of (non-unitary) operators as follows. Let $S_W \in \text{Sp}_{(0)}(2n, \mathbb{R})$ be a free symplectic matrix; we know from Lemma 4.1 that every $S \in \text{Sp}(2n, \mathbb{R})$ is the product of two such matrices. Let $\widehat{S}_{W,m} \in \text{Mp}(2n, \mathbb{R})$ be any one of the two metaplectic operators covering S_W . We now define

$$(4.19) \quad R_\tau^{\nu(S_W)}(S_W) = i^{\nu(S_W)} R_\tau(S_W),$$

$$(4.20) \quad \nu(S_W) \equiv n\pi + \arg \det(S - I) \pmod{2\pi}.$$

The operators $R_\tau^{\nu(S_W)}(S_W)$ are invertible, and

$$(4.21) \quad R_\tau^{\nu(S_W)}(S_W)^{-1} = R_\tau^{-\nu(S_W)}(S_W^{-1}) = R_\tau^{-\nu(S_W)}(S_{W^*})$$

(the first equality in view of formula (3.10) in Proposition 3, and the second since the free symplectic matrix S_W^{-1} is generated by $W^*(x, x') = -W(x', x)$; cf. (4.4)).

By definition $\text{Mp}_\tau(2n, \mathbb{R})$ is the group of operators $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ generated by $R_\tau^{\nu(S_W)}(S_W)$: for an arbitrary $S \in \text{Sp}(2n, \mathbb{R})$ the operator $R_\tau^\nu(S) \in \text{Mp}_\tau(2n, \mathbb{R})$ is the product of a finite number of $R_\tau^{\nu(S_W)}(S_W)$ with $S_W \in \text{Sp}_{(0)}(2n, \mathbb{R})$. We have moreover the following equivalence of the fact that each $\widehat{S} \in \text{Mp}(2n, \mathbb{R})$ can be written as the product of two metaplectic operators of the type $\widehat{S}_{W,m}$:

Proposition 9. *Let $S \in \text{Sp}_{(0)}(2n, \mathbb{R})$ and choose $S_W, S_{W'} \in \text{Sp}_{(0)}(2n, \mathbb{R})$ such that $S = S_W S_{W'}$. We have*

$$(4.22) \quad R_\tau^{\nu(S_W)}(S_W) R_\tau^{\nu(S_{W'})}(S_{W'}) = R_\tau^{\nu(S)}(S_W S_{W'}),$$

where the index $\nu(S)$ is given by the formula

$$(4.23) \quad \nu(S) = \nu(S_W) + \nu(S_{W'}) + \frac{1}{2} \text{sign } M(SS').$$

Proof. It follows from formula (3.9) in Proposition 3 that we have

$$R_\tau^{\nu(S_W)}(S_W) R_\tau^{\nu(S_{W'})}(S_{W'}) = e^{i\Phi} R_\tau(S) R_\tau(S'),$$

$$\Phi = \frac{\pi}{2} (\nu(S_W) + \nu(S_{W'}) + \frac{1}{2} \text{sign } M(S_W S_{W'})),$$

hence the assertion. □

Proposition 10. *For every $R_\tau^\nu(S) \in \text{Mp}_\tau(2n, \mathbb{R})$ we have the symplectic covariance formulae*

$$(4.24) \quad R_\tau^\nu(S) \text{Op}_\tau(a) = \text{Op}_\tau(a \circ S) R_\tau^\nu(S)$$

and, for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$(4.25) \quad W_\tau(R_\tau(S)f, R_{1-\tau}(S)g)(z) = W_\tau(f, g)(S^{-1}z).$$

Proof. The intertwining formula (4.24) follows from Proposition 3 (formula (3.8)), and formula (4.25) follows from Proposition 4 (formula (3.14)). □

5. THE CASE OF BORN–JORDAN OPERATORS

5.1. Born–Jordan operators.

5.1.1. *Motivations.* Concurrently with Weyl, the physicists Born and Jordan [4] elaborated on Heisenberg’s seminal paper [18] on “matrix mechanics” and proposed the quantization rule

$$(5.1) \quad x_j^m p_j^\ell \xrightarrow{\text{BJ}} \frac{1}{\ell + 1} \sum_{k=0}^\ell \widehat{P}_j^{\ell-k} \widehat{X}_j^m \widehat{P}_j^k$$

which coincides with (1.1) when $m + \ell \leq 2$. We now make the following fundamental remark: the Born–Jordan prescription (5.1) is obtained by averaging the τ -ordering (1.2) on the interval $[0, 1]$; this is immediately seen using the property

$$B(k + 1, \ell - k + 1) = \int_0^1 (1 - \tau)^k \tau^{\ell-k} d\tau = \frac{k!(\ell - k)!}{(k + \ell + 1)!}$$

of the beta function. On the other hand, as already mentioned, Boggiatto and his collaborators have indirectly rediscovered the Born–Jordan prescription by introducing in [1], via the τ -Wigner function, the pseudo-differential operators

$$A_{\text{BJ}} = \int_0^1 A_\tau d\tau.$$

These facts show that this scheme certainly deserves to be studied, both from a quantum mechanical and a time-frequency analysis perspective.

5.1.2. *Definition of Born–Jordan operators.* In [1] Boggiatto et al. (also see [2, 3]) define a transform $Q : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ by integrating over $[0, 1]$ the τ -cross Wigner transforms (2.15). We will use the notation $Q = W_{\text{BJ}}$; thus, for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$(5.2) \quad W_{\text{BJ}}(f, g) = \int_0^1 W_\tau(f, g) d\tau.$$

They show that $W_{\text{BJ}}(f, g)$ has properties similar to those of the usual cross-Wigner transform, and that it belongs to the Cohen class. In particular, $W_{\text{BJ}}f = W_{\text{BJ}}(f, f)$ satisfies the marginal properties

$$(5.3) \quad \int W_{\text{BJ}}f(x, p) dp = |f(x)|^2, \quad \int W_{\text{BJ}}f(x, p) dx = |Ff(p)|^2$$

for every $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ (this immediately follows from property (2.16) of the τ -Wigner transform). We refer to [1] for a discussion of the advantages of using $W_{\text{BJ}}(f, g)$ instead of $W(f, g)$ in time-frequency analysis.

For $a \in \mathcal{S}'(\mathbb{R}^n)$ we can define an operator, which we denote A_{BJ} , by the formula

$$(5.4) \quad (A_{\text{BJ}}f|g)_{L^2} = \langle a, W_{\text{BJ}}(f, g) \rangle$$

$f, g \in \mathcal{S}(\mathbb{R}^n)$ (cf. (2.14)). We will call A_{BJ} the Born–Jordan operator with symbol a and write $A_{\text{BJ}} = \text{Op}_{\text{BJ}}(a)$.

Proposition 11. *The Born–Jordan operator $A_{\text{BJ}} = \text{Op}_{\text{BJ}}(a)$ is given by*

$$(5.5) \quad A_{\text{BJ}} = \int a_\sigma(z) \Theta(z) \widehat{T}(z) dz,$$

where Θ is the real function defined by

$$(5.6) \quad \Theta(z) = \frac{\sin(2\pi px)}{2\pi px}.$$

The operator A_{BJ} is a continuous operator $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ for every $a \in \mathcal{S}'(\mathbb{R}^{2n})$. That this formula really defines a continuous operator $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ follows from the fact that $\Theta \in L^\infty(\mathbb{R}^{2n})$.

Proof. Using the representation (2.10) of the τ -operators we have

$$(5.7) \quad A_{BJ} = \text{Op}_{BJ}(a) = \int a_\sigma(z) \widehat{T}_{BJ}(z) dz,$$

where $\widehat{T}_{BJ}(z)$ is the unitary operator defined by

$$(5.8) \quad \widehat{T}_{BJ}(z) = \int_0^1 \widehat{T}_\tau(z) d\tau = \frac{\sin(2\pi px)}{2\pi px} \widehat{T}(z).$$

Integrating both sides of formula (2.6) for $\widehat{T}_\tau(z)$ in the interval $[0, 1]$ we have $\widehat{T}_{BJ}(z) = \Theta(z)\widehat{T}(z)$, hence the expression (5.5). \square

In view of the relation (2.3) between a τ -operator and its adjoint we have

$$(5.9) \quad \text{Op}_{BJ}(a)^* = \text{Op}_{BJ}(\bar{a}).$$

Hence the Born–Jordan operators share with Weyl operators the property of being (formally) self-adjoint if and only if their symbol is real. This makes Born–Jordan operators good alternative candidates for quantization.

The reader is urged to notice that while every Born–Jordan operator is a Weyl operator, the converse property is not true because an arbitrary distribution $b_\sigma \in \mathcal{S}'(\mathbb{R}^{2n})$ cannot in general be written in the form $a_\sigma\Theta$ (see de Gosson and Luef [17]; see also the discussion in Kauffmann [20] where conceptual questions from quantum mechanics are taken up).

5.2. Partial metaplectic covariance of Born–Jordan operators. The intertwining properties for τ operators do not carry over to the Born–Jordan case. It is meaningless to expect a relation such as $R_{BJ}(S)\widehat{T}_{BJ}(z) = \widehat{T}_{BJ}(Sz)R_{BJ}(S)$ which would lead to a symplectic covariance property of the type (3.8). The good news is, however, that Born–Jordan operators enjoy a symplectic covariance property for operators belonging to a subgroup of the standard metaplectic group $\text{Mp}(2n, \mathbb{R})$. Recall that $\text{Mp}(2n, \mathbb{R})$ is generated by the modified Fourier transform $\widehat{J} = e^{i\frac{n\pi}{4}}F$, the multiplication operators $\widehat{V}_{-P}f = e^{i\pi Px^2}f$ ($P = P^T$) and the unitary scaling operators $\widehat{M}_{L,m}f(x) = i^m \sqrt{|\det L|}f(Lx)$ ($\det L \neq 0$, $m\pi = \arg \det L$). The projections of these operators on $\text{Sp}(2n, \mathbb{R})$ are, respectively, J , $V_{-P} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$, and

$$M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^2 \end{pmatrix}.$$

Proposition 12. *Let $A_{BJ} = \text{Op}_{BJ}(a)$ with $a \in \mathcal{S}'(\mathbb{R}^{2n})$. We have*

$$(5.10) \quad \widehat{S} \text{Op}_{BJ}(a) = \text{Op}_{BJ}(a \circ S^{-1})\widehat{S}$$

for every $\widehat{S} \in \text{Mp}(2n, \mathbb{R})$, which is a product of a (finite number) of operators \widehat{J} and $\widehat{M}_{L,m}$.

Proof. It suffices to prove formula (5.10) for $\widehat{S} = \widehat{J}$ and $\widehat{S} = \widehat{M}_{L,m}$. First let \widehat{S} be an arbitrary element of $\text{Mp}(2n, \mathbb{R})$. We have

$$\begin{aligned} \widehat{S} \text{Op}_{\text{BJ}}(a) &= \int a_\sigma(z) \Theta(z) \widehat{S} \widehat{T}(z) dz \\ &= \left(\int a_\sigma(z) \Theta(z) \widehat{T}(Sz) dz \right) \widehat{S}, \end{aligned}$$

where the second equality follows from the usual symplectic covariance property $\widehat{S} \widehat{T}(z) = \widehat{T}(Sz) \widehat{S}$ of the Heisenberg operators. Making the change of variables $z' = Sz$ in the integral we get, since $\det S = 1$,

$$\int a_\sigma(z) \Theta(z) \widehat{T}(Sz) dz = \int a_\sigma(S^{-1}z) \Theta(S^{-1}z) \widehat{T}(z) dz.$$

Now, by definition of the symplectic Fourier transform we have

$$a_\sigma(S^{-1}z) = \int e^{-2\pi i \sigma(S^{-1}z, z')} a(z') dz' = (a \circ S^{-1})_\sigma(z).$$

Now let $\widehat{S} = \widehat{M}_{L,m}$; we have

$$\Theta(M_L^{-1}z) = \frac{\sin(2\pi Lp \cdot (L^T)^{-1}x)}{2\pi Lp \cdot (L^T)^{-1}x} = \Theta(z).$$

Similarly $\Theta(J^{-1}z) = \Theta(z)$; hence in both cases

$$\begin{aligned} \widehat{S} \text{Op}_{\text{BJ}}(a) &= \left(\int (a \circ S^{-1})_\sigma \Theta(z) \widehat{T}(z) dz \right) \widehat{S} \\ &= \text{Op}_{\text{BJ}}(a \circ S^{-1}) \widehat{S}, \end{aligned}$$

whence formula (5.10). □

The proof above shows that the essential step consists in noting that $\Theta(S^{-1}z) = \Theta(z)$ when $S = J$ or $S = M_L$. It is clear that this property fails if one takes $S = V_P$ with $P \neq 0$, so we cannot expect to have full symplectic covariance for Born–Jordan operators since $\text{Sp}(2n, \mathbb{R})$ is generated by the set of all matrices J , M_L and V_P . Full symplectic covariance is excluded anyway in view of our discussion in the Introduction to this paper, since this property is characteristic of Weyl calculus, and thus cannot be satisfied by any other pseudo-differential calculus.

5.3. The case of “physical” Hamiltonians. Weyl and Born–Jordan quantization are the same for monomials $x_j^m p_j^\ell$ with $m + \ell \leq 2$. It follows that every Born–Jordan operator with polynomial symbol

$$a(z) = Q(z) + a \cdot z + b,$$

where $Q(z)$ is a complex quadratic form in the coordinates x_j, p_k , $a \in \mathbb{C}^{2n}$, $b \in \mathbb{C}$, is symplectically covariant with respect to the full metaplectic group $\text{Mp}(2n, \mathbb{R})$.

In fact, Weyl and Born–Jordan quantization are also identical for symbols arising from physical Hamiltonians

$$(5.11) \quad H(z) = \sum_{j=1}^n \frac{1}{2m_j} (p_j - A_j(x))^2 + V(x),$$

where A_j and V are real C^∞ functions. This easily follows from the two observations below: any τ -operator with symbol $a = a(x)$ only depending on x is multiplication

by $a(x)$ (this immediately follows from formula (2.2)); applying the same formula to the term $p_j A_j$ appearing in the expansion of $(p_j - A_j(x))^2$ in (5.11) we have

$$\text{Op}_\tau(p_j A_j)f(x) = \frac{1}{2\pi i} [(1 - \tau)\partial_{x_j}(A_j f) + \tau A_j \partial_{x_j} f],$$

and hence, integrating from 0 to 1,

$$\text{Op}_{\text{BJ}}(p_j A_j) = \frac{1}{2\pi i} [\partial_{x_j}(A_j f) + A_j \partial_{x_j} f].$$

Collecting these facts we get

$$(5.12) \quad H_{\text{BJ}} = \sum_{j=1}^n \frac{1}{2m_j} \left(\frac{1}{2\pi i} \partial_{x_j} - A_j(x) \right)^2 + V(x),$$

which is identical with $\widehat{H} = H_{\text{Weyl}}$, as can be directly verified using the expression (1.4) for Weyl pseudo-differential operators. It follows that this operator satisfies full symplectic covariance with respect to the metaplectic group.

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