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SPECTRAL STRUCTURE OF DIGIT SETS OF SELF-SIMILAR TILES ON $\mathbb{R}^1$

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ABSTRACT. We study the structure of the digit sets $\mathcal{D}$ for the integral self-similar tiles $T(b,\mathcal{D})$ (we call such a $\mathcal{D}$ a *tile digit set* with respect to b). So far the only available classes of such tile digit sets are the complete residue sets and the product-forms. Our investigation here is based on the spectrum of the mask polynomial $P_{\mathcal{D}}$, i.e., the zeros of $P_{\mathcal{D}}$ on the unit circle. By using the Fourier criteria of self-similar tiles of Kenyon and Protasov, as well as the algebraic techniques of cyclotomic polynomials, we characterize the tile digit sets through some product of cyclotomic polynomials (kernel polynomials), which is a generalization of the product-form to higher order.

1. INTRODUCTION

Let A be an $s \times s$ expanding matrix (i.e., all eigenvalues have moduli > 1) with integral entries and $|\det A| = b$. Also let $\mathcal{D} \subset \mathbb{Z}^s$ with $\#\mathcal{D} = b$; we call $\mathcal{D}$ a *digit set*. Consider the iterated function system (IFS) defined by $\{S_j\}_{j=0}^{m-1}$, where

$$S_j(x) = A^{-1}(x + d_j), \qquad 0 \le j \le m-1. \tag{1.1}$$

Then these affine maps are contractive under certain norms in $\mathbb{R}^s$, and it is well known that there exists a unique compact set $T := T(A,\mathcal{D}) \subset \mathbb{R}^s$ satisfying the set-valued relation $AT = T + \mathcal{D}$. Alternatively, $T(A,\mathcal{D})$ can be expressed as a set of radix expansions with base A and digits in $\mathcal{D}$:

$$T(A,\mathcal{D}) = \{\sum_{k=1}^{\infty} A^{-k} d_k : d_k \in \mathcal{D}\}. \tag{1.2}$$

In [B], Bandt proved that if $|\det A| = b = \#\mathcal{D}$ and $T^o \neq \emptyset$, then $T(A,\mathcal{D})$ is a *translational tile*. This means that there exists $\mathcal{J} \subset \mathbb{R}^s$ such that $\bigcup_{t\in\mathcal{J}}(T+t) = \mathbb{R}^s$, and the Lebesgue measure of $(T+t)\cap(T+t')$ is zero for all $t, t' \in \mathcal{J}$, $t \neq t'$. In that case we call T an *(integral) self-affine tile*, and a *self-similar tile* if the matrix A is, in addition, a scalar times an orthonormal matrix. The digit set $\mathcal{D}$ is called a *(self-affine) tile digit set* with respect to A, and $\mathcal{J}$ is called a *tiling set* for T.

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The study of self-affine tiles and the tiling theory was initiated by Thurston [Th] and Kenyon [K], and the foundation and basic properties were laid down in a series of papers by Lagarias and Wang ([LW1], [LW2], [LW3], [LW4]). By now there is a wealth of literature on this subject, with topics including the tiling theory, the geometric and fractal structures, the topological properties, the classification problems and applications to wavelet theory ([AT], [GH], [HLR], [KL1], [KL2], [LW1], [LW2], [LW3], [LW4], [LL], [LR], [SW], [V]). The related topic of the Fuglede's problem on tiles and spectral sets has also received a lot of attention recently ([Ko], [L], [LaW], [T]). However, despite such intensive studies, many of these aspects are still not fully understood. Among them is the following basic question:

(Q) *Given an expanding integral matrix A, can we classify all the tile digit sets $\mathcal{D}$, and what is the structure of such a $\mathcal{D}$?*

The question turns out to be rather intriguing and challenging; it is not clear even in $\mathbb{R}^1$. A well-known sufficient condition (it is also true in any dimension) for $\mathcal{D}$ to be a tile digit set is: $\mathcal{D}$ *is a complete residue set* $(\bmod A)$ [B] (which is called a *standard tile digit set* [LW3]). This condition is also necessary in $\mathbb{R}^1$ if $b = p$ is a prime [K]. It is also true in higher dimensions when an additional hypothesis is assumed ([LW3], [HL]).

The situation is far more complicated if b is not a prime number. For example in $\mathbb{R}^1$ if $A = [b]$ with $b = 4$ and $\mathcal{D} = \{0, 1, 8, 9\}$, then $\mathcal{D}$ is not a complete residue (mod 4), but $T(4, \mathcal{D}) = [0, 1] \cup [2, 3]$ is a tile with a tiling set $\mathcal{J} = \{0, 1\} \oplus 4\mathbb{Z}$. In reference to the work of Odlyzko [O], Lagarias and Wang proposed a class of digit sets in $\mathbb{R}^s$ called a *product-form* [LW3] as a generalization of the complete residue class. They also showed that for $b = p^\ell$ a prime power, then a tile digit set $\mathcal{D}$ of b must be a "product-form-like" set. More recently, Lau and Rao introduced a *weak product-form* and used this idea to classify all tile digit sets $\mathcal{D}$ $(\subset \mathbb{Z})$ for $b = pq$, a product of two primes [LR].

In this paper, we only focus on self-similar tiles in $\mathbb{R}^1$. We assume that $b > 1$, $A = [b]$ and $\#\mathcal{D} = b$. We also assume, without loss of generality, that $0 \in \mathcal{D} \subset \mathbb{Z}^+$ and g.c.d.$(\mathcal{D}) = 1$. We first recall that a cyclotomic polynomial $\Phi_d(x)$ is the minimal polynomial of $e^{2\pi i/d}$. It is easy to see that a product-form can be expressed into a product of cyclotomic polynomials (Section 3). Our main result is to obtain a characterization of the tile digit sets as a product of cyclotomic polynomials. We call

$$P_{\mathcal{D}}(x) = \sum_{d \in \mathcal{D}} x^d = 1 + x^{d_1} + \cdots + x^{d_{b-1}}$$

a *mask polynomial* of $\mathcal{D}$. A well-known necessary and sufficient condition for $T(b, \mathcal{D})$ to be a self-similar tile, due to Kenyon [K], is that the mask polynomial satisfies: *for any $m > 0$, there exists k such that*

$$P_{\mathcal{D}}(e^{2\pi i m/b^k}) = 0.$$

The Kenyon criterion shows a close link of the zero set of the mask polynomials with the cyclotomic polynomial. Note that $P_{\mathcal{D}}(e^{2\pi i/d}) = 0$ if and only if $\Phi_d(x)|P_{\mathcal{D}}(x)$. We call the set $\{d > 1 : \Phi_d(x)|P_{\mathcal{D}}(x)\}$ the *spectrum* of $\mathcal{D}$, and

$$S_{\mathcal{D}} = \{p^\alpha > 1 : p \text{ prime},\ \Phi_{p^\alpha}(x)|P_{\mathcal{D}}(x)\}$$

the *prime-power spectrum* of $\mathcal{D}$. They play an important role in our consideration.

The spectrum was used extensively by Coven and Meyerowitz [CM] in their study of integer tiles $\mathcal{A}$ (i.e., $\#\mathcal{A}$ is finite, and there exists $\mathcal{L}$ such that $\mathcal{A} \oplus \mathcal{L} = \mathbb{Z}$). They gave an inductive characterization of such an $\mathcal{A}$ when $\#\mathcal{A} = p^\alpha q^\beta$, p, q are primes. Later, Łaba and Wang [L], [LaW] used the spectrum to investigate the spectral sets and spectral measures. In the context of self-similar tile digit sets $\mathcal{D}$, we showed that $\mathcal{D}$ is an integer tile (see [LLR]). In the present paper, we prove the following theorem, which characterizes the prime-power spectrum of $\mathcal{D}$ (Theorem 2.4).

Theorem 1.1. *Let $b = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be a product of prime powers and let $\mathcal{D}$ be a tile digit set of b. Then*

$$S_{\mathcal{D}} = \bigcup_{j=1}^{k} S_{p_j},$$

where $S_{p_j} = \{p_j^{a_{j,1}}, \ldots, p_j^{a_{j,\alpha_j}}\}$ and $\{a_{j,1}, \ldots, a_{j,\alpha_j}\}$ is a complete residue set modulo α_i.

It is well known that if $\mathcal{D} = \{0, 1, \ldots, b-1\}$, then the mask polynomial

$$P_{\mathcal{D}}(x) = 1 + x + \cdots + x^{b-1} = \prod_{d|b, d>1} \Phi_d(x);$$

if $\mathcal{D}$ is a complete residue set modulo b, then $P_{\mathcal{D}}(x) = \Big(\prod_{d|b,d>1} \Phi_d(x)\Big) Q(x)$ for some integral polynomial $Q(x)$. If $\mathcal{D}$ is a product-form, i.e. $\mathcal{D} = \mathcal{E}_0 \oplus b^{\ell_1}\mathcal{E}_1 \oplus \cdots \oplus b^{\ell_k}\mathcal{E}_k$, where $\mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_k \equiv \mathbb{Z}_b$, we can see that $\mathcal{D}$ satisfies

(P_1): *for any factor d (> 1) of b, there exists $j(d) \geq 0$ such that $\Phi_d(x^{b^j}) | P_{\mathcal{D}}(x)$.*

In fact, if $\mathcal{D}$ is a tile digit set that satisfies (P_1), we can write the mask polynomial as

$$P_{\mathcal{D}}(x) = \Big(\prod_{d|b,d>1} \Phi_d(x^{b^{j(d)}})\Big) Q(x) \tag{1.3}$$

for some integral polynomial $Q(x)$. It is also easy to check that if $\mathcal{D}$ is a weak product-form [LR], then $\mathcal{D}$ satisfies condition (P_1) (Corollary 3.3). Actually, condition (P_1) is satisfied for the more general *modulo product-form*, which is defined through the product-form by allowing certain modulo action of each factor (see Definition 3.4).

On the other hand, such a simple expression of $P_{\mathcal{D}}$ in (1.3) is not sufficient to cover all tile digit sets. For this we introduce the notion of $(P_k), k \geq 1$ and, correspondingly, the more concrete k^{th}*-order modulo product-forms* (Section 4). The case $k = 1$ corresponds to the ordinary product-forms. We have the following theorem:

Theorem 1.2. *If $\mathcal{D}$ is a kth-order modulo product-forms, then $\mathcal{D}$ satisfies the condition (P_k) and it is a tile digit set.*

Roughly speaking, a second-order product-form can be constructed as follows: we take a modulo product-form and rearrange the digits to form a product, then use this to construct a new product-form. By doing so, we are able to construct new tile digit sets that are not known in the literature.

By using a device of Protasov [P], we can represent the mask polynomial by a "tree" of cyclotomic polynomials (a Φ-*tree*) with respect to b, and a *blocking* $\mathcal{N}$ of

the tree: a finite subset such that every path on the tree meets one and only one element of $\mathcal{N}$ (Section 5). Our main theorem is:

Theorem 1.3. *Let $\mathcal{D}$ be a set of non-negative integers with $\#\mathcal{D} = b$. Then the following are equivalent:*

(i) $\mathcal{D}$ is a tile digit set;

(ii) there is a blocking $\mathcal{N}$ in the Φ-tree such that

$$P_{\mathcal{D}}(x) = \Big(\prod_{\Phi_d \in \mathcal{N}} \Phi_d(x)\Big) Q(x);$$

(iii) $P_{\mathcal{D}}(x)$ satisfies condition (P_k) for some $k \geq 1$.

We call the product in (ii) a *kernel polynomial* of $\mathcal{D}$. It plays a central role on the structure of the tile digit sets. For $b = pq$ with the prime-power spectrum equal to $\{p, q^\ell\}$, the digit set $\mathcal{D}$ is a (1st-order) modulo product-form, and the kernel polynomial is of the form $\Phi_p(x)\Phi_{q^\ell}(x^{p^\ell})$. This was proved in [LR] (an errata in the statement of Theorem 1.5 and 5.1 in [LR]: $\mathcal{E}_0$ should be $\{0, 1, \ldots, p-1\}$ (mod p); it was shown in the proof there). For the more general case where $b = p^\alpha q$, we determine, in a separate paper [LLR] all the kernel polynomials that generate tile digit sets and we also show that all tile digit sets are of modulo product-forms up to some order depending on the prime-power spectrum.

For the organization of the paper, we first provide some basic properties of the cyclotomic polynomials and integer tilings in Section 2; we also bring in the Kenyon criterion to study the mask polynomial $P_{\mathcal{D}}$ and prove Theorem 1.1. In Section 3, we consider the condition (P_1) and the various kinds of product-forms, and in Section 4, we set up the (P_k) condition and the higher-order product-form and prove Theorem 1.2. In Section 5, we make use of Protasov's device to study the Φ-tree and prove Theorem 1.3. We conclude in Section 6 with some remarks and open questions.

2. Preliminaries

In this section, we give a brief summary on the cyclotomic polynomials which is needed for later discussions. We use $\Phi_d(x)$ to denote the *d-th cyclotomic polynomial*. It is the minimal polynomial of the primitive d-th root of unity, i.e., $\Phi_d(e^{2\pi i/d}) = 0$. Note that the degree of $\Phi_d(x)$ is equal to the Euler-phi function $\varphi(d)$ (the number of relatively prime integers in $1, ..., d-1$). It is well known that

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{2.1}$$

The formula provides a constructive way to find Φ_d inductively. The following basic properties of cyclotomic polynomials will be used throughout the paper.

Proposition 2.1. *The cyclotomic polynomials satisfy the following:*

(i) If p is a prime, then $\Phi_p(x) = 1 + x + ... + x^{p-1}$ and $\Phi_{p^{\alpha+1}}(x) = \Phi_p(x^{p^\alpha})$;

(ii) $\Phi_s(x^p) = \Phi_{sp}(x)$ if p is prime and $p|s$, and
$\Phi_s(x^p) = \Phi_s(x)\Phi_{sp}(x)$ if p is prime but $p \nmid s$;

(iii) $\Phi_s(1) = \begin{cases} 0, & \text{if } s = 1; \\ p, & \text{if } s = p^\alpha; \\ 1, & \text{otherwise.} \end{cases}$

Proposition 2.2. *Let $b \geq 2$ be an integer. Then for any two distinct factors d_1, d_2 $(\neq 1)$ of b, and for any integers $u_1, u_2 \geq 0$, $\Phi_{d_1}(x^{b^{u_1}})$ and $\Phi_{d_2}(x^{b^{u_2}})$ have no common factor.*

Proof. Note that if $f(x)$ and $g(x)$ have no common divisor, then so do $f(x^n)$ and $g(x^n)$ (use $a(x)f(x)+b(x)g(x) = 1$ for some polynomials $a(x), b(x)$). Hence, we only need to show that $\Phi_{d_1}(x)$ and $\Phi_{d_2}(x^{b^{u_2-u_1}})$ (assuming $u_1 \leq u_2$) have no common divisor. This is simple as $e^{2\pi i/d_1}$ is not a root of the polynomial $\Phi_{d_2}(x^{b^k})$ for all $k \geq 0$. □

Let $\mathbb{Z}^+$ be the set of non-negative integers. For $\mathcal{A} \subset \mathbb{Z}^+$, we let

$$P_{\mathcal{A}}(x) = \sum\nolimits_{a\in\mathcal{A}} x^a$$

and call it the *mask polynomial* of $\mathcal{A}$. For a finite set $\mathcal{A} \subset \mathbb{Z}^+$, we use

$$S_{\mathcal{A}} = \{p^\alpha > 1 : \ p \text{ prime}, \ \Phi_{p^\alpha}(x)|P_{\mathcal{A}}(x)\} \tag{2.2}$$

to denote the *prime-power spectrum* of $\mathcal{A}$, and $\widetilde{S}_{\mathcal{A}} = \{s > 1 : \ \Phi_s(x)|P_{\mathcal{A}}(x)\}$ the *spectrum* of $\mathcal{A}$.

We call a finite set $\mathcal{A} \subset \mathbb{Z}$ an *integer tile* if there exists $\mathcal{L}$ such that $\mathcal{A} \oplus \mathcal{L} = \mathbb{Z}$. The class of integer tiles on $\mathbb{Z}$ has been studied in depth in connection with the factorization of cyclic groups and cyclotomic polynomials ([CM], [deB]). In [CM], Coven and Meyerowitz introduced the following two conditions to study the integer tiles:

(T1) $\#\mathcal{A} = P_{\mathcal{A}}(1) = \prod_{s\in S_{\mathcal{A}}} \Phi_s(1)$.

(T2) *For any distinct prime powers $s_1, \ldots, s_n \in S_{\mathcal{A}}$, then $s_1 \ldots s_n \in \widetilde{S}_{\mathcal{A}}$.*

They showed that *if $\mathcal{A} \subset \mathbb{Z}^+$ is a finite set, and suppose $(T1)$ and $(T2)$ hold, then $\mathcal{A}$ tiles $\mathbb{Z}$ with period $n = l.c.m.(S_{\mathcal{A}})$. Conversely, if $\mathcal{A}$ is an integer tile, then $(T1)$ holds; if in addition $\#\mathcal{A} = p^\alpha q^\beta$, $\alpha, \beta \geq 0$, then $(T2)$ holds.* It is still an open question whether an integer tile must satisfy (T2) in general. The class of integer tiles and the (self-affine) tile digit sets are closely related. In fact it is shown in [LLR] that *all tile digit sets (in any dimension) are integer tiles.*

In the following, we start to consider the tile digit sets of b in $\mathbb{R}$. We assume that $\mathcal{D} \subset \mathbb{Z}^+$, $0 \in \mathcal{D}$, and $g.c.d.(\mathcal{D}) = 1$. For $T := T(b, \mathcal{D})$, we let χ_T denote the characteristic function of T. The self-similar identity $bT = T + \mathcal{D}$ can be expressed as a refinement equation

$$\chi_T(x) = \sum_{d\in\mathcal{D}} \chi_T(bx - d), \text{ for a.e. } x \in \mathbb{R},$$

and the Fourier transform is $\widehat{\chi}_T(\xi) = \prod_{k=1}^{\infty} P_{\mathcal{D}}(e^{2\pi i\xi/b^k})$. By using the Riemann-Lebesgue lemma, it is not difficult to derive from the product that for χ_T to be an L^1-function (i.e., T is a self-similar tile since it has positive Lebesgue measure), then $\widehat{\chi}_T(m) = 0$ for any integer $m \neq 0$, so that one of the factors is 0. The converse can be proved by considering a certain tempered distribution (see, e.g., [HL]). This is the basic idea of the following criterion due to Kenyon (actually holds in $\mathbb{R}^s$).

Theorem 2.3 ([K])**.** *$T(b, \mathcal{D})$ is a self-similar tile if and only if for each integer $m > 0$, there exists $k \geq 1$ (depending on m) such that*

$$P_{\mathcal{D}}(e^{2\pi i m/b^k}) = 0.$$

The factorization of $P_{\mathcal{D}}(x)$ is closely connected with the cyclotomic polynomials. By using the Kenyon criterion, we prove the following theorem, which characterizes the prime power spectrum of $\mathcal{D}$.

Theorem 2.4. *Let $b = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the product of prime powers and let $\mathcal{D}$ be a tile digit set of b. Then for each p_j, and $0 \le \ell \le \alpha_j - 1$, there exists a unique $a_{j,\ell} \equiv \ell \pmod{\alpha_j}$ such that $\Phi_{p_j^{a_{j,\ell}}}(x)$ divides $P_{\mathcal{D}}(x)$, and the prime power spectrum of $\mathcal{D}$ is*

$$S_{\mathcal{D}} = \bigcup_{j=1}^{k} \{p_j^{a_{j,1}}, \ldots, p_j^{a_{j,\alpha_j}}\}.$$

Proof. We prove the theorem for p_1. For simplicity, we write $b = p_1^{\alpha_1} t$ and $e(x) = e^{2\pi i x}$. For a fixed $0 \le \ell \le \alpha_j - 1$ and for each $n \ge 1$, we set $a_n = p_1^{\alpha_1 - \ell} t^n$. By the Kenyon criterion, for any n, there exists $m = m(n)$ such that $e(a_n/b^m)$ is a root of $P_{\mathcal{D}}(x)$.

We claim that there exists n such that $n \ge m(n)$. Suppose otherwise; for any $n \ge 1$, we have $n < m(n)$. Let $u_n = p_1^{\ell + (m-1)\alpha_1} t^{m-n}$. Then $e(a_n/b^m) = e(1/u_n)$ are roots of $P_{\mathcal{D}}(x)$. Note that $u_n > p_1^{\ell + (m-1)\alpha_1}$; hence u_n tends to infinity as n tends to infinity (since $n < m(n) := m$). This means that u_n contains infinitely many distinct numbers, which is a contradiction since $P_{\mathcal{D}}(x)$ is a finite degree polynomial.

Hence, we can find n such that $n \ge m(n)$ and

$$e(a_n/b^m) = e(t^{n-m}/p_1^{\ell + (m-1)\alpha_1})$$

is a root of $P_{\mathcal{D}}(x)$. Letting $a_{1,\ell} = \ell + (m-1)\alpha_1$, we have $\Phi_{p_1^{a_{1,\ell}}}(x) | P_{\mathcal{D}}(x)$. This shows the existence part in the theorem.

For the uniqueness, we note that we have $P_{\mathcal{D}}(1) = \#\mathcal{D} = b$. On the other hand, each $\Phi_{p_j^{a_{j,\ell}}}(1) = p_j$ (by Proposition 2.1), and the product of all these $\Phi_{p_j^{a_{j,\ell}}}(1)$ is $p_1^{\alpha_1} \cdots p_k^{\alpha_k} = b$. Moreover, $\Phi_{p_j^{a_{j,\ell}}}(x)$ divides $P_{\mathcal{D}}(x)$. It follows that the set of all these $p_j^{a_{j,\ell}}$ is precisely the prime power spectrum $S_{\mathcal{D}}$ and the uniqueness follows. □

It follows from (2.1) that if $\mathcal{D}$ is a complete residue set modulo b (*standard digit set* [LW3]), then $P_{\mathcal{D}}(x) = \Big(\prod_{d|b, d>1} \Phi_d(x)\Big) Q(x)$ for some integral polynomial $Q(x)$. We see that the $S_{\mathcal{D}}$ in Theorem 2.4 is simply the set p_j^i, with $i = 1, \ldots, \alpha_i$. The theorem implies that if we change to the other tile digit sets of b, the prime power factors are preserved in a modulo way.

Example 2.5. Let $\mathcal{D} = \{0, 1, 8, 9\}$ and $b = 4 = 2^2$. It is known that $T(b, \mathcal{D}) = [0, 1] \cup [2, 3]$ is a tile. Its mask polynomial is

$$P_{\mathcal{D}}(x) = 1 + x + x^8 + x^9 = \Phi_2(x)\Phi_{2^4}(x).$$

We see that $\{1, 4\}$ is a complete residue (mod 2).

If $\mathcal{A} = \{0, 1, 4, 5\}$ (it is an integer tile), we have

$$P_{\mathcal{A}}(x) = 1 + x + x^4 + x^5 = \Phi_2(x)\Phi_{2^3}(x).$$

Note that $\{1, 3\}$ is not a complete residue (mod 2); this shows that $\mathcal{A}$ is not a tile digit set by Theorem 2.4.

3. Modulo product-forms

We denote $\mathbb{Z}_b$ to be the complete residue class of integers (mod b). It is known that if b is a prime, then $\mathcal{D}$ is a tile digit set if and only if $\mathcal{D} \equiv \mathbb{Z}_b$ [K]. But when b is not a prime, the problem is vastly more complicated. The following classes of $\mathcal{D}$ are some of the basic tile digit sets that are known.

Definition 3.1. $\mathcal{D}$ is called a product-form digit set (with respect to b) [LW3] if

$$\mathcal{D} = \mathcal{E}_0 \oplus b^{l_1}\mathcal{E}_1 \oplus \ldots \oplus b^{l_k}\mathcal{E}_k, \tag{3.1}$$

where $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_k \equiv \mathbb{Z}_b$, and $0 \le l_1 \le l_2 \le \ldots \le l_k$; if $\mathcal{E} = \{0, 1, 2, \ldots, b-1\}$, then $\mathcal{D}$ is called a strict product-form [O].

Furthermore if $\mathcal{D}' \equiv \mathcal{D}$ (mod b^{l_k+1}) for $\mathcal{D}$ in (3.1), then $\mathcal{D}'$ is called a weak product-form [LR].

In terms of the mask polynomials, a product-form can be expressed as

$$P_{\mathcal{D}}(x) = P_{\mathcal{E}_0}(x)P_{\mathcal{E}_1}(x^{b^{l_1}})\ldots P_{\mathcal{E}_k}(x^{b^{l_k}}), \tag{3.2}$$

where

$$P_{\mathcal{E}}(x) = P_{\mathcal{E}_0}(x)\ldots P_{\mathcal{E}_k}(x) \equiv 1 + x + \ldots + x^{b-1} (\text{mod } x^b - 1). \tag{3.3}$$

For the weak product-form,

$$P_{\mathcal{D}}(x) = P_{\mathcal{E}_0}(x)P_{\mathcal{E}_1}(x^{b^{l_1}})\ldots P_{\mathcal{E}_k}(x^{b^{l_k}}) + (x^{b^{l_k+1}} - 1)Q(x) \tag{3.4}$$

for some integral polynomial $Q(x)$.

In view of the expressions in (3.2)-(3.4), we introduce a new condition on $\mathcal{D}$:
($\mathbf{P}_1$) *for any $d > 1$ and $d|b$, there exists $j \ge 0$ (depends on d) such that $\Phi_d(x^{b^j})|P_{\mathcal{D}}(x)$.*
We also say that a polynomial $P(x)$ satisfies (P_1) if the above holds with $P_{\mathcal{D}}(x)$ replaced by $P(x)$.

Theorem 3.2. *Suppose $\mathcal{D}$ satisfies condition (P_1). Then $\mathcal{D}$ is a tile digit set, and*

$$P_{\mathcal{D}}(x) = \prod_{d|b, d>1} \Phi_d(x^{b^{j(d)}})\ Q(x), \tag{3.5}$$

where $j(d)$ is an integer depending on d.

Proof. Letting $m > 0$, we claim that there exists $d|b,\ d > 1$ such that $\Phi_d(e^{2\pi i m/b^k}) = 0$. Consider the standard tile digit set $\mathcal{E} = \{0, 1, \ldots, b-1\}$. By Theorem 2.3, there exists $k \ge 1$ such that

$$P_{\mathcal{E}}(e^{2\pi i m/b^k}) = 0.$$

Let $a = g.c.d.(m, b^k)$ and let $s = m/a,\ \ell = b^k/a$; then $s,\ \ell\ (> 1)$ are relatively prime and $P_{\mathcal{E}}(e^{2\pi i s/\ell}) = 0$. It follows that

$$\Phi_\ell(x)|P_{\mathcal{E}}(x). \tag{3.6}$$

As $P_{\mathcal{E}}(x) = \prod_{d|b,d>1} \Phi_d(x)$, we have $\ell = d$ for some $d > 1$ and the claim follows.

By condition (P_1), there exists $j := j(d) \ge 0$ such that

$$P_{\mathcal{D}}(x) = \Phi_d(x^{b^j})Q_d(x) \tag{3.7}$$

for some polynomial $Q_d(x)$. Hence we have

$$P_{\mathcal{D}}(e^{2\pi i m/b^{k+j}}) \;=\; \Phi_d(e^{2\pi i m/b^k})Q_d(e^{2\pi i m/b^{k+j}}) \;=\; 0.$$

This implies that $\mathcal{D}$ satisfies the Kenyon criterion and hence it is a tile digit set.

For the expression in (3.5), we only need to use (3.7) and observe that those $\Phi_d(x^{b^{j(d)}})$'s are distinct factors of $P_{\mathcal{D}}(x)$ (by Proposition 2.2). □

Corollary 3.3. *A product-form (or a weak product-form) digit set is a tile digit set.*

Proof. It follows from (3.3) that

$$P_{\mathcal{E}_0}(x)\dots P_{\mathcal{E}_k}(x) = P_{\mathcal{E}}(x) = 1 + x + \dots + x^{b-1} + (x^b - 1)Q(x)$$

for some polynomial $Q(x)$. This implies that for any $d|b$ and $d > 1$, $\Phi_d(x)|P_{\mathcal{E}_i}(x)$ for some $i = 0, \dots, k$. Hence $\Phi_d(x^{b^{l_i}})|P_{\mathcal{E}_i}(x^{b^{l_i}})$. On the other hand, by (3.2),

$$P_{\mathcal{D}}(x) = P_{\mathcal{E}_0}(x)P_{\mathcal{E}_1}(x^{b^{l_1}})\dots P_{\mathcal{E}_k}(x^{b^{l_k}}),$$

we obtain $\Phi_d(x^{b^{l_i}})|P_{\mathcal{D}}(x)$, and hence $\mathcal{D}$ is a tile digit set by Proposition 3.2.

For the weak product-form, we need to observe that the last factor in (3.4) is divisible by $\Phi_d(x^{b^{l_i}})$. □

We now introduce a more general kind of digit set that satisfies (P_1). Consider the product-form $\mathcal{D}$ in Definition 3.1. We define a decomposition of the spectrum: let $S_i = \{d > 1 : d|b,\ \Phi_d(x)|P_{\mathcal{E}_i}(x)\}$ and let

$$\Psi_i(x) = \prod_{d\in S_i} \Phi_d(x). \tag{3.8}$$

Then $\Psi_i(x)|P_{\mathcal{E}_i}(x)$; hence $\Psi_i(x^{b^{l_i}})|P_{\mathcal{E}_i}(x^{b^{l_i}})$. Let

$$K_{\mathcal{D}}^{(i)}(x) = \Psi_0(x)\Psi_1(x^{b^{l_1}})...\Psi_i(x^{b^{l_i}}), \quad 0 \le i \le k\ , \tag{3.9}$$

and denote $K_{\mathcal{D}}^{(k)}(x)$ by $K_{\mathcal{D}}(x)$. Then $K_{\mathcal{D}}(x)|P_{\mathcal{D}}(x)$. Moreover, $K_{\mathcal{D}}(x)$ satisfies (P_1). This is because for any $d > 1$ a factor of b, we have $d \in \mathcal{S}_i$ for some i; hence $\Phi_d(x^{b^{l_i}})|\Psi_i(x^{b^{l_i}})$ so that $\Phi_d(x^{b^{l_i}})|K_{\mathcal{D}}(x)$ also.

Definition 3.4. We say that $\mathcal{D}$ is a modulo product-form if $\mathcal{D} := \mathcal{D}^{(k)}$ is defined through a product-form $\mathcal{D}' = \mathcal{E}_0 \oplus b^{l_1}\mathcal{E}_1 \oplus \dots \oplus b^{l_k}\mathcal{E}_k,\ 0 \le l_1 \le \dots \le l_k$ as follows: let

$$n_i \;=\; l.c.m.\ \{s :\ \Phi_s(x) \mid K_{\mathcal{D}'}^{(i)}(x)\}$$

and

$$\begin{cases} \mathcal{D}^{(0)} \equiv \mathcal{E}_0 \pmod{n_0}, \\ \vdots \\ \mathcal{D}^{(i)} \equiv \mathcal{D}^{(i-1)} \oplus b^{l_i}\mathcal{E}_i \pmod{n_i}, \\ \vdots \\ \mathcal{D}^{(k)} \equiv \mathcal{D}^{(k-1)} \oplus b^{l_k}\mathcal{E}_k \pmod{n_k}. \end{cases} \tag{3.10}$$

Remarks. (1) Note that $b^{l_i}|n_i$ and $n_i|b^{l_i+1}$. Indeed let $\mathcal{R}_i = \{s : \Phi_s(x)|K^{(i)}_{\mathcal{D}'}(x)\}$, and let $d \in S_i$; then $\Phi_d(x^{b^{l_i}})|\Psi_i(x^{b^{l_i}})$. Using Proposition 2.1(ii), we have $db^{l_i} \in \mathcal{R}_i$. This means that $db^{l_i}|n_i$; therefore $b^{l_i}|n_i$.

Also for any $s \in \mathcal{R}_i$, $\Phi_s(x)|\Phi_d(x^{b^{l_j}})$ for some $0 \le j \le i$ and $d \in S_j$. Since $d|b$, we have $\Phi_d(x^{b^{l_j}})|(x^{b^{l_i+1}} - 1)$. Hence $\Phi_s(x)|(x^{b^{l_i+1}} - 1)$, so that $s|b^{l_i+1}$. The definition of *l.c.m.* implies that $n_i|b^{l_i+1}$.

(2) It is clear that a product-form is a modulo product-form by ignoring all the modulo actions, and so is the weak product-form by keeping the last modulo.

Theorem 3.5. *Let $\mathcal{D}$ be a modulo product-form derived from the product form $\mathcal{D}'$. Then the polynomial $K_{\mathcal{D}'}(x)$ in* (3.9) *divides $P_{\mathcal{D}}(x)$; hence $\mathcal{D}$ satisfies condition* (P_1) *and is a tile digit set.*

Proof. For each $i = 0, ..., k$, by the definition of n_i, $x^{n_i} - 1 = \prod_{d|n_i} \Phi_d(x)$ and $\Psi_i(x^{b^{l_i}})$ is a product of cyclotomic polynomials, we have $\Psi_i(x^{b^{l_i}}) \mid (x^{n_i} - 1)$. Moreover, we know $\Psi_i(x^{b^{l_i}})|P_{\mathcal{E}_i}(x^{b^{l_i}})$. Hence in view of

$$P_{\mathcal{D}^{(i+1)}}(x) = P_{\mathcal{D}^{(i)}}(x)P_{\mathcal{E}_i}(x^{b^{l_i}}) + (x^{n_i} - 1)Q_{i+1}(x),$$

we conclude that $\Psi_d(x^{b^{l_i}}) \mid P_{\mathcal{D}^{(i+1)}}(x)$.

Note that the definition of n_{i+1} implies that $\Psi_i(x^{b^{l_i}}) \mid (x^{n_{i+1}} - 1)$; this together with the expression of $P_{\mathcal{D}^{(i+2)}}(x)$ implies that $\Psi_i(x^{b^{l_i}}) \mid P_{\mathcal{D}^{(i+2)}}(x)$. Continuing the process, we obtain $\Psi_i(x^{b^{l_i}}) \mid P^{(k)}_{\mathcal{D}}(x)$. This shows that $K_{\mathcal{D}'} = K^{(k)}_{\mathcal{D}'}$ in (3.9) divides $P_{\mathcal{D}}$. As $K_{\mathcal{D}'}$ satisfies (P_1), we have that $\mathcal{D} = \mathcal{D}^{(k)}$ satisfies (P_1) also, and $\mathcal{D}$ is a tile digit set following from Theorem 3.2. □

We end this section by giving an example to illustrate the construction of a modulo product-form digit set.

Example 3.6. Let $b = 12$ and write $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_2 = \{0, 1\} \oplus \{0, 4, 8\} \oplus \{0, 2\}$. Let

$$\mathcal{D}' = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus 12\mathcal{E}_2.$$

Then $\mathcal{D}'$ is a (strict) product-form digit set. Let $\mathcal{D}^{(0)} = \{0, 1\}$; then $n_0 = 2$. To construct $\mathcal{D}^{(1)}$, we first evaluate n_1. Since $1 + x^4 + x^8 = \Phi_3(x^4)$, we have

$$K^{(1)}_{\mathcal{D}'}(x) = \Psi_0(x)\Psi_1(x) = \Phi_2(x)\Phi_3(x^4) = \Phi_2(x)\Phi_3(x)\Phi_6(x)\Phi_{12}(x).$$

Hence $n_1 = 12$. We can choose

$$\mathcal{D}^{(1)} = \{0, 1, 4, 8, 9, 17\} \equiv \mathcal{D}^{(0)} \oplus \mathcal{E}_1 \pmod{12}.$$

Next we observe that $K^{(2)}_{\mathcal{D}'}(x) = \Psi_0(x)\Psi_1(x)\Psi_2(x^{12})$ is given by

$$\Phi_2(x)\Phi_3(x^4)\Phi_{2^2}(x^{12}) = \Phi_2(x)\Phi_3(x)\Phi_6(x)\Phi_{12}(x)\Phi_{16}(x)\Phi_{48}(x).$$

This implies that $n_2 = 48$. We let

$$\mathcal{D} = \{0, 1, 4, 8, 9, 17, 25, 33, 41, 72, 76, 80\} \equiv \mathcal{D}^{(1)} \oplus 12\mathcal{E}_2 \pmod{48}.$$

Then $\mathcal{D}$ is a modulo product-form.

On the other hand, $\mathcal{D}$ is not a product-form. For if so, then $\mathcal{D}$ can only be of the form $\mathcal{E}_0 \oplus 12\mathcal{E}_1$ with $\mathcal{E}_0 \oplus \mathcal{E}_1 \equiv \mathbb{Z}_{12}$. It is direct to check that $\mathcal{E}_0 = \{0, 1, 4, 8, 9, 17\}$ necessarily. But then it cannot be written in the product-form as needed.

It is clear that $\mathcal{D}$ is not a weak product-form either, since taking modulo 12^n with $n \geq 2$ reduces back to the original set which is not a product-form, while taking modulo 12 contains only 6 digits.

4. Higher-order product-forms

The condition (P_1) and the modulo product-form in the last section do not cover all tile digit sets (see Example 4.4). In this regard, we set up a higher-order analog in this section, which will be studied in detail in the following sections.

Let $\Phi_d(x^{b^j})$ be as in the definition of (P_1). Then by Proposition 2.1(ii),

$$\Phi_d(x^{b^j}) = \Phi_{t_1}(x)\dots\Phi_{t_n}(x),$$

and each $\Phi_{t_j}(x)$ is a factor of $P_{\mathcal{D}}(x)$ (by Proposition 3.2). We observe that a more relaxed condition that $\Phi_{t_j}(x^{b^{i_j}})$ is a factor of $P_{\mathcal{D}}(x)$ also suffices for the Kenyon criterion to hold. We formulate this in the following. For $d|b$, let

$$j_1 = j_1(d) := \min\{j : \exists \text{ a factor } \Phi_t(x) \text{ of } \Phi_d(x^{b^j}) \text{ s.t. } \Phi_t(x)|P_{\mathcal{D}}(x)\} \tag{4.1}$$

(and $j_1 = \infty$ if no factor $\Phi_t(x)$ exists). Define

($\mathbf{P}_2$) *For each $d|b, d > 1$, $j_1(d) < \infty$ and for any factor $\Phi_{t_1}(x)$ of $\Phi_d(x^{b^{j_1}})$, there exists $j_2 \geq 0$ (depends on t_1) with $\Phi_{t_1}(x^{b^{j_2}}) \mid P_{\mathcal{D}}(x)$.*

Likewise we can repeat the same procedure of defining $j_2 = j_2(t_1)$ and find j_3 for the factors of $\Phi_{t_1}(x^{b^{j_2}})$ to define (P_3), and inductively for $(P_k), k \geq 3$.

It is clear that $(P_1) \Rightarrow (P_2)$ by putting $j_1 = j$ and $j_2 = 0$. Also $(P_{k-1}) \Rightarrow (P_k)$.

Proposition 4.1. *Suppose $\mathcal{D}$ satisfies (P_k). Then $\mathcal{D}$ is a tile digit set.*

Proof. We will check for the case (P_2) that the Kenyon criterion is fulfilled; the general case follows from the same idea. As in Theorem 3.2, for $m > 0$, there exists $\ell_1 > 1$ with $\ell_1|b$ and $\Phi_{\ell_1}(e^{2\pi i m/b^k}) = 0$. Taking j_1 as in the assumption of (P_2), note that

$$\Phi_{\ell_1}((e^{2\pi i m/b^{j_1+k}})^{b^{j_1}}) = \Phi_{\ell_1}(e^{2\pi i m/b^k}) = 0.$$

Let $a = g.c.d.(m, b^{j_1+k})$ and let $\ell_2 = b^{j_1+k}/a$, $c = m/a$; they are relatively prime. From the above, we have $\Phi_{\ell_1}((e^{2\pi i c/l_2})^{b^{j_1}}) = 0$, which implies that $\Phi_{\ell_2}(x)|\Phi_{l_1}(x^{b^{j_1}})$. The (P_2) assumption implies there exists $j_2 \geq 0$ such that $P_{\mathcal{D}}(x) = \Phi_{\ell_2}(x^{b^{j_2}})Q(x)$ for some polynomial $Q(x)$. Let $j_1 + j_2 + k \geq 1$. Then we have

$$P_{\mathcal{D}}(e^{2\pi i m/b^{j_1+j_2+k}}) = \Phi_{\ell_2}(e^{2\pi i m/b^{j_1+k}})Q(e^{2\pi i m/b^{j_1+j_2+k}}) = 0.$$

This verifies the Kenyon criterion for $\mathcal{D}$, and hence $\mathcal{D}$ is a tile digit set. □

Next we give a concrete class of digit sets that satisfies the (P_k) condition. We regard the product-form and modulo product-form in Definitions 3.1 and 3.4 as 1st-order. We define

Definition 4.2. $\mathcal{D}$ is called a 2nd-order product-form (with respect to b) if

$$\mathcal{D} = \mathcal{G}_0 \oplus b^{l_1}\mathcal{G}_1 \oplus \dots \oplus b^{l_k}\mathcal{G}_k, \tag{4.2}$$

where $0 \leq l_1 \leq l_2 \leq \dots \leq l_k$, $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_k$, and $\mathcal{G}$ itself is a modulo product-form as in Definition 3.4 (possibly in another decomposition different from the $\mathcal{G}_i$).

For the above $\mathcal{G}$, let $S_i^{(2)} = \{s : \Phi_s(x)|P_{\mathcal{G}_i}(x),\ \Phi_s(x)|K_{\mathcal{G}}(x)\}$, where $K_{\mathcal{G}}$ is defined in (3.9). We use the same procedure as in Definition 3.4 to define the 2nd-order

modulo product-form and the corresponding $K_{\mathcal{D}}$. Inductively we can define the kth-order modulo product-form.

Remark. Roughly speaking, we can produce new digit sets as follows: we start with a modulo product-form $\mathcal{G}$, rearrange its digits to form a product, then use it to construct the 2nd-order product-form, and then the 2nd-order modulo product-form.

Theorem 4.3. *Let $\mathcal{D}$ be a kth-order product-form (or a kth-order modulo product-form). Then it satisfies the (P_k) condition and hence a tile digit set.*

Proof. We only prove the case when $\mathcal{D}$ is a 2nd-order product-form; the other cases are similar. Let $d|b$ and $d > 1$. Since $\mathcal{G}$ satisfies (P), by Theorem 3.2, there exists j such that $\Phi_d(x^{b^j})|P_{\mathcal{G}}(x)$. As $P_{\mathcal{G}}(x) = P_{\mathcal{G}_0}(x)\dots P_{\mathcal{G}_k}(x)$, we let i be the first index such that there exists a factor $\Phi_t(x)$ of $\Phi_d(x^{b^j})$ that divides $P_{\mathcal{G}_i}(x)$. Letting j_1 be as in (4.1) corresponding to $\mathcal{D}$, it follows from (4.2) that $j_1 = j_1(d) = j + l_i$.

Now for each factor $\Phi_e(x)$ of $\Phi_d(x^{b^{j_1}})$, by

$$\Phi_d(x^{b^{j_1}}) = \prod_{\Phi_{e'}(x)|\Phi_d(x^{b^j})} \Phi_{e'}(x^{b^{l_i}}) \ ,$$

we have $\Phi_e(x)|\Phi_{e'}(x^{b^{l_i}})$ for some e' in the above product. In view of $\Phi_d(x^{b^j})|P_{\mathcal{G}}(x)$, there exists m such that $\Phi_{e'}(x)|P_{\mathcal{G}_m}(x)$. By the choice of i, we have $m \geq i$. Let $j_2 = j_2(e) = l_m - l_i$. Then

$$\Phi_e(x^{b^{j_2}}) \mid \Phi_{e'}(x^{b^{l_m}}) \quad \text{and} \quad \Phi_{e'}(x^{b^{l_m}}) \mid P_{\mathcal{G}_m}(x^{b^{l_m}}).$$

As $P_{\mathcal{G}_m}(x^{b^{l_m}})|P_{\mathcal{D}}(x)$, this shows that $\Phi_e(x^{b^{j_2}})|P_{\mathcal{D}}(x)$.

The following diagram indicates the implications of the new classes of tile digit sets for a given b:

$$\begin{array}{ccccccc}
1^{\text{st}}\text{-order mpf} & \Rightarrow & 2^{\text{nd}}\text{-order mpf} & \Rightarrow & \dots & & \\
\Downarrow & & \Downarrow & & & & \\
(P_1) & \Rightarrow & (P_2) & \Rightarrow & \dots & \Rightarrow & \mathcal{D} \text{ is a tile digit set}
\end{array}$$

(where mpf means modulo product-form). The (P_k) can be understood better through the tree structure of cyclotomic polynomials to be developed in the next section. We show that the converse of the last implication also holds; i.e., every tile digit set of b must satisfy condition (P_k) for some k. On the other hand, it is still unclear that if a digit set satisfies (P_k) for some k, then the digit set must be a kth-order product-form.

We conclude this section by constructing a non-trivial 2nd-order product-form digit set.

Example 4.4. Let $b = 12$ and

$$\mathcal{D} = \{0,1\} \oplus 2^4\{0,6\} \oplus 2^7 \cdot 3^2\{0,2,4\}.$$

Then $\mathcal{D}$ is a 2nd-order product-form and it satisfies (P_2), but not (P_1).

Indeed by rearranging the terms, we can write $\mathcal{D}$ as

$$\mathcal{D} = \{0,1\} \oplus 12\{0,8\} \oplus (12)^2\{0,16,32\} := \mathcal{G}_0 \oplus 12\mathcal{G}_1 \oplus (12)^2\mathcal{G}_2.$$

It is clear that $\mathcal{G} := \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 = \{0,1,8,9,16,17\} \oplus 12\{0,2\}$ is a 1st-order product-form. Therefore $\mathcal{D}$ is a 2nd-order product-form.

On the other hand, $\mathcal{D}$ does not satisfy (P_1). Suppose otherwise, consider

$$\begin{aligned} P_{\mathcal{D}}(x) &= \Phi_2(x)\Phi_2(x^{6\cdot 2^4})\Phi_3(x^{2^8\cdot 3^2}) \\ &= \Phi_2(x)\cdot\Phi_{2^6}(x)\Phi_{2^6\cdot 3}(x)\cdot\Phi_{3^3}(x)\Phi_{2\cdot 3^3}(x)\dots\Phi_{2^8\cdot 3^3}(x) \end{aligned} \tag{4.3}$$

and for the factor $d = 4$, there exists $j \geq 0$ such that $\Phi_4(x^{12^j})|P_{\mathcal{D}}(x)$. Since $\Phi_{2^{2j+2}}(x)|\Phi_4(x^{12^j})$, hence $\Phi_{2^{2j+2}}(x)|P_{\mathcal{D}}(x)$. From (4.3), we must have $2j+2 = 6$, so that $j = 2$. Thus $\Phi_4(x^{12^2})|P_{\mathcal{D}}(x)$. But this is impossible because

$$\Phi_4(x^{12^2}) = \Phi_{2^6}(x)\Phi_{2^6\cdot 3}(x)\Phi_{2^6\cdot 3^2}(x),$$

and $\Phi_{2^6\cdot 3^2}(x)$ does not divide $P_{\mathcal{D}}(x)$. □

It is also interesting to observe the following:

Example 4.5. If we multiply one more factor $\Phi_{2^6\cdot 3^2}(x)$ to (4.3), then it is the mask polynomial for another tile digit set $\mathcal{D} = \{0,1\} \oplus 12^2\{0,2\} \oplus 12^2\{0,16,32\}$, which is a 1st-order product-form of $b = 12$.

The expression of $\mathcal{D}$ follows from

$$\begin{aligned} P_{\mathcal{D}}(x) &= \Phi_2(x)\cdot\Phi_{2^6}(x)\Phi_{2^6\cdot 3}(x)\Phi_{2^6\cdot 3^2}(x)\cdot\Phi_{3^3}(x)\Phi_{2\cdot 3^3}(x)\dots\Phi_{2^8\cdot 3^3}(x) \\ &= \Phi_2(x)\Phi_2(x^{12^2\cdot 2})\Phi_3(x^{12^2\cdot 2^4})\ . \end{aligned}$$

It is a 1st-order product form as $\{0,1\} \oplus \{0,2\} \oplus \{0,16,32\} \equiv \mathbb{Z}_{12}$. □

We will come back to these two examples in Section 6 to explain some situations.

5. Φ-tree, blocking and kernel polynomials

In this section, we will study the mask polynomial by using a graph-theoretic consideration, and then use it to consider the condition (P_k). We make use of a setup by Protasov on the refinement equations [P]. Let $V_0 = \{\vartheta\}$ be the root,

$$V_k = \{\mathbf{j} = j_k\dots j_1 :\ j_\ell \in \{0,1,\dots,b-1\},\ j_1 \neq 0\}, \quad k \geq 1,$$

and $V = \bigcup_{k\geq 0} V_k$. (We reverse the usual ordering on the index as we are dealing with the integers in the b-adic expansion instead of the decimals.) For any $\mathbf{j} \in V_k,\ k \neq 0$, it has b offsprings $j_{k+1}\mathbf{j}$ (note that by assumption, ϑ has only $b-1$ offsprings in V, which are the elements of V_1). Let E be the set of edges connecting those $\mathbf{j}$ and $j_{k+1}\mathbf{j}$. Then (V,E) is a tree with ϑ as the root, and we call it a *Protasov tree* (associated with b).

For each $\mathbf{j} \in V_k$, we let $m_{\mathbf{j}} = j_k b^{k-1} + \dots + j_2 b + j_1$, which is the b-adic expansion determined by $\mathbf{j}$. Note that $j_1 \neq 0$ by the assumption on the Protasov tree. It follows that there is a one-to-one correspondence between V_k and the set of integers in $\{1,\dots,b^k-1\}$ which are not divisible by b.

We call $B \subset V \setminus \{\vartheta\}$ a *blocking* if it is a finite set and every infinite path starting from ϑ must intersect exactly one element of B. The following criterion is due to Protasov [P] on the refinement equation adjusted to the present situation. For each $\mathbf{j} \in V_k$, we use $e(m_{\mathbf{j}})$ to denote $e^{2\pi i m_{\mathbf{j}}/b^k}$ for simplicity.

Theorem 5.1. *$\mathcal{D}$ is a tile digit set if and only if there is a blocking B such that for any $\mathbf{j} \in B$,*

$$P_{\mathcal{D}}\big(e(m_{\mathbf{j}})\big) = 0. \tag{5.1}$$

(We call such a B a $P_{\mathcal{D}}$-blocking.)

Remark. The Kenyon criterion involves checking all integers $m > 0$, while the Protasov criterion only involves checking finitely many m's (although the finding of the blocking set B is not direct). The seemingly weaker tree criterion actually implies the Kenyon criterion in the following way: Suppose B is such a blocking. Let $m \geq 1$ be an integer such that $b \nmid m$. We write m in b-adic expansion as $m = m_{\mathbf{j}}$ with $\mathbf{j} = j_t \cdots j_1 \in V$. If $\mathbf{j}$ has an ancestor $\mathbf{i} = j_k \cdots j_1$ belonging to B, then we choose this k for the b^k in the Kenyon criterion; otherwise there is an ℓ so that $\mathbf{i} = \underbrace{0 \cdots 0}_{\ell} j_t \ldots j_1 \in B$, and then $k = \ell + t$ satisfies the Kenyon criterion.

In the following we will convert the Protasov tree into a tree of cyclotomic polynomials, which is more tractable to study the structure of the tile digit sets.

For $\mathbf{j} \in V_k$, we let $a_{\mathbf{j}} = \text{g.c.d.}(m_{\mathbf{j}}, b^k)$, $d = d_{\mathbf{j}} = b^k/a_{\mathbf{j}}$, and associate with $\mathbf{j}$ a Φ_d so that $\Phi_d(e^{2\pi i m_{\mathbf{j}}/b^k}) = 0$. In this way, we define a map τ from V to the set of all cyclotomic polynomials by mapping $\mathbf{j}$ to $\Phi_{d_{\mathbf{j}}}$. (By convention $\tau(\vartheta) = \vartheta$.)

Letting Φ_d be in the range of τ, we define

$$C_d = \{\mathbf{i} : \tau(\mathbf{i}) = \Phi_d\} = \tau^{-1}(\Phi_d);$$

if $\mathbf{j} \neq \vartheta$ is such that $\tau(\mathbf{j}) = \Phi_d$, we define

$$L_{\mathbf{j}} = \{\ell \mathbf{j} : 0 \leq \ell \leq b-1\} \text{ and } C_d^* = \bigcup_{\mathbf{i} \in C_d} L_{\mathbf{i}}.$$

For $\mathbf{i} = i_k ... i_1$, we denote $\mathbf{i}^- = i_{k-1} ... i_1$. The following proposition gives some basic properties of the map τ.

Proposition 5.2. *With the above notation, we have*

(i) if $\tau(\mathbf{i}) = \tau(\mathbf{j})$, then $\mathbf{i}$ and $\mathbf{j}$ lie in the same V_k, and $\tau(\mathbf{i}^-) = \tau(\mathbf{j}^-)$;

(ii) $C_d \subset V_k$ for some k; $\#C_d = \deg \Phi_d$ and

$$\Phi_d(x) = \prod\nolimits_{\mathbf{j} \in C_d} \big(x - e(m_{\mathbf{j}})\big); \tag{5.2}$$

(iii) if $\tau(\mathbf{j}) = \Phi_d$, then $\tau(L_{\mathbf{j}}) = \{\Phi_e : \Phi_e(x) | \Phi_d(x^b)\}$ and

$$\Phi_d(x^b) = \prod\nolimits_{\mathbf{j}' \in C^*} \big(x - e(m_{\mathbf{j}'})\big). \tag{5.3}$$

Proof. (i) Suppose that $\mathbf{i} \in V_{k_1}$, $\mathbf{j} \in V_{k_2}$ with $k_1 < k_2$ and $\tau(\mathbf{i}) = \tau(\mathbf{j}) = \Phi_d$. Let $a_{\mathbf{i}} = \text{g.c.d.}(b^{k_1}, m_{\mathbf{i}})$, $a_{\mathbf{j}} = \text{g.c.d.}(b^{k_2}, m_{\mathbf{j}})$. Then

$$d \;=\; b^{k_1}/a_{\mathbf{i}} \;=\; b^{k_2}/a_{\mathbf{j}}.$$

Hence $a_{\mathbf{j}} = b^{k_2 - k_1} a_{\mathbf{i}}$, so that b divides $m_{\mathbf{j}}$, which implies $j_1 = 0$. This contradicts the assumption that $j_1 \neq 0$ for $\mathbf{j} \in V$.

To prove the second part, we let $\mathbf{i} = i_k \cdots i_1$ and $\mathbf{j} = j_k \cdots j_1$. Since $\tau(\mathbf{i}) = \tau(\mathbf{j})$, we have $\text{g.c.d.}(b^k, m_{\mathbf{i}}) = \text{g.c.d.}(b^k, m_{\mathbf{j}})$. This is equivalent to

$$m_{\mathbf{j}} = c_1 b^k + c_2 m_{\mathbf{i}} \quad \text{and} \quad m_{\mathbf{i}} = c_1' b^k + c_2' m_{\mathbf{j}}$$

for some integers c_1, c_2, c_1', c_2'. Hence

$$m_{\mathbf{j}} = (c_1 b) b^{k-1} + c_2 m_{\mathbf{i}} \quad \text{and} \quad m_{\mathbf{i}} = (c_1' b) b^{k-1} + c_2' m_{\mathbf{j}},$$

so that $\text{g.c.d.}(b^{k-1}, m_{\mathbf{i}}) = \text{g.c.d.}(b^{k-1}, m_{\mathbf{j}})$. So $\text{g.c.d.}(b^{k-1}, m_{\mathbf{i}'}) = \text{g.c.d.}(b^{k-1}, m_{\mathbf{j}'})$.

(ii) $C_d \subset V_k$ for some k follows from (i). From this, we have $C_d = \{\mathbf{j} \in V_k : \Phi_d(e(m_{\mathbf{j}})) = 0\}$, so that $\#C_d \leq \deg \Phi_d$. On the other hand, let $a_{\mathbf{j}} = \text{g.c.d.}(b^k, m_{\mathbf{j}})$.

Then $d = b^k/a_{\mathbf{j}}$ for some $\mathbf{j} \in C$. Note that $\deg \Phi_d$ equals the Euler-phi function $\varphi(d)$, i.e.,

$$\deg \Phi_d = \#\{r : 1 \le r < d,\ \text{g.c.d.}(r, d) = 1\}. \tag{5.4}$$

Hence for each such r, we have $r < d = b^k/a_{\mathbf{j}}$. This implies that $ra_{\mathbf{j}} < b^k$ and we can write $ra_{\mathbf{j}} = m_{\mathbf{j}'}$ for a unique $\mathbf{j}' \in V_k$. In this case, $e^{2\pi i r/d} = e(m_{\mathbf{j}'})$. We have shown that each r in (5.4) corresponds to exactly one $\mathbf{j}' \in V_k$ with $\tau(\mathbf{j}') = \Phi_d$. Hence $\deg \Phi_d \le \#C_d$.

For $\mathbf{j} \in C_d \subset V_k$, $e(m_{\mathbf{j}}) := e^{2\pi i m_{\mathbf{j}}/b^k}$ is a root of $\Phi_d(x)$. This implies that the product in (5.2) is a factor of $\Phi_d(x)$. Since $\deg \Phi_d = \#C_d$ and the polynomials involved are monic, identity (5.2) must hold.

(iii) Given $\mathbf{j}$ such that $\tau(\mathbf{j}) = \Phi_d$, by noting that $e^{2\pi i(\ell b^k + m_{\mathbf{j}})/b^{k+1}}$ with $0 \le \ell \le b-1$ and $\mathbf{j} \in C_d$ are roots of $\Phi_d(x^b)$, we have $\tau(L_{\mathbf{j}}) \subset \{\Phi_e : \Phi_e(x) | \Phi_d(x^b)\}$.

Conversely, letting $a_{\mathbf{j}} = \text{g.c.d.}(b^k, m_{\mathbf{j}})$, we have $m_{\mathbf{j}} = ra_{\mathbf{j}}$, $b^k = da_{\mathbf{j}}$ and r, d are relatively prime. We then write $b = sd'$, where d' contains all prime factors appearing in d. We note that

$$\frac{\ell b^k + m_{\mathbf{j}}}{b^{k+1}} = \frac{\ell d a_{\mathbf{j}} + r a_{\mathbf{j}}}{b d a_{\mathbf{j}}} = \frac{\ell d + r}{bd} = \frac{\ell d + r}{s d' d} \tag{5.5}$$

and $\text{g.c.d.}(\ell d + r, d'd) = 1$. Now, given any $\Phi_e(x) | \Phi_d(x^b)$, $e = s'dd'$, where s' is a factor of s (by Proposition 2.1(ii)). Since $\text{g.c.d.}(d, s) = 1$, $\{\ell d + r : 0 \le \ell \le b-1\}$ contains a complete residue set (mod s). This implies that there exists ℓ_0 such that $\ell_0 d + r \equiv s/s' (\text{mod } s)$ and so that $\frac{\ell_0 d + r}{s} = \frac{c}{s'}$ with $\text{g.c.d.}(c, s') = 1$. By (5.5),

$$\frac{\ell_0 b^k + m_{\mathbf{j}}}{b^{k+1}} = \frac{c}{s' d' d}$$

and $\text{g.c.d.}(c, s'd'd) = 1$. This implies that $\tau(\ell_0 m_{\mathbf{j}}) = \Phi_e$, and $\{\Phi_e : \Phi_e(x) | \Phi_d(x^b)\} \subset \tau(L_{\mathbf{j}})$ follows.

The identity (5.3) follows from $e^{2\pi i(\ell b^k + m_{\mathbf{j}})/b^{k+1}}$ with $0 \le \ell \le b-1$ and $\mathbf{j} \in C_d$ are roots of $\Phi_d(x^b)$, and the degree of the two polynomials in the identity is $b \deg(\Phi_d)$. □

By this proposition, we can associate a Protasov tree with a *tree of cyclotomic polynomials (with respect to b)*, which we call a *Φ-tree* (see Figure 1). The set of vertices of this tree at level 1 are $\tau(V_1)$. The offsprings of Φ_d are the cyclotomic factors of $\Phi_d(x^b)$; they are determined by Proposition 2.1(ii). By (iii), an edge joining $\mathbf{j}$ to its offspring $\ell\mathbf{j}$ corresponds to an edge joining Φ_d and a cyclotomic factor of $\Phi_d(x^b)$. We can see easily from (iii) that the map τ is surjective on the Φ-tree. Moreover, all Φ_d in the tree are different by Proposition 2.2.

A blocking B in V is called a *symmetric blocking* if $\tau^{-1}(\tau(\mathbf{j})) \subset B$ for every $\mathbf{j} \in B$.

Lemma 5.3. *Let B be a symmetric blocking in the Protasov tree; then $\mathcal{N}_B := \tau(B)$ is a blocking in the Φ-tree. Conversely, if $\mathcal{N}$ is a blocking in the Φ-tree, then $B_{\mathcal{N}} = \tau^{-1}(\mathcal{N})$ is a symmetric blocking in the Protasov tree.*

Proof. It is clear that $\mathcal{N}_B$ is a finite set. Let $\Gamma = \{\vartheta, \Phi_{d_1}, \Phi_{d_2}, \cdots\}$ be an infinite path in the Φ-tree (i.e. $\Phi_{d_k}(x)$ is a factor of $\Phi_{d_{k-1}}(x^b)$). Since B is symmetric, by Proposition 5.2(iii), there exists an infinite path $\gamma = \{\vartheta, \mathbf{j}_1, \mathbf{j}_2, \cdots\}$ in the Protasov tree such that $\tau(\mathbf{j}_k) = \Phi_{d_k}$ for all k. Since B is a blocking, we can find a vertex $\mathbf{j}_r$

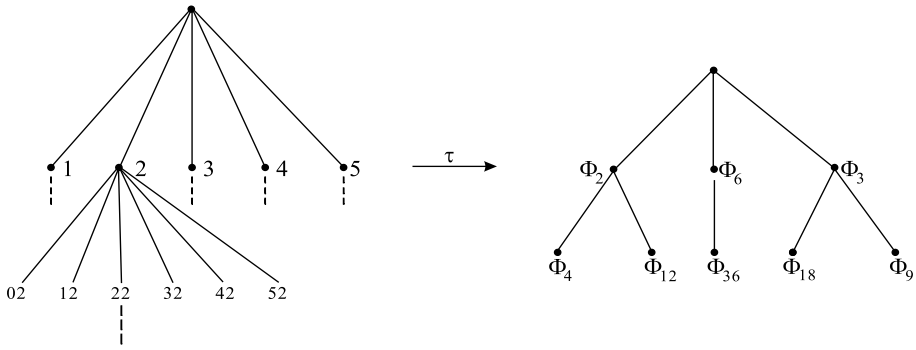

FIGURE 1. An illustration of the Protasov tree and the associated Φ-tree for the case $b = 6$. On the first level, $\tau(1) = \tau(5) = \Phi_6$; $\tau(2) = \tau(4) = \Phi_3$; $\tau(3) = \Phi_2$. On the second level, $\Phi_3(x^6) = \Phi_9(x)\Phi_{18}(x)$; hence Φ_3 has two descendants Φ_9 and Φ_{18}. The map τ acts on the descendant of 2 as: $\tau(02) = \tau(22) = \tau(32) = \Phi_9$; $\tau(12) = \tau(32) = \tau(52) = \Phi_{18}$.

on γ such that $\mathbf{j}_r \in B$. Hence $\Phi_{d_r} \in \mathcal{N}_B$. That Γ hits $\mathcal{N}_B$ exactly once is an easy consequence from the blocking B in V.

Conversely, let $\gamma = \{\vartheta, \mathbf{j}_1, \mathbf{j}_2, ...\}$ be an infinite path in V. Then Proposition 5.2(iii) implies that $\tau(\gamma) = \{\vartheta, \tau(\mathbf{j}_1), ...\}$ is a path in the Φ-tree. Hence, there is a unique $\tau(\mathbf{j}_k) = \Phi_d \in \mathcal{N} \cap \tau(\gamma)$. It follows that $\mathbf{j}_k \in B_{\mathcal{N}}$, i.e., γ meets $B_{\mathcal{N}}$ at $\mathbf{j}_k$ and it is the unique point by Proposition 5.2(i). This shows that $B_{\mathcal{N}}$ is a blocking, and it is clearly symmetric. □

Definition 5.4. Letting $\mathcal{N}$ be a blocking of the Φ-tree, we call

$$K(x) = \prod_{\Phi_d \in \mathcal{N}} \Phi_d(x)$$

a kernel polynomial (with respect to b).

Note that the most basic kernel polynomial is $K(x) = \prod_{d|b,d>1} \Phi_d(x)$. We can generate all the kernel polynomials by adopting the following simple procedure step by step, starting from $K_{\tau(\mathcal{E})}(x)$: let $K_{\mathcal{N}}$ be as in the definition, and let $\Phi_d \in \mathcal{N}$. The descendants of Φ_d, denoted by $\mathcal{N}_d$, are the factors of $\Phi_d(x^b)$. It follows that $\mathcal{N}' = (\mathcal{N} \setminus \{\Phi_d\}) \cup \mathcal{N}_d$ is again a blocking, and the corresponding kernel polynomial is

$$K_{\mathcal{N}'}(x) = \frac{\Phi_d(x^b) K_{\mathcal{N}}(x)}{\Phi_d(x)} \ . \tag{5.6}$$

Lemma 5.5. *A minimal $P_{\mathcal{D}}$-blocking (i.e., smallest cardinality) is symmetric.*

Proof. Let B be a minimal $P_{\mathcal{D}}$-blocking and let $\mathbf{j}$ be an element of B. Suppose there exists $\mathbf{i}$ such that $\tau(\mathbf{i}) = \tau(\mathbf{j})$, but $\mathbf{i} \notin B$. If there is no ancestor of $\mathbf{i}$ belonging to B, then we can form a B' by including $\mathbf{i}$ into B and dropping all its offsprings in B. This B' is a $P_{\mathcal{D}}$-blocking since $\tau(\mathbf{i}) = \tau(\mathbf{j})$, and has smaller cardinality than B, so it is impossible by the minimality assumption. Hence there is an ancestor $\mathbf{i}'$ of $\mathbf{i}$ such that $\mathbf{i}' \in B$. Let $\mathbf{j}'$ be the ancestor of $\mathbf{j}$ such that $|\mathbf{j}'| = |\mathbf{i}'|$. Then by Proposition 5.2(i), $\tau(\mathbf{i}') = \tau(\mathbf{j}')$. By dropping all the offsprings of $\mathbf{j}'$ and adding $\mathbf{j}'$

to B, we obtain a smaller $P_{\mathcal{D}}$-blocking. We see that it is again impossible, and the lemma follows. □

Our main conclusion is the following characterization of the tile digit sets.

Theorem 5.6. *Let $b > 1$ be an integer and let $\mathcal{D}$ be a digit set with $\#\mathcal{D} = b$. Then the following are equivalent:*

(i) $\mathcal{D}$ is a tile digit set of b;

(ii) there is a symmetric $P_{\mathcal{D}}$-blocking in the Protasov tree of b;

(iii) there is a blocking $\mathcal{N}$ in the Φ-tree of b such that

$$K(x) := \prod_{\Phi_d \in \mathcal{N}} \Phi_d(x)$$

is a kernel polynomial and $K(x)|P_{\mathcal{D}}(x)$;

(iv) $P_{\mathcal{D}}(x)$ satisfies condition (P_k) for some $k \geq 1$.

Proof. $(i) \Rightarrow (ii)$. Since we can always find a minimal blocking, the assertion follows from the Protasov criterion (Theorem 5.1) and Lemma 5.5.

$(ii) \Rightarrow (iii)$. For a symmetric blocking B in (ii), Lemma 5.3 implies that $\mathcal{N}_B$ is a blocking in the Φ-tree. For any $\Phi_d \in \mathcal{N}_B$, $\Phi_d = \tau(\mathbf{j})$ for some $\mathbf{j} \in B$. By (ii), $P_{\mathcal{D}}\big(e(m_{\mathbf{j}})\big) = 0$; this shows that $\Phi_d(x)|P_{\mathcal{D}}(x)$. Since $\mathcal{N}_B$ is a blocking, $K(x)$ is a kernel polynomial by definition, and clearly it divides $P_{\mathcal{D}}(x)$.

$(iii) \Rightarrow (iv)$. We can localize the (P_k) condition by checking (P_{k_d}) for each $d|b$, then take $k = \max\{k_d : d|n\}$. Note that by assumption, $P_{\mathcal{D}}(x) = K(x)Q(x)$; hence it suffices to prove that for each $d|b$, $K(x)$ satisfies some (P_{k_d}) condition.

We trace through the steps of producing $K(x) := K_{\mathcal{N}}(x)$ as in (5.6). For a fixed $d|b$, there exists a smallest j_1 such that some factors of $\Phi_d(x^{b^{j_1}})$ divide $K(x)$ (as $\mathcal{N}$ is a blocking). If all factors of $\Phi_d(x^{b^{j_1}})$ divide $K(x)$, then $K(x)$ satisfies (P_1) for d.

Otherwise, there exists at least one factor in $\Phi_d(x^{b^{j_1}})$ that does not divide $K(x)$, say e_1. We repeat the same procedure on Φ_{e_1} as for Φ_d and find j_2 such that some factors in $\Phi_{e_1}(x^{b^{j_2}})$ will divide $K(x)$. We check whether all factors of $\Phi_{e_1}(x^{b^{j_2}})$ will divide $K(x)$, i.e, to satisfy condition (P_2). If not, we continue on with the same procedure. Since the blocking $\mathcal{N}$ is a finite set, the process must stop finally. This factor d must satisfy some (P_{k_d}) condition.

$(iv) \Rightarrow (i)$. This has been proved in Proposition 4.1. □

Remarks. (1) For the 1st-order product-form $\mathcal{D}$, we have seen that $K_{\mathcal{D}}$ defined in (3.9) is a polynomial satisfying (P_1) and it divides $P_{\mathcal{D}}(x)$; it is a kernel polynomial for $\mathcal{D}$. Similarly, for the $\mathcal{D}$ in (4.2), the polynomial $K_{\mathcal{D}}$ is also a kernel polynomial of $\mathcal{D}$; it satisfies (P_2).

(2) For a tile digit set $\mathcal{D}$, $P_{\mathcal{D}}$ may contain more than one kernel polynomial. For example, let $P_{\mathcal{D}}(x)$ be the mask polynomial of $\mathcal{D}$ in Example 4.5. It contains the factors $\Phi_{2^7 3^3}(x)$, $\Phi_{2^8 3^3}(x)$ and $\Phi_{2^6.3^2}(x)$. We write

$$P_{\mathcal{D}}(x) = K_1(x)\Phi_{2^7 3^3}(x)\Phi_{2^6.3^2}(x) = K_2(x)\Phi_{2^7 3^3}(x)\Phi_{2^8 3^3}(x)$$

with $K_1(x), K_2(x)$ defined in an obvious way. It is seen that $K_1(x)\Phi_{2^7 3^3}(x)$ is the mask polynomial of the tile digit set in Example 4.4. This $K_1(x)$ forms a blocking in the Φ-tree of $b = 12$ and is thus a kernel polynomial of $\mathcal{D}$ (the blocking property can be checked easily using the diagram of Figure 2). Moreover, note that both factors $\Phi_{2^6.3^2}(x)$ and $\Phi_{2^8 3^3}(x)$ are on the same path in the Φ-tree, and the path

has no branches. We can interchange the two factors to form $K_2(x)$, which is still a blocking, and therefore is another kernel polynomial of $\mathcal{D}$.

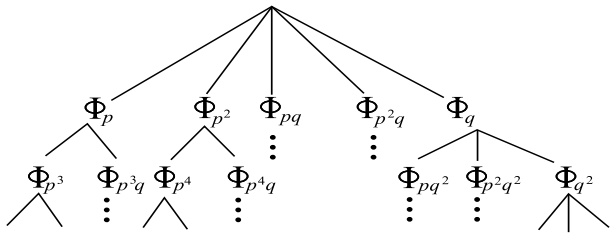

FIGURE 2. The Φ-tree for p^2q.

(3) It is possible that a kernel polynomial $K(x)$ does not generate any tile digit set at all. For example, let $b = pq$ and consider

$$K(x) = \Phi_p(x)\Phi_q(x)\Phi_{p^kq^k}(x), \quad k \geq 1. \tag{5.7}$$

It is clear that $K(x)$ is a blocking (see Figure 1). For $k = 1$, $K(x)$ corresponds to $\{0, \ldots, b-1\}$. However for $k > 1$, $K(x)$ cannot generate a tile digit set (i.e., $K(x)$ does not divide $P_{\mathcal{D}}(x)$ for any tile digit set $\mathcal{D}$. To see this, we note that from [LR] the weak product-form characterization implies that the kernel polynomial can only be the form $\Phi_p(x)\Phi_{q^\ell}(x^{p^\ell})$. Now from (5.7) the prime power spectrum is $\{p, q\}$. Hence, $\ell = 1$ and k in (5.7) can only be 1 in order to obtain a tile digit set for $b = pq$.

It is not clear which kernel polynomials can generate a tile digit set. A complete answer is known only for $b = p^\alpha$ (every kernel polynomial can generate tile digit sets) and pq (see Remark 3). For the more general case that $b = p^\alpha q$, we can use Theorem 5.6 and the technique of integer tiling in [CM] to determine all the kernel polynomials that generate tile digit sets. Moreover, we will give a partial answer to $b = p^\alpha q^\beta$. These details will be given in [LLR].

In Section 4, we have shown that a kth-order modulo product-form satisfies the (P_k) condition. Actually for $b = p^\alpha q$, the converse is also true. This is contained in our classification of the kernel polynomials for $b = p^\alpha q$ in [LLR]. In here we provide a simple sufficient condition to the kernel polynomial that generates tile digit sets.

Proposition 5.7. *Let $K(x)$ be a kernel polynomial associated with b and let $Q(x)$ be a polynomial in $\mathbb{Z}[x]$ with $Q(1) = 1$. If $P(x) = K(x)Q(x)$ has non-negative coefficients, then $P(x) = P_{\mathcal{D}}(x)$ for some tile digit set $\mathcal{D}$.*

Proof. Write $P(x)$ explicitly as

$$P(x) = a_0 + a_1x^{d_1} + \cdots + a_mx^{d_m},$$

where $a_j \geq 1$ are integers and $\sum_{\ell=1}^m a_\ell = P(1) = K(1)Q(1) = b$. Hence $m \leq b$.

Let $\mathcal{D} = \{0 = d_0, d_1, \ldots, d_m\}$. We claim that $m = b$. Supposing $m < b$, we note that $(b, \mathcal{D})$ define an IFS (as in (1.1)) and a self-similar set $T(b, \mathcal{D})$. Let

$$p_j = \frac{a_j}{\sum_{\ell=1}^m a_\ell}$$

be a set of probability on the IFS. It induces a self-similar measure μ with support on $T(b, \mathcal{D})$ [H]. Note that $P(x)$ is the mask polynomial of μ, and $K(x)$ is a kernel

polynomial. They imply that $P(x)$ satisfies the Protasov criterion for a refinement equation [P]. It follows that μ is absolutely continuous, and its support $T(b,\mathcal{D})$ has positive Lebesgue measure. However, $\#\mathcal{D} = m < b$ implies that $T(b,\mathcal{D})$ has zero Lebesgue measure. This is a contradiction. Hence $m = b$. This means that a_j must all be 1 and $\mathcal{D}$ must be a tile digit set with $P(x) = P_{\mathcal{D}}(x)$. □

6. Remarks and open questions

Our consideration in this paper is in terms of tile digit sets; in fact, many of the theorems also hold for the absolutely continuous self-similar measures. Let $\mathcal{D} \subset \mathbb{Z}^+$ such that $\mathcal{D} = m$ (may not equal b), and let $\phi_d(x) = b^{-1}(x+d)$ be the corresponding affine map. It is well known that there exists a unique probability measure of equal weight $\mu = \mu(b,m,\mathcal{D})$ which satisfies

$$\mu(E) = \sum_{d\in\mathcal{D}} \frac{1}{m}\mu(\phi_d^{-1}(E)), \tag{6.1}$$

for any Borel set E.

A natural question is to determine when μ is absolutely continuous with respect to the Lebesgue measure. It is known that the general criterion of Kenyon and Protasov for the mask polynomial of $\mathcal{D}$ is necessary and sufficient for μ to be absolutely continuous ([DFW], [P]). We can modify Theorem 2.4 and Theorem 5.6 as follows:

Let $\mu = \mu(b,\mathcal{D})$ be the self-similar measure defined in (6.1)*. Then:*

(i) If μ is absolutely continuous with respect to the Lebesgue measure, then Theorem 2.4 *holds except for the uniqueness of the prime-power spectrum.*

(ii) If μ is absolutely continuous, then the equivalence of Theorem 5.6 *holds.*

It is easy to show that if the above μ is absolutely continuous, then $m \geq b$. As an application, we prove the following interesting proposition.

Proposition 6.1. *If $\mu = \mu(b,m,\mathcal{D})$ is absolutely continuous, then b divides m.*

Proof. Let $b = p_1^{\alpha_1}...p_k^{\alpha_k}$ be its unique prime factorization. If $\mu(b,m,\mathcal{D})$ is absolutely continuous, then Theorem 2.4 shows that for each p_j there exists $\{a_{j,\ell}\}_{\ell=0}^{\alpha_j-1}$ with $a_{j,\ell} \equiv \ell (\mathrm{mod}\ \alpha_j)$ and $\Phi_{p_j^{a_{j,\ell}}}(x)$ divides $P_{\mathcal{D}}(x)$. Let

$$S(x) = \prod_{j=1}^{k}\prod_{\ell=0}^{\alpha_j-1} \Phi_{p_j^{a_{j,\ell}}}(x).$$

We have $P_{\mathcal{D}}(x) = S(x)Q(x)$ for some integral polynomial $Q(x)$. Note that the cyclotomic factors are all distinct. By Proposition 2.1(iii) we have $S(1) = p_1^{\alpha_1}...p_k^{\alpha_k} = b$. Hence,

$$m = P_{\mathcal{D}}(1) = S(1)Q(1)$$

and $Q(1)$ is an integer since $Q(x)$ is integral polynomial. This completes the proof. □

There are also many other unanswered basic questions on the self-affine tiles and the structure of the tile digit sets. So far most of our consideration is on $\mathbb{R}^1$, and very little is known on $\mathbb{R}^s$. One of the difficulties is that there is no analogue for the cyclotomic polynomials in higher dimensions. In [HL], it was proved that for $|\det(A)| = \#\mathcal{D} = p$ a prime, if the linear span span$(\mathcal{D})$ is $\mathbb{R}^s$, then $T(A,\mathcal{D})$ is a self-affine tile if and only if $\mathcal{D}$ is a complete residue set with respect to A.

Q1: Can the condition $\mathrm{span}(\mathcal{D}) = \mathbb{R}^s$ be omitted in the above characterization?

It is easy to generalize the product-form to higher dimension [LW3]. It will be interesting to set up the following:

Q2: Find the corresponding generalization for condition (P_k) and the modulo product-forms in $\mathbb{Z}^s$, and use them to study the tile digit sets.

Our study on the structure of the tile digit sets is closely related to the spectral set problems. Recall that a closed subset $\Omega \subset \mathbb{R}^s$ is called a *spectral set* if $L^2(\Omega)$ admits an exponential orthonormal basis $\{e^{2\pi i\langle\lambda,\cdot\rangle}\}_{\lambda\in\Lambda}$. The well-known Fuglede conjecture asserted that Ω is a spectral set if and only if it is a translational tile. Recently Tao [T] gave a counterexample that the conjecture fails in $\mathbb{R}^s$ with $s \geq 5$, and the example was eventually modified to $s \geq 3$ [KM]. Now the conjecture remains open in $\mathbb{R}^1$ and $\mathbb{R}^2$ (see, e.g., [L], [LaW]). For the more restrictive class of self-affine tiles, the following is still open:

Q3: Supposing that $T(A, \mathcal{D})$ is a self-affine tile in $\mathbb{R}^s$, is $T(A, \mathcal{D})$ a spectral set?

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