EXTENSIONS OF FINITE CYCLIC GROUP ACTIONS
ON NON-ORIENTABLE SURFACES

E. BUJALANCE, F.J. CIRRE, AND M.D.E. CONDER

Abstract. Conditions are derived for the extension of an action of a cyclic group on a compact non-orientable surface to the faithful action of some larger group on the same surface. It is shown that such a cyclic action is realised by means of a non-maximal NEC signature, then the action always extends. The special case where the full automorphism group is cyclic of the largest possible order (for given genus) is also considered. This extends previous work by the authors for group actions on orientable surfaces. In addition, the smallest algebraic genus of a non-orientable surface on which a given cyclic group acts as the full automorphism group is determined.

Introduction

A natural extension of the definition of a compact Riemann surface, which is orientable, is to allow dianalytic transition functions — that is, functions which are either analytic or the composite of complex conjugation with an analytic function. This applies to compact non-orientable surfaces, which have been called non-orientable Riemann surfaces (by Singerman in [14]), or non-orientable Klein surfaces without boundary (using the terminology introduced by Alling and Greenleaf in [1]). Under the well-known correspondence between compact Klein surfaces and real algebraic curves explained in [1], the non-orientable surfaces we will deal with correspond to the so-called purely imaginary real algebraic curves — that is, real algebraic curves with no real points.

An automorphism of a non-orientable surface $S$ is a homeomorphism $v : S \to S$ which is dianalytic in local coordinates. The group of all such homeomorphisms is called the automorphism group of $S$, and denoted by Aut($S$). A group $G$ is said to act on $S$ if it is a subgroup of Aut($S$). Here we consider the question of whether a given cyclic group acting on $S$ is the full group of all automorphisms of $S$. The same question for Riemann surfaces was considered in [5].

A compact non-orientable surface $S$ of algebraic genus $g \geq 2$ can be represented as the quotient $H/\Lambda$ of the hyperbolic plane $H$ under the action of a fixed point-free non-Euclidean crystallographic (NEC) group $\Lambda$. A finite group $G$ then acts as a group of automorphisms of $S$ if and only if $G$ is isomorphic to $\Gamma/\Lambda$ for some NEC group $\Gamma$ containing $\Lambda$ as a normal subgroup. If $G \neq \text{Aut}(S)$, then $\Gamma$ is properly contained with finite index in some other NEC group $\Gamma'$ which also normalises $\Lambda$.
The converse holds as well, and accordingly, the above question is closely related to the finite-index extendability of NEC groups.

The extendability of $\Gamma$ depends mainly on the geometry of a fundamental region for $\Gamma$. In particular, although $\Gamma$ could be contained in an NEC group $\Gamma'$ normalising $\Lambda$, the group $\Gamma$ might be abstractly isomorphic to a maximal NEC group — that is, to a group which is not contained as a subgroup of finite index in any other NEC group. If this happens and $f : \Gamma \to f(\Gamma)$ is such an isomorphism, then $f(\Gamma)/f(\Lambda)$ is the group of all automorphisms of the surface $S = H/f(\Lambda)$. For some signatures, however, it can happen that every NEC group $\Gamma$ with signature $\sigma$ is properly contained in another NEC group $\Gamma'$ with finite index, and the dimensions of the Teichmüller spaces of $\Gamma$ and $\Gamma'$ coincide. Such a signature $\sigma$ is called non-maximal, and the pair $(\sigma, \sigma')$ of signatures of $\Gamma$ and $\Gamma'$ is called a normal pair if $\Gamma$ is normal in $\Gamma'$, and non-normal otherwise.

The same question for Fuchsian signatures was analysed by Greenberg in [9] and answered completely by Singerman in [15]. Using the list of non-maximal Fuchsian signatures produced by Singerman, the first author produced a complete list of normal pairs of NEC signatures in [3], and subsequently Estévez and Izquierdo gave the list of the non-normal ones in [8]. These lists play a key role in this paper, since it follows from the above remarks that if a finite group $G$ can be written as $\Gamma/\Lambda$ and the signature of $\Gamma$ does not appear in either list, then $G$ is the group of all automorphisms of some surface homeomorphic to $H/\Lambda$. If, on the other hand, the signature of $\Gamma$ is non-maximal, then the action of $G$ in any surface homeomorphic to $H/\Lambda$ could possibly be extended. The main result of this paper shows that such an extension always occurs when the group $G$ is cyclic.

The paper is organised as follows. In Section 1 we recall basic facts and introduce notation for NEC groups and non-orientable surfaces to be used in the paper. In Section 2 we analyse the 15 normal pairs of NEC signatures (from Table 1 in Section 1) in terms of the possible extension of the action of a cyclic group. The analysis yields our main result, namely Theorem 2.1. In Section 3 we describe some applications concerning non-orientable surfaces whose full automorphism group is cyclic of largest possible order. A further application is considered in Section 4, where we determine for each $n$ the smallest algebraic genus of a non-orientable surface on which the cyclic group of order $n$ acts as the full automorphism group of the surface. We call this the full cross-cap genus of the cyclic group, following the definition by May [13] of the symmetric cross-cap number as the smallest topological genus of a non-orientable surface on which the cyclic group acts effectively. We also show that for each integer $n > 1$, there exists some $g_0$ such that $C_n$ is the full automorphism group of some non-orientable surface of algebraic genus $g$ for every $g \geq g_0$. Finally, we give a table showing the spectrum of all algebraic genera of the non-orientable surfaces on which $C_n$ acts as the full automorphism group, for $2 \leq n \leq 10$.

1. Preliminaries

We will use combinatorial theory of non-Euclidean crystallographic groups (which are known as NEC groups, for short). We refer the reader to [6] Section 0.2 for a general account of this topic, but we briefly recall the main facts we need and fix the notation we will use here. An NEC group is a cocompact discrete subgroup of the group of orientation-preserving or -reversing isometries of the hyperbolic plane.
The defining relations are as follows:

\begin{equation}
\sigma(\Gamma) = (\gamma; \pm; [m_1, \ldots, m_r]; \{ (n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k}) \}).
\end{equation}

The integers \( m_1, \ldots, m_r \) are called proper periods; each bracket \( (n_{ij}, \ldots, n_{is_j}) \) is a period cycle and the integers \( n_{ij} \) are called link periods. An empty set of proper periods (where \( r = 0 \)) will be denoted by \([-] \), an empty period-cycle (where \( s_i = 0 \)) by \((-)\), and the fact that \( \sigma \) has no period-cycles (that is, where \( k = 0 \)) by \(-\).

The signature of \( \Gamma \) gives some topological features of the projection \( H \rightarrow H/\Gamma \), and also provides a presentation of \( \Gamma \). The generators are as follows:

- elliptic elements \( x_i \), for \( 1 \leq i \leq r \);
- reflections \( c_{i0}, \ldots, c_{is_i} \), for \( 1 \leq i \leq k \);
- orientation-preserving elements \( e_i \), for \( 1 \leq i \leq k \);
- hyperbolic elements \( a_i \) and \( b_i \), for \( 1 \leq i \leq \gamma \), if the sign is +;
- glide reflections \( d_i \), for \( 1 \leq i \leq \gamma \), if the sign is -.

The defining relations are as follows:

- \( x_i^{m_i} = 1 \) for \( 1 \leq i \leq r \);
- \( c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1 \) for \( 1 \leq j \leq s_i \), for \( 1 \leq i \leq k \);
- \( c_{is_i} = e_i c_{i0} e_i^{-1} \) for \( 1 \leq i \leq k \);
- \( x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_{\gamma} b_{\gamma} a_{\gamma}^{-1} b_{\gamma}^{-1} = 1 \) if the sign is +;
- \( x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_{\gamma}^2 = 1 \) if the sign is -.

The orientation-preserving elements of \( \Gamma \) are those expressible as words (on the generators) in which the total number of occurrences of the reflections \( c_{ij} \) and glide reflections \( d_i \) is even. These elements constitute the canonical Fuchsian subgroup \( \Gamma^+ \), which has index 1 or 2 in \( \Gamma \), with the former case occurring precisely when \( \Gamma \) itself is a Fuchsian group.

The area of a fundamental region for an NEC group \( \Gamma \) with signature \((1.1)\) is \( 2\pi \mu(\Gamma) \), given by

\[
\mu(\Gamma) = \alpha\gamma + k - 2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left( 1 - \frac{1}{n_{ij}} \right),
\]

where \( \alpha = 2 \) if the sign is + and \( \alpha = 1 \) otherwise. If \( \Gamma' \) is a subgroup of finite index in \( \Gamma \), then \( \Gamma' \) is also an NEC group, and its area is given by the Riemann-Hurwitz formula:

\[
\mu(\Gamma') = [\Gamma : \Gamma'] \cdot \mu(\Gamma).
\]

The algebraic genus \( g \) of a compact surface of topological genus \( \gamma \) with \( k \) boundary components, is defined as

\[
g = \begin{cases} 
2\gamma + k - 1 & \text{if the surface is orientable,} \\
\gamma + k - 1 & \text{otherwise.}
\end{cases}
\]

It follows from the Uniformisation Theorem that any non-orientable surface without boundary of algebraic genus \( g \geq 2 \) is of the form \( H/\Lambda \), for some torsion-free proper NEC group \( \Lambda \) with signature \((g + 1; -; [-]; \{-}\)); see \[14\]. A finite group \( G \) acts (faithfully) as a group of automorphisms of such a surface if and only if there exist an NEC group \( \Gamma \) and an epimorphism \( \theta: \Gamma \rightarrow G \) whose kernel is \( \Lambda \). Epimorphisms whose kernel is a torsion-free proper NEC group are called smooth epimorphisms, for short.
Non-orientability requires that the canonical Fuchsian subgroup $\Gamma^+$ is taken by the smooth epimorphism $\theta$ to $G$, rather than to some subgroup of index 2 in $G$. In particular, this implies that $G$ is generated by elements expressible as words in which the total number of occurrences of the images of the reflections $c_{ij}$ and glide reflections $d_i$ is even.

The following proposition shows that there is a strong restriction in the signature of $\Gamma$ when $G$ is cyclic.

**Proposition 1.1.** Let $\Gamma$ be an NEC group admitting a smooth epimorphism onto a cyclic group. Then the signature of $\Gamma$ has no link periods.

**Proof.** Assume the contrary, namely that $\sigma(\Gamma)$ has a non-trivial link period $n_{ij} > 1$. Let $c$ and $c'$ be the canonical reflections associated with it, and let $\theta : \Gamma \to C_n = \langle v \mid v^n = 1 \rangle$ be a smooth epimorphism. Clearly $n$ has to be even since $c$ and $c'$ have order 2, and ker $\theta$ is torsion-free. Moreover, $\theta(c) = \theta(c') = v^{n/2}$, since this is the unique element of order 2 in $C_n$. But then $\theta(cc') = v^{n/2}v^{n/2} = 1$, which is a contradiction, since the elliptic element $cc'$ (of order $n_{ij} > 1$) cannot lie in ker $\theta$. □

Proposition 1.1 rules out all of the non-normal pairs of NEC signatures, and 13 of the 36 normal pairs, namely the pairs 5, 7, 12, 13, 15, 16, 17, 18, 22, 23, 27, 31 and 33 listed in Table 3. (Incidentally, that table has typographical errors in its 3rd and 26th entries.) The pairs 24, 25, 26, 28, 32, 34, 35 and 36 are also ruled out, because in those cases, $\Gamma$ is not a proper NEC group (but is a Fuchsian group). This leaves the 15 normal pairs in Table 1 to be analysed.

**Table 1.** Normal pairs of NEC signatures to be analysed.

| Signature $\sigma = \sigma(\Gamma)$ | Signature $\sigma' = \sigma(\Gamma')$ | $|\Gamma' : \Gamma|$ |
|-------------------------------------|-------------------------------------|------------------|
| Case 1 (3; −; [−]; {−}) | (0; +; [2, 2, 2]; {(−)}) | 2 |
| Case 2 (2; −; [t]; {−}) | (0; +; [2, 2]; {(t)}) | 2 |
| Case 3 (2; −; [−]; {(−)}) | (0; +; [2, 2]; {(2, 2)}) | 2 |
| Case 4 (1; +; [−]; {(−)}) | (0; +; [2, 2, 2]; {(−)}) | 2 |
| Case 5 (1; −; [t]; {(−)}) | (0; +; [2]; {(2, 2, t)}) | 2 |
| Case 6 (1; −; [−]; {(−), (−)}) | (0; +; [2]; {(2, 2, 2)}) | 2 |
| Case 7 (1; −; [t, u]; {−}), max(t, u) ≥ 3 | (0; +; [2]; {(t, u)}) | 2 |
| Case 8 (1; −; [t, t]; {−}), t ≥ 3 | (0; +; [2, t]; {(−)}) | 2 |
| Case 9 (0; +; [−]; {(−), (−), (−)}) | (0; +; [−]; {(2, 2, 2, 2, 2)}) | 2 |
| Case 10 (0; +; [t]; {(−), (−)}) | (0; +; [−]; {(2, 2, 2, 2, t)}) | 2 |
| Case 11 (0; +; [t, u]; {(−)}, max(t, u) ≥ 3 | (0; +; [−]; {(2, 2, t, u)}) | 2 |
| Case 12 (0; +; [t, t]; {(−)}) | (0; +; [t]; {(2, 2)}) | 2 |
| Case 13 (0; +; [t, t]; {(−)}, t ≥ 3 | (0; +; [t, 2]; {(−)}) | 2 |
| Case 14 (1; −; [t, t]; {−}), t ≥ 3 | (0; +; [−]; {(2, 2, 2, t)}) | 4 |
| Case 15 (0; +; [t, t]; {(−)}, t ≥ 3 | (0; +; [−]; {(2, 2, 2, t)}) | 4 |

Necessary and sufficient conditions for the existence of a smooth epimorphism $\theta : \Gamma \to C_n$ can be found from Theorems 3.5, 3.6 and 3.7 of [4] and Theorems 3.1.3, 3.1.6 and 3.1.8 of [5]. These also yield some arithmetic conditions involving the
periods of $\Gamma$ and $n$, which will be described in each case in the next section; for instance, in all but Cases 1, 2, 7, 8 and 14, the group $\Gamma$ contains reflections, which implies that $n$ has to be even.

2. Analysis of cases

For each normal pair $(\sigma, \sigma')$ in Table $[1]$ we consider the possibility of some extension of a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ to a smooth epimorphism $\theta': \Gamma' \rightarrow G'$, where $\Gamma$ is a proper NEC group with signature $\sigma$, and $\Gamma'$ is an NEC group with signature $\sigma'$, such that $\Gamma$ can be embedded as a subgroup of $\Gamma'$ with finite index $m$, and $G'$ is a finite group of order $mn$ containing $C_n$ as a subgroup of index $m$. Two embeddings $i_1, i_2: \Gamma \rightarrow \Gamma'$ are said to be equivalent if there exists an automorphism $\beta \in \text{Aut}(\Gamma')$ such that $\beta i_1 = i_2$. In the cases that follow, inequivalent embeddings were found and analysed with the help of the MAGMA system $[2]$.

We will use the following standard notation: $|H:K|$ denotes the index of a subgroup $K$ in a group $H$, and $[a, b]$ denotes the commutator $a^{-1}b^{-1}ab$ of any pair $(a, b)$ of elements of a group, while $a^x$ denotes the conjugate $x^{-1}ax$ of $a$ by the element $x$ (or the image of $a$ under the automorphism $x$, depending on the context). Note that the index $|\Gamma':\Gamma|$ is 2 in all cases other than cases 14 and 15, where it is 4.

Case 1: $\sigma = (3; -; [-]; \{ - \})$, $\sigma' = (0; +; [2, 2, 2]; \{ (-) \})$. In this case $\Gamma$ is generated by elements $d_1, d_2$ and $d_3$ such that $d_1^2d_2^2d_3^2 = 1$, while $\Gamma'$ is generated by involutions $x_1, x_2, x_3$ and $c$ such that $[x_1x_2x_3, c] = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists for all values of $n$. Up to equivalence there is a unique embedding of $\Gamma$ as a subgroup of index 2 in $\Gamma'$, given by $d_1 = x_1c$, $d_2 = cx_2$ and $d_3 = x_2cx_3x_2$; see $[3]$ Proposition 4.8. Conjugation by $c$ is an involutory automorphism of the embedded copy of $\Gamma$ with the property that $d_1c = d_1^{-1}$, and $d_2^c = d_2^{-1}$, and $d_3^c = d_1^{-2}d_3^{-1}d_1^2$. Hence in any extension of $\theta: \Gamma \rightarrow C_n$ to a smooth epimorphism $\theta': \Gamma' \rightarrow G'$, with $|G':C_n| = 2$, conjugation by the image of $c$ must invert the images of all the $d_i$, and so $G'$ is the dihedral group $D_n$. Such an extension is always possible, so the given action of $C_n$ is never maximal.

Case 2: $\sigma = (2; -; [t]; \{ - \})$, $\sigma' = (0; +; [2, 2]; \{ t \})$. The analysis of this case is similar to Case 1. Here $\Gamma$ is generated by elements $d_1, d_2$ and $x$ such that $d_1^2d_2^2x = 1$ and $x$ has order $t$, while $\Gamma'$ is generated by involutions $x_1, x_2, c$ and $c$ such that $[x_1x_2, c]$ has order $t$. An embedding of $\Gamma$ into $\Gamma'$ is given by taking $d_1 = x_1c$, $d_2 = cx_2$ and $c = x_2cx_1x_2$, because the latter has order $t$ (being conjugate to $[x_1x_2, c]$). This embedding is unique up to equivalence, since $\Gamma'$ has only one subgroup of index 2 with the same abelianisation (first homology group) $\mathbb{Z} \oplus \mathbb{Z}_{2t}$ as $\Gamma$. Let us write $n = 2^n n'$ and $t = 2^e t'$, where $n'$ and $t'$ are odd. It follows from the results of $[4, 6]$ that there exists a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ if and only if $t$ is a divisor of $n$, and $e' = e - 1$ if $n$ is even. Conjugation by $c$ is an involutory automorphism with the property that $d_1^c = d_1^{-1}$, and $d_2^c = d_2^{-1}$, and $x^c = d_2^2d_1^2 = d_2^2x^{-1}d_2^{-2}$. Hence an extension of $\theta: \Gamma \rightarrow C_n$ is always possible, to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Case 3: $\sigma = (2; -; [-]; \{ (-) \})$, $\sigma' = (0; +; [2, 2]; \{ (2, 2) \})$. Here $\Gamma$ is generated by elements $d_1, d_2$ and $c$ such that $c^2 = [d_1^2d_2^2, c] = 1$, while $\Gamma'$ is generated by involutions $x_1, x_2, c_0$ and $c_1$ such that $(c_0c_1)^2 = (c_1(x_1x_2)^{-1}c_0(x_1x_2))^2 = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if $n$ is even. Up to equivalence
(including the outer automorphism of \( \Gamma' \) that interchanges \( x_1 \) with \( x_2 \) and \( c_0 \) with \( c_1 \)), there is a unique embedding of \( \Gamma \) into \( \Gamma' \), given by \( d_1 = c_1 x_1 \), \( d_2 = x_1 c_1 x_2 x_1 \) and \( c = c_0 \); this is equivalent to the one given in [3 Proposition 4.4]. Conjugation by \( x_1 \) is an involutory automorphism with the property that \( d_1 x_1 = d_1^{-1} \) and \( d_2 x_1 = d_2^{-1} d_1^{-1} \) while \( c x_1 = d_1^{-1} c d_1 = d_1^{-1} c^{-1} d_1 \). Hence an extension of \( \theta : \Gamma \to C_n \) is always possible, to a smooth epimorphism \( \theta' : \Gamma' \to D_n \).

**Case 4:** \( \sigma = (1;+;[-]; \{(-)\}) \), \( \sigma' = (0;+;[2,2,2]; \{(-)\}) \). In this case \( \Gamma \) is generated by elements \( a, b \) and \( c \) such that \( c = 0 \) and \( \Gamma' \) is generated by involutions \( x_1, x_2, x_3 \) and \( c_0 \) such that \( [x_1 x_2 x_3, c_0] = 1 \). As above, \( \theta : \Gamma \to C_n \) exists if and only if \( \Gamma \) is even. Up to equivalence there is a unique embedding of \( \Gamma \) into \( \Gamma' \), given by \( a = x_2 x_1 \), \( b = x_1 x_2 x_3 \) and \( c = c_0 \); see also [3 Proposition 4.5]. Conjugation by \( x_1 \) is an involutory automorphism with the property that \( a x_1 = a^{-1} \) and \( b x_1 = b^{-1} \) while \( c x_1 = b c a^{-1} b = b c a^{-1} (b a)^{-1} \). Once again it follows that an extension of \( \theta : \Gamma \to C_n \) is always possible, to a smooth epimorphism \( \theta' : \Gamma' \to D_n \).

**Case 5:** \( \sigma = (1;−;[t]; \{−\}) \), \( \sigma' = (0;+;[2,2]; \{(2,2,t)\}) \). The analysis of this case is similar to Case 2. The group \( \Gamma \) is generated by elements \( d, e_1, e_2, c_1 \) and \( c_2 \) such that \( d^2 x_1 = x_1^4 = c_1^2 = 1 \), while \( \Gamma' \) is generated by involutions \( x_1, c_0, c_1 \) and \( c_2 \) such that \( (c_0 c_1)^2 = (c_1 c_2)^2 = (c_2 x_1 c_0 x_1)^t = 1 \). A smooth epimorphism \( \theta : \Gamma \to C_n \) exists if and only if \( n \) is divisible by both \( 2 \) and \( t \). An embedding of \( \Gamma \) into \( \Gamma' \) is given by \( d = c_0 x_1 \), \( x = x_1 c_0 x_1 c_2 \) and \( c = c_1 \), and this is unique up to equivalence for all \( t \geq 2 \) (allowing for an outer automorphism when \( t = 2 \)). Conjugation by \( c_0 \) is an involutory automorphism with the property that \( d c_0 = d^{-1} \) and \( x c_0 = d^2 x_1 d^{-2} \) while \( c_0 = c = c_1 \), so again \( \theta : \Gamma \to C_n \) always extends to a smooth epimorphism \( \theta' : \Gamma' \to D_n \).

**Case 6:** \( \sigma = (1;−;[−]; \{−\}) \), \( \sigma' = (0;+;[2,2,2]; \{(2,2,2)\}) \). The analysis of this case is similar to Case 3. The group \( \Gamma \) is generated by elements \( d, e_1, e_2, c_1 \) and \( c_2 \) such that \( d^2 e_1 e_2 = c_1^2 = c_2^2 = [e_1, c_1] = [e_2, c_2] = 1 \), while \( \Gamma' \) is generated by involutions \( x, c_0, c_1, c_2 \) and \( c_3 \) such that \( (c_0 c_1)^2 = (c_1 c_2)^2 = (c_2 c_3)^2 = (c_3 x_1 c_0 x_1)^2 = 1 \). A smooth epimorphism \( \theta : \Gamma \to C_n \) exists if and only if \( n \) is even. Up to equivalence (including the outer automorphism of \( \Gamma' \) that centralises \( x \) and interchanges \( c_0 \) with \( c_3 \) and \( c_1 \) with \( c_2 \)), there is a unique embedding of \( \Gamma \) into \( \Gamma' \), given by \( d = x c_0 \), \( e_1 = c_0 c_2 \), \( e_2 = x c_0 x \), \( c_{10} = c_1 \) and \( c_{20} = c_3 \); see also [3 Proposition 4.2]. Conjugation by \( c_0 \) is an involutory automorphism with the property that \( d c_0 = d^{-1} \) and \( e_1 c_0 = e_1^{-1} \) and \( e_2 c_0 = e_1 e_2^{-1} e_1^{-1} \), while \( c_{10} c_0 = c_{10}^{-1} c_2 \) and \( c_{20} c_0 = e_1 c_{20}^{-1} e_1^{-1} \). Therefore once again, an extension of \( \theta : \Gamma \to C_n \) is always possible, to a smooth epimorphism \( \theta' : \Gamma' \to D_n \).

**Case 7:** \( \sigma = (1;−;[t,u]; \{−\}) \), \( \sigma' = (0;+;[2]; \{(t,u)\}) \), \( \max(t,u) \geq 3 \). The analysis of this case is again similar to that in earlier cases. The group \( \Gamma \) is generated by elements \( d, x \) and \( y \) such that \( d^2 x y = x^t = y^u = 1 \), while \( \Gamma' \) is generated by involutions \( x_1, c_0 \) and \( c_1 \) such that \( (c_0 c_1)^t = (c_1 x_1 c_0 x_1)^u = 1 \). A smooth epimorphism \( \theta : \Gamma \to C_n \) exists if and only if \( n \) is the least common multiple of \( t \) and \( u \), and \( n/t \) and \( n/u \) are both odd. Up to equivalence there is a unique embedding of \( \Gamma \) into \( \Gamma' \), given by \( d = x_1 c_0 \), \( x = c_0 x_1 \) and \( y = x_1 c_0 x_1 \). Conjugation by \( c_0 \) is an involutory automorphism with the property that \( d c_0 = d^{-1} \) and \( x c_0 = x^{-1} \) and \( y c_0 = x y^{-1} x^{-1} \), so an extension of \( \theta : \Gamma \to C_n \) is always possible to a smooth epimorphism \( \theta' : \Gamma' \to D_n \).
Before proceeding with other cases, we consider what happens when \( C_n \) acts with signature \( \sigma = (1; -; [t, t]; \{-\}) \). According to Case 7 with \( t = u \), this action always extends to one of a larger group, namely a dihedral group acting with signature \( \sigma' = (0; +; [2]; \{(t, t)\}) \). The possibility of other extensions with different signatures \( \sigma' \) will be encountered below, in Case 8 with \( \sigma' = (0; +; [2]; \{-\}) \), and Case 14 with \( \sigma' = (0; +; [-]; \{2, 2, 2, t\}) \). We will see the same kind of thing occurring in Cases 12, 13 and 15, compared with Case 11.

**Case 8:** \( \sigma = (1; -; [t, t]; \{-\}) \), \( \sigma' = (0; +; [2]; \{-\}) \), \( t \geq 3 \). Here the group \( \Gamma \) is generated by elements \( d, x \) and \( y \) such that \( d^2xy = x^t = y^t = 1 \), while \( \Gamma' \) is generated by elements \( x_1, x_2 \) and \( c \) such that \( x_1^2 = x_2^2 = c^2 = [x_1x_2, c] = 1 \). A smooth epimorphism \( \theta : \Gamma \to C_n \) exists if and only if \( t = n \). Up to equivalence there is a unique embedding of \( \Gamma \) into \( \Gamma' \), given by \( d = x_1cx_2^{-1} \), \( x = x_2 \) and \( y = x_1x_2x_1 \).

(Note: when \( t \) is odd, there are two subgroups of index 2 in \( \Gamma' \) with the same abelianisation \( \mathbb{Z}_{2t} \oplus \mathbb{Z}_4 \) as \( \Gamma \), but one of them is isomorphic to the NEC group with signature \( (0; +; [t, t]; \{-\}) \), and so we must take the other one.) Conjugation by \( x_1 \) is an involutory automorphism with the property that \( d^{x_1} = x^{-1}dx \) while \( x^{x_1} = y \) and \( y^{x_1} = x \). Hence the epimorphism \( \theta : \Gamma \to C_n \) can be extended if and only if there exists an involutory automorphism of \( C_n \) that interchanges \( \theta(x) \) and \( \theta(y) \) while centralising \( \theta(d) \).

Now this is possible in many cases. For example, if \( \theta(y) = \theta(x)^{-1} \) while \( \theta(d) = 1 \) (or \( \theta(d) = \theta(x)^{n/2} \) when \( n \) is even), then \( \theta \) extends to a smooth epimorphism \( \theta' : \Gamma' \to D_n \). In other cases where such an automorphism exists, \( \theta \) will extend to the action of a dicyclic group \( C_n : C_2 \) (of order \( 2n \)). But there are also many cases where this is not possible, such as when \( n > 6 \) and either \( \theta(y), \theta(d) = (\theta(x)^2, \theta(x)^{2n/3} \) for \( n \) odd, or \( (\theta(x)^3, \theta(x)^{2n/3}) \) for \( n \) even. Hence \( \theta : \Gamma \to C_n \) does not always extend in this way.

**Case 9:** \( \sigma = (0; +; [-]; \{(\cdot), (\cdot), (\cdot), (\cdot)\}) \), \( \sigma' = (0; +; [-]; \{(2, 2, 2, 2, 2, 2)\}) \). In this case, which is similar to Case 6, the group \( \Gamma \) is generated by elements \( e_1, e_2, e_3 \) and involutions \( c_{10}, c_{20} \) and \( c_{30} \) such that \( e_1e_2e_3 = e_{11, 10} = e_{2, 20} = [e_3, c_{30}] = 1 \), while \( \Gamma' \) is generated by involutions \( c_0, c_1, c_2, c_3, c_4 \) and \( c_5 \) such that \( (c_0c_1)^2 = (c_2c_3)^2 = \cdots = (c_4c_5)^2 = (c_5c_0)^2 = 1 \). A smooth epimorphism \( \theta : \Gamma \to C_n \) exists if and only if \( n \) is even. Up to equivalence there is just one embedding of \( \Gamma \) into \( \Gamma' \), given by \( e_1 = c_0c_2, e_2 = c_2c_4, e_3 = c_4c_0, c_{10} = c_1, c_{20} = c_3, \) and \( c_{30} = c_5 \); see also [3 Proposition 4.1]. Conjugation by \( c_0 \) is an involutory automorphism with the property that \( e_{10} = e_{1}^{-1}, e_{20} = e_{1}e_{2}^{-1}e_{1}^{-1} \) and \( e_{30} = e_{3}^{-1}, \) while \( c_{10} = c_{10} = c_{10}^{-1}, \) and \( c_{20} = c_{1}c_{20}^{-1}e_{1}^{-1} \) and \( c_{30} = c_{30}^{-1} \). Once more, an extension of \( \theta : \Gamma \to C_n \) is always possible, to a smooth epimorphism \( \theta' : \Gamma' \to D_n \).

**Case 10:** \( \sigma = (0; +; [-]; \{(\cdot), (\cdot), (\cdot)\}) \), \( \sigma' = (0; +; [-]; \{(2, 2, 2, 2, 2)\}) \). The analysis here is again similar to earlier cases. The group \( \Gamma \) is generated by elements \( x, e_1, e_2, c_1, c_10, \) \( c_{20} \) and \( c_{30} \) such that \( xc_{1}c_{2} = x^t = c_{10}^2 = c_{20}^2 = [e_1, c_{10}] = [e_2, c_{20}] = 1 \), while \( \Gamma' \) is generated by involutions \( c_0, c_1, c_2, c_3, c_4 \) and \( c_5 \) such that \( (c_0c_1)^2 = (c_2c_3)^2 = \cdots = (c_4c_5)^2 = (c_5c_0)^2 = 1 \). As in Case 5, a smooth epimorphism \( \theta : \Gamma \to C_n \) exists if and only if \( t \) divides \( n \) and \( n \) is even. Up to equivalence (including an outer automorphism when \( t = 2 \)), there is just one embedding of \( \Gamma \) into \( \Gamma' \), given by \( x = c_1c_0, e_1 = c_0c_2, e_2 = c_2c_4, c_{10} = c_1, \) and \( c_{20} = c_3 \). Conjugation by \( c_0 \) is an involutory automorphism such that \( x^c = x^{-1} \) and \( e_{1}^c = e_{1}^{-1} \) and \( e_{2}^c = x_{e_2}^{-1}x_{-1} \),
while $c_{10}^{-c_0} = c_{10} = c_{10}^{-1}$ and $c_{20}^{-c_0} = xc_{20}^{-1}x^{-1}$. Hence $\theta : \Gamma \to C_n$ is always extendable to a smooth epimorphism $\theta' : \Gamma' \to D_n$.

Case 11: $\sigma = (0;+;[t,u];\{(-)\}), \quad \sigma' = (0;+;[-];\{(2,2,t,u)\}), \quad \max(t,u) \geq 3$. Here the analysis is similar to Case 7. The group $\Gamma$ is generated by elements $x, y$ and $c$ such that $x^t = y^u = c^2 = [xy,c] = 1$, while $\Gamma'$ is generated by involutions $c_0, c_1, c_2$ and $c_3$ such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^t = (c_3c_0)^u = 1$. Up to equivalence (including an outer automorphism when $t = 2$ or $u = 2$) there is a unique embedding of $\Gamma$ into $\Gamma'$, given by $x = c_2c_3$, $y = c_3c_0$ and $c = c_1$. A smooth epimorphism $\theta : \Gamma \to C_n$ exists if and only if $n$ is even and $n = \text{lcm}(t,u)$. Conjugation by $c_0$ is an involutory automorphism such that $x^{-c_0} = y^{-1}x^{-1}y$ and $y^{-c_0} = y^{-1}$ while $c_1^{-c_0} = c_1 = c_1^{-1}$, so $\theta : \Gamma \to C_n$ is always extendable to a smooth epimorphism $\theta' : \Gamma' \to D_n$.

Case 12: $\sigma = (0;+;[t,t];\{(-)\}), \quad \sigma' = (0;+;[t];\{(2,2)\}), \quad t \geq 3$. Here the group $\Gamma$ is generated by elements $x, y$ and $c$ such that $x^t = y^t = c^2 = [xy,c] = 1$, while $\Gamma'$ is generated by elements $x_1, c_0$ and $c_1$ such that $x_1^t = c_0^2 = c_1^2 = (c_0c_1)^2 = (c_1x_1^{-1}c_0x_1)^2 = 1$. A smooth epimorphism $\theta : \Gamma \to C_n$ exists if and only if $n$ is even and $t = n$. Up to equivalence (including an outer automorphism of $\Gamma'$ that inverts $x_1$ and interchanges $c_0$ with $c_1$), there is a unique embedding of $\Gamma$ into $\Gamma'$, given by $x = x_1^{-1}, y = c_0x_1c_0$ and $c = c_1$. Conjugation by $c_0$ is an involutory automorphism such that $x^{-c_0} = y^{-1}$ (and $y^{-c_0} = x^{-1}$) while $c_0^{-c_0} = c$.

Hence the epimorphism $\theta : \Gamma \to C_n$ can be extended if and only if there exists an involutory automorphism of $\Gamma_0$ that interchanges $\theta(x)$ and $\theta(y)^{-1}$ while centralising $\theta(c)$. As in Case 8, this can happen sometimes but not always. For example, if $\theta(y) = \theta(x)$ while $\theta(d) = 1$ or $\theta(x)^u/2$, then $\theta$ extends to a smooth epimorphism $\theta' : \Gamma' \to D_n$. In other cases where such an automorphism exists, $\theta$ will extend to the action of a dicyclic group $C_n : C_2$ (of order $2n$). But there are clearly many other examples where this is not possible.

Case 13: $\sigma = (0;+;[t,t];\{(-)\}), \quad \sigma' = (0;+;[t,2];\{(-)\}), \quad t \geq 3$. In this case the analysis is very similar to Case 12, except that $\Gamma'$ is generated by elements $x_1, x_2$ and $c_0$ such that $x_1^t = x_2^2 = c_0^2 = [x_1x_2, c_0] = 1$. Again we require $n$ to be even and $t = n$. Up to equivalence there is a unique embedding of $\Gamma$ into $\Gamma'$, given by $x = x_1, y = x_2x_1x_2$ and $c = c_0$. Conjugation by $x_2$ is an involutory automorphism with the property that $x^{-x_2} = y$ (and $y^{-x_2} = x$) while $c^{x_2} = yc_{x_2}^{-1}$. Hence the epimorphism $\theta : \Gamma \to C_n$ can be extended if and only if there exists an involutory automorphism of $\Gamma_0$ that interchanges $\theta(x)$ and $\theta(y)$ while centralising $\theta(c)$. When such an extension occurs, it gives an action of the dihedral group $D_n$ or some other dicyclic group $C_n : C_2$ (of order $2n$), but there are clearly many cases where the extension is impossible.

Case 14: $\sigma = (1;[-;[t,t];\{(-)\}), \quad \sigma' = (0;+;[-];\{(2,2,2,t)\}), \quad t \geq 3$. In this case the group $\Gamma$ is generated by elements $d, x$ and $y$ such that $d^2xy = x^t = y^t = 1$, while $\Gamma'$ is generated by involutions $c_0, c_1, c_2$ and $c_3$ such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^t = 1$, but (in contrast to the previous cases) the index $|\Gamma' : \Gamma|$ is 4. As in Case 8, a smooth epimorphism $\theta : \Gamma \to C_n$ exists if and only if $t = n$. There are three subgroups of index 4 in $\Gamma'$ with the same abelianisation $\mathbb{Z}_{2t} \oplus \mathbb{Z}_t$ as $\Gamma$, but two of them contain an element of order 2 lying outside the subgroup generated by the elements of order $t$, and so we must take the third one. This gives an embedding of $\Gamma$ into $\Gamma'$ via $d = c_2c_1c_0, x = c_0c_3$ and $y = c_2c_3c_0c_2$, which is
unique up to equivalence. Conjugation by $c_0$ is an involutory automorphism with the property that $d^{c_0} = d^{-1}$ and $x^{c_0} = x^{-1}$ and $y^{c_0} = xy^{-1}x^{-1}$, and conjugation by $c_2$ is another involutory automorphism with the property that $d^{c_2} = d^{-1}$ and $x^{c_2} = y^{-1}$ and $y^{c_2} = x^{-1}$. (Also conjugation by $c_0c_2y^{-1}$ is the same automorphism as considered in Case 8.) Now $\theta$ can always be extended to a smooth epimorphism from $(\Gamma, c_0)$ to the dihedral group $D_n$, but the full extension to an epimorphism $\theta': \Gamma' \to C_n: (C_2 \times C_2)$ depends on the existence of an involutory automorphism of $C_n$ interchanging $\theta(x)$ and $\theta(y)^{-1}$ while inverting $\theta(d)$. Such a full extension is possible frequently, but not always.

Case 15: $\sigma = (0; +; [t, t]; \{(-)\})$, $\sigma' = (0; +; [-]; \{(2, 2, 2, t)\})$, $t \geq 3$. Here the group $\Gamma$ is generated by elements $x, y$ and $c$ such that $x^t = y^t = c^2 = [xy, c] = 1$, while $\Gamma'$ is generated by involutions $c_0, c_1, c_2$ and $c_3$ such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0) = 1$, and again the index $[\Gamma': \Gamma]$ is 4. As in Case 12, a smooth epimorphism $\theta: \Gamma \to C_n$ exists if and only if $n$ is even and $t = n$. Up to equivalence there is just one embedding of $\Gamma$ into $\Gamma'$, given by $x = c_0c_3$, $y = c_2c_3c_0c_2$, and $c = c_1$. Conjugation by $c_0$ is an involutory automorphism such that $x^{c_0} = x^{-1}$ and $y^{c_0} = xy^{-1}x^{-1}$ while $c^{c_0} = c = c^{-1}$, and conjugation by $c_2$ is another involutory automorphism such that $x^{c_2} = y^{-1}$ and $y^{c_2} = x^{-1}$ while $c^{c_2} = c = c^{-1}$. Hence $\theta$ can always be extended to a smooth epimorphism from $(\Gamma, c_0)$ to $D_n$, but the full extension to an epimorphism $\theta': \Gamma' \to C_n: (C_2 \times C_2)$ depends on the existence of an involutory automorphism of $C_n$ interchanging $\theta(x)$ and $\theta(y)^{-1}$ while centralising $\theta(c)$. Such a full extension is possible frequently, but not always.

As a consequence of the analysis of the above 15 cases we have the following.

**Theorem 2.1.** Let $C_n$ be a cyclic group acting with non-maximal signature on a non-orientable surface $S$. Then the action of $C_n$ always extends to the action of a larger group on $S$.

**Proof.** The only cases above in which the expected extension does not always occur are Cases 8 and 14 (where $\Gamma$ has signature $\{1; -; [t, t]; \{-\}\}$) and Cases 12, 13 and 15 (where $\Gamma$ has signature $\{0; +; [t, t]; \{(-)\}\}$). The actions in those cases, however, extend in the way described in Cases 7 and 11, when we take $u = t$. □

**Remark 2.2.** Observe that in all cases, the action of $C_n$ extends at least to an action of the dihedral group $D_n$ on the same surface.

**Remark 2.3.** It is worth mentioning that Theorem 2.1 contrasts with the analogous theorem for cyclic actions on orientable surfaces. For in the orientable case, a cyclic group acting with a non-maximal Fuchsian signature does not always extend; this was shown by the first and third authors of the present paper in [5, Theorem 4.1].

3. **Applications**

It has been known for some time that the maximum order of a cyclic group of automorphisms acting on a non-orientable surface of algebraic genus $g$ is $2g + 2$ when $g$ is even, and $2g$ when $g$ is odd. This was shown by Wendy Hall in [10], and again by the first author of the present paper in [4, Corollary 4.4]. Moreover, if $\theta: \Gamma \to C_{2g+2}$ or $C_{2g}$ is a smooth epimorphism realising such a maximal cyclic
action, then the signature of $\Gamma$ is

- $(0;+;[2, g + 1];\{(-)\})$ if $g$ is even, or
- $(0;+;[2, 2g];\{(-)\})$ or $(1;−;[2, 2g];\{−\})$ if $g$ is odd.

This is an easy consequence of the Riemann-Hurwitz formula and the fact that there is no smooth epimorphism from an NEC group with signature $(1;−;[2, g + 1];\{−\})$ to $C_{2g+2}$.

It is also known that such a maximal cyclic action on a non-orientable surface always extends to a larger group of automorphisms; see [7]. We can now give a direct proof of this result, from our analysis of the cases in Section 2.

**Corollary 3.1.** Let $S$ be a compact non-orientable surface of algebraic genus $g \geq 2$ admitting an automorphism $v$ of maximum possible order (namely, $2g + 2$ if $g$ is even, or $2g$ if $g$ is odd). Then the full automorphism group of $S$ properly contains $\langle v \rangle$.

**Proof.** This is a consequence of the fact that such a maximal cyclic group $\langle v \rangle$ acts with signature of the form $(0;+;[t,u];\{−\})$ or $(1;−;[t,u];\{−\})$, and all such actions were shown to extend in Cases 7 and 11 of Section 2. □

Next, it is a natural question to ask what is the largest order of a cyclic group that acts as the full group of automorphisms of some non-orientable surface $S$ of given genus.

**Theorem 3.2.** For every integer $g \geq 2$, the order of the largest cyclic group $C_n$ that acts as the full group of automorphisms of a compact non-orientable surface of algebraic genus $g$ is given in Table 2, together with all possibilities for the corresponding signature.

<table>
<thead>
<tr>
<th>Genus</th>
<th>Largest $n$</th>
<th>Possible signatures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g \equiv 1 \mod 4$</td>
<td>$g + 1$</td>
<td>$(0;+;[2, 2, 2g+1];{−})$ or $(1;−;[2, 2, 2g+1];{−})$</td>
</tr>
<tr>
<td>$g$ even</td>
<td>$g$</td>
<td>$(0;+;[2, 2, g];{−})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>or $(0;+;[2, 3, 3];{−})$ when $g = 6$,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>or $(0;+;[2, 3, 4];{−})$ when $g = 12$,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>or $(0;+;[2, 3, 5];{−})$ when $g = 30$.</td>
</tr>
<tr>
<td>$g \equiv 3 \mod 4$</td>
<td>$g − 1$</td>
<td>$(0;+;[2, 2];{−})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>or $(0;+;[2, 2, 2];{−})$ or $(0;+;[2, 2, 2];{−})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>or $(1;−;[2, 2, 2];{−})$ when $g = 3$,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>or $(0;+;[2, 3, 6];{−})$ or $(1;−;[2, 3, 6];{−})$ when $g = 7$.</td>
</tr>
</tbody>
</table>

**Proof.** First, suppose $\theta : \Gamma \to C_n = \langle v \vert v^n = 1 \rangle$ is a smooth epimorphism such that $\ker \theta$ has signature $(g + 1;−;[−];\{−\})$. Then by Proposition [1,1] we know that $\Gamma$ has signature $\sigma = (\gamma;±;[m_1, \ldots, m_r];\{−, \ldots, \{−\})$ for a suitable choice
of non-negative integers $\gamma$, $r$ and $k$ and integers $m_1, \ldots, m_r \geq 2$. Moreover, by the Riemann-Hurwitz formula these must satisfy

$$\alpha \gamma + k - 2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) = \frac{g - 1}{n},$$

where $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise.

Now let us consider all possible signatures $\sigma$ for which $n \geq g - 1$. Since $1 - \frac{1}{m_i} \geq \frac{1}{2}$ for all $i$, we require

$$\alpha \gamma + k + \frac{r}{2} \leq 3.$$

Also for the action to be non-orientable, we require either $\gamma > 0$ or $k > 0$, with $\alpha = 1$ (for sign $-$) if $k = 0$, and of course $\alpha = 2$ (for sign $+$) if $\gamma = 0$.

A straightforward calculation shows there are 17 different types of NEC signatures that satisfy these conditions, namely the following:

$$(0; +; [2, 2, 2, 2]; \{-\}), \quad (1; -; [2, 2, 2, 2]; \{-\}), \quad (0; +; [m_1, m_2, m_3]; \{-\}),$$

$$(1; -; [2, 2]; \{-\}), \quad (0; +; [2, 2]; \{-\}), \quad (0; +; [m_1, m_2]; \{-\}),\quad (2; -; [2, 2]; \{-\}),$$

$$(0; +; [m_1]; \{-\}, \{-\}), \quad (1; -; [m_1]; \{-\}), \quad (2; -; [m_1]; \{-\}),$$

$$(0; +; [-]; \{-\}, \{-\}), \quad (1; -; [-]; \{-\}, \{-\}), \quad (1; +; [-]; \{-\}).$$

Among these, 10 give non-maximal NEC signatures, in which case Theorem 2.3 shows that the cyclic action is never the full group of automorphisms of the surface. We deal with the remaining 7 signature types below. Since each is a maximal NEC signature, we may choose a maximal NEC group $\Gamma$ with such signature, so that if $\theta : \Gamma \to C_n$ is a smooth epimorphism with non-orientable kernel, then the maximality of $\Gamma$ prevents $\theta$ from being extended to a larger group $\Gamma'$. Accordingly, this guarantees that $C_n$ acts as the full group of automorphisms of the non-orientable surface $H/\ker \theta$.

If $\sigma = (0; +; [2, 2, 2]; \{-\})$, then $n = g - 1$, and the group $\Gamma$ is generated by elements $x_1, x_2, x_3, x_4$ and $c$ such that $x_1^2 = x_2^2 = x_3^2 = x_4^2 = c^2 = [x_1x_2x_3x_4, c] = 1$. Here $\theta$ must take each generator to the unique involution of $C_n$, and therefore $n = 2$ (so $g = 3$).

Similarly, if $\sigma = (1; -; [2, 2, 2]; \{-\})$, then $n = g - 1$, and $\Gamma$ is generated by elements $d, x_1, x_2, x_3$ and $x_4$ such that $x_1^2 = x_2^2 = x_3^2 = x_4^2 = d^2x_1x_2x_3x_4 = 1$. Again $\theta$ must take each $x_i$ to the unique involution of $C_n$, and it also follows that the $\theta$-image of $d$ has order at most 2, giving $n = 2$ (and $g = 3$).

Next, if $\sigma = (0; +; [m_1, m_2, m_3]; \{-\})$, we have

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 2 - \frac{g - 1}{n} \geq 1,$$

and hence $(m_1, m_2, m_3)$ are the parameters of a spherical or Euclidean triangle group. Also $\Gamma$ is generated by elements $x_1, x_2, x_3, x_4$ and $c$ such that $x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = c^2 = [x_1x_2x_3, c] = 1$, and so for a non-orientable action of $C_n$ we need $\text{lcm}(m_1, m_2, m_3) = n$. Again by the Riemann-Hurwitz formula, it follows that we
have the following possibilities:

\[
\begin{align*}
[m_1, m_2, m_3] &= [3, 3, 3], \quad n = 3, \quad g = 4; && [m_1, m_2, m_3] &= [2, 4, 4], \quad n = 4, \quad g = 5; \\
[m_1, m_2, m_3] &= [2, 3, 3], \quad n = g = 6; && [m_1, m_2, m_3] &= [2, 3, 4], \quad n = g = 12; \\
[m_1, m_2, m_3] &= [2, 3, 5], \quad n = g = 30; && [m_1, m_2, m_3] &= [2, 3, 6], \quad n = 6, \quad g = 7; \\
[m_1, m_2, m_3] &= [2, 2, g], \quad n = g, \quad \text{even}; && [m_1, m_2, m_3] &= [2, 2, \frac{g+1}{2}], \\
&\quad n = g + 1 \equiv 2 \pmod{4}.
\end{align*}
\]

The first two give smaller values of \(n\) for the given \(g\) than we find in other cases, so we ignore them. The next four give sporadic examples for the cases where \(g\) is even and \(g \equiv 3 \pmod{4}\), and it is an easy exercise to show that these four are all achievable. The seventh one is achievable when \(\theta(x_1) = \theta(x_2) = \theta(c) = v^\frac{2}{3}\) and \(\theta(x_3) = v\), and the eighth when \(\theta(x_1) = \theta(x_2) = \theta(c) = v^{\frac{3+1}{2}}\) and \(\theta(x_3) = v^2\).

The same possibilities for \([m_1, m_2, m_3]\) arise when \(\sigma = (1; -; [m_1, m_2, m_3]; \{-\})\). In that case, however, \(\Gamma\) is generated by elements \(d, x_1, x_2\) and \(x_3\) such that \(x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = d^2 x_1 x_2 x_3 = 1\), and so for a non-orientable action of \(C_n\) we also need the \(\theta\)-image of \(x_1 x_2 x_3\) to be a square in \(C_n\). This restriction leaves the only possibilities of interest as \([m_1, m_2, m_3] = [2, 3, 6]\) and \([2, 2, \frac{g+1}{2}]\), with \(n = g - 1 = 6\) and \(n = g + 1\) respectively. Both are achievable — the latter when \(\theta(d) = v\), \(\theta(x_1) = \theta(x_2) = \theta(c) = v^{\frac{3+1}{2}}\) and \(\theta(x_3) = v^2\).

If \(\sigma = (0; +; [2, 2]; \{(-), (-)\})\), then \(n = g - 1\), and \(\Gamma\) is generated by elements \(x_1, x_2, e_1, c_1\) and \(c_2\) such that \(x_1^2 = x_2^2 = e_1^2 = c_2^2 = 1\). This is achievable for \(g \equiv 3 \pmod{4}\) whenever \(\theta(x_1) = \theta(x_2) = \theta(e_1) = \theta(c_2) = v^{\frac{3+1}{2}}\) and \(\theta(c_1) = v\).

Similarly, if \(\sigma = (1; -; [2, 2]; \{(-)\})\), then \(n = g - 1\), and \(\Gamma\) is generated by elements \(d, x_1, x_2\) and \(c\) such that \(x_1^2 = x_2^2 = c^2 = [d^2 x_1 x_2, c] = 1\). This time we can take \(\theta(d) = v\) and \(\theta(x_1) = \theta(x_2) = \theta(c) = v^{\frac{3+1}{2}}\) whenever \(g \equiv 3 \pmod{4}\).

Finally, if \(\sigma = (2; -; [2, 2]; \{-\})\), then \(n = g - 1\), and \(\Gamma\) is generated by elements \(d_1, d_2, x_1\) and \(x_2\) such that \(x_1^2 = x_2^2 = d_1^2 d_2 x_1 x_2 = 1\). Here we can take \(\theta(d_1) = v\), \(\theta(d_2) = v^{\frac{3+1}{2}}\) and \(\theta(x_1) = \theta(x_2) = v^{\frac{3+1}{2}}\) whenever \(g \equiv 3 \pmod{4}\).

As a consequence of Remark 2.2 and the proof of Theorem 3.2 we also have the following.

Corollary 3.3. If the cyclic group \(C_n\) acts faithfully on a non-orientable surface \(S\) of algebraic genus \(g \geq 2\), and either \(n > g + 1\) where \(g \equiv 1 \pmod{4}\), or \(n > g\) where \(g\) is even, or \(n > g - 1\) where \(g \equiv 3 \pmod{4}\), then also the dihedral group \(D_n\) acts faithfully on \(S\).

Next, it is well-known that every compact Riemann surface of genus \(g\) whose full automorphism group is cyclic of the largest possible order, namely \(4g + 2\), is hyperelliptic (see [11]). For non-orientable surfaces, however, the situation is slightly different, as we show below.

First we recall the definition of \(q\)-hyperellipticity. A surface \(S\) of algebraic genus \(g \geq 2\) is said to be \(q\)-hyperelliptic if it admits an involutory automorphism \(\phi\) such that the quotient surface \(S/\langle \phi \rangle\) has algebraic genus \(q\). If \(q = 0\), then \(S\) is hyperelliptic, while \(1\)-hyperelliptic surfaces are usually called elliptic-hyperelliptic.

Proposition 3.4. Let \(S\) be a compact non-orientable surface of algebraic genus \(g \geq 2\) whose full automorphism group is cyclic of the largest possible order for its genus \(g\) (as given in Table 2), but \(g \not\equiv 3, 6, 7, 12, 30\).
(1) If \( g \not\equiv 3 \mod 4 \), then \( S \) is hyperelliptic.
(2) If \( g \equiv 3 \mod 4 \), then \( S \) is elliptic-hyperelliptic but not hyperelliptic.

Proof. Let \( S = H/\Lambda \) where \( \Lambda \) is a torsion-free proper NEC group, and \( \text{Aut} \ S = \Gamma/\Lambda \), where \( \Gamma \) is an NEC group whose signature is one of those given in Table 2. Now \( \text{Aut} \ S \cong C_n = \langle v \mid v^n = 1 \rangle \), where \( n \) is even. Let \( \phi = v^{n/2} \) be the unique involution in \( \text{Aut} \ S \), and let \( \langle \phi \rangle = \Lambda_1/\Lambda \) where \( \Lambda_1 \) is an NEC group contained in \( \Gamma \). Then the algebraic genus of the quotient surface \( S/\langle \phi \rangle = H/\Lambda_1 \) can be read from the signature of \( \Lambda_1 \).

Observe that \( \sigma(\Lambda_1) \) has no link periods (by Proposition 11.1), and that its proper periods are all equal to 2 (since \( \Lambda \) is torsion-free and has index 2 in \( \Lambda_1 \)). It follows that

\[ \sigma(\Lambda_1) = (\gamma', +; [2, r', 2]; \{(-), k', (-)} \]

for non-negative integers \( \gamma', r' \) and \( k' \). The algebraic genus \( q \) of \( S/\langle \phi \rangle \) is then \( \alpha' \gamma' + k' - 1 \), where \( \alpha' = 1 \) or 2 according to the sign of \( \sigma(\Lambda_1) \). Applying the Riemann-Hurwitz formula to \( \Lambda_1 \subseteq \Gamma \), we obtain

\[ q - 1 + \frac{r'}{2} = \mu(\Lambda_1) = |\Gamma : \Lambda_1| \cdot \mu(\Gamma) = \frac{n}{2} \cdot \mu(\Gamma), \]

and thus everything reduces to computing the number \( r' \) of proper periods in \( \sigma(\Lambda_1) \).

To do that, we apply Theorems 2.2.3 and 2.2.4 of [6], which reveal the relationship between \( r' \) and the proper periods of \( \Gamma \).

If \( g \equiv 1 \mod 4 \), then \( n = g + 1 \), and the proper periods of \( \Gamma \) are \( [2, 2, \frac{g+1}{2}] \), with \( \frac{g+1}{2} \) odd. Moreover, the Riemann-Hurwitz formula gives \( \mu(\Gamma) = \frac{g-1}{g+1} \) for each of the two possible signatures of \( \Gamma \) in this case (as given in Table 2). Letting \( x_i \) (for \( i = 1, 2, 3 \)) be the canonical elliptic generators of \( \Gamma \) that have these orders, we find that the orders of the cosets \( \Lambda_1 x_i \in \Gamma/\Lambda_1 \) are 1, 1, \( \frac{g+1}{2} \), respectively. A direct application of [6, Theorem 2.2.3] then shows that \( r' = g + 1 = n \), and we deduce from (3.1) above that \( q = 0 \); in other words, \( S \) is hyperelliptic.

A similar argument also shows that if \( g \) is even, then \( q = 0 \), so \( S \) is hyperelliptic. On the other hand, if \( g \equiv 3 \mod 4 \), then \( q = 1 \), so \( S \) is elliptic-hyperelliptic. \[ \square \]

4. The Full Cross-cap Genus of a Cyclic Group

The symmetric cross-cap number of a finite group \( G \) is the minimum topological genus of any compact non-orientable surface \( S \) (with empty boundary) on which \( G \) acts effectively as a group of automorphisms; see [13] (and also [10] for the original concept). A variant of this number was determined for all cyclic \( G \) by the first author in [4], namely the smallest such genus greater than 2. For instance, if \( n \equiv 2 \mod 4 \), then this number is \( n/2+1 \), attainable via a smooth epimorphism \( \theta: \Gamma \rightarrow C_n \) where \( \Gamma \) has signature \( (0; +; [2, n/2]; \{(-)} \). This signature was investigated in Case 11 in Section 2 where we found that the smooth epimorphism \( \theta \) always extends to a larger group action. Hence for all such \( n \), whenever \( C_n \) acts on a non-orientable surface of the minimum genus, the full automorphism group of the surface is strictly larger than \( C_n \). In this section we completely answer the following question: What is the minimum algebraic genus of a non-orientable surface \( S \) on which \( C_n \) acts effectively as the full automorphism group of \( S \)? We call this number the full cross-cap genus of the group.
Suppose the cyclic group $G$ acts on a non-orientable surface $H/\Lambda$ of algebraic genus $g$, and let $\theta: \Gamma \to G$ be the corresponding epimorphism, with $\ker \theta = \Lambda$. Since the signature of $\Lambda$ is $(g+1; -; [-]; \{-\})$, the Riemann-Hurwitz formula $\mu(\Lambda) = |G|\mu(\Gamma)$ gives
\[ g = 1 + |G|\mu(\Gamma). \]
In order to find the full cross-cap genus of $G$, we have to minimize the area of $\Gamma$ among all NEC groups admitting a smooth epimorphism onto $G$ which cannot be extended to a larger NEC group $\Gamma'$.

For $G = C_n$, Theorem 2.1 shows that $\Gamma$ cannot have any of the signatures $\sigma$ occurring in the second column of Table 1. In the following lemma, we find the only signatures for which the full cross-cap genus can be attained.

**Lemma 4.1.** If the full cross-cap genus of $C_n$ is attained by means of a smooth epimorphism $\theta: \Gamma \to C_n$, then the signature of $\Gamma$ is of the form
\[(0; +; [m_1, m_2, m_3]; \{-\}) \text{ or } (1; -; [m_1, m_2, m_3]; \{-\}),\]
where $2 \leq m_1 \leq m_2 \leq m_3$.

**Proof.** We first show that the area of $\Gamma$ is bounded above by
\[(4.1) \quad \mu(\Gamma) \leq 2 - \frac{2}{p} - \frac{1}{n}, \quad \text{where } p \text{ is the smallest prime divisor of } n.\]

To see this, note that for $n$ even, we may choose a maximal NEC group $\Delta$ with signature $(0; +; [2, 2, n]; \{-\})$ and define $\theta: \Delta \to C_n = \langle v \rangle$ by
\[ \theta(x_1) = \theta(x_2) = v^{n/2}, \quad \theta(x_3) = v, \quad \theta(e_1) = v^{-1}, \quad \theta(e_1) = v^{n/2}, \]
while for $n$ odd, we may choose a maximal NEC group $\Delta$ with signature $(1; -; [p, p, n]; \{-\})$ and define $\theta: \Delta \to C_n = \langle v \rangle$ by
\[ \theta(d_1) = v^{(n-1)/2-n/p}, \quad \theta(x_1) = \theta(x_2) = v^{n/p}, \quad \theta(x_3) = v. \]

It is easy to check that in both cases, $\theta$ is a smooth epimorphism, and then by the maximality of $\Delta$ it follows that $C_n$ acts as the full automorphism group of the surface $H/\ker \theta$. Also $\mu(\Delta) = 2 - 2/p - 1/n$ in both cases, with $p = 2$ if $n$ is even. Thus $\mu(\Gamma) \leq \mu(\Delta) = 2 - 2/p - 1/n$, giving inequality (4.1).

Next we consider signatures $\sigma$ for which $C_n$ acts as the full group and for which the area $\mu(\Gamma)$ satisfies the above inequality. Let $\sigma = (\gamma; \pm; [m_1, \ldots, m_r]; \{-\}, \{k, \}; \{-\})$.

When $n$ is even, we have $p = 2$, and then since $1/m_i \leq 1/2$ for all $i$, we find
\[ \alpha \gamma + k - 2 + \frac{r}{2} \leq \alpha \gamma + k - 2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) = \mu(\Gamma) \leq 1 - 1/n < 1, \]
which gives $\alpha \gamma + k + \frac{r}{2} < 3$. As in the proof of Theorem 3.2, we find there are seven different types of NEC signatures satisfying this condition, namely $(0; +; [m_1, m_2, m_3]; \{-\}), (1; -; [m_1, m_2, m_3]; \{-\}), (0; +; [m_1, m_2]; \{-\}), (1; -; [m_1, m_2]; \{-\}), (0; +; [m_1]; \{-\}), (1; -; [m_1]; \{-\})$ and $(2; -; [m_1]; \{-\})$. 

---

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
The last five of these, however, give non-maximal NEC signatures, so by Theorem 2.1 we may ignore them. This leaves only the two signature types $(0; +; [m_1, m_2, m_3]; \{-\})$ and $(1; -; [m_1, m_2, m_3]; \{-\})$, as required.

When $n$ is odd, the NEC group $\Gamma$ cannot contain reflections, so $k = 0$, and $\sigma$ is of the form $(\gamma; -; [m_1, \ldots, m_r]; \{-\})$ for some $\gamma > 0$. Then since $m_i \geq p$ for all $i$, we have
\[
2\left(1 - \frac{1}{p}\right) > 2 - \frac{2}{p} - \frac{1}{n} \geq \mu(\Gamma) = \gamma - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) \geq \gamma - 2 + r\left(1 - \frac{1}{p}\right),
\]
and further, since $1 - \frac{1}{p} \geq 1 - \frac{1}{3} = \frac{2}{3}$, this gives $6 > 3\gamma + 2(r-2)$; that is, $3\gamma + 2r < 10$.

The latter inequality has just four solutions for which the area $\mu(\Gamma)$ is positive, namely $(\gamma, r) = (3, 0), (2, 1), (1, 2)$ and $(1, 3)$. In the first three cases, we have non-maximal signatures $(3; -; [-]; \{-\}), (2; -; [m_1]; \{-\})$ and $(1; -; [m_1, m_2]; \{-\})$, which we can eliminate using Theorem 2.1. This leaves just the fourth case, which gives signature type $(1; -; [m_1, m_2, m_3]; \{-\})$. □

Conditions for an NEC group $\Gamma$ with given signature to admit a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ were given in [4]. For signature type $(0; +; [m_1, m_2, m_3]; \{-\})$, one requires only that $n$ is even and lcm$(m_1, m_2, m_3) = n$; see [4] Thm. 3.5. On the other hand, for signature type $(1; -; [m_1, m_2, m_3]; \{-\})$, the condition lcm$(m_1, m_2, m_3) = n$ is both necessary and sufficient for odd $n$, but an extra condition is required for even $n$; see [4] Thm. 3.7 and [6] Thm. 3.1.8. Observe, however, that these two signature types give the same area, namely $\mu(\Gamma) = 2 - (1/m_1 + 1/m_2 + 1/m_3)$. Hence all we need to do (for each $n$, whether even or odd) is find the maximum value of $1/m_1 + 1/m_2 + 1/m_3$ over the set
\[
T(n) = \{(m_1, m_2, m_3) : \ 2 \leq m_1 \leq m_2 \leq m_3 \text{ and lcm}(m_1, m_2, m_3) = n\}.
\]
The next lemma exhibits the triples $(m_1, m_2, m_3)$ for which this maximum is attained.

**Lemma 4.2.** Let $n = p_1^{e_1}p_2^{e_2} \cdots p_k^{e_k}$ be the prime-power decomposition of $n$, such that $p_1 < p_2 < \cdots < p_k$. Then the maximum value of $1/m_1 + 1/m_2 + 1/m_3$ over all triples $(m_1, m_2, m_3)$ in the set $T(n)$ is attained by:

(a) $(p_1, p_2, p_3)$, when $n$ is of the form $p_1p_2p_3$ with $3 < p_1 < p_2 < p_3 < \frac{p_1(p_2-1)}{p_2-p_1}$,

(b) $(p_1, p_1, \frac{n}{p_1})$, when $s > 1$ and $e_1 = 1$ and $n$ is not of the form in (a), and

(c) $(p_1, p_1, n)$ otherwise.

**Proof.** First observe that if $a, b$ and $c$ are rational numbers such that $a < c$ and $b > 1$, then $\frac{1}{ab} + \frac{1}{c} < \frac{1}{a} + \frac{1}{bc}$, since the latter is equivalent to $c + ab < bc + a$ and hence to $c - a < b(c-a)$. It follows that in order to maximize the expression $1/m_1 + 1/m_2 + 1/m_3$ over all triples $(m_1, m_2, m_3) \in T(n)$, we require $m_1$ and $m_2$ to be prime; moreover, if $m_1$ and $m_2$ are equal, then $m_1 = m_2 = p_1$, while if $m_1$ and $m_2$ are distinct, then $(m_1, m_2) = (p_1, p_2)$.

Now if $m_1 = m_2 = p_1$, then since $n = \text{lcm}(m_1, m_2, m_3) = \text{lcm}(p_1, m_3)$, we have $m_3 = n$ or $n/p_1$, and hence the maximum value for all triples in this case is attained when $m_3 = n/p_1$ or $n$. When $m_3 = n/p_1$, we need $s > 1$ and $e_1 = 1$ (because lcm$(m_1, m_2, m_3) = n$ is divisible by $p_1^{e_1}$), and so this possibility is covered by (b). Otherwise $m_3 = n$, as in (c).
On the other hand, if \((m_1, m_2) = (p_1, p_2)\), then \(m_3\) must be divisible by \(n/p_1p_2\), so \(m_3 = n/p_1p_2\) or \(n/p_2\) or \(n/p_1\) or \(n\). We can eliminate the last two possibilities, because in each case a larger value of \(1/m_1 + 1/m_2 + 1/m_3\) is obtainable by taking \(m_2 = p_1\) (and maintaining the condition \(\text{lcm}(m_1, m_2, m_3) = n\)).

If \((m_1, m_2, m_3) = (p_1, p_2, n/p_2)\), then \(e_2 = 1\) (since \(\text{lcm}(m_1, m_2, m_3) = n\) is divisible by \(p_2^{e_2}\)), so \(n = p_1^{e_1}p_2^q\) where \(q\) is coprime to \(p_1\) and \(p_2\). Now if the maximum is not also attained by \((p_1, p_1, n)\), then

\[
0 > \left( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{n} \right) - \left( \frac{1}{p_1} + \frac{1}{p_2} + \frac{p_2}{n} \right) = \frac{n(p_2 - p_1) + p_1p_2(1 - p_2)}{np_1p_2},
\]

and therefore \(n(p_2 - p_1) < p_1p_2(p_2 - 1)\). Dividing by \(p_1p_2(p_2 - 1)\), we find

\[
p_1^{e_1-1}q = \frac{n}{p_1p_2} < \frac{p_2 - 1}{p_2 - p_1} = 1 + \frac{p_1 - 1}{p_2 - p_1} \leq 1 + (p_1 - 1) = p_1,
\]

which forces \(e_1 = 1\) and \(q = 1\), so that \(n = p_1p_2\). But then \(m_3 = n/p_2 = p_1 < p_2 = m_2\), so this case is also impossible.

Finally, if \((m_1, m_2, m_3) = (p_1, p_2, n/p_1p_2)\), then \(e_1 = e_2 = 1\) (since \(\text{lcm}(m_1, m_2, m_3) = n\) is divisible by \(p_1^{e_1}\) and \(p_2^{e_2}\)), so \(n = p_1p_2q\) where \(q\) is coprime to \(p_1\) and \(p_2\). Moreover, \(q = n/p_1p_2 = m_3 > 1\), so \(s \geq 3\), and hence \(q = m_3 \geq p_3 > p_2 \geq 3\). Now if the maximum is not attained by \((p_1, p_1, n/p_1)\), then

\[
0 > \left( \frac{1}{p_1} + \frac{1}{p_2} + \frac{p_2}{n} \right) - \left( \frac{1}{p_1} + \frac{1}{p_2} + \frac{p_2}{n} \right) = \frac{n(p_2 - p_1) + p_1^2p_2(1 - p_2)}{np_1p_2},
\]

and therefore \(n(p_2 - p_1) < p_1^2p_2(p_2 - 1)\). Dividing by \(p_1p_2(p_2 - 1)\), we find

\[
q < \frac{p_1(p_2 - 1)}{p_2 - p_1} = p_1 + \frac{p_1(p_1 - 1)}{p_2 - p_1} \leq p_1 + p_1(p_1 - 1) = p_1^2.
\]

This shows that \(q\) cannot be divisible by the square of a prime, or by the product of two primes, so \(q\) itself is a prime. Thus \(n = p_1p_2p_3\), where \(3 < p_3 < \frac{p_1(p_2 - 1)}{p_2 - p_1}\), as given in (a).

By Lemma 4.3, and the Riemann-Hurwitz formula, these maxima give the full cross-cap genus of \(C_n\) for all \(n\), as follows.

**Theorem 4.3.** Let \(n = p_1^{e_1}p_2^{e_2}\ldots p_s^{e_s}\) be the prime-power decomposition of \(n\), such that \(p_1 < p_2 < \cdots < p_s\). Then the full cross-cap genus of the cyclic group \(C_n\) is:

(a) \(2p_1p_2p_3 - p_1p_2 - p_1p_3 - p_2p_3 + 1\) when \(n = p_1p_2p_3\) with \(3 < p_1 < p_2 < p_3 < \frac{p_1(p_2 - 1)}{p_2 - p_1}\),

(b) \(2n - \frac{2n}{p_1} - p_1 + 1\) when \(s > 1\) and \(e_1 = 1\) and \(n\) is not of the form in (a), and

(c) \(2n - \frac{2n}{p_1}\) otherwise.

We can also easily obtain the following theorem.
**Theorem 4.4.** For each integer $n > 1$, there exists some $g_0$ such that $C_n$ is the full automorphism group of some non-orientable surface of algebraic genus $g$ for every $g \geq g_0$.

**Proof.** For $n$ even, let $\Delta$ be a maximal NEC group with signature $(\gamma; -; [n, \ldots, n]; \{(-)\})$ where $\gamma > 0$ and $\gamma + r > 2$, and define $\theta : \Delta \to C_n = \langle v \rangle$ by

$$\theta(d_i) = v \text{ for } 1 \leq i \leq \gamma, \quad \theta(x_j) = v \text{ for } 1 \leq j \leq r,$$

while for $n$ odd, let $\Delta$ be a maximal NEC group with signature $(\gamma; -; [n, \ldots, n]; \{-\})$ where $\gamma > 0$ and $\gamma + r > 3$, and define $\theta : \Delta \to C_n = \langle v \rangle$ by

$$\theta(d_i) = v^{\alpha_i} \text{ for } 1 \leq i \leq \gamma \quad \text{and} \quad \theta(x_j) = v \text{ for } 1 \leq j \leq r,$$

where the $\alpha_i$ are chosen such that twice their sum is congruent to $-r \mod n$. (Note that this is always possible since $n$ is odd.) It is easy to check that in both cases, $\theta$ is a smooth epimorphism, and then by the maximality of $\Delta$ it follows that $C_n$ acts as the full automorphism group of the non-orientable surface $H/\ker \theta$.

The algebraic genus of this surface is given by

$$g = 1 + |G|\mu(\Gamma)$$

$$= \begin{cases} 1 + n(\gamma + 1 - 2 + r(1-1/n)) = n(\gamma-1) + r(n-1) + 1 & \text{for } n \text{ even,} \\ 1 + n(\gamma - 2 + r(1-1/n)) = n(\gamma-2) + r(n-1) + 1 & \text{for } n \text{ odd.} \end{cases}$$

By allowing $\gamma$ to take values from $1$ to $n - 1$ and letting $r$ be arbitrary (but non-negative and subject to $\gamma + r > 2$ or $\gamma + r > 3$ respectively), we can make this genus equal to any integer greater than $(n-1)^2$. \hfill \Box

Note that $g_0$ may be taken to be smaller than $(n-1)^2$ when $n > 3$. Table 3 shows the exact range of genera of non-orientable surfaces on which $C_n$ acts as the full automorphism group, for small $n$. We leave the verification of details as an exercise for the reader.
Table 3. Algebraic genera of non-orientable surfaces on which $C_n$ acts as the full automorphism group, for $2 \leq n \leq 10$.

<table>
<thead>
<tr>
<th>$C_n$</th>
<th>Coverage</th>
<th>Signature type</th>
<th>Genera</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$</td>
<td>All $g \geq 2$</td>
<td>$(0;+;[2,\ldots,2];{(-)})$</td>
<td>$g = r - 1$, for $r \geq 3$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>All $g \geq 4$</td>
<td>$(1;-;[3,\ldots,3];{(-)})$</td>
<td>$g = 2r-2$ (even), for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2;-;[3,\ldots,3];{(-)})$</td>
<td>$g = 2r+1$ (odd), for $r \geq 2$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>All $g \geq 4$</td>
<td>$(0;+;[2,\ldots,2,2,4];{(-)})$</td>
<td>$g = 2r-2$ (even), for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(0;+;[2,\ldots,2,2,4,4];{(-)})$</td>
<td>$g = 2r-1$ (odd), for $r \geq 3$</td>
</tr>
<tr>
<td>$C_5$</td>
<td>All $g \geq 8$</td>
<td>$(1;-;[5,\ldots,5];{(-)})$</td>
<td>$g = 4r-4 \equiv 0 \text{ mod } 4$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2;-;[5,\ldots,5];{(-)})$</td>
<td>$g = 4r+1 \equiv 1 \text{ mod } 4$, for $r \geq 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(3;-;[5,\ldots,5];{(-)})$</td>
<td>$g = 4r+6 \equiv 2 \text{ mod } 4$, for $r \geq 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(4;-;[5,\ldots,5];{(-)})$</td>
<td>$g = 4r+11 \equiv 3 \text{ mod } 4$, for $r \geq 0$</td>
</tr>
<tr>
<td>$C_6$</td>
<td>All $g \geq 5$</td>
<td>$(0;+;[2,\ldots,2,2,3];{(-)})$</td>
<td>$g = 3r-4 \equiv 2 \text{ mod } 3$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(0;+;[2,\ldots,2,2,6];{(-)})$</td>
<td>$g = 3r-3 \equiv 0 \text{ mod } 3$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(0;+;[2,\ldots,2,2,3,6];{(-)})$</td>
<td>$g = 3r-2 \equiv 1 \text{ mod } 3$, for $r \geq 3$</td>
</tr>
<tr>
<td>$C_7$</td>
<td>All $g \geq 12$, $g \neq 16, 17, 23$</td>
<td>$(1;-;[7,\ldots,7];{(-)})$</td>
<td>$g = 6r-6 \equiv 0 \text{ mod } 6$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2;-;[7,\ldots,7];{(-)})$</td>
<td>$g = 6r+1 \equiv 1 \text{ mod } 6$, for $r \geq 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(3;-;[7,\ldots,7];{(-)})$</td>
<td>$g = 6r+8 \equiv 2 \text{ mod } 6$, for $r \geq 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(4;-;[7,\ldots,7];{(-)})$</td>
<td>$g = 6r+15 \equiv 3 \text{ mod } 6$, for $r \geq 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(5;-;[7,\ldots,7];{(-)})$</td>
<td>$g = 6r+22 \equiv 4 \text{ mod } 6$, for $r \geq 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(6;-;[7,\ldots,7];{(-)})$</td>
<td>$g = 6r+29 \equiv 5 \text{ mod } 6$, for $r \geq 0$</td>
</tr>
<tr>
<td>$C_8$</td>
<td>All $g \geq 8$</td>
<td>$(0;+;[2,\ldots,2,2,8];{(-)})$</td>
<td>$g = 4r-4 \equiv 0 \text{ mod } 4$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1;-;[2,\ldots,2];{(-)})$</td>
<td>$g = 4r+1 \equiv 1 \text{ mod } 4$, for $r \geq 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(0;+;[2,\ldots,2,4,8];{(-)})$</td>
<td>$g = 4r-2 \equiv 2 \text{ mod } 4$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1;-;[2,\ldots,2,4];{(-)})$</td>
<td>$g = 4r+3 \equiv 3 \text{ mod } 4$, for $r \geq 2$</td>
</tr>
<tr>
<td>$C_9$</td>
<td>All $g \geq 12$</td>
<td>$(1;-;[3,\ldots,3,3,9];{(-)})$</td>
<td>$g = 6r-6 \equiv 0 \text{ mod } 6$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2;-;[3,\ldots,3,9];{(-)})$</td>
<td>$g = 6r+1 \equiv 1 \text{ mod } 6$, for $r \geq 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1;-;[3,\ldots,3,9,9];{(-)})$</td>
<td>$g = 6r-4 \equiv 2 \text{ mod } 6$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2;-;[3,\ldots,3,9];{(-)})$</td>
<td>$g = 6r+3 \equiv 3 \text{ mod } 6$, for $r \geq 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1;-;[3,\ldots,3,3,9,9];{(-)})$</td>
<td>$g = 6r-2 \equiv 4 \text{ mod } 6$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2;-;[3,\ldots,3,9,9];{(-)})$</td>
<td>$g = 6r+5 \equiv 5 \text{ mod } 6$, for $r \geq 2$</td>
</tr>
<tr>
<td>$C_{10}$</td>
<td>All $g \geq 9$</td>
<td>$(0;+;[2,\ldots,2,2,5];{(-)})$</td>
<td>$g = 5r-6 \equiv 4 \text{ mod } 5$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(0;+;[2,\ldots,2,10];{(-)})$</td>
<td>$g = 5r-5 \equiv 0 \text{ mod } 5$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1;-;[2,\ldots,2];{(-)})$</td>
<td>$g = 5r+1 \equiv 1 \text{ mod } 5$, for $r \geq 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(0;+;[2,\ldots,2,5,5];{(-)})$</td>
<td>$g = 5r-3 \equiv 2 \text{ mod } 5$, for $r \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(0;+;[2,\ldots,2,2,5,10];{(-)})$</td>
<td>$g = 5r-2 \equiv 3 \text{ mod } 5$, for $r \geq 3$</td>
</tr>
</tbody>
</table>
REFERENCES


DEPARTAMENTO DE MATEMÁTICAS FUNDAMENTALES, FACULTAD DE CIENCIAS, UNED, C/ SENDA DEL REY S/N, 28040 MADRID, SPAIN

E-mail address: eb@mat.uned.es

DEPARTAMENTO DE MATEMÁTICAS FUNDAMENTALES, FACULTAD DE CIENCIAS, UNED, C/ SENDA DEL REY S/N, 28040 MADRID, SPAIN

E-mail address: jcirre@mat.uned.es

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AUCKLAND, PRIVATE BAG 92019, AUCKLAND 1142, NEW ZEALAND

E-mail address: m.conder@auckland.ac.nz

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use