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MINIMAL SURFACE SYSTEMS, MAXIMAL SURFACE SYSTEMS AND SPECIAL LAGRANGIAN EQUATIONS

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Abstract. We extend Calabi's correspondence between minimal graphs in Euclidean space $\mathbb{R}^3$ and maximal graphs in Lorentz-Minkowski space $\mathbb{L}^3$. We establish the twin correspondence between 2-dimensional minimal graphs in Euclidean space $\mathbb{R}^{n+2}$ carrying a positive area-angle function and 2-dimensional maximal graphs in pseudo-Euclidean space $\mathbb{R}^{n+2}_n$ carrying the same positive area-angle function.

We generalize Osserman's Lemma on degenerate Gauss maps of entire 2-dimensional minimal graphs in $\mathbb{R}^{n+2}$ and offer several Bernstein-Calabi type theorems. A simultaneous application of the Harvey-Lawson Theorem on special Lagrangian equations and our extended Osserman's Lemma yield a geometric proof of Jörgens' Theorem on the 2-variable unimodular Hessian equation.

We introduce the correspondence from 2-dimensional minimal graphs in $\mathbb{R}^{n+2}$ to special Lagrangian graphs in $\mathbb{C}^2$, which induces an explicit correspondence from 2-variable symplectic Monge-Ampère equations to the 2-variable unimodular Hessian equation.

Contents

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1. Motivation and main results

The point of this paper is to investigate the minimal surface system in Euclidean space $\mathbb{R}^{n+2}$ endowed with the metric ${dx_1}^2 + \cdots + {dx_{n+2}}^2$ and the maximal surface system in pseudo-Euclidean space $\mathbb{R}^{n+2}_n$ equipped with the pseudo-Euclidean metric ${dx_1}^2 + {dx_2}^2 - {dx_3}^2 - \cdots - {dx_{n+2}}^2$.

The research on the minimal surface system is initiated in [22, 29, 30]. Lawson and Osserman [22] studied non-existence, non-uniqueness and irregularity of solutions of the minimal surface system. There has been extensive work in extending Bernstein's Theorem to higher codimensions [11, 14, 18, 19, 25, 28, 34, 35, 36, 40] and studying the minimal surface system [23, 24, 37, 38].

About one hundred years ago, Bernstein proved a truly beautiful theorem that the only entire solutions of the minimal surface equation in $\mathbb{R}^3$

$$0 = \left(1 + {f_y}^2\right) f_{xx} - 2 f_x f_y f_{xy} + \left(1 + {f_x}^2\right) f_{yy}$$

are affine functions. As illustrated in [27], extending Bernstein's Theorem in $\mathbb{R}^3$ is one of the central themes in the modern theory of minimal submanifolds.

Calabi introduced a new extension of Bernstein's Theorem in $\mathbb{R}^3$. In 1970, he showed that the only entire maximal graphs in $(2+1)$-dimensional space-time $\mathbb{L}^3$ are spacelike planes. In 1976, Cheng and Yau generalized Calabi's Theorem in Lorentz space $\mathbb{L}^n$ for all dimensions $n \geq 3$.

Furthermore, Calabi [6] introduced an interesting duality between the minimal surface equation in $\mathbb{R}^3$ and the maximal surface equation in $\mathbb{L}^3$. Alías and Palmer [4] employed Calabi's correspondence to show that the non-existence of entire non-planar minimal graphs in $\mathbb{R}^3$ is equivalent to the non-existence of entire non-planar maximal graphs in $\mathbb{L}^3$. Recently, Araújo and Leite [1] discovered interesting results on the duality which is equivalent to Calabi's correspondence.

In this paper, we extend the geometric integrability in the Calabi correspondence to higher codimensions $n \geq 2$ by constructing the twin correspondence between 2-dimensional minimal graphs having a positive area-angle function (introduced in Section 2.1) in Euclidean space $\mathbb{R}^{n+2}$ and 2-dimensional maximal graphs having the same positive area-angle function in pseudo-Euclidean space $\mathbb{R}^{n+2}_n$.

More explicitly, for the codimension $n \geq 2$, by the twin correspondence, a minimal graph $x\mathbf{e}_1 + y\mathbf{e}_2 + f_1(x,y)\mathbf{e}_3 + \cdots + f_n(x,y)\mathbf{e}_{n+2}$ in $\mathbb{R}^{n+2}$ over the simply connected domain Ω satisfying the positive area-angle condition $\sum_{1\leq i<j\leq n} \frac{\partial(f_i,f_j)}{\partial(x,y)}^2 < 1$ associates a maximal graph $x\mathbf{e}_1 + y\mathbf{e}_2 + g_1(x,y)\mathbf{e}_3 + \cdots + g_n(x,y)\mathbf{e}_{n+2}$ in $\mathbb{R}^{n+2}_n$ over the domain Ω obeying the positive area-angle condition $\sum_{1\leq i<j\leq n} \frac{\partial(g_i,g_j)}{\partial(x,y)}^2 < 1$.

Our twin correspondence shows that the minimal surface system in $\mathbb{R}^{n+2}$ becomes the integrability condition for the maximal surface system in $\mathbb{R}^{n+2}_n$. The twin correspondence with $n = 2$ induces an explicit duality between the 2-variable special Lagrangian equation and the 2-variable split special Lagrangian equation. We also construct the correspondence from minimal graphs in $\mathbb{R}^{n+2}$ to special Lagrangian graphs in $\mathbb{C}^2$.

The Bernstein problem for surfaces in higher codimension is to find conditions under which extremal graphs of functions from $\mathbb{R}^2$ to $\mathbb{R}^n$ are affine functions. Unlike Bernstein's Theorem in $\mathbb{R}^3$, for the higher codimensions $n \geq 2$, there exist plenty of entire 2-dimensional minimal non-planar graphs in $\mathbb{R}^{n+2}$.

Recently, Hasanis, Savas-Halilaj and Vlachos [14] showed that Bernstein's Theorem for 2-dimensional entire minimal graphs in $\mathbb{R}^{n+2}$ with a positive area-angle function holds. The twin correspondence induces a Calabi type theorem for entire 2-dimensional maximal graphs in $\mathbb{R}^{n+2}_n$ with a positive area-angle function.

We briefly sketch the structure of the rest of this paper. In Section 2, we quickly review some notation, the minimal surface system in Euclidean space $\mathbb{R}^{n+2}$, and the maximal surface system in pseudo-Euclidean space $\mathbb{R}^{n+2}_n$.

In Section 3.1, we study the generalized Gauss map of minimal surfaces in $\mathbb{R}^{n+2}$ to investigate their global properties. Our extended Osserman's Lemma on degenerate Gauss maps of entire 2-dimensional minimal graphs in $\mathbb{R}^{n+2}$ indicates that pairs of their height functions having nowhere zero Jacobian determinant on the whole plane contribute to the degeneracy of their Gauss maps.

Employing our extended Osserman's Lemma, we are able to prove several Bernstein type theorems for minimal surfaces with codimensions $n \geq 2$. If the height functions of entire 2-dimensional minimal graphs in $\mathbb{R}^{n+2}$ are strictly monotone, they are planes. We present a minimal-surface proof of Jörgens' Theorem that the only entire solutions of the unimodular Hessian equation are quadratic polynomial functions.

Section 3.2 is devoted to the construction of the correspondence from minimal graphs in $\mathbb{R}^{n+2}$ to special Lagrangian graphs in $\mathbb{C}^2$.

In Section 4.1, we construct the twin correspondence between 2-dimensional minimal graphs in $\mathbb{R}^{n+2}$ with a positive area-angle function and 2-dimensional maximal graphs in $\mathbb{R}^{n+2}_n$ with the same positive area-angle function. This generalizes the classical Calabi correspondence between minimal graphs in $\mathbb{R}^3$ and maximal graphs in $\mathbb{L}^3$. An application of the twin correspondence yields a higher codimension extension of Calabi's Theorem in $\mathbb{L}^3$.

In Section 4.2, we introduce the special Lagrangian equation and the split special Lagrangian equation and see why they are geometrically equivalent to each other. Exploiting Jörgens' Theorem on the unimodular Hessian equation, we can classify all entire solutions of these two symplectic Monge-Ampère equations. In particular, our Calabi type theorem states that any entire maximal gradient graph (x, y, f_x, f_y) in $\mathbb{R}^4_2$ for some potential function $f : \mathbb{R}^2 \to \mathbb{R}$ should be a spacelike plane.

Finally, in Section 4.3, we prove the existence of simultaneous conformal coordinate transformations for twin graphs in $\mathbb{R}^{n+2}$ and $\mathbb{R}^{n+2}_n$. The twin correspondence admits an explicit description via the induced holomorphic curves in their Weierstrass representation formulas.

2. Preliminaries

2.1. **Notation and assumptions.** Throughout this paper, $\Omega \subset \mathbb{R}^2$ denotes a simply connected domain. All real-valued functions here are at least of class $\mathcal{C}^2$. Our ambient spaces are $\mathbb{R}^{n+2}$ and $\mathbb{R}^{n+2}_n$. Here, $\mathbb{R}^{n+2}$ denotes the Euclidean space endowed with the metric ${dx_1}^2 + \cdots + {dx_{n+2}}^2$ and $\mathbb{R}^{n+2}_n$ the pseudo-Euclidean space equipped with the psuedo-Euclidean metric ${dx_1}^2 + {dx_2}^2 - {dx_3}^2 - \cdots - {dx_{n+2}}^2$.

Definition 1. Let $n \geq 2$. Given a function $\phi = (\phi_1, \cdots, \phi_n) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^n$, we use the notation

$$\|\mathcal{J}\| := \sqrt{\sum_{1 \leq i < j \leq n} {\mathcal{J}_{i,j}}^2}, \quad \mathcal{J}_{i,j} := \frac{\partial\left(\phi_i, \phi_j\right)}{\partial(x, y)}.$$

When the estimation $\|\mathcal{J}\| < 1$ holds on the domain Ω, we say that $\phi = (\phi_1, \cdots, \phi_n)$ is a positive area-angle map and that the graph of $\phi = (\phi_1, \cdots, \phi_n)$ has a positive area-angle. We call the function $\Theta_\phi = \cos^{-1}(\|\mathcal{J}\|) \in \left(0, \frac{\pi}{2}\right]$ the area-angle of ϕ.

2.2. **Minimal surface system in Euclidean space $\mathbb{R}^{n+2}$.** Let $\mathbf{e}_1 = (1, 0, \cdots, 0)$, $\cdots$, $\mathbf{e}_{n+2} = (0, 0, \cdots, 1)$ denote the standard basis in $\mathbb{R}^{n+2}$.

Proposition 1 (Osserman, [29, 32])**.** *The graph $\mathbf{\Phi}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + f_1(x, y)\mathbf{e}_3 + \cdots + f_n(x, y)\mathbf{e}_{n+2}$ of the height function $f = (f_1, \cdots, f_n) : \Omega \to \mathbb{R}^n$ becomes a minimal surface in $\mathbb{R}^{n+2}$ if and only if the minimal surface system holds:*

$$G\frac{\partial^2 f_k}{\partial x^2} - 2F\frac{\partial^2 f_k}{\partial x \partial y} + E\frac{\partial^2 f_k}{\partial y^2} = 0, \quad k \in \{1, \cdots, n\}.$$

Every $\mathcal{C}^2$ solution of the minimal surface system is real analytic. Here, we set

$$\begin{cases} E = 1 + \left(\frac{\partial f_1}{\partial x}\right)^2 + \cdots + \left(\frac{\partial f_n}{\partial x}\right)^2, \\ F = \quad \frac{\partial f_1}{\partial x}\frac{\partial f_1}{\partial y} + \cdots + \frac{\partial f_n}{\partial x}\frac{\partial f_n}{\partial y}, \\ G = 1 + \left(\frac{\partial f_1}{\partial y}\right)^2 + \cdots + \left(\frac{\partial f_n}{\partial y}\right)^2. \end{cases}$$

The patch $\mathbf{\Phi}$ induces the metric $ds^2_{\mathbf{\Phi}} = Edx^2 + 2Fdxdy + Gdy^2$.

Definition 2. Let $\omega = \sqrt{EG - F^2} > 0$ denote the area element. As in the codimension-one case, we call $\frac{1}{\omega}$ the angle function of the minimal graph $\mathbf{\Phi}$ in $\mathbb{R}^{n+2}$.

Proposition 2. *Let $(\alpha_k, \beta_k) = \left(\frac{\partial f_k}{\partial x}, \frac{\partial f_k}{\partial y}\right)$, $k \in \{1, \cdots, n\}$.*

(a) We can rewrite the minimal surface system in the divergence-zero form:

$$\frac{\partial}{\partial x}\left(\frac{G}{\omega}\alpha_k - \frac{F}{\omega}\beta_k\right) + \frac{\partial}{\partial y}\left(\frac{E}{\omega}\beta_k - \frac{F}{\omega}\alpha_k\right) = 0, \quad k \in \{1, \cdots, n\}.$$

(b) We also have $\frac{\partial}{\partial x}\left(\frac{G}{\omega}\right) = \frac{\partial}{\partial y}\left(\frac{F}{\omega}\right)$ and $\frac{\partial}{\partial x}\left(\frac{F}{\omega}\right) = \frac{\partial}{\partial y}\left(\frac{E}{\omega}\right)$.

2.3. **Maximal surface system in pseudo-Euclidean space $\mathbb{R}^{n+2}_n$.** We consider $\mathbf{e}_1 = (1, 0, \cdots, 0)$, $\cdots$, $\mathbf{e}_{n+2} = (0, 0, \cdots, 1)$ as the standard basis in $\mathbb{R}^{n+2}_n$.

Proposition 3. *The spacelike graph $\widehat{\mathbf{\Phi}}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + g_1(x, y)\mathbf{e}_3 + \cdots + g_n(x, y)\mathbf{e}_{n+2}$ of the height function $g = (g_1, \cdots, g_n) : \Omega \to \mathbb{R}^n$ becomes a maximal surface in $\mathbb{R}^{n+2}_n$ if and only if g satisfies the maximal surface system:*

$$\widehat{G}\frac{\partial^2 g_k}{\partial x^2} - 2\widehat{F}\frac{\partial^2 g_k}{\partial x \partial y} + \widehat{E}\frac{\partial^2 g_k}{\partial y^2} = 0, \quad k \in \{1, \cdots, n\},$$

or equivalently,

$$\frac{\partial}{\partial x}\left(\frac{\widehat{G}}{\widehat{\omega}}\widehat{\alpha}_k - \frac{\widehat{F}}{\widehat{\omega}}\widehat{\beta}_k\right) + \frac{\partial}{\partial y}\left(\frac{\widehat{E}}{\widehat{\omega}}\widehat{\beta}_k - \frac{\widehat{F}}{\widehat{\omega}}\widehat{\alpha}_k\right) = 0, \quad \left(\widehat{\alpha}_k, \widehat{\beta}_k\right) = \left(\frac{\partial g_k}{\partial x}, \frac{\partial g_k}{\partial y}\right).$$

Here, we write

$$\begin{cases} \widehat{E} = 1 - \left(\frac{\partial g_1}{\partial x}\right)^2 - \cdots - \left(\frac{\partial g_n}{\partial x}\right)^2, \\ \widehat{F} = \quad -\frac{\partial g_1}{\partial x}\frac{\partial g_1}{\partial y} - \cdots - \frac{\partial g_n}{\partial x}\frac{\partial g_n}{\partial y}, \\ \widehat{G} = 1 - \left(\frac{\partial g_1}{\partial y}\right)^2 - \cdots - \left(\frac{\partial g_n}{\partial y}\right)^2. \end{cases}$$

We assume the spacelike condition $\widehat{E}\widehat{G} - \widehat{F}^2 > 0$. *The patch* $\widehat{\mathbf{\Phi}}$ *induces the Riemannian metric* $ds^2_{\widehat{\mathbf{\Phi}}} = \widehat{E}dx^2 + 2\widehat{F}dxdy + \widehat{G}dy^2$. *We set* $\widehat{\omega} = \sqrt{\widehat{E}\widehat{G} - \widehat{F}^2}$. *We call* $\frac{1}{\widehat{\omega}}$ *the angle function of the maximal graph* $\widehat{\mathbf{\Phi}}$ *in* $\mathbb{R}^{n+2}_n$. *Also, we obtain two identities* $\frac{\partial}{\partial x}\left(\frac{\widehat{G}}{\widehat{\omega}}\right) = \frac{\partial}{\partial y}\left(\frac{\widehat{F}}{\widehat{\omega}}\right)$ *and* $\frac{\partial}{\partial x}\left(\frac{\widehat{F}}{\widehat{\omega}}\right) = \frac{\partial}{\partial y}\left(\frac{\widehat{E}}{\widehat{\omega}}\right)$.

The proofs of the above results are analogous to the minimal surface system.

3. Two dimensional minimal graphs in $\mathbb{R}^{n+2}$

3.1. **Extension of Osserman's Lemma.** Bernstein's Theorem in $\mathbb{R}^3$ is essentially concerned with the Gauss map. We first introduce the generalized Gauss map of minimal surfaces in $\mathbb{R}^{n+2}$ [9, 15, 16, 29, 32]. Inside the $(n+1)$-dimensional complex projective space $\mathbb{CP}^{n+1}$, we take the complex hyperquadric

$$\mathcal{Q}_n := \{z = [z_1 : \cdots : z_{n+2}] \in \mathbb{CP}^{n+1} : {z_1}^2 + \cdots + {z_{n+2}}^2 = 0\}.$$

Definition 3 (Generalized Gauss map of minimal surfaces in $\mathbb{R}^{n+2}$)**.** Let Σ be a minimal surface in $\mathbb{R}^{n+2}$. Consider a conformal harmonic immersion $X : \Sigma \to \mathbb{R}^{n+2}$, $\xi \mapsto X(\xi)$. The Gauss map of Σ is the map $\mathcal{G} : \Sigma \to \mathcal{Q}_n \subset \mathbb{CP}^{n+1}$ defined by

$$\mathcal{G}(\xi) = \left[\overline{\frac{\partial X}{\partial \xi}}\right] = \left[\frac{\partial X}{\partial \xi_1} + i\frac{\partial X}{\partial \xi_2}\right] \in \mathbb{CP}^{n+1}.$$

The map $\mathcal{G}$ is independent of the choice of the conformal parameter ξ.

Lemma 4 (Osserman's Lemma, [30, 32])**.** *Let* Σ *be an entire minimal graph of the function* $f = (f_1, \cdots, f_n) : \mathbb{R}^2 \to \mathbb{R}^n$, $n \geq 1$,

$$x\mathbf{e}_1 + y\mathbf{e}_2 + f_1(x,y)\mathbf{e}_3 + \cdots + f_n(x,y)\mathbf{e}_{n+2}.$$

(a) The Gauss map image $\mathcal{G}(\Sigma)$ *lies on a hyperplane* $z_2 = \lambda z_1$ *for some* $\lambda \in \mathbb{C} - \mathbb{R}$.

(b) There exists a non-singular linear transformation $(u,v) \mapsto (x,y) = (u, au+b)$ *for some* $a, b \in \mathbb{R}$, $b \neq 0$ *such that* $u + iv$ *is a global isothermal parameter for* Σ.

Osserman's Lemma on Gauss maps is highly useful. It is used in [11, 14, 19, 28] for the proofs of various Bernstein type theorems for entire minimal graphs in $\mathbb{R}^4$. In Lemma 5, we present an extension of Osserman's Lemma. It indicates that, given an entire two dimensional minimal graph in $\mathbb{R}^{n+2}$, pairs of its height functions having non-zero Jacobian determinant on the whole plane also contribute to the degeneracy of its Gauss map. The following simple observation on the sign of the Jacobian determinants of entire holomorphic graphs is the motivation of Lemma 5:

Remark 1. We consider an entire holomorphic graph Σ in $\mathbb{R}^{2k+2}$ given by

$$\begin{aligned} & F_1(x,y)\mathbf{e}_1 + F_2(x,y)\mathbf{e}_2 + F_3(x,y)\mathbf{e}_3 + \cdots + F_{2k+2}(x,y)\mathbf{e}_{2k+2} \\ = \quad & x\mathbf{e}_1 + y\mathbf{e}_2 + \mathrm{Re}(\phi_1)\mathbf{e}_3 + \mathrm{Im}(\phi_1)\mathbf{e}_4 + \cdots + \mathrm{Re}(\phi_k)\mathbf{e}_{2k+1} + \mathrm{Im}(\phi_k)\mathbf{e}_{2k+2}, \end{aligned}$$

where $\phi_1(z), \cdots, \phi_k(z) : \mathbb{C} \to \mathbb{C}$, $z = x + iy \in \mathbb{R} + i\mathbb{R}$ are entire holomorphic functions. The entire graph Σ is minimal in $\mathbb{R}^{2k+2}$. We observe that the Jacobian $\mathcal{J}_{1,2} := \frac{\partial(F_1,F_2)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(x,y)} = 1$ is positive on the whole plane $\mathbb{R}^2$ and that, for each $i \in \{1, \cdots, k\}$, the Jacobian $\mathcal{J}_{2i+1,2i+2} := \frac{\partial(F_{2i+1},F_{2i+2})}{\partial(x,y)}$ is always non-negative on the whole plane $\mathbb{R}^2$. This is because, for any differentiable function $\phi : \mathbb{C} \to \mathbb{C}$

given by $\xi = \xi_1 + i\xi_2 \in \mathbb{R} + i\mathbb{R} \mapsto \phi_1(\xi_1, \xi_2) + i\phi_2(\xi_1, \xi_2) \in \mathbb{R} + i\mathbb{R}$, we have the identity

$$\frac{\partial(\phi_1, \phi_2)}{\partial(\xi_1, \xi_2)} = \left|\frac{\partial\phi}{\partial\xi}\right|^2 - \left|\frac{\partial\phi}{\partial\bar{\xi}}\right|^2.$$

If ϕ is holomorphic, then $\frac{\partial(\phi_1,\phi_2)}{\partial(\xi_1,\xi_2)} \geq 0$. If ϕ is anti-holomorphic, then $\frac{\partial(\phi_1,\phi_2)}{\partial(\xi_1,\xi_2)} \leq 0$.

Lemma 5 (Extended Osserman's Lemma)**.** *Let Σ be an entire minimal graph of the function $f = (f_1, \cdots, f_n) : \mathbb{R}^2 \to \mathbb{R}^n$, $n \geq 1$,*

$$\begin{aligned}\mathbf{\Phi}(x, y) &= F_1(x, y)\mathbf{e}_1 + F_2(x, y)\mathbf{e}_2 + F_3(x, y)\mathbf{e}_3 + \cdots + F_{n+2}(x, y)\mathbf{e}_{n+2} \\ &= x\mathbf{e}_1 + y\mathbf{e}_2 + f_1(x, y)\mathbf{e}_3 + \cdots + f_n(x, y)\mathbf{e}_{n+2}.\end{aligned}$$

Whenever there exists a pair (i, j) with $1 \leq i < j \leq n + 2$ such that the Jacobian

$$\mathcal{J}_{i,j} := \frac{\partial(F_i, F_j)}{\partial(x, y)}$$

is positive or negative on the entire plane $\mathbb{R}^2$, the following two statements hold:

(a) The image $\mathcal{G}(\Sigma)$ of Σ under the Gauss map lies on a hyperplane of the form

$$z_i = \lambda_{(i,j)} z_j$$

for some constant $\lambda_{(i,j)} \in \mathbb{C} - \mathbb{R}$.

(b) For some constant $\lambda \in \mathbb{C} - \mathbb{R}$, we have an analogue of the Cauchy-Riemann equations

$$\frac{\partial F_i}{\partial x} + \lambda\frac{\partial F_i}{\partial y} = \lambda_{(i,j)}\left(\frac{\partial F_j}{\partial x} + \lambda\frac{\partial F_j}{\partial y}\right).$$

Proof. Step A. Following the arguments in [32], we prepare the global conformal coordinate chart for the entire graph Σ. We begin with the two identities:

$$\frac{\partial}{\partial x}\left(\frac{F}{\omega}\right) = \frac{\partial}{\partial y}\left(\frac{E}{\omega}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{G}{\omega}\right) = \frac{\partial}{\partial y}\left(\frac{F}{\omega}\right).$$

Since $\mathbb{R}^2$ is simply connected, Poincaré's Lemma guarantees the existence of functions $M, N : \mathbb{R}^2 \to \mathbb{R}$ satisfying the equalities

$$M_x = \frac{E}{\omega}, M_y = \frac{F}{\omega}, N_x = \frac{F}{\omega}, N_y = \frac{G}{\omega}.$$

We then introduce the coordinate transformation

$$\Psi : (x, y) \mapsto (\xi_1, \xi_2) := (x + M(x, y), y + N(x, y)).$$

One computes the Jacobian determinant of the transformation Ψ:

$$J_\Psi = \frac{\partial(\xi_1, \xi_2)}{\partial(x, y)} = \det\begin{pmatrix} 1 + \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & 1 + \frac{G}{\omega}\end{pmatrix} = 2 + \frac{E + G}{\omega} > 2.$$

Since $J_\Psi = \frac{\partial(\xi_1,\xi_2)}{\partial(x,y)} > 0$, we obtain the existence of the local inverse $(\xi_1, \xi_2) \mapsto (x, y)$. Furthermore, Osserman [32] proved that the mapping

$$\Psi : (x, y) \mapsto (\xi_1, \xi_2)$$

becomes a diffeomorphism from $\mathbb{R}^2$ to itself. A straightforward computation shows that $\xi = \xi_1 + i\xi_2$ becomes a global isothermal parameter on Σ:

$$ds^2_\Sigma = \frac{\omega}{J_\Psi}\left(d\xi_1{}^2 + d\xi_2{}^2\right).$$

Now, the conformal immersion $X := \mathbf{\Phi} \circ \Psi^{-1} : \mathbb{R}^2 \to \mathbb{R}^{n+2}$:

$$X(\xi_1, \xi_2) = \sum_{k=1}^{n+2} F_k(x(\xi_1, \xi_2), y(\xi_1, \xi_2))\mathbf{e}_k$$

induces the entire holomorphic functions

$$\phi_k(\xi) = 2\frac{\partial F_k}{\partial \xi}, \quad k \in \{1, \cdots, n\}.$$

Step B. (Proof of (a)) According to the assumption that the Jacobian determinant

$$\mathcal{J}_{i,j} = \frac{\partial(F_i, F_j)}{\partial(x, y)}$$

is positive or negative on the entire plane $\mathbb{R}^2$, we find that

$$\mathrm{Im}\left(\phi_i \overline{\phi_j}\right) = \frac{\partial(F_i, F_j)}{\partial(\xi_1, \xi_2)} = \frac{\partial(F_i, F_j)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(\xi_1, \xi_2)} = \frac{\mathcal{J}_{i,j}}{J_\Psi}$$

is positive or negative on $\mathbb{C}$. In particular, two holomorphic functions ϕ_i and ϕ_j have no zeros. So, the holomorphic function $\frac{\phi_i}{\phi_j}$ is entire. Furthermore, the function

$$\mathrm{Im}\left(\frac{\phi_i}{\phi_j}\right) = \frac{1}{|\phi_j|^2}\mathrm{Im}\left(\phi_i \overline{\phi_j}\right)$$

has the same (nowhere-zero) sign on the entire plane $\mathbb{C}$. Since the holomorphic function $\frac{\phi_i}{\phi_j}$ is entire, this means that $\frac{\phi_i}{\phi_j}$ becomes a non-zero constant function. Write $\phi_i = c\phi_j$ for some $c \in \mathbb{C}^* = \mathbb{C} - \{0\}$. Since $\mathrm{Im}\, c = \mathrm{Im}\left(\frac{\phi_i}{\phi_j}\right) \neq 0$, we know $c \in \mathbb{C} - \mathbb{R}$. By the definition of the Gauss map $\mathcal{G}$:

$$[z_1 : \cdots : z_{n+2}] = \mathcal{G}(\xi) = \left[\overline{\frac{\partial X}{\partial \xi}}\right] = \left[\overline{\phi_1}, \cdots, \overline{\phi_{n+2}}\right],$$

we see that the image $\mathcal{G}(\Sigma)$ lies on the hyperplane $z_i = \lambda z_j$ with $\lambda = \overline{c} \in \mathbb{C} - \mathbb{R}$.

Step C. (Proof of (b)) We recall that the coordinate change $\Psi : (x, y) \mapsto (\xi_1, \xi_2)$ gives the conformal immersion $X = \mathbf{\Phi} \circ \Psi^{-1}$. The Chain Rule yields

$$\frac{G}{\omega}\frac{\partial \mathbf{\Phi}}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial \mathbf{\Phi}}{\partial y} = \left(1 + \frac{G}{\omega} + i\frac{F}{\omega}\right)\left(\frac{\partial X}{\partial \xi_1} + i\frac{\partial X}{\partial \xi_2}\right),$$

which means that the graph $\mathbf{\Phi}(x, y)$ has the Gauss map

$$\begin{aligned}
\mathcal{G}(x, y) &= \left[\overline{\frac{\partial X}{\partial \xi}}\right] \\
&= \left[\frac{G}{\omega}\frac{\partial \mathbf{\Phi}}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial \mathbf{\Phi}}{\partial y}\right] \\
&= \left[\frac{G}{\omega}\frac{\partial F_1}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial F_1}{\partial y}, \cdots, \frac{G}{\omega}\frac{\partial F_{n+2}}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial F_{n+2}}{\partial y}\right] \\
&= \left[\frac{G}{\omega}, i - \frac{F}{\omega}, \frac{G}{\omega}\frac{\partial f_1}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial f_1}{\partial y}, \cdots, \frac{G}{\omega}\frac{\partial f_n}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial f_n}{\partial y}\right].
\end{aligned}$$

Since $z_2 = \lambda z_1$ on the Gauss map image $\mathcal{G}(\Sigma)$ for some constant $\lambda \in \mathbb{C} - \mathbb{R}$, the equality $i - \frac{F}{\omega} = \lambda \frac{G}{\omega}$ holds on $\mathbb{C}$. Since $z_i = \lambda_{(i,j)} z_j$ on $\mathcal{G}(\Sigma)$, we obtain

$$\frac{G}{\omega}\frac{\partial f_i}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial f_i}{\partial y} = \lambda_{(i,j)}\left(\frac{G}{\omega}\frac{\partial f_j}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial f_j}{\partial y}\right).$$

Since $i - \frac{F}{\omega} = \lambda \frac{G}{\omega}$ and $\frac{G}{\omega} > 0$, this reduces to $\frac{\partial F_i}{\partial x} + \lambda \frac{\partial F_i}{\partial y} = \lambda_{(i,j)}\left(\frac{\partial F_j}{\partial x} + \lambda \frac{\partial F_j}{\partial y}\right)$. □

The above proof of Lemma 5 contains an explicit formula for the Gauss map:

Proposition 6. *The minimal graph $(x,y) \in \Omega \subset \mathbb{R}^2 \mapsto \mathbf{\Phi}(x,y) = x\mathbf{e}_1 + y\mathbf{e}_2 + f_1(x,y)\mathbf{e}_3 + \cdots + f_n(x,y)\mathbf{e}_{n+2}$ in $\mathbb{R}^{n+2}$ has the Gauss map $\mathcal{G} : \Omega \to \mathcal{Q}_n \subset \mathbb{CP}^{n+1}$:*

$$\begin{aligned}\mathcal{G}(x,y) &= \left[\frac{G}{\omega},\ i - \frac{F}{\omega}, \frac{G}{\omega}\frac{\partial f_1}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial f_1}{\partial y}, \cdots, \frac{G}{\omega}\frac{\partial f_n}{\partial x} + \left(i - \frac{F}{\omega}\right)\frac{\partial f_n}{\partial y}\right] \\ &= \left[1 - i\frac{F}{\omega}, i\frac{E}{\omega}, \left(1 - i\frac{F}{\omega}\right)\frac{\partial f_1}{\partial x} + i\frac{E}{\omega}\frac{\partial f_1}{\partial y}, \cdots, \left(1 - i\frac{F}{\omega}\right)\frac{\partial f_n}{\partial x} + i\frac{E}{\omega}\frac{\partial f_n}{\partial y}\right].\end{aligned}$$

Here, $ds^2_{\mathbf{\Phi}} = Edx^2 + 2Fdxdy + Gdy^2$ is the induced metric of the graph $\mathbf{\Phi}$.

Proof. In the notation in the proof of Lemma 5, the second identity follows from

$$\left(1 - i\frac{F}{\omega}\right)\frac{\partial \mathbf{\Phi}}{\partial x} + i\frac{E}{\omega}\frac{\partial \mathbf{\Phi}}{\partial y} = \left(1 + \frac{E}{\omega} - i\frac{F}{\omega}\right)\left(\frac{\partial X}{\partial \xi_1} + i\frac{\partial X}{\partial \xi_2}\right).$$

□

Using Lemma 5 and Proposition 6, we present a minimal-surface proof of

Corollary 7 (Jörgens Theorem, [17])**.** *The only entire $\mathcal{C}^3$ functions satisfying the unimodular Hessian equation $F_{xx}F_{yy} - {F_{xy}}^2 = 1$ are quadratic polynomial functions.*

Proof. According to the Harvey-Lawson Theorem [12], the surface Σ given by

$$F_1(x,y)\mathbf{e}_1 + F_2(x,y)\mathbf{e}_2 + F_3(x,y)\mathbf{e}_3 + F_4(x,y)\mathbf{e}_4 = x\mathbf{e}_1 + y\mathbf{e}_2 + F_x\mathbf{e}_3 + F_y\mathbf{e}_4$$

becomes an entire minimal graph in $\mathbb{R}^4$. We can use Proposition 6 to show that its Gauss map $\mathcal{G} : \Sigma \to \mathcal{Q}_2 \subset \mathbb{CP}^3$ reads, for some constant $\epsilon \in \{-1, 1\}$,

$$\mathcal{G}(x,y) = [z_1, z_2, z_3, z_4] = [\epsilon F_{yy},\ i - \epsilon F_{xy}, \epsilon + iF_{xy}, iF_{yy}].$$

The Gauss map image $\mathcal{G}(\Sigma)$ lies on two hyperplanes $z_2 = i\epsilon z_3$ and $z_4 = i\epsilon z_1$. Since the Jacobians $\mathcal{J}_{1,2} := \frac{\partial(F_1,F_2)}{\partial(x,y)} = 1$ and $\mathcal{J}_{3,4} := \frac{\partial(F_3,F_4)}{\partial(x,y)} = F_{xx}F_{yy} - {F_{xy}}^2 = 1$ are positive on the whole plane $\mathbb{R}^2$, (a) in Lemma 5 implies that the Gauss map image $\mathcal{G}(\Sigma)$ lies on two hyperplanes $z_1 = \lambda_{(1,2)} z_2$ and $z_3 = \lambda_{(3,4)} z_4$ for some constants $\lambda_{(1,2)} \in \mathbb{C} - \mathbb{R}$ and $\lambda_{(3,4)} \in \mathbb{C} - \mathbb{R}$. We conclude that its Gauss map $\mathcal{G}$ is constant. □

Remark 2. Here we verify the claim on the Gauss map appearing in the above proof:

$$\mathcal{G}(x,y) = [\epsilon F_{yy},\ i - \epsilon F_{xy}, \epsilon + iF_{xy}, iF_{yy}].$$

We need to compute the pullback metric ds^2 of the graph Σ:

$$\begin{aligned} ds^2 &= Edx^2 + 2Fdxdy + Gdy^2 \\ &= \left(1 + {F_{xx}}^2 + {F_{xy}}^2\right)dx^2 + 2F_{xy}\left(F_{xx} + F_{yy}\right)dxdy + \left(1 + {F_{xy}}^2 + {F_{yy}}^2\right)dy^2. \end{aligned}$$

Write $\omega = \sqrt{EG - F^2} > 0$. It follows from $F_{xx}F_{yy} - {F_{xy}}^2 = 1$ that

$$\begin{aligned}\omega &= \sqrt{\left(1 + {F_{xx}}^2 + {F_{xy}}^2\right)\left(1 + {F_{xy}}^2 + {F_{yy}}^2\right) - \left(F_{xx}F_{xy} + F_{xy}F_{yy}\right)^2} \\ &= \sqrt{\left(F_{xx} + F_{yy}\right)^2 + \left(1 - \left(F_{xx}F_{yy} - {F_{xy}}^2\right)\right)^2} \\ &= |F_{xx} + F_{yy}|.\end{aligned}$$

Since $F_{xx}F_{yy} - {F_{xy}}^2 > 0$, the continuous function $F_{xx} + F_{yy}$ vanishes nowhere. We can find a constant $\epsilon \in \{-1, 1\}$ satisfying $F_{xx} + F_{yy} = \epsilon\omega$ on $\mathbb{R}^2$ and obtain

$$\frac{F}{\omega} = \frac{F_{xy}\left(F_{xx} + F_{yy}\right)}{\omega} = \epsilon F_{xy}$$

and

$$\frac{G}{\omega} = \frac{1 + {F_{xy}}^2 + {F_{yy}}^2}{\omega} = \frac{F_{yy}\left(F_{xx} + F_{yy}\right)}{\omega} = \epsilon F_{yy}.$$

Now, by Proposition 6, the Gauss map of the entire minimal graph Σ in $\mathbb{R}^4$ reads

$$\begin{aligned}\mathcal{G}(x, y) &= \left[\frac{G}{\omega},\ i - \frac{F}{\omega}, \frac{G}{\omega}F_{xx} + \left(i - \frac{F}{\omega}\right)F_{xy}, \frac{G}{\omega}F_{yx} + \left(i - \frac{F}{\omega}\right)F_{yy}\right] \\ &= \left[\epsilon F_{yy},\ i - \epsilon F_{xy}, \epsilon + iF_{xy}, iF_{yy}\right].\end{aligned}$$

Remark 3. The proof of Corollary 7 used the Harvey-Lawson Theorem [12] that the gradient graph of a solution to the unimodular Hessian equation in two variables is a special Lagrangian surface in $\mathbb{C}^2$, which has the area-minimizing property. On the other hand, Warren [39, Theorem 3.2] showed that the gradient graph of a convex solution to the unimodular Hessian equation becomes a spacelike volume-maximizing submanifold of the pseudo-Euclidean space endowed with a suitably chosen pseudo-metric. See also Mealy's earlier result [26], [13, Theorem 6.3] and Section 4.2.

Remark 4. In Section 4.2, we show that Bernstein type theorems for symplectic Monge-Ampère equations in two variables can be reduced to Jörgens' Theorem. Other proofs of Jörgens' Theorem or Bernstein type theorems for special Lagrangian equations are given by Fu [11] and Yuan [40].

There are many applications of our extended Osserman's Lemma. One can show that if the height functions of entire 2-dimensional minimal graphs in $\mathbb{R}^{n+2}$ are strictly monotone, they are planes:

Theorem 8. *Let Σ be an entire minimal graph of $f = (f_1, \cdots, f_n) : \mathbb{R}^2 \to \mathbb{R}^n$.*

(a) Suppose that, for each $k \in \{1, \cdots, n\}$, one of the following inequalities holds:

$$\frac{\partial f_k}{\partial x} > 0,\ \ \frac{\partial f_k}{\partial x} < 0,\ \ \frac{\partial f_k}{\partial y} > 0,\ \ \frac{\partial f_k}{\partial y} < 0$$

on the whole plane $\mathbb{R}^2$. Then, the entire minimal graph Σ in $\mathbb{R}^{n+2}$ is a plane.

(b) Suppose that, for some fixed $k \in \{1, \cdots, n\}$, either $\frac{\partial f_k}{\partial x}$ or $\frac{\partial f_k}{\partial y}$ is a non-zero constant function on the whole plane $\mathbb{R}^2$. Then, the function f_k is affine.

Proof. (a) By Lemma 4 or Lemma 5, there exists a constant $\lambda \in \mathbb{C}^*$ such that the equality $z_2 = \lambda z_1$ holds on the Gauss map image $\mathcal{G}(\Sigma)$. We fix $k \in \{1, \cdots, n\}$.

(a1) When $\frac{\partial f_k}{\partial y} = \frac{\partial(x, f_k)}{\partial(x, y)}$ never vanishes, take $(i, j) = (k + 2, 1)$ in (a) of Lemma 5. Thus, $\mathcal{G}(\Sigma)$ lies on a hyperplane $z_{k+2} = \lambda_{k+2} z_1$ for some constant $\lambda_{k+2} \in \mathbb{C}^*$.

(a2) In the case when $\frac{\partial f_k}{\partial x} = \frac{\partial(f_k,y)}{\partial(x,y)}$ never vanishes, take $(i,j) = (k+2,2)$ in (a) of Lemma 5. So, the Gauss map $\mathcal{G}$ takes its values in the hyperplane $z_{k+2} = \lambda_k z_2$ for some constant $\lambda_{k+2} \in \mathbb{C}^*$. We see that $\mathcal{G}(\Sigma)$ lies on a hyperplane $z_{k+2} = (\lambda_k \lambda) z_1$.

We conclude that, for each $k \in \{2, \cdots, n\}$, the Gauss map image $\mathcal{G}(\Sigma)$ lies on some hyperplane $z_k = \alpha_k z_1$. Hence, the Gauss map $\mathcal{G} = [z_1 : \cdots : z_{n+2}]$ is constant.

(b) We consider the two cases.

(b1) If $\frac{\partial f_k}{\partial y}$ is a non-zero constant function, we can take $(i,j) = (k+2,1)$ in (b) of Lemma 5 to obtain $\frac{\partial f_k}{\partial x} + \lambda \frac{\partial f_k}{\partial y} = \frac{1}{\lambda_{k+2}}$ for some constant $\lambda \in \mathbb{C}^*$. Hence, $\frac{\partial f_k}{\partial x}$ is also a constant function.

(b2) In the case when $\frac{\partial f_k}{\partial x}$ is a non-zero constant function, we take $(i,j) = (k+2,2)$ in (b) of Lemma 5 to write $\frac{\partial f_k}{\partial x} + \lambda \frac{\partial f_k}{\partial y} = \frac{\lambda}{\lambda_{k+2}}$ for some constant $\lambda \in \mathbb{C}^*$. Thus, $\frac{\partial f_k}{\partial y}$ is also constant. □

Corollary 9. *Let Σ be an entire minimal graph $\mathbf{\Phi}(x,y) = x\mathbf{e}_1 + y\mathbf{e}_2 + f(x,y)\mathbf{e}_3 + g(x,y)\mathbf{e}_4$ in $\mathbb{R}^4$. If $\frac{\partial f}{\partial x}$ is a non-zero constant function, then Σ is a plane.*

Proof. By (b) of Theorem 8, the function $f(x,y)$ is affine. So, the surface Σ can be viewed as an entire minimal graph in some affine subspace $\mathbb{R}^3$. Bernstein's Theorem says that Σ is a plane. (Alternatively, since the function f has a bounded gradient, Simon's Theorem [14, 34] implies that the graph Σ is a plane.) □

Corollary 10. *Let Σ be an entire minimal graph of $f = (f_1, \cdots, f_n) : \mathbb{R}^2 \to \mathbb{R}^n$, $n \geq 2$. If the mappings $(x,y) \mapsto (f_i(x,y), f_j(x,y))$ are diffeomorphisms from $\mathbb{R}^2$ to itself for all pairs (i,j) with $1 \leq i < j \leq n$, then Σ is a plane.*

Proof. By (a) in Lemma 4, we can take a linear diffeomorphism from $\mathbb{R}^2$ to itself,

$$(u,v) \mapsto (x,y) = (u, au + bv),$$

such that $u + iv$ becomes a global isothermal parameter for the graph Σ. It follows that the composition map

$$(u,v) \mapsto (x,y) \mapsto (f_i(x,y), f_j(x,y))$$

is the harmonic diffeomorphism from $\mathbb{R}^2$ to itself. Thus, it is linear. Since the first map $(u,v) \mapsto (x,y)$ is a linear diffeomorphism, the second map $(x,y) \mapsto (f_i(x,y), f_j(x,y))$ is also linear. □

Remark 5. Schoen [33] and Ni [28] proved the case $n = 2$ in Corollary 10. Brendle and Warren [5] proved that, given two uniformly convex domains in $\mathbb{R}^n$ with smooth boundary, there exists a diffeomorphism from the first domain to the second one such that its graph becomes a minimal Lagrangian submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

3.2. **Construction of special Lagrangian graphs in $\mathbb{C}^2$.** The Harvey-Lawson Theorem [12] indicates that the special Lagrangian equation in $\mathbb{C}^2$ is the first integral of the minimal surface system for the gradient graph in $\mathbb{R}^4$.

Theorem 11 (From minimal graphs in $\mathbb{R}^{n+2}$ to special Lagrangian graphs in $\mathbb{C}^2 = \mathbb{R}^2 \times i\mathbb{R}^2$)**.** *Let $x\mathbf{e}_1 + y\mathbf{e}_2 + f_1(x,y)\mathbf{e}_3 + \cdots + f_n(x,y)\mathbf{e}_{n+2}$ be a minimal graph in $\mathbb{R}^{n+2}$ defined on the simply connected domain $\Omega \subset \mathbb{R}^2$ with the induced metric $ds^2 = Edx^2 + 2Fdxdy + Gdy^2$, where $\omega = \sqrt{EG - F^2}$. Then, there exists a minimal gradient graph $x\mathbf{e}_1 + y\mathbf{e}_2 + M(x,y)\mathbf{e}_3 + N(x,y)\mathbf{e}_4$ satisfying the equalities*

$$M_x = \frac{E}{\omega}, M_y = \frac{F}{\omega},\ N_x = \frac{F}{\omega},\ N_y = \frac{G}{\omega}.$$

Proof. As in the proof of Lemma 5, we begin with the two identities:

$$\frac{\partial}{\partial x}\left(\frac{F}{\omega}\right) = \frac{\partial}{\partial y}\left(\frac{E}{\omega}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{G}{\omega}\right) = \frac{\partial}{\partial y}\left(\frac{F}{\omega}\right).$$

Since Ω is simply connected, Poincaré's Lemma guarantees the existence of functions $N, M : \Omega \to \mathbb{R}$ satisfying the equalities

$$M_x = \frac{E}{\omega}, M_y = \frac{F}{\omega}, N_x = \frac{F}{\omega}, N_y = \frac{G}{\omega}.$$

The map $(x, y) \mapsto (M(x, y), N(x, y))$ is area-preserving because of the equality

$$\frac{\partial(M, N)}{\partial(x, y)} = \frac{EG - F^2}{\omega^2} = 1.$$

The equality $M_y = \frac{F}{\omega} = N_x$ shows that, by Poincaré's Lemma again, $(x, y) \mapsto (M(x, y), N(x, y))$ is a gradient map $(M, N) = (h_x, h_y)$ for some function $h : \Omega \to \mathbb{R}$. The function h satisfies the unimodular Hessian equation

$$h_{xx}h_{yy} - {h_{xy}}^2 = \frac{\partial(M, N)}{\partial(x, y)} = 1.$$

We then observe that the function h solves the special Lagrangian equation

$$\cos\theta\,(h_{xx} + h_{yy}) + \sin\theta(1 - h_{xx}h_{yy} + {h_{xy}}^2) = 0$$

with $\theta = \frac{\pi}{2}$. The Harvey-Lawson Theorem [12] implies that the gradient graph

$$x\mathbf{e}_1 + y\mathbf{e}_2 + M(x, y)\mathbf{e}_3 + N(x, y)\mathbf{e}_4 = x\mathbf{e}_1 + y\mathbf{e}_2 + h_x\mathbf{e}_3 + h_y\mathbf{e}_4$$

becomes a special Lagrangian surface in $\mathbb{C}^2$. □

Remark 6. In Section 4.2, we show that this correspondence from minimal graphs to special Lagrangian graphs induces the symplectic graph rotation in Lemma 18, which says that a solution of the symplectic Monge-Ampère equations in two variables induces a solution of the unimodular Hessian equation in two variables.

Example 1 (From catenoids to Lagrangian catenoids)**.** We consider the catenoid $z = \rho \operatorname{arccosh}\left(\frac{\sqrt{x^2+y^2}}{\rho}\right)$ in $\mathbb{R}^3$, where $\rho > 0$ is a constant. Applying Theorem 11 to the catenoid, we find that the gradient map

$$(x, y) \mapsto \left(\sqrt{1 - \frac{\rho^2}{x^2 + y^2}}\, x, \sqrt{1 - \frac{\rho^2}{x^2 + y^2}}\, y\right)$$

is area-preserving. Its graph is known as a Lagrangian catenoid. Castro and Urbano [7] presented several geometric characterizations of the Lagrangian catenoids [12, Theorem 3.5] and [31, Example 2].

Example 2 (From helicoids to Lagrangian catenoids)**.** Let $\rho > 0$ be a constant. The application of Theorem 11 to the helicoid $z = \rho \arctan\left(\frac{y}{x}\right)$ in $\mathbb{R}^3$ gives the area-preserving gradient map

$$(x, y) \mapsto \left(\sqrt{1 + \frac{\rho^2}{x^2 + y^2}}\, x, \sqrt{1 + \frac{\rho^2}{x^2 + y^2}}\, y\right).$$

Its graph is also a Lagrangian catenoid. The area-preserving map in Example 1 is the inverse of the area-preserving map in Example 2.

Example 3 (From Scherk surfaces to Lagrangian Scherk surfaces)**.** Let $\rho > 0$ be a constant. Under the correspondence in Theorem 11, Scherk's graph

$$z = \frac{1}{\rho}\left[\ln\left(\cos\left(\rho x\right)\right) - \ln\left(\cos\left(\rho y\right)\right)\right]$$

yields a special Lagrangian graph

$$x\mathbf{e}_1 + y\mathbf{e}_2 + \frac{1}{\rho}\operatorname{arcsinh}\left[\tan\left(\rho x\right)\cos\left(\rho y\right)\right]\mathbf{e}_3 + \frac{1}{\rho}\operatorname{arcsinh}\left[\tan\left(\rho y\right)\cos\left(\rho x\right)\right]\mathbf{e}_4.$$

It might be interesting to find some geometric characterizations of this surface.

4. Twin correspondence and its applications

4.1. **Existence of twin correspondence.** Our aim is to generalize Calabi's correspondence [6] between the minimal surface equation in $\mathbb{R}^3$ and the maximal surface equation in $\mathbb{L}^3$ to higher codimensions $n \geq 2$. We recall that the notion of the positive area-angle map was introduced in Section 2.1. Our ambient spaces are the Euclidean space $\mathbb{R}^{n+2}$ endowed with the metric ${dx_1}^2 + \cdots + {dx_{n+2}}^2$ and the pseudo-Euclidean space $\mathbb{R}^{n+2}_n$ equipped with the pseudo-Euclidean metric ${dx_1}^2 + {dx_2}^2 - {dx_3}^2 - \cdots - {dx_{n+2}}^2$.

Theorem 12 (Twin correspondence - version A)**.** *There exists a twin correspondence (up to vertical translations) between two dimensional minimal graphs in $\mathbb{R}^{n+2}$ carrying a positive area-angle function and two dimensional maximal graphs in $\mathbb{R}^{n+2}_n$ carrying the same positive area-angle function. More explicitly, we have the twin correspondence between the minimal graph in $\mathbb{R}^{n+2}$,*

$$x\mathbf{e}_1 + y\mathbf{e}_2 + f_1(x,y)\mathbf{e}_3 + \cdots + f_n(x,y)\mathbf{e}_{n+2}$$

over the simply connected domain Ω satisfying the positive area-angle condition

$$\sum_{1\leq i<j\leq n} \frac{\partial\left(f_i, f_j\right)}{\partial(x,y)}^2 < 1$$

and the maximal graph in $\mathbb{R}^{n+2}_n$,

$$x\mathbf{e}_1 + y\mathbf{e}_2 + g_1(x,y)\mathbf{e}_3 + \cdots + g_n(x,y)\mathbf{e}_{n+2}$$

over the same domain Ω satisfying the positive area-angle condition

$$\sum_{1\leq i<j\leq n} \frac{\partial\left(g_i, g_j\right)}{\partial(x,y)}^2 < 1.$$

For the construction of the twin correspondence in Theorem 12, we need to restate it even more explicitly:

Theorem 13 (Twin correspondence - version B)**.** *(a) If the graph $\mathbf{\Phi}$ of a positive area-angle map $f = (f_1, \cdots, f_n) : \Omega \to \mathbb{R}^n$ with the area-angle function $\Theta \in \left(0, \frac{\pi}{2}\right]$ becomes a minimal surface in $\mathbb{R}^{n+2}$, then there exists a positive area-angle map $g = (g_1, \cdots, g_n) : \Omega \to \mathbb{R}^n$ with the same area-angle function $\Theta \in \left(0, \frac{\pi}{2}\right]$ such that its graph $\widehat{\mathbf{\Phi}}$ becomes a maximal surface in $\mathbb{R}^{n+2}_n$.*

(b) Conversely, if the graph $\widehat{\mathbf{\Phi}}$ of a positive area-angle map $g = (g_1, \cdots, g_n) : \Omega \to \mathbb{R}^n$ with the area-angle function $\Theta \in \left(0, \frac{\pi}{2}\right]$ is a maximal surface in $\mathbb{R}^{n+2}_n$, then we can associate a positive area-angle map $f = (f_1, \cdots, f_n) : \Omega \to \mathbb{R}^n$ with the

same area-angle function $\Theta \in \left(0, \frac{\pi}{2}\right]$ *satisfying that its graph* $\mathbf{\Phi}$ *becomes a minimal surface in* $\mathbb{R}^{n+2}$.

(c) The twin correspondence $\mathbf{\Phi} \Leftrightarrow \widehat{\mathbf{\Phi}}$ *in (a) and (b) fulfills the following relations:*

(c1) The twin relations hold:

$$\left(\frac{\partial g_k}{\partial x}, \frac{\partial g_k}{\partial y}\right) = \left(-\frac{E}{\omega}\frac{\partial f_k}{\partial y} + \frac{F}{\omega}\frac{\partial f_k}{\partial x}, \frac{G}{\omega}\frac{\partial f_k}{\partial x} - \frac{F}{\omega}\frac{\partial g_k}{\partial y}\right), \quad k \in \{1, \cdots, n\},$$

or equivalently,

$$\left(\frac{\partial f_k}{\partial x}, \frac{\partial f_k}{\partial y}\right) = \left(\frac{\widehat{E}}{\widehat{\omega}}\frac{\partial g_k}{\partial y} - \frac{\widehat{F}}{\widehat{\omega}}\frac{\partial g_k}{\partial x}, -\frac{\widehat{G}}{\widehat{\omega}}\frac{\partial g_k}{\partial x} + \frac{\widehat{F}}{\widehat{\omega}}\frac{\partial g_k}{\partial y}\right), \quad k \in \{1, \cdots, n\}.$$

Here, $ds^2_{\mathbf{\Phi}} = Edx^2 + 2Fdxdy + Gdy^2$ *denotes the induced metric and* $\omega = \sqrt{EG - F^2}$. *Likewise, we write* $ds^2_{\widehat{\mathbf{\Phi}}} = \widehat{E}dx^2 + 2\widehat{F}dxdy + \widehat{G}dy^2$ *and* $\widehat{\omega} = \sqrt{\widehat{E}\widehat{G} - \widehat{F}^2}$.

(c2) They share the area-angle. In fact each Jacobian determinant is preserved:

$$\frac{\partial\left(f_i, f_j\right)}{\partial(x, y)} = \frac{\partial\left(g_i, g_j\right)}{\partial(x, y)}, \quad i, j \in \{1, \cdots, n\}.$$

(c3) The angle duality holds:

$$\widehat{\omega}\omega = \sin^2\Theta > 0.$$

(c4) Two induced metrics are conformally equivalent. In fact, $ds^2_{\mathbf{\Phi}} = \frac{\omega}{\widehat{\omega}} ds^2_{\widehat{\mathbf{\Phi}}}$.

(d) Since two integrability conditions in (c1) are equivalent, we see that the twin correspondence is involutive. The twin minimal surface $\widehat{\widehat{\Sigma}}$ *of the twin surface* $\widehat{\Sigma}$ *of a minimal surface* Σ *is congruent to* Σ *up to vertical translations.*

Proof. We show (a) and (c) simultaneously. Working backwards gives (b).

Let $(x, y) \in \Omega \mapsto \mathbf{\Phi}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + f_1(x, y)\mathbf{e}_3 + \cdots + f_n(x, y)\mathbf{e}_{n+2}$ be a minimal surface in $\mathbb{R}^{n+2}$. Write $ds^2_{\mathbf{\Phi}} = Edx^2 + 2Fdxdy + Gdy^2$ and $\omega = \sqrt{EG - F^2} > 0$. We further assume the positive area-angle condition

$$1 > \cos\Theta = \|\mathcal{J}\| := \sqrt{\sum_{1 \le i < j \le n} {\mathcal{J}_{i,j}}^2}, \quad \mathcal{J}_{i,j} := \frac{\partial\left(f_i, f_j\right)}{\partial(x, y)}.$$

After setting $(\alpha_k, \beta_k) = \left(\frac{\partial f_k}{\partial x}, \frac{\partial f_k}{\partial y}\right)$, we obtain

$$E = 1 + \sum_{k=1}^{n} {\alpha_k}^2, \ F = \sum_{k=1}^{n} \alpha_k\beta_k, \ G = 1 + \sum_{k=1}^{n} {\beta_k}^2, \ \mathcal{J}_{i,j} = \alpha_i\beta_j - \alpha_j\beta_i.$$

Then, Lagrange's Identity gives

$$\omega^2 = EG - F^2 = 1 + \sum_{k=1}^{n} {\alpha_k}^2 + \sum_{k=1}^{n} {\beta_k}^2 + \sum_{1 \le i < j \le n} {\mathcal{J}_{i,j}}^2$$

and thus

$$\|\mathcal{J}\|^2 = \sum_{1 \le i < j \le n} {\mathcal{J}_{i,j}}^2 = \omega^2 - 1 - (E - 1) - (G - 1) = \omega^2 + 1 - E - G.$$

Since the minimal surface system reads

$$\frac{\partial}{\partial x}\left(\frac{G}{\omega}\alpha_k - \frac{F}{\omega}\beta_k\right) + \frac{\partial}{\partial y}\left(\frac{E}{\omega}\beta_k - \frac{F}{\omega}\alpha_k\right) = 0, \quad k \in \{1, \cdots, n\},$$

and since Ω is simply connected, Poincaré's Lemma guarantees the existence of functions $g_1, \cdots, g_n : \Omega \to \mathbb{R}$ satisfying the following integrability condition:

$$\left(\frac{\partial g_k}{\partial x}, \frac{\partial g_k}{\partial y}\right) = \left(-\frac{E}{\omega}\beta_k + \frac{F}{\omega}\alpha_k, \frac{G}{\omega}\alpha_k - \frac{F}{\omega}\beta_k\right), \quad k \in \{1, \cdots, n\}.$$

Our goal is to show that

$$\widehat{\mathbf{\Phi}}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + g_1(x, y)\mathbf{e}_3 + \cdots + g_n(x, y)\mathbf{e}_{n+2}$$

becomes a maximal surface in $\mathbb{R}^{n+2}_n$. Write $ds^2_{\widehat{\mathbf{\Phi}}} = \widehat{E}dx^2 + 2\widehat{F}dxdy + \widehat{G}dy^2$. Then,

$$\widehat{E} = 1 - \sum_{k=1}^{n} \widehat{\alpha}_k^2,\ \widehat{F} = -\sum_{k=1}^{n} \widehat{\alpha}_k\widehat{\beta}_k,\ \widehat{G} = 1 - \sum_{k=1}^{n} \widehat{\beta}_k^2, \quad \left(\widehat{\alpha}_k, \widehat{\beta}_k\right) = \left(\frac{\partial g_k}{\partial x}, \frac{\partial g_k}{\partial y}\right).$$

The first part of (c1) is now finished because the above condition reads

$$\left(\widehat{\alpha}_k, \widehat{\beta}_k\right) = \left(-\frac{E}{\omega}\beta_k + \frac{F}{\omega}\alpha_k, \frac{G}{\omega}\alpha_k - \frac{F}{\omega}\beta_k\right).$$

Claim A. We show the spacelike condition $\widehat{E}\widehat{G} - \widehat{F}^2 > 0$. It requires several steps.

Step A1. We compute Jacobian determinants. The above twin relation reads

$$\begin{pmatrix} \widehat{\beta}_k \\ -\widehat{\alpha}_k \end{pmatrix} = \begin{pmatrix} \frac{G}{\omega} & -\frac{F}{\omega} \\ -\frac{F}{\omega} & \frac{E}{\omega} \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} = \begin{pmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{pmatrix} \begin{pmatrix} \widehat{\beta}_k \\ -\widehat{\alpha}_k \end{pmatrix}.$$

Equating the determinant of both sides in

$$\begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix} = \begin{pmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{pmatrix} \begin{pmatrix} \widehat{\beta}_i & \widehat{\beta}_j \\ -\widehat{\alpha}_i & -\widehat{\alpha}_j \end{pmatrix}$$

yields the equality $\alpha_i\beta_j - \alpha_j\beta_i = \widehat{\alpha}_i\widehat{\beta}_j - \widehat{\alpha}_j\widehat{\beta}_i$, which gives the assertion (c2):

$$\mathcal{J}_{i,j} = \frac{\partial\left(f_i, f_j\right)}{\partial(x, y)} = \alpha_i\beta_j - \alpha_j\beta_i = \widehat{\alpha}_i\widehat{\beta}_j - \widehat{\alpha}_j\widehat{\beta}_i = \frac{\partial\left(g_i, g_j\right)}{\partial(x, y)}, \quad i, j \in \{1, \cdots, n\}.$$

Step A2. Here, our aim is to prove the equality

$$\left(\widehat{\alpha}_1^2 + \cdots + \widehat{\alpha}_n^2\right) + \left(\widehat{\beta}_1^2 + \cdots + \widehat{\beta}_n^2\right) = \frac{(E + G + 2)\omega^2 - (E + G)^2}{\omega^2}.$$

Using the twin relations and the definition $\omega^2 = EG - F^2$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} \left[\widehat{\alpha}_k^2 + \widehat{\beta}_k^2\right] \\
&\quad = \sum_{k=1}^{n} \left[\left(-\frac{E}{\omega}\beta_k + \frac{F}{\omega}\alpha_k\right)^2 + \left(\frac{G}{\omega}\alpha_k - \frac{F}{\omega}\beta_k\right)^2\right] \\
&\quad = \frac{1}{\omega^2}\left[\left(F^2 + G^2\right)\sum_{k=1}^{n} {\alpha_k}^2 + \left(F^2 + E^2\right)\sum_{k=1}^{n} {\beta_k}^2 - 2(E + G)F\sum_{k=1}^{n} \alpha_k\beta_k\right] \\
&\quad = \frac{1}{\omega^2}\left[\left(F^2 + G^2\right)(E - 1) + \left(F^2 + E^2\right)(G - 1) - 2(E + G)F^2\right] \\
&\quad = \frac{(E + G + 2)\left(EG - F^2\right) - (E + G)^2}{\omega^2}.
\end{aligned}$$

Step A3. Here we deduce the identity

$$\widehat{E}\widehat{G}-\widehat{F}^2=\frac{\left(1-\|\mathcal{J}\|^2\right)^2}{\omega^2}.$$

Then, the desired estimation $\widehat{E}\widehat{G}-\widehat{F}^2>0$ immediately follows from our assumption $1>\|\mathcal{J}\|$. We recall that the Jacobian determinant equality $\mathcal{J}_{i,j}=\widehat{\alpha}_i\widehat{\beta}_j-\widehat{\alpha}_j\widehat{\beta}_i$ holds. Using Lagrange's Identity again, we deduce

$$\begin{aligned}
\widehat{E}\widehat{G}-\widehat{F}^2 &= \left(1-\sum_{k=1}^n\widehat{\alpha}_k^2\right)\left(1-\sum_{k=1}^n\widehat{\beta}_k^2\right)-\left(-\sum_{k=1}^n\widehat{\alpha}_k\widehat{\beta}_k\right)^2\\
&= 1-\sum_{k=1}^n\left[\widehat{\alpha}_k^2+\widehat{\beta}_k^2\right]+\left(\sum_{k=1}^n\widehat{\alpha}_k^2\right)\left(\sum_{k=1}^n\widehat{\beta}_k^2\right)-\left(\sum_{k=1}^n\widehat{\alpha}_k\widehat{\beta}_k\right)^2\\
&= 1-\sum_{k=1}^n\left[\widehat{\alpha}_k^2+\widehat{\beta}_k^2\right]+\sum_{1\le i<j\le n}{\mathcal{J}_{i,j}}^2\\
&= 1-\frac{(E+G+2)\omega^2-(E+G)^2}{\omega^2}+\left(\omega^2+1-E-G\right)\\
&= \frac{\left(E+G-\omega^2\right)^2}{\omega^2}\\
&= \frac{\left(1-\|\mathcal{J}\|^2\right)^2}{\omega^2}.
\end{aligned}$$

Since $1>\cos\Theta=\|\mathcal{J}\|$, we deduce $\widehat{E}\widehat{G}-\widehat{F}^2>0$. Now, set $\widehat{\omega}:=\sqrt{\widehat{E}\widehat{G}-\widehat{F}^2}>0$. Then, the above equalities give the angle duality in (c3):

$$\widehat{\omega}=\sqrt{\widehat{E}\widehat{G}-\widehat{F}^2}=\frac{E+G-\omega^2}{\omega}=\frac{1-\|\mathcal{J}\|^2}{\omega}=\frac{\sin^2\Theta}{\omega}>0.$$

Claim B. We verify that the graph $(x,y)\in\Omega\mapsto\widehat{\mathbf{\Phi}}(x,y)=x\mathbf{e}_1+y\mathbf{e}_2+g_1(x,y)\mathbf{e}_3+\cdots+g_n(x,y)\mathbf{e}_{n+2}$ becomes a maximal surface in pseudo-Euclidean space $\mathbb{R}^{n+2}_n$.

Step B1. We check (c4), which implies that the metric $ds^2_{\widehat{\mathbf{\Phi}}}=\frac{\widehat{\omega}}{\omega}ds^2_{\mathbf{\Phi}}$ is positive definite. We need to prove the three identities $\frac{E}{\omega}=\frac{\widehat{E}}{\widehat{\omega}}$, $\frac{F}{\omega}=\frac{\widehat{F}}{\widehat{\omega}}$ and $\frac{G}{\omega}=\frac{\widehat{G}}{\widehat{\omega}}$. The twin relations and the definition $\omega^2=EG-F^2$ yield

$$\begin{aligned}
\sum_{k=1}^n\widehat{\alpha}_k^2 &= \sum_{k=1}^n\left(-\frac{E}{\omega}\beta_k+\frac{F}{\omega}\alpha_k\right)^2\\
&= \frac{E^2}{\omega^2}\sum_{k=1}^n{\beta_k}^2+\frac{F^2}{\omega^2}\sum_{k=1}^n{\alpha_k}^2-2\frac{EF}{\omega^2}\sum_{k=1}^n\alpha_k\beta_k\\
&= \frac{E^2(G-1)+F^2(E-1)-2EF^2}{\omega^2}\\
&= \frac{E\omega^2-E^2-F^2}{\omega^2}
\end{aligned}$$

and so, from the angle duality $\widehat{\omega} = \frac{1-\|\mathcal{J}\|^2}{\omega} = \frac{E+G-\omega^2}{\omega}$,

$$\widehat{E} - \frac{\widehat{\omega}}{\omega}E = 1 - \sum_{k=1}^{n}\widehat{\alpha}_k^2 - \frac{\widehat{\omega}}{\omega}E = 1 - \frac{E\omega^2 - E^2 - F^2}{\omega^2} - \frac{E+G-\omega^2}{\omega^2}E = 0.$$

The remaining two identities in (c4) can be proved similarly. Thus, (c4) is proved.

Step B2. We show that the height function g satisfies the maximal surface system. We use the three identities in Step B1 to rewrite the twin relation as

$$\begin{pmatrix} \frac{\partial f_k}{\partial x} \\ \frac{\partial f_k}{\partial y} \end{pmatrix} = \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} = \begin{pmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{pmatrix} \begin{pmatrix} \widehat{\beta}_k \\ -\widehat{\alpha}_k \end{pmatrix} = \begin{pmatrix} \frac{\widehat{E}}{\widehat{\omega}} & \frac{\widehat{F}}{\widehat{\omega}} \\ \frac{\widehat{F}}{\widehat{\omega}} & \frac{\widehat{G}}{\widehat{\omega}} \end{pmatrix} \begin{pmatrix} \widehat{\beta}_k \\ -\widehat{\alpha}_k \end{pmatrix} = \begin{pmatrix} \frac{\widehat{E}}{\widehat{\omega}}\widehat{\beta}_k - \frac{\widehat{F}}{\widehat{\omega}}\widehat{\alpha}_k \\ -\frac{\widehat{G}}{\widehat{\omega}}\widehat{\alpha}_k + \frac{\widehat{F}}{\widehat{\omega}}\widehat{\beta}_k \end{pmatrix},$$

which is the second part of (c1). It therefore follows that, for all $k \in \{1, \cdots, n\}$,

$$\frac{\partial}{\partial x}\left(\frac{\widehat{G}}{\widehat{\omega}}\widehat{\alpha}_k - \frac{\widehat{F}}{\widehat{\omega}}\widehat{\beta}_k\right) + \frac{\partial}{\partial y}\left(\frac{\widehat{E}}{\widehat{\omega}}\widehat{\beta}_k - \frac{\widehat{F}}{\widehat{\omega}}\widehat{\alpha}_k\right) = \frac{\partial}{\partial x}\left(-\frac{\partial f_k}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f_k}{\partial x}\right) = 0,$$

which is the maximal surface system. This completes the proof of (a). □

Bernstein's Theorem for two dimensional entire minimal graphs in Euclidean space with a positive area-angle holds [14]. As a consequence of this and the twin correspondence, we obtain

Corollary 14. *The only* 2*-dimensional entire maximal graphs in* $\mathbb{R}^{n+2}_n$*,* $n \geq 2$ *with a positive area-angle function are spacelike planes.*

Proof. Let $\widehat{\Sigma}$ be a 2-dimensional entire maximal graph in $\mathbb{R}^{n+2}_n$ with a positive area-angle. Taking $\Omega = \mathbb{R}^2$ in Theorem 13, we find that its twin minimal graph Σ in $\mathbb{R}^{n+2}$ is also entire. By a theorem of Hasanis, Savas-Halilaj and Vlachos [14], we find that the entire minimal graph Σ with bounded Jacobian is a plane. Since the twin correspondence transforms the planes in $\mathbb{R}^{n+2}$ to the spacelike planes in $\mathbb{R}^{n+2}_n$, we conclude that $\widehat{\Sigma}$ becomes a spacelike plane. □

Corollary 15 (Calabi's Theorem, [6])**.** *The only entire maximal graphs in Lorentz space* $\mathbb{R}^3_1$ *are spacelike planes.*

Proof. An entire maximal graph $x\mathbf{e}_1 + y\mathbf{e}_2 + g(x,y)\mathbf{e}_3$ in $\mathbb{R}^3_1$ can be viewed as an entire maximal graph $x\mathbf{e}_1 + y\mathbf{e}_2 + g(x,y)\mathbf{e}_3 + 0\mathbf{e}_4$ in $\mathbb{R}^4_2$ with area-angle $\Theta = \frac{\pi}{2}$. □

Let D^2 denote the Hessian matrix operator. The twin correspondence induces the following duality:

Corollary 16 (Duality between minimal gradient graphs in $\mathbb{R}^4$ and maximal gradient graphs in $\mathbb{R}^4_2$)**.** *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected domain. If the gradient graph* $x\mathbf{e}_1 + y\mathbf{e}_2 + \widehat{p}_x\mathbf{e}_3 + \widehat{p}_y\mathbf{e}_4$ *of the function* $\widehat{p} : \Omega \to \mathbb{R}$ *such that* $|\det D^2\widehat{p}| < 1$ *is a maximal surface in* $\mathbb{R}^4_2$*, then the height function of the twin minimal graph in* $\mathbb{R}^4$ *of the maximal graph* $x\mathbf{e}_1 + y\mathbf{e}_2 - \widehat{p}_y\mathbf{e}_3 + \widehat{p}_x\mathbf{e}_4$ *also becomes a gradient map of some function* $p : \Omega \to \mathbb{R}$ *with* $|\det D^2 p| < 1$*.*

Remark 7. We omit the proof of Corollary 16 because the next section contains a more general explanation. Indeed, Lemma 18 in Section 4.2 indicates that the special Lagrangian equation is equivalent to the split special Lagrangian equation.

4.2. **Symplectic Monge-Ampère equations.** We are interested in two symplectic Monge-Ampère equations. The first one is the well-known equation of special Lagrangian graphs in the complex space $\mathbb{C}^2$,

$$\cos\alpha\,(F_{xx}+F_{yy})+\sin\alpha\left(1-F_{xx}F_{yy}+{F_{xy}}^2\right)=0,$$

and the second one is the equation of split special Lagrangian graphs in the para-complex space $\mathbb{D}^2$

$$\cosh\beta\,(F_{xx}+F_{yy})+\sinh\beta\left(1+F_{xx}F_{yy}-{F_{xy}}^2\right)=0.$$

Remark 8. The even dimensional real vector space $\mathbb{R}^{2n}$ admits two different special Lagrangian geometries:

(a) As in [2, 3, 10], we may view $\mathbb{R}^{2n}$ as the complex space $\mathbb{C}^n$ endowed with the complex structure J satisfying $J^2=-Id$ and the pseudo-Hermitian form

$$-\sum_{j=1}^{p}dz_jd\overline{z}_j+\sum_{j=p+1}^{n}dz_jd\overline{z}_j,$$

where $0\le p\le n$. The signature of their special Lagrangian submanifolds is $(p,n-p)$, so in particular, for $p\neq 0,n$, they have indefinite induced metric. Recently, Dong [10] proved the instability of such special Lagrangian submanifolds and Anciaux [3] showed that they minimize the volume in their Lagrangian homology classes.

(b) On the other hand, adopting the split Kähler structure [13, 26], we here regard $\mathbb{R}^{2n}$ as the para-complex space $\mathbb{D}^n$ endowed with the para-complex structure I satisfying $I^2=Id$. Mealy [26] proved that the split special Lagrangian submanifolds (or equivalently, spacelike Lagrangian submanifolds of zero mean curvature) are homology maximizing. See also [13, Theorem 5.3].

Recently, Kim, McCann and Warren [20] studied an interesting relationship between the split special Lagrangian geometry and the classical mass transport problem. The readers should consult [13, 26] for more detailed background on the split special Lagrangian geometry.

Lemma 17 (Split special Lagrangian equation)**.** *In pseudo-Euclidean space $\mathbb{R}^4_2$, we consider a spacelike graph Σ given by $(x,y,z,w)=(x,y,h_x(x,y),h_y(x,y))$ for some $\mathcal{C}^3$ function $h:\Omega\to\mathbb{R}$ defined on a connected xy-domain Ω. The following two statements are equivalent.*

(a) The graph Σ has zero mean curvature in $\mathbb{R}^4_2$.

(b) The potential function h satisfies the split special Lagrangian equation

$$\cosh\theta\,(h_{xx}+h_{yy})+\sinh\theta(1+h_{xx}h_{yy}-{h_{xy}}^2)=0$$

for some constant angle $\theta\in\mathbb{R}$.

Proof. We give the details of the equivalence of (a) and (b). Since the spacelike graph Σ in $\mathbb{R}^4_2$ admits the positive definite metric

$$ds^2=\left(1-{h_{xx}}^2-{h_{yx}}^2\right)dx^2-2\left(h_{xx}h_{xy}+h_{yx}h_{yy}\right)dxdy+\left(1-{h_{xy}}^2-{h_{yy}}^2\right)dy^2,$$

we require the inequality

$$\begin{aligned}0 &< \left(1-{h_{xx}}^2-{h_{yx}}^2\right)\left(1-{h_{xy}}^2-{h_{yy}}^2\right)-\left(h_{xx}h_{xy}+h_{yx}h_{yy}\right)^2\\ &= \left(1+h_{xx}h_{yy}-{h_{xy}}^2\right)^2-\left(h_{xx}+h_{yy}\right)^2.\end{aligned}$$

This guarantees that $1+h_{xx}h_{yy}-{h_{xy}}^2$ never vanish on the domain Ω. Furthermore, we obtain the well-defined function $\phi:\Omega\to(-1,1)$ given by

$$\phi := \frac{h_{xx}+h_{yy}}{1+h_{xx}h_{yy}-{h_{xy}}^2}.$$

The gradient graph $(x,y,f(x,y),g(x,y))=(x,y,h_x,h_y)$ has zero mean curvature in $\mathbb{R}^4_2$ when its height function $(f,g)=(h_x,h_y)$ solves the maximal surface system

$$\begin{cases} 0=\left(1-{f_y}^2-{g_y}^2\right)f_{xx}+2\left(f_xf_y+g_xg_y\right)f_{xy}+\left(1-{f_x}^2-{g_x}^2\right)f_{yy}, \\ 0=\left(1-{f_y}^2-{g_y}^2\right)g_{xx}+2\left(f_xf_y+g_xg_y\right)g_{xy}+\left(1-{f_x}^2-{g_x}^2\right)g_{yy}, \end{cases}$$

or equivalently,

$$\begin{cases} 0=\left(1-{h_{xy}}^2-{h_{yy}}^2\right)h_{xxx}+2\left(h_{xx}+h_{yy}\right)h_{xy}h_{xxy}+\left(1-{h_{xx}}^2-{h_{xy}}^2\right)h_{xyy}, \\ 0=\left(1-{h_{xy}}^2-{h_{yy}}^2\right)h_{yxx}+2\left(h_{xx}+h_{yy}\right)h_{xy}h_{yxy}+\left(1-{h_{xx}}^2-{h_{xy}}^2\right)h_{yyy}. \end{cases}$$

This can be rewritten as

$$\begin{cases} 0=\left(1+h_{xx}h_{yy}-{h_{xy}}^2\right)\frac{\partial}{\partial x}(h_{xx}+h_{yy})-(h_{xx}+h_{yy})\frac{\partial}{\partial x}\left(1+h_{xx}h_{yy}-{h_{xy}}^2\right), \\ 0=\left(1+h_{xx}h_{yy}-{h_{xy}}^2\right)\frac{\partial}{\partial y}(h_{xx}+h_{yy})-(h_{xx}+h_{yy})\frac{\partial}{\partial y}\left(1+h_{xx}h_{yy}-{h_{xy}}^2\right). \end{cases}$$

This is equivalent to saying that the gradient of ϕ vanishes on Ω:

$$0=\phi_x=\frac{\partial}{\partial x}\left(\frac{h_{xx}+h_{yy}}{1+h_{xx}h_{yy}-{h_{xy}}^2}\right) \text{ and } 0=\phi_y=\frac{\partial}{\partial y}\left(\frac{h_{xx}+h_{yy}}{1+h_{xx}h_{yy}-{h_{xy}}^2}\right).$$

Since its domain Ω is connected, this system holds only when the quotient function $\phi:\Omega\to(-1,1)$ is a constant $-\tanh\theta$ for some $\theta\in\mathbb{R}$. □

Lemma 18 (Symplectic graph rotations)**.** *Let $F:\Omega\to\mathbb{R}$ be a function of class $\mathcal{C}^2$ satisfying the symplectic Monge-Ampère equation*

$$\lambda_1\left(F_{xx}+F_{yy}\right)+\lambda_2\left(1-\epsilon\left(F_{xx}F_{yy}-{F_{xy}}^2\right)\right)=0,$$

where $\lambda_1,\lambda_2\in\mathbb{R}$ and $\epsilon\in\{-1,1\}$ are constants satisfying that ${\lambda_1}^2+\epsilon{\lambda_2}^2=1$. Then, the induced function $h:\Omega\to\mathbb{R}$ defined by

$$h(x,y)=\lambda_2F(x,y)-\epsilon\lambda_1\frac{x^2+y^2}{2}$$

satisfies the unimodular Hessian equation $h_{xx}h_{yy}-{h_{xy}}^2=1$.

Proof. We compute

$$\begin{aligned} h_{xx}h_{yy}-{h_{xy}}^2 &= \left(\lambda_2F_{xx}-\epsilon\lambda_1\right)\left(\lambda_2F_{yy}-\epsilon\lambda_1\right)-\left(\lambda_2F_{xy}\right)^2 \\ &= {\lambda_2}^2\left(F_{xx}F_{yy}-{F_{xy}}^2\right)-\epsilon\lambda_2\lambda_1\left(F_{xx}+F_{yy}\right)+{\lambda_1}^2 \\ &= {\lambda_2}^2\left(F_{xx}F_{yy}-{F_{xy}}^2\right)+\epsilon{\lambda_2}^2\left(1-\epsilon\left(F_{xx}F_{yy}-{F_{xy}}^2\right)\right)+{\lambda_1}^2 \\ &= {\lambda_1}^2+\epsilon{\lambda_2}^2 \\ &= 1. \end{aligned}$$

□

Remark 9. By the correspondence in Theorem 11, the minimal graph (x,y,F_x,F_y) in $\mathbb{R}^4$ corresponds to the minimal gradient graph (x,y,h_x,h_y) in $\mathbb{R}^4$. This observation induces Lemma 18. Under the correspondence in Lemma 18, any harmonic function $F(x,y)$ corresponds to the same function $h(x,y)=\pm\frac{1}{2}\left(x^2+y^2\right)$.

Theorem 19 (Calabi type theorem for entire split special Lagrangian equation)**.** *If a maximal surface in $\mathbb{R}^4_2$ is an entire gradient graph (x, y, f_x, f_y) for some $\mathcal{C}^3$ function $f : \mathbb{R}^2 \to \mathbb{R}$, then it is a spacelike plane.*

Proof. First, Lemma 17 guarantees that, for some constant $\theta \in \mathbb{R}$, the potential function f satisfies the split special Lagrangian equation

$$\cosh\theta\,(f_{xx} + f_{yy}) + \sinh\theta(1 + f_{xx}f_{yy} - {f_{xy}}^2) = 0.$$

(a) When $\sinh\theta \neq 0$, since Lemma 18 says that the entire function

$$h(x, y) = \sinh\theta f(x, y) + \cosh\theta\frac{x^2 + y^2}{2}$$

satisfies the unimodular Hessian equation $h_{xx}h_{yy} - {h_{xy}}^2 = 1$, Jörgens' Theorem guarantees that h is quadratic and that f is also quadratic.

(b) When $\sinh\theta = 0$, we see that the entire function f is harmonic. In other words, the function $\phi : \mathbb{C} \to \mathbb{C}$, $z = x + iy \mapsto f_x - if_y$ is holomorphic. The induced metric of the maximal graph (x, y, f_x, f_y) is conformal:

$$ds^2 = \left(1 - {f_{xx}}^2 - {f_{yx}}^2\right)\left(dx^2 + dy^2\right).$$

Since our graph (x, y, f_x, f_y) is spacelike, we have $1 - {f_{xx}}^2 - {f_{yx}}^2 > 0$ on the whole plane $\mathbb{R}^2$. Since the entire holomorphic function $\phi'(z) = f_{xx} - if_{yx}$ is bounded:

$$|\phi'(z)| = \sqrt{{f_{xx}}^2 + {f_{yx}}^2} < 1,$$

Liouville's Theorem guarantees that three entire functions f_{xx}, f_{yx} and $f_{yy} = -f_{xx}$ are constants. Hence, f is quadratic. □

Corollary 20 (Bernstein type theorem for entire special Lagrangian equation, Fu [11], Yuan [40])**.** *If a minimal surface in $\mathbb{R}^4$ becomes an entire gradient graph (x, y, f_x, f_y) for some function $f : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^3$, then the potential function f is harmonic or quadratic.*

Proof. The Harvey-Lawson Theorem shows that there exists a constant $\theta \in \mathbb{R}$ such that the potential function f satisfies the special Lagrangian equation

$$\cos\theta\,(f_{xx} + f_{yy}) + \sin\theta(1 - f_{xx}f_{yy} + {f_{xy}}^2) = 0.$$

When $\sin\theta \neq 0$, since the entire function $h(x, y) = \sin\theta f(x, y) - \cos\theta\frac{x^2+y^2}{2}$ satisfies $h_{xx}h_{yy} - {h_{xy}}^2 = 1$, Jörgens' Theorem guarantees that h and f are quadratic. □

Remark 10. We provide two comments on the *reverse* unimodular Hessian equation $h_{xx}h_{yy} - {h_{xy}}^2 = -1$. First, the graph rotation, which is analogous to the construction in Lemma 18, is available for the following two Monge-Ampère equations:

$$\sinh\theta\,(F_{xx} + F_{yy}) + \cosh\theta\left(1 + F_{xx}F_{yy} - {F_{xy}}^2\right) = 0$$

and

$$\sin\theta\,(-F_{xx} + F_{yy}) + \cos\theta\left(1 + F_{xx}F_{yy} - {F_{xy}}^2\right) = 0.$$

Indeed, whenever a function F of class $\mathcal{C}^2$ satisfies

$$\lambda_1\,(\epsilon\,F_{xx} + F_{yy}) + \lambda_2\left(1 + F_{xx}F_{yy} - {F_{xy}}^2\right) = 0,$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\epsilon \in \{-1, 1\}$ are constants with $-\epsilon{\lambda_1}^2 + {\lambda_2}^2 = 1$, the new function $h(x, y) = \lambda_2 F(x, y) + \lambda_1\frac{x^2+\epsilon y^2}{2}$ satisfies $h_{xx}h_{yy} - {h_{xy}}^2 = -1$. Second, we

observe that it admits entire solutions of the form $h(x,y) = xy + f(x)$. Chamberland [8, Theorem 3.1] constructed a new entire solution of $h_{xx}h_{yy} - {h_{xy}}^2 = -1$.

4.3. **Twin surfaces in simultaneous conformal coordinates.** We are able to read the twin surfaces in simultaneous conformal coordinates and obtain the twin relations with respect to their Weierstrass representation formulas.

Theorem 21 (Reading twin surfaces in conformal coordinates)**.** *Let $\mathbf{\Phi}(x,y)$ be the minimal graph in $\mathbb{R}^{n+2}$ of a positive area-angle map $f : \Omega \to \mathbb{R}^n$ and $\widehat{\mathbf{\Phi}}(x,y)$ its twin maximal graph in $\mathbb{R}^{n+2}_n$ of a positive area-angle map $g : \Omega \to \mathbb{R}^n$.*

(a) There exists a simultaneous conformal coordinate $\xi = \xi_1 + i\xi_2$ for the minimal graph $\mathbf{\Phi}(x,y)$ in $\mathbb{R}^{n+2}$ and its twin maximal graph $\widehat{\mathbf{\Phi}}(x,y)$ in $\mathbb{R}^{n+2}_n$.

(b) Letting $\Psi : (x,y) \mapsto (\xi_1, \xi_2)$ be the coordinate transformation in (a), we see that the conformal harmonic immersion $\mathbf{\Phi} \circ \Psi^{-1}$ induces the holomorphic curve

$$2\frac{\partial}{\partial \xi}\mathbf{\Phi} \circ \Psi^{-1} = (\phi_1, \phi_2, \phi_3, \cdots, \phi_{n+2})$$

and the conformal harmonic immersion $\widehat{\mathbf{\Phi}} \circ \Psi^{-1}$ induces the holomorphic curve

$$2\frac{\partial}{\partial \xi}\widehat{\mathbf{\Phi}} \circ \Psi^{-1} = \left(\widehat{\phi}_1, \widehat{\phi}_2, \widehat{\phi}_3, \cdots, \widehat{\phi}_{n+2}\right).$$

Then, they obey the twin relations

$$\phi_1 = \widehat{\phi}_1, \quad \phi_2 = \widehat{\phi}_2, \quad \phi_{k+2} = -i\widehat{\phi}_{k+2}, \quad k \in \{1, \cdots, n\}.$$

Proof. As in the proof of (a) in Lemma 5, we take the coordinate transformation

$$\Psi : (x,y) \mapsto (\xi_1, \xi_2)$$

such that

$$J_\Psi = \frac{\partial(\xi_1, \xi_2)}{\partial(x,y)} = \det\begin{pmatrix} 1 + \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & 1 + \frac{G}{\omega} \end{pmatrix} = 2 + \frac{E+G}{\omega} > 2.$$

Since $J_\Psi = \frac{\partial(\xi_1,\xi_2)}{\partial(x,y)} > 0$, we have the existence of the local inverse

$$(\xi_1, \xi_2) \mapsto (x,y).$$

Now, using the Chain Rule, we obtain the conformal metrics:

$$ds^2_{\mathbf{\Phi}} = \frac{\omega}{J_\Psi}\left({d\xi_1}^2 + {d\xi_2}^2\right)$$

and

$$ds^2_{\widehat{\mathbf{\Phi}}} = \frac{\widehat{\omega}}{J_\Psi}\left({d\xi_1}^2 + {d\xi_2}^2\right) = \frac{\widehat{\omega}}{\omega} ds^2_{\mathbf{\Phi}}.$$

Proof of (a) is finished. Next, we prove (b). We consider two conformal immersions

$$\mathbf{\Phi} \circ \Psi^{-1}(\xi_1, \xi_2) = x(\xi_1, \xi_2)\,\mathbf{e}_1 + y(\xi_1, \xi_2)\,\mathbf{e}_2 + \sum_{k=1}^{n} f_k(x(\xi_1, \xi_2), y(\xi_1, \xi_2))\mathbf{e}_{k+2}$$

and

$$\widehat{\mathbf{\Phi}} \circ \Psi^{-1}(\xi_1, \xi_2) = x(\xi_1, \xi_2)\,\mathbf{e}_1 + y(\xi_1, \xi_2)\,\mathbf{e}_2 + \sum_{k=1}^{n} g_k(x(\xi_1, \xi_2), y(\xi_1, \xi_2))\mathbf{e}_{k+2}.$$

By the definition of the induced holomorphic curves, obviously, we meet

$$\widehat{\phi}_1 = \phi_1 \text{ and } \widehat{\phi}_2 = \phi_2.$$

It now remains to check the equality

$$\phi_{k+2} = -i\widehat{\phi}_{k+2}, \quad k \in \{1, \cdots, n\}.$$

We need to show that

$$\frac{\partial}{\partial \xi} g_k(x(\xi_1, \xi_2), y(\xi_1, \xi_2)) = -i\left(\frac{\partial}{\partial \xi} f_k(x(\xi_1, \xi_2), y(\xi_1, \xi_2))\right), \ \xi = \xi_1 + i\xi_2,$$

or equivalently,

$$\left(\frac{\partial f_k}{\partial \xi_1}, \frac{\partial f_k}{\partial \xi_2}\right) = \left(\frac{\partial g_k}{\partial \xi_2}, -\frac{\partial g_k}{\partial \xi_1}\right).$$

We only check the second components. The twin relations in Theorem 13 read

$$\left(\widehat{\alpha}_k, \widehat{\beta}_k\right) = \left(-\frac{E}{\omega}\beta_k + \frac{F}{\omega}\alpha_k, \frac{G}{\omega}\alpha_k - \frac{F}{\omega}\beta_k\right),$$

where

$$\left(\alpha_k, \beta_k, \widehat{\alpha}_k, \widehat{\beta}_k\right) = \left(\frac{\partial f_k}{\partial x}, \frac{\partial f_k}{\partial y}, \frac{\partial g_k}{\partial x}, \frac{\partial g_k}{\partial y}\right).$$

Also, we prepare the equality

$$\begin{pmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi_1}{\partial x} & \frac{\partial \xi_1}{\partial y} \\ \frac{\partial \xi_2}{\partial x} & \frac{\partial \xi_2}{\partial y} \end{pmatrix}^{-1} = \frac{1}{J_\Psi}\begin{pmatrix} 1 + \frac{G}{\omega} & -\frac{F}{\omega} \\ -\frac{F}{\omega} & 1 + \frac{E}{\omega} \end{pmatrix}.$$

We then use this together with the Chain Rule to deduce

$$\begin{aligned}
\frac{\partial g_k}{\partial \xi_1} &= \frac{\partial x}{\partial \xi_1}\frac{\partial g_k}{\partial x} + \frac{\partial y}{\partial \xi_1}\frac{\partial g_k}{\partial y} \\
&= \frac{\partial x}{\partial \xi_1}\left[-\frac{E}{\omega}\beta_k + \frac{F}{\omega}\alpha_k\right] + \frac{\partial y}{\partial \xi_1}\left[\frac{G}{\omega}\alpha_k - \frac{F}{\omega}\beta_k\right] \\
&= \left[\frac{\partial x}{\partial \xi_1}\frac{F}{\omega} + \frac{\partial y}{\partial \xi_1}\frac{G}{\omega}\right]\alpha_k - \left[\frac{\partial x}{\partial \xi_1}\frac{E}{\omega} + \frac{\partial y}{\partial \xi_1}\frac{F}{\omega}\right]\beta_k \\
&= \frac{1}{J_\Psi}\left[\left(1+\frac{G}{\omega}\right)\frac{F}{\omega} + \left(-\frac{F}{\omega}\right)\frac{G}{\omega}\right]\alpha_k - \frac{1}{J_\Psi}\left[\left(1+\frac{G}{\omega}\right)\frac{E}{\omega} + \left(-\frac{F}{\omega}\right)\frac{F}{\omega}\right]\beta_k \\
&= \frac{1}{J_\Psi}\frac{F}{\omega}\alpha_k - \frac{1}{J_\Psi}\left(1 + \frac{E}{\omega}\right)\beta_k \\
&= -\frac{\partial x}{\partial \xi_2}\frac{\partial f_k}{\partial x} - \frac{\partial y}{\partial \xi_2}\frac{\partial f_k}{\partial y} \\
&= -\frac{\partial f_k}{\partial \xi_2}.
\end{aligned}$$

□

Remark 11. In the case when $n = 1$, the twin correspondence in conformal coordinates, described in (b) of Theorem 21, is also observed in [21]. Araújo and Leite [1] illustrated several interesting results on Calabi's correspondence.

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