

## NON-UNIFORM HYPERBOLICITY AND NON-UNIFORM SPECIFICATION

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ABSTRACT. In this paper we deal with an invariant ergodic hyperbolic measure  $\mu$  for a diffeomorphism  $f$ , assuming that  $f$  is either  $C^{1+\alpha}$  or  $C^1$  and the Oseledec splitting of  $\mu$  is dominated. We show that this system  $(f, \mu)$  satisfies a weaker and non-uniform version of specification, related with notions studied in several recent papers.

Our main results have several consequences: as corollaries, we are able to improve the results about quantitative Poincaré recurrence, removing the assumption of the non-uniform specification property in the main theorem of “Recurrence and Lyapunov exponents” by Saussol, Troubetzkoy and Vaienti that establishes an inequality between Lyapunov exponents and local recurrence properties. Another consequence is the fact that any such measure is the weak limit of averages of Dirac measures at periodic points, as in a paper by Sigmund. One can show that the topological pressure can be calculated by considering the convenient weighted sums on periodic points whenever the dynamic is positive expansive and every measure with pressure close to the topological pressure is hyperbolic.

### 1. INTRODUCTION

In seminal works, Bowen [3, 4] and Sigmund [25] introduced in the 1970s the notion of *specification* and used this to show many ergodic properties for dynamical systems satisfying this property, including subshifts of finite type and sofic subshifts, the restriction of an axiom A diffeomorphism to its non-wandering set, expanding differentiable maps, and geodesic flows on manifolds with negative curvature. Before we continue, let us recall the definition of the *uniform* specification property (SP) as defined by Bowen in [3].

We say that  $f : X \rightarrow X$  has the *specification property* (SP) if given  $\theta > 0$  there exists  $K_\theta \in \mathbb{N}$  such that for all  $x \in X$ , there exists  $p \in \mathbb{N}$  such that the dynamical ball

$$B_m^n(x, \theta) := \bigcap_{k=-m}^n f^{-k}(B(f^k(x), \theta))$$

contains a periodic point with period  $p \leq n + m + K_\theta$ .

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While the specification property is easily checked for full shifts and subshifts of finite type, it is a strong hypothesis for dynamical systems. For instance, in the one dimensional setting, a very interesting result shows that for continuous maps  $f : [0, 1] \rightarrow [0, 1]$ , specification holds if and only if  $f$  is topologically mixing (see [2]). On the other hand, it is known that for  $\beta$ -shifts, specification occurs rarely: it is verified only on a set of  $\beta$  of Lebesgue measure zero (see [6]).

A remarkable result is that given  $r \geq 1$ , every elementary set of a  $C^r$  Axiom A map has specification property. In the proof of this result shadowing and mixing properties play a central role. Let us discuss the shadowing property in detail.

Given a map  $f : M \rightarrow M$  and a real number  $\beta$ , we say that a (possibly infinite) string  $(x_1, x_2, \dots, x_n)$  of points  $x_i \in M$  is a  $\beta$ -pseudo-orbit, if for every  $1 \leq i < n$  we have that  $d(f(x_i), x_{i+1}) \leq \beta$ . A pseudo-orbit  $(x_i)_{i=1}^n$  is  $\alpha$ -shadowed by a point  $x$ , if for every  $0 \leq i < n$  we have that  $d(f^i(x), x_{i+1}) \leq \alpha$ . Finally, we say that  $f$  has the *shadowing property* if given  $\alpha > 0$  there exists  $\beta > 0$  such that every  $\beta$ -pseudo-orbit is  $\alpha$ -shadowed by some point in  $M$ . A classical theorem in the hyperbolic dynamics states that

**Theorem 1.1.** *If  $\Lambda$  is a hyperbolic set for a  $C^r$  map  $f$ ,  $r \geq 1$ , then  $f : \Lambda \rightarrow \Lambda$  has the shadowing property.*

The shadowing property has powerful consequences in hyperbolic dynamics. Just to illustrate, one consequence of this property is the so-called *closing lemma*. We say that a  $\beta$ -pseudo-orbit  $(x_i)_{i=1}^n$  is *periodic* if  $x_1 = x_n$ . The closing lemma states that if given  $\alpha > 0$  there exists  $\beta > 0$  such that for every periodic  $\beta$ -pseudo-orbit, there exists a *periodic* orbit that  $\alpha$ -shadows the pseudo-orbit. More precisely,

**Theorem 1.2.** *If  $\Lambda$  is a hyperbolic set for a diffeomorphism  $f$ , then given  $\alpha > 0$  there exists  $\beta > 0$  such that for every periodic  $\beta$ -pseudo-orbit  $(x_i)_{i=1}^n$  in  $\Lambda$  there is an  $n$ -periodic point  $x$  of  $f$  such that  $d(x_i, f^i(x)) < \alpha$  whenever  $0 \leq i < n$ . If  $\Lambda$  is locally maximal, then  $x \in \Lambda$ .*

Since several interesting features of a hyperbolic dynamical system can be deduced using shadowing, it attracts, on its own, the interest and attention of several recent studies. In particular, several attempts were made in order to generalize this notion beyond the hyperbolic setting.

An important result that provided shadowing in the non-uniformly hyperbolic setting was obtained by Pollicott in [22] for  $C^{1+\alpha}$  maps preserving a measure with non-zero Lyapunov exponents. From this result it is possible to obtain the so-called *Katok's Closing Lemma*, as in [13]. Since this version of shadowing property is more subtle and technical than the usual shadowing, making strong use of Lyapunov exponents, Oseledec's Theorem and Pesin's Theory, we postpone its discussion until Section 2, where we define the elements and state this theorem in detail.

If we relax the differentiability class and assume that the map studied is only  $C^1$ , Pesin's Theory is no longer available. In this case, in order to obtain non-uniform shadowing property, it is necessary to assume some additional regularity condition to control the stable and unstable manifolds, even for a measure with non-zero Lyapunov exponents. This was done successfully in a recent paper by Sun and the second author ([26]). In this work, they were able to prove a version of non-uniform

shadowing, assuming that the map is only  $C^1$  and that the Oseledec splitting is dominated. We will discuss this result in detail in Section 6.

Since for hyperbolic maps the specification property is deduced from the shadowing property, we could ask if some of the non-uniform shadowing properties previously discussed could lead us to equivalent versions of (non-uniform) specification. Before we continue this reasoning, let us point out that the search for non-uniform specification is as old as the specification itself. For instance, to the best of our knowledge, the first weaker version was introduced by Marcus in [18], showing that periodic measures are weakly dense for every ergodic toral automorphism and extending a previous work of Sigmund that proved the same for hyperbolic toral automorphisms.

For the non-uniformly hyperbolic setting, several versions of non-uniform specification, including [9, 14, 15, 26], were introduced and had been used to show many ergodic properties. For instance, Saussol, Troubetzkoy and Vaienti in [24] also introduced another non-uniform specification property and showed that every hyperbolic ergodic measure having non-uniform specification property satisfies an inequality between Lyapunov exponents and local recurrence rates. This notion is defined using *slow varying* functions that appear in Pesin’s Theory, i.e., functions  $q$  that satisfy  $q(f^{\pm 1}(x)) \leq e^\eta q(x)$  for every  $x \in M$ . Let us define the non-uniform specification precisely.

**Definition 1.3.** Let  $\mu$  be an invariant measure of  $f$ . We say that  $(f, \mu)$  satisfies non-uniform specification property (NS) if, for  $\mu$  almost every  $x$ , any small  $\eta > 0$ , any  $\eta$ -slowing varying positive function  $q$ , any integers  $m, n$ , and any  $\theta > 0$  there exists  $K := K(\eta, \theta, x, m, n)$  such that:

- (i) the non-uniform dynamical ball

$$\tilde{B}_m^n(x, \theta) := \bigcap_{k=-m}^n f^{-k} B(f^k(x), \theta q(f^k(x))^{-2})$$

contains a periodic point with period  $p \leq n + m + K$ ;

- (ii) the dependence of  $K$  on  $m, n$  satisfies

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} = 0.$$

In other words, pieces of orbits with length  $n$  are shadowed by periodic orbits whose period is less than  $n$  plus a sublinear term with respect to  $n$ .

A natural question arises as to whether this non-uniform specification property is valid for non-uniformly hyperbolic systems.

In [19], the first author had proved that every expanding ergodic measure for maps with non-flat critical set, i.e., a strongly mixing measure with positive exponents, has this non-uniform specification property. In this paper, we give a positive answer for hyperbolic ergodic measures to show that every hyperbolic ergodic measure naturally has the non-uniform specification property of [24] and thus we can remove the assumption of non-uniform specification property in the main theorem of [24] to establish an inequality between Lyapunov exponents and local recurrence properties and some other corollaries. Now we state our main results.

**Theorem 1.4.** *Let  $f$  be a  $C^{1+\alpha}$  ( $\alpha > 0$ ) diffeomorphism. Then, every  $f$  ergodic hyperbolic measure  $\mu$  satisfies NS.*

*Remark 1.5.* In particular, if  $\mu$  is a mixing hyperbolic measure, then we have a stronger version of the non-uniform specification property: for any  $p \geq n + m + K$ , the non-uniform dynamical ball  $\tilde{B}_m^n(x, \theta)$  contains a periodic point with period  $p$ .

*Remark 1.6.* Note that if the function  $q(\cdot) \equiv 1$ , then the non-uniform dynamical ball  $\tilde{B}_m^n(x, \theta, q)$  is the general well-known dynamical ball  $B_m^n(x, \theta)$ . So, in the conclusion of non-uniform specification property we can replace non-uniform dynamical balls by dynamical balls and  $\eta$  can be omitted.

We observe that the specification property that we just discussed has many similar different forms, as was presented in [31]. Let us discuss a slight generalization of the non-uniform specification above, which we call generalized non-uniform specification property with respect to several orbit segments (being a generalization of NS introduced in Definition 1.3).

**Definition 1.7** (Generalized non-uniform specification GNS). We say that  $\mu$  has the *generalized non-uniform specification property* if for  $\mu$  almost every  $x$ , any small  $\eta > 0$ , any  $\eta$ -slowing varying function  $q$  (that is,  $q(f^{\pm 1}(x)) \leq e^\eta q(x)$ ), any integer  $m, n$ , and any  $\theta > 0$  there exists  $K := K(\eta, \theta, x, m, n)$  satisfying

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} = 0$$

and so that the following holds: given points  $x_1, x_2, \dots, x_k$  in a full  $\mu$ -measure set and positive integers  $m_1, \dots, m_k, n_1, \dots, n_k$ , there is  $p_i \geq 0$  with

$$\sum_{i=1}^k p_i \leq \sum_{i=1}^k K(\eta, \theta, x_i, m_i, n_i)$$

(in particular, if  $\mu$  is mixing, for any  $t \geq \sum_{i=1}^k K(\eta, \theta, x_i, m_i, n_i)$  there is  $p_i$  with  $\sum_{i=1}^k p_i = t$ ) and a periodic point  $z$  with period  $p = \sum_{j=1}^k (n_j + m_j + p_j)$  such that

$$z \in \tilde{B}_{m_1}^{n_1}(x_1, \theta) := \bigcap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta q(f^j(x_1))^{-2}),$$

and for  $2 \leq i \leq k$ ,

$$f^{\sum_{j=1}^{i-1} (n_j + p_j) + \sum_{j=2}^i m_j}(z) \in \tilde{B}_{m_i}^{n_i}(x_i, \theta) := \bigcap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta q(f^j(x_i))^{-2}).$$

A potential application of the GNS property is the computation of the topological pressure or the Hausdorff dimension (for the one dimensional maps) of the set of irregular points in [27], i.e., the set of points such that the Birkhoff averages do not converge. Let us discuss this in more detail. For  $\alpha \in \mathbb{R}$ , we define

$$X(\phi, \alpha) = \{x \in X : \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = \alpha\}.$$

We define the multifractal spectrum for  $\phi$  to be  $L_\phi := \{\alpha \in \mathbb{R} : X(\phi, \alpha) \neq \emptyset\}$ .

**Theorem 1.8** (see [27]). *Suppose  $f : X \rightarrow X$  has specification,  $\phi, \varphi \in C(X, \mathbb{R})$  and  $\alpha \in L_\phi$ . Then the pressure*

$$P_{X(\phi, \alpha)}(\varphi) = \sup\{h_\mu(f) + \int \phi d\mu, \mu \in M_f(X) \text{ and } \int \phi d\mu = \alpha\},$$

where  $M_f(X)$  denotes the space of all invariant measures.

In principle, the absence of shadowing could prevent a  $C^1$  diffeomorphism from having the GNS property. For example, it was proved by Díaz and Abdenur [1] that shadowing property does not hold for an open and dense subset of the  $C^1$  non-hyperbolic robust transitive diffeomorphisms. However, the existence of a hyperbolic invariant measure could help to produce the shadowing. Thus, one natural question that arises from Theorem 1.4 above is

**Question 1.9.** Let  $f$  be a  $C^1$  diffeomorphism. Does every  $f$  ergodic hyperbolic measure  $\mu$  satisfy GNS?

In particular, if  $q(x) \equiv 1$ , the required result in Question 1.9 is obviously valid for uniformly hyperbolic systems [25], since it can be deduced from classical (uniform) specification property [25] for dynamical balls. Moreover,  $K(\theta, x, m, n)$  can be chosen only dependent on  $\theta$  from [25].

Here we show that the answer to Question 1.9 is positive, either if  $f$  is  $C^{1+\alpha}$  or if  $f$  is  $C^1$  with dominated Oseledec splitting.

**Theorem 1.10.** *Let  $f$  be a  $C^{1+\alpha}$  ( $\alpha > 0$ ) diffeomorphism. Then every  $f$  ergodic hyperbolic measure  $\mu$  satisfies GNS.*

*Remark 1.11.* In particular, if  $q(x) \equiv 1$ , we also have a generalization of Remark 1.6 for the general well-known dynamical ball  $B_m^n(x, \theta)$ . I.e., in the conclusion of GNS we can replace non-uniform dynamical balls by dynamical balls and  $\eta$  can be omitted.

At the end of this section, we point out that the above results are also valid for  $C^1$  non-uniformly hyperbolic systems with dominated splitting which is based on a recent result of [28]. Before that we recall the notion of dominated splitting. Let  $\Delta$  be an  $f$ -invariant set and  $T_\Delta M = E \oplus F$  be a  $Df$ -invariant splitting on  $\Delta$ .  $T_\Delta M = E \oplus F$  is called  $(S_0, \lambda)$ -dominated on  $\Delta$  (or simply dominated) if there exist two constants  $S_0 \in \mathbb{Z}^+$  and  $\lambda > 0$  such that

$$\frac{1}{S} \log \frac{\|Df^S|_{E(x)}\|}{m(Df^S|_{F(x)})} \leq -2\lambda, \quad \forall x \in \Delta, \quad S \geq S_0.$$

According to the Oseledec Theorem [21], every ergodic hyperbolic measure  $\mu$  has  $s$  ( $s \leq d = \dim M$ ) non-zero Lyapunov exponents

$$\lambda_1 < \dots < \lambda_r < 0 < \lambda_{r+1} < \dots < \lambda_s$$

with associated Oseledec splitting

$$T_x M = E_x^1 \oplus \dots \oplus E_x^s, \quad x \in O(\mu),$$

where we recall that  $O(\mu)$  denotes an Oseledec basin of  $\mu$ . Here in the above Oseledec splitting we denote  $E_x^1 \oplus \dots \oplus E_x^r$  to be a stable bundle and  $E_x^{r+1} \oplus \dots \oplus E_x^s$  to be an unstable bundle, respectively.

**Theorem 1.12.** *Let  $f$  be a  $C^1$  diffeomorphism. Then every ergodic hyperbolic measure  $\mu$  in whose Oseledec splitting the stable bundle dominates the unstable bundle on  $\text{supp}(\mu)$  satisfies GNS.*

## 2. $C^{1+\alpha}$ PESIN THEORY

In this section we give a quick review concerning some notions and results of  $C^{1+\alpha}$  Pesin Theory. We point the reader to [12, 13, 22] for more details. Let  $f : M \rightarrow M$  be a  $C^{1+\alpha}$  diffeomorphism. We recall the concept of a Pesin set and recall Katok’s shadowing lemma in this section.

**2.1. Pesin set.** Given  $\lambda, \mu \gg \varepsilon > 0$ , and for all  $k \in \mathbb{Z}^+$ , we define  $\Lambda_k = \Lambda_k(\lambda, \mu; \varepsilon)$  to be all points  $x \in M$  for which there is a splitting  $T_x M = E_x^s \oplus E_x^u$  with invariant property  $D_x f^m(E_x^s) = E_{f^m x}^s$  and  $D_x f^m(E_x^u) = E_{f^m x}^u$  satisfying:

- (a)  $\|Df^n|_{E_{f^m x}^s}\| \leq e^{\varepsilon k} e^{-(\lambda-\varepsilon)n} e^{\varepsilon|m|}, \forall m \in \mathbb{Z}, n \geq 1;$
- (b)  $\|Df^{-n}|_{E_{f^m x}^u}\| \leq e^{\varepsilon k} e^{-(\mu-\varepsilon)n} e^{\varepsilon|m|}, \forall m \in \mathbb{Z}, n \geq 1;$
- (c)  $\tan(\angle(E_{f^m x}^s, E_{f^m x}^u)) \geq e^{-\varepsilon k} e^{-\varepsilon|m|}, \forall m \in \mathbb{Z}.$

We set  $\Lambda = \Lambda(\lambda, \mu; \varepsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k$  and call  $\Lambda$  a Pesin set.

If for an ergodic hyperbolic measure  $\mu$  we denote by  $\lambda$  the absolute value of the largest negative Lyapunov exponent and  $\mu$  the smallest positive Lyapunov exponent, then for any  $0 < \varepsilon < \min\{\lambda, \mu\}$  one has a  $\mu$  full-measure Pesin set  $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$  (see, for example, Proposition 4.2 in [22]). Also, for any point  $x \in O(\mu) \cap \Lambda$ ,  $E_x^s$  and  $E_x^u$  coincide with  $E_x^1 \oplus \dots \oplus E_x^r$  and  $E_x^{r+1} \oplus \dots \oplus E_x^s$ , respectively.

The following statements are elementary properties of Pesin blocks (see [22]):

- (a)  $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \subseteq \dots;$
- (b)  $f(\Lambda_k) \subseteq \Lambda_{k+1}, f^{-1}(\Lambda_k) \subseteq \Lambda_{k+1};$
- (c)  $\Lambda_k$  is compact for  $\forall k \geq 1;$
- (d) for  $\forall k \geq 1$  the splitting  $x \rightarrow E_x^u \oplus E_x^s$  depends continuously on  $x \in \Lambda_k.$

**2.2. Shadowing lemma.** We recall Katok’s shadowing lemma [22] in this subsection. Let  $(\delta_k)_{k=1}^{+\infty}$  be a sequence of positive real numbers. Let  $(x_n)_{n=-\infty}^{+\infty}$  be a sequence of points in  $\Lambda = \Lambda(\lambda, \mu, \varepsilon)$  for which there exists a sequence  $(s_n)_{n=-\infty}^{+\infty}$  of positive integers satisfying:

- (a)  $x_n \in \Lambda_{s_n}, \forall n \in \mathbb{Z};$
- (b)  $|s_n - s_{n-1}| \leq 1, \forall n \in \mathbb{Z};$
- (c)  $d(fx_n, x_{n+1}) \leq \delta_{s_n}, \forall n \in \mathbb{Z}.$

Then we call  $(x_n)_{n=-\infty}^{+\infty}$  a  $(\delta_k)_{k=1}^{+\infty}$ -pseudo-orbit. Given  $\tau > 0$ , a point  $x \in M$  is a  $\tau$ -shadowing point for the  $(\delta_k)_{k=1}^{+\infty}$ -pseudo-orbit if  $d(f^n x, x_n) \leq \tau \varepsilon_{s_n}, \forall n \in \mathbb{Z}$ , where  $\varepsilon_k = \varepsilon_0 e^{-\varepsilon k}$  and  $\varepsilon_0$  is a constant only dependent on the system of  $f$ .

**Lemma 2.1** (Shadowing lemma). *Let  $f : M \rightarrow M$  be a  $C^{1+\alpha}$  diffeomorphism with a non-empty Pesin set  $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$  and fixed parameters,  $\lambda, \mu \gg \varepsilon > 0$ . For  $\forall \tau > 0$  there exists a sequence  $(\delta_k)_{k=1}^{+\infty}$  such that for any  $(\delta_k)_{k=1}^{+\infty}$ -pseudo-orbit there exists a unique  $\tau$ -shadowing point.*

3. NON-UNIFORM SPECIFICATION AND RECURRENCE TIMES

Given  $x \in M$  and  $r > 0$ , denote the first return time of a ball  $B(x, r)$  radius  $r$  at  $x$  by

$$\tau(B(x, r)) := \min\{k > 0 \mid f^k(B(x, r)) \cap B(x, r) \neq \emptyset\}.$$

Then we can use our Theorem 1.4 to obtain as a corollary the following improvement of the main theorem in [24]. That is, we can delete the assumption of NS in [24].

**Corollary 3.1.** *Let  $f$  be a  $C^{1+\alpha}$  ( $\alpha > 0$ ) diffeomorphism. Then for every  $f$  ergodic hyperbolic measure  $\mu$ , one has for  $\mu$  a.e.  $x \in M$ ,*

$$\limsup_{r \rightarrow 0} \frac{\tau(B(x, r))}{-\log r} \leq \frac{1}{\lambda_u} - \frac{1}{\lambda_s},$$

where  $\lambda_u, \lambda_s$  are the minimal positive Lyapunov exponent and maximal negative Lyapunov exponent of  $\mu$ , respectively.

*Remark 3.2.* From the main theorem in [24], if  $h_\mu(f) > 0$ , then for  $\mu$  a.e.  $x \in M$ ,

$$\liminf_{r \rightarrow 0} \frac{\tau(B(x, r))}{-\log r} \geq \frac{1}{\Lambda_u} - \frac{1}{\Lambda_s},$$

where  $\Lambda_u, \Lambda_s$  are the maximal positive Lyapunov exponent and minimal negative Lyapunov exponent of  $\mu$ , respectively. Therefore, using our Corollary 3.1 we can get that if  $M$  is two dimensional and  $h_\mu(f) > 0$ , then for  $\mu$  a.e.  $x \in M$ ,

$$\lim_{r \rightarrow 0} \frac{\tau(B(x, r))}{-\log r} = \frac{1}{\lambda_u} - \frac{1}{\lambda_s} = \frac{1}{\Lambda_u} - \frac{1}{\Lambda_s}.$$

For completeness, here we recall the result of the uniformly hyperbolic case in [24].

**Theorem 3.3.** *Let  $f$  be a  $C^1$  diffeomorphism and  $\text{supp}(\mu)$  be a compact locally maximal hyperbolic set. Then for  $\mu$  a.e.  $x \in M$ ,*

$$\limsup_{r \rightarrow 0} \frac{\tau(B(x, r))}{-\log r} \leq \frac{1}{\lambda_u} - \frac{1}{\lambda_s},$$

where  $\lambda_u, \lambda_s$  are the minimal positive Lyapunov exponent and maximal negative Lyapunov exponent of  $\mu$ , respectively.

Now, we assume that  $f : X \rightarrow X$  is a homeomorphism on a compact metric space,  $\mu$  is an invariant measure and  $\Gamma$  is a subset of  $X$  with positive measure for  $\mu$ . For  $x \in \Gamma$ , define

$$\dots < t_{-2}(x) < t_{-1}(x) < t_0(x) = 0 < t_1(x) < t_2(x) < \dots$$

to be the all times such that  $f^{t_i}(x) \in \Gamma$  (called recurrence times). By the Poincaré Recurrence Theorem, the sequence above is well defined at  $\mu$  a.e.  $x \in \Gamma$ . Note that

$$t_1(f^{t_i}(x)) = t_{i+1}(x) - t_i(x).$$

In general, we call  $t_1$  the first recurrence time. An increasing sequence of natural numbers  $\{N_i\}_{i \geq 1}$  is said to be *non-lacunary* if  $\lim_{i \rightarrow +\infty} \frac{N_{i+1}}{N_i} = 1$ . For recurrence times, we have a basic proposition of their non-lacunary as follows.

**Proposition 3.4.** For  $\mu$  a.e  $x \in \Gamma$ ,

$$\lim_{i \rightarrow +\infty} \frac{t_{i+1}(x)}{t_i(x)} = 1 \quad \text{and} \quad \lim_{i \rightarrow -\infty} \frac{t_{i-1}(x)}{t_i(x)} = 1.$$

In particular,

$$\lim_{i,j \rightarrow +\infty} \frac{t_{i+1}(x) - t_{j-1}(x)}{t_i(x) - t_j(x)} = 1.$$

*Proof.* It can be proved by using the Borel-Cantelli lemma and the Kač lemma (see [20], Proposition 3.8). Here we give another direct proof of the first equality which only depends on the Birkhoff Ergodic Theorem. The proof of the remaining equalities are similar.

By the Birkhoff Ergodic Theorem, for any subset  $A \subseteq M$  the limit function

$$\chi_A^*(x) := \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_A(f^j(x))$$

exists for  $\mu$  a.e.  $x$  and  $f$ -invariant. Moreover,  $\int \chi_A^*(x) d\mu = \int \chi_A(x) d\mu = \mu(A)$ .

If  $A := \Gamma$ , we claim that for  $\mu$  a.e.  $x \in \Gamma$ ,  $\chi_\Gamma^*(x) > 0$  (this is a basic fact of recurrence, and we will give the proof below). Then, by the definition of  $t_i(x)$ , for  $\mu$  a.e  $x \in \Gamma$ ,

$$\lim_{i \rightarrow +\infty} \frac{i}{t_i(x)} = \lim_{t_i(x) \rightarrow +\infty} \frac{1}{t_i(x)} \sum_{j=0}^{t_i(x)-1} \chi_\Gamma(f^j(x)) = \chi_\Gamma^*(x) > 0.$$

Thus, for  $\mu$  a.e  $x \in \Gamma$ ,

$$\lim_{i \rightarrow +\infty} \frac{t_{i+1}(x)}{t_i(x)} = \lim_{i \rightarrow +\infty} \frac{t_{i+1}(x)}{i+1} \cdot \frac{i+1}{i} \cdot \frac{i}{t_i(x)} = \frac{1}{\chi_\Gamma^*(x)} \cdot 1 \cdot \chi_\Gamma^*(x) = 1.$$

Now we start to prove the claim. If  $\mu$  is ergodic, it is obvious since  $\chi_\Gamma^*(x) \equiv \int \chi_\Gamma(x) d\mu = \mu(\Gamma) > 0$  holds for a.e. point  $x$ . For the general invariant case, we prove by contradiction. Assume that the set  $\Gamma_1 := \{x \in \Gamma \mid \chi_\Gamma^*(x) = 0\}$  has  $\mu$  positive measure. Consider  $A := \Gamma_1$ ; then  $\chi_{\Gamma_1}^*(x)$  exists for  $\mu$  a.e.  $x$  and  $\int \chi_{\Gamma_1}^*(x) d\mu = \int \chi_{\Gamma_1}(x) d\mu = \mu(\Gamma_1) > 0$ . Thus there exists  $x \in \bigcup_{n \in \mathbb{Z}} f^n(\Gamma_1)$  such that  $\chi_{\Gamma_1}^*(x) > 0$ , since by definition  $\chi_{\Gamma_1}^*(x) \equiv 0$  for all  $x \in (\bigcup_{n \in \mathbb{Z}} f^n(\Gamma_1))^c$ . Recall that  $\chi_{\Gamma_1}^*(x)$  is  $f$ -invariant, and thus if we let  $y \in \Gamma_1$  such that  $f^m(y) = x$  for some integer  $m$ , then  $\chi_{\Gamma_1}^*(y) = \chi_{\Gamma_1}^*(f^{-m}(x)) = \chi_{\Gamma_1}^*(x) > 0$ . Note that  $\Gamma_1 \subset \Gamma$  implies  $\chi_{\Gamma_1}(\cdot) \leq \chi_\Gamma(\cdot)$ , and thus  $\chi_{\Gamma_1}^*(\cdot) \leq \chi_\Gamma^*(\cdot)$ . So

$$\chi_\Gamma^*(y) \geq \chi_{\Gamma_1}^*(y) > 0.$$

This contradicts the choice of  $\Gamma_1$  since  $y \in \Gamma_1$ . □

*Remark 3.5.* We emphasize the mainly used technique in Proposition 3.4 that for an increasing sequence of integers  $\{N_i\}_{i \geq 1}$ , one sufficient condition to get non-lacunary is that  $\lim_{i \rightarrow +\infty} \frac{i}{N_i} > 0$ . Thus this can also be used to characterize the non-lacunary of hyperbolic times [19] instead of using the Borel-Cantelli lemma.

*Remark 3.6.* We point out another equivalent statement of Proposition 3.4. That is,  $x$  satisfies  $\lim_{i \rightarrow +\infty} \frac{t_{i+1}(x)}{t_i(x)} = 1 \Leftrightarrow$  for any  $\epsilon > 0$  and there exists a large integer  $N(x)$  such that for all  $n \geq N(x)$  there is  $t \in [n, n + n\epsilon)$  such that  $t = t_i(x)$  for some  $i$ . The proof of this relation is trivial, but the technique of the latter version is also useful and has been essentially used in the proof to establish an (in)equality between

metric entropy of hyperbolic ergodic measure and the number of hyperbolic periodic points in [13, 5, 16]. One main technique used in [13, 5, 16] is that the recurrence time varies so slowly (linearly) that if the cardinality of a sequence of sets (for example, separated sets) is growing exponentially, then there exists a new sequence composed of their subsets such that the cardinality still grows exponentially and in every subset the recurrence times are the same for all points.

4. PROOF OF THEOREM 1.4: NS

Set  $\tilde{\Lambda}_k = \text{supp}(\mu|_{\Lambda_k})$  and  $\tilde{\Lambda} = \bigcup_{k=1}^{\infty} \tilde{\Lambda}_k$ . Clearly,  $f^{\pm 1}(\tilde{\Lambda}_k) \subset \tilde{\Lambda}_{k+1}$ , and the sub-bundles  $E^s(x)$ ,  $E^u(x)$  depend continuously on  $x \in \tilde{\Lambda}_k$ . Moreover,  $\tilde{\Lambda}$  is  $f$ -invariant with  $\mu$ -full measure. Let  $\Delta_k \subseteq \tilde{\Lambda}_k$  be the set of all points of  $x$  satisfying the fact that

- (i) recurrence times of  $x$  are well defined for the set  $\Gamma := \tilde{\Lambda}_k$  and
- (4.1) (ii) 
$$\lim_{i,j \rightarrow +\infty} \frac{t_{i+1}(x) - t_{-j-1}(x)}{t_i(x) - t_{-j}(x)} = 1.$$

By the Poincaré Recurrence Theorem and Proposition 3.4,  $\mu(\Delta_k) = \mu(\tilde{\Lambda}_k)$ , and thus  $\bigcup_{k \geq 1} \Delta_k$  is also a subset of  $M$  with full measure. So we only need to prove that for every fixed  $\Delta_k$  with positive measure, all points in  $\Delta_k$  satisfy the conditions of non-uniform specification property. Take and fix a point  $x \in \Delta_k$ .

Recall  $\varepsilon$  to be the number that appeared in the definition of a Pesin set. Let  $0 < \eta \leq \varepsilon/2$  and  $q$  be an  $\eta$ -slowing varying positive function,  $\theta > 0$  and let  $m, n$  be two positive integers. Let  $\tau = \frac{\theta q^{-2}(x)}{\varepsilon_0} > 0$ . By Lemma 2.1 for this  $\tau$  there exists a sequence  $(\delta_k)_{k=1}^{+\infty}$  such that for any  $(\delta_k)_{k=1}^{+\infty}$ -pseudo-orbit there exists a unique  $\tau$ -shadowing point.

Take and fix for  $\tilde{\Lambda}_k$  a finite cover  $\alpha_k = \{V_1, V_2, \dots, V_{r_k}\}$  by non-empty open balls  $V_i$  in  $M$  such that  $\text{diam}(U_i) < \delta_{k+1}$  and  $\mu(U_i) > 0$ , where  $U_i = V_i \cap \tilde{\Lambda}_k$ ,  $i = 1, 2, \dots, r_k$ . This is obtained from the definition of  $\tilde{\Lambda}_k$ . By the Birkhoff Ergodic Theorem and the ergodicity of  $\mu$  we have

$$(4.2) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{h=0}^{n-1} \mu(f^{-h}(U_i) \cap U_j) = \mu(U_i)\mu(U_j) > 0.$$

Then one can take

$$X_{i,j} := \min\{h \in \mathbb{N} \mid h \geq 1, \mu(f^{-h}(U_i) \cap U_j) > 0\}.$$

By (4.2),  $1 \leq X_{i,j} < +\infty$ . Let

$$M_k = \max_{1 \leq i,j \leq r_k} X_{i,j}.$$

Note that  $M_k$  is a positive integer dependent on  $k, \theta$  and the  $\eta$ -slowing varying positive function  $q$ , but independent of  $m, n$ . So

$$(4.3) \quad \frac{M_k}{m}, \frac{M_k}{n} \rightarrow 0$$

as  $m, n \rightarrow \infty$ , respectively.

Now we start to find the needed shadowing periodic orbit. Take positive integers  $l_1$  and  $l_2$  such that

$$(4.4) \quad t_{-l_1} < -m \leq t_{-l_1+1} \quad \text{and} \quad t_{l_2} > n \geq t_{l_2-1},$$

and take positive integers  $s_1$  and  $s_2$  such that

$$(4.5) \quad t_{-l_1-s_1} \leq (1 + \frac{2\eta}{\varepsilon})t_{-l_1} < t_{-l_1-s_1+1} \quad \text{and} \quad t_{l_2+s_2} \geq (1 + \frac{2\eta}{\varepsilon})t_{l_2} > t_{l_2+s_2-1}.$$

Since  $f^{t_{-l_1-s_1}}(x), f^{t_{l_2+s_2}}(x) \in \tilde{\Lambda}_k$ , we can take  $U_i$  and  $U_j$  such that

$$f^{t_{-l_1-s_1}}(x) \in U_i, f^{t_{l_2+s_2}}(x) \in U_j.$$

By (4.2), there exist  $y \in U_j$  and  $0 \leq N \leq M_k$  such that  $f^N(y) \in U_i$ .

Recall from the property of Pesin blocks that  $f^\pm(\Lambda_k) \subseteq \Lambda_{k+1}$ . Thus, if  $u \in \Lambda_k$ , then  $f^i(u) \in \Lambda_{k+|i|}, \forall i \in \mathbb{Z}$ . Note that

$$f^{t_{-l_1-s_1}}(x), x, f^{t_{l_2+s_2}}(x), y, f^N(y) \in \tilde{\Lambda}_k \subseteq \Lambda_k$$

and

$$d(f^{t_{l_2+s_2}}(x), y) < \delta_{k+1}, \quad d(f^{t_{-l_1-s_1}}(x), f^N(y)) < \delta_{k+1}.$$

So

$$\begin{aligned} f^{t_{-l_1-s_1}}(x) &\in \Lambda_k, f^{t_{-l_1-s_1+1}}(x) \in \Lambda_{k+1}, \dots, f^{t_{-l_1-s_1+i}}(x) \in \Lambda_{\min\{k+i, k+l_1+s_1-i\}}, \\ \dots, f^{-1}(x) &\in \Lambda_{k+1}, x \in \Lambda_k, f(x) \in \Lambda_{k+1}, \dots, f^i(x) \in \Lambda_{\min\{k+i, k+l_2+s_2-i\}}, \\ \dots, f^{t_{l_2+s_2-1}}(x) &\in \Lambda_{k+1}, y \in \Lambda_k, \dots, f^{i-1}(y) \in \Lambda_{\min\{k+i, k+N-i\}}, \\ \dots, f^{N-1}(y) &\in \Lambda_{k+1}. \end{aligned}$$

Repeat the above sequence of points infinitely many times, and thus we get a  $(\delta_k)_{k=1}^{+\infty}$ -pseudo-orbit. Then there exists a unique  $\tau$ -shadowing point  $z$ . Note that  $f^{t_{l_2+s_2}-t_{-l_1-s_1}+N}(z)$  is also a  $\tau$ -shadowing point. So

$$f^p(z) = z,$$

where  $p = t_{l_2+s_2} - t_{-l_1-s_1} + N$ .

Now we start to verify the conditions of non-uniform specification property for the above chosen shadowing point  $z$ . Define  $K := K(\eta, \theta, x, m, n) = t_{l_2+s_2} - t_{-l_1-s_1} + M_k - m - n$ . Clearly we have  $p \leq m + n + K$  since  $N \leq M_k$ , and thus for the first condition of NS we only need to show

$$z \in \tilde{B}_m^n(x, \theta) := \bigcap_{i=-m}^n f^{-i} B(f^k(x), \theta q(f^k(x))^{-2}).$$

First we consider  $0 \leq i \leq t_{l_2} - 1$  and calculate  $d(f^i(x), f^i(z))$ . More precisely,  $\tau$ -shadowing implies that

$$\begin{aligned} &d(f^i(x), f^i(z)) \\ &\leq \max\{\tau\varepsilon_{k+i}, \tau\varepsilon_{k+t_{l_2+s_2}-i}\} \\ &\leq \max\{\tau\varepsilon_i, \tau\varepsilon_{t_{l_2+s_2}-t_{l_2}}\} \quad (\text{note that } i \leq t_{l_2} \text{ and } \varepsilon_k \text{ is a decreasing sequence}) \\ &= \max\{\tau\varepsilon_0 e^{-i\varepsilon}, \tau\varepsilon_0 e^{-(t_{l_2+s_2}-t_{l_2})\varepsilon}\} \\ &\leq \max\{\tau\varepsilon_0 e^{-i\varepsilon}, \tau\varepsilon_0 e^{-2t_{l_2}\eta}\} \quad (\text{using (4.5)}) \\ &\leq \max\{\tau\varepsilon_0 e^{-2i\eta}, \tau\varepsilon_0 e^{-2i\eta}\} \quad (\text{using } 2\eta < \varepsilon \text{ and } i \leq t_{l_2}) \\ &= \tau\varepsilon_0 e^{-2i\eta} \\ &= \theta q^{-2}(x) e^{-2i\eta} \quad (\text{by the choice of } \tau) \\ &\leq \theta q(f^i(x))^{-2} \quad (\text{using } q(f^i(x)) \leq q(x) e^{i\eta}). \end{aligned}$$

Secondly we can follow the similar method to show that for  $t_{-l_1} + 1 \leq i \leq 0$ ,

$$d(f^i(x), f^i(z)) \leq \theta q(f^i(x))^{-2}.$$

Notice that  $t_{-l_1} < -m$  and  $n < t_{l_2}$ , and thus  $z \in \tilde{B}_m^n(x, \theta)$ .

At the end we prove the second condition of NS:

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} = 0.$$

In fact, by using (4.1), (4.3), (4.4) and (4.5), we have

$$\begin{aligned} & \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} \\ \leq & \limsup_{n \rightarrow +\infty} \frac{t_{l_2+s_2} - t_{-l_1-s_1}}{m + n} + \limsup_{m, n \rightarrow +\infty} \frac{M_k}{m + n} - 1 \\ \leq & \limsup_{m, n \rightarrow +\infty} \frac{t_{l_2+s_2} - t_{-l_1-s_1}}{t_{l_2-1} - t_{-l_1+1}} + 0 - 1 \quad (\text{using (4.4), (4.3)}) \\ = & \limsup_{m, n \rightarrow +\infty} \left( \frac{t_{l_2+s_2} - t_{-l_1-s_1}}{t_{l_2+s_2-1} - t_{-l_1-s_1+1}} \cdot \frac{t_{l_2+s_2-1} - t_{-l_1-s_1+1}}{t_{l_2} - t_{-l_1}} \cdot \frac{t_{l_2} - t_{-l_1}}{t_{l_2-1} - t_{-l_1+1}} \right) - 1 \\ = & \limsup_{m, n \rightarrow +\infty} \frac{t_{l_2+s_2-1} - t_{-l_1-s_1+1}}{t_{l_2} - t_{-l_1}} - 1 \quad (\text{using (4.1)}) \\ \leq & 1 + \frac{2\eta}{\varepsilon} - 1 = \frac{2\eta}{\varepsilon} \quad (\text{using (4.5)}). \end{aligned}$$

Letting  $\eta \rightarrow 0$ , one has

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} = 0,$$

so we complete the proof. □

*Remark 4.1.* In particular, if  $\mu$  is a mixing hyperbolic measure, we can replace inequality (4.2) by

$$(4.6) \quad \lim_{n \rightarrow +\infty} \mu(f^{-n}(U_i) \cap U_j) = \mu(U_i)\mu(U_j) > 0.$$

Then by (4.6) we can take a finite integer

$$X_{i, j} = \max\{n \in \mathbb{N} \mid n \geq 1, \mu(f^{-n}(U_i) \cap U_j) = 0\} + 1.$$

Let

$$M_k = \max_{1 \leq i, j \leq r_k} X_{i, j}.$$

Then for any  $N \geq M_k$  there exist  $y \in U_j$  such that  $f^N(y) \in U_i$ . So we can follow the above proof, and then the non-uniform specification can be stronger: for any  $p \geq n + m + K$ , the non-uniform dynamical ball  $\tilde{B}_m^n(x, \theta)$  contains a periodic point with period  $p$ .

### 5. PROOF OF THEOREM 1.10: GNS

In this section we prove Theorem 1.10. Before that, we show two propositions as follows.

**Proposition 5.1.** *Let  $f$  be a  $C^{1+\alpha}$  ( $\alpha > 0$ ) diffeomorphism. Then for any small  $0 < \sigma < 1$ , there is a subset  $\Lambda_\sigma^*$  with  $\mu(\Lambda_\sigma^*) > 1 - \sigma$  such that for every  $x \in \Lambda_\sigma^*$ , any small  $\eta > 0$ , any  $\theta_* > 0$  and any integer  $m, n$ , there exists  $K := K(\eta, \theta_*, x, m, n)$  satisfying*

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta_*, x, m, n)}{m + n} = 0,$$

and so that the following holds. Given points  $x_1, x_2, \dots, x_k$  in  $\Lambda_\sigma^*$  and positive integers  $m_1, \dots, m_k, n_1, \dots, n_k$ , there are numbers  $p_{*i} \geq 0$  ( $1 \leq i \leq k$ ) with

$$\sum_{i=1}^k p_{*i} \leq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$$

(in particular, if  $\mu$  is mixing, for any  $t \geq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$  there is  $p_{*i}$  with  $\sum_{i=1}^k p_{*i} = t$ ) and there exists a periodic point  $z_*$  with period  $p_* = \sum_{j=1}^k (n_j + m_j + p_{*j})$  such that

$$z_* \in \bigcap_{j=-m_1}^{n_1} f^{-j}(B(f^j(x_1), \theta_* e^{-2|j|\eta})),$$

and for  $2 \leq i \leq k$ ,

$$f^{\sum_{j=1}^{i-1} (n_j + p_{*j}) + \sum_{j=2}^i m_j}(z_*) \in \bigcap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta_* e^{-2|j|\eta}).$$

This property is also valid for one side case.

*Proof.* The proof is a generalization of that of Theorem 1.4. Recall  $\varepsilon$  to be the number that appeared in the definition of a Pesin set. Recall  $\Delta_{k_*}$  to be the set introduced in the proof of Theorem 1.4 and note that if  $k_*$  is large, then the measure of  $\Delta_{k_*}$  is close to 1. Let  $k_*$  be large enough such that  $\Delta_{k_*}$  satisfies  $\mu(\Delta_{k_*}) > 1 - \sigma$ . We will prove this  $\Delta_{k_*}$  is the required  $\Lambda_\sigma^*$ .

Let  $\tau = \frac{\theta_*}{\varepsilon_0} > 0$ . By Lemma 2.1 there exists a sequence  $(\delta_k)_{k=1}^{+\infty}$  such that for any  $(\ )_{k=1}^{+\infty}$ -pseudo-orbit there exists a unique  $\tau$ -shadowing point.

Let  $x \in \Delta_{k_*}$ ,  $0 < \eta \leq \varepsilon/2$  and  $m, n$  be two positive integers. Note that  $\delta_{k_*}$  is dependent on  $\tau = \frac{\theta_*}{\varepsilon_0}$  and  $\Lambda_{k_*}$ , but independent of  $x$ . This is different from the one in the proof of Theorem 1.4. Let  $K(\eta, \theta_*, x, m, n)$  be the number defined as in the proof of Theorem 1.4. (Note that the choices of  $l_1, s_1$  and  $l_2, s_2$  only depend on  $\Delta_{k_*}$ ,  $x, m, n$ , and the choice of  $M_{k_*}$  only depends on  $\delta_{k_*+1}$ , and thus  $K(\eta, \theta_*, x, m, n)$  only depends  $\Delta_{k_*}$  and  $\tau = \frac{\theta_*}{\varepsilon_0}$ .) So this  $K(\eta, \theta_*, x, m, n)$  also satisfies

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta_*, x, m, n)}{m + n} = 0.$$

Re-denote the  $l_1, s_1$  and  $l_2, s_2$  with respect to  $x$  in the proof of Theorem 1.4 by  $l_1(x), s_1(x)$  and  $l_2(x), s_2(x)$ .

Given points  $x_1, x_2, \dots, x_k$  in  $\Delta_{k_*}$  and positive integers  $m_1, \dots, m_k, n_1, \dots, n_k$ , similar to the proof of Theorem 1.4, we can take

$$y_i, f^{N_i}(y_i) \in \tilde{\Lambda}_{k_*}$$

with  $0 \leq N_i \leq M_{k_*}$  such that

$$d(f^{t_{l_2(x_i)+s_2(x_i)}}(x_i), y_i) < \delta_{k_*+1}, \quad d(f^{t_{-l_1(x_{i+1})-s_1(x_{i+1})}}(x_{i+1}), f^{N_i}(y_i)) < \delta_{k_*+1}, \quad \forall 1 \leq i \leq k_*,$$

where  $x_{k+1} = x_1$ . Note that

$$f^{t_{-l_1(x_i)-s_1(x_i)}}(x_i), x, f^{t_{l_2(x_i)+s_2(x_i)}}(x_i), y_i, f^{N_i}(y_i) \in \tilde{\Lambda}_{k_*} \subseteq \Lambda_{k_*}.$$

Similar to the proof of Theorem 1.4, by the shadowing lemma there is a periodic point  $z_*$  with period  $p_* = \sum_{j=1}^k (t_{l_2(x_j)+s_2(x_j)} - t_{l_1(x_j)+s_1(x_j)} + N_j)$  such that

$$z_* \in \bigcap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \tau \varepsilon_0 e^{-2|j|\eta}) = \bigcap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta_* e^{-2|j|\eta}),$$

and for  $2 \leq i \leq k$ ,

$$f^{\sum_{j=1}^{i-1} (t_{l_2(x_j)+s_2(x_j)} + N_j) - \sum_{j=2}^i t_{l_1(x_j)+s_1(x_j)}}(z_*) \in \bigcap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta_* e^{-2|j|\eta}).$$

Let  $p_{*i} = t_{l_2(x_i)+s_2(x_i)} - n_i + N_i - t_{-l_1(x_{i+1})-s_1(x_{i+1})} - m_{i+1}$  where  $m_{k+1} = m_1$ , and thus  $\sum_{i=1}^k p_{*i} = \sum_{i=1}^k (t_{l_2(x_i)+s_2(x_i)} - t_{-l_1(x_i)-s_1(x_i)} + N_i - m_i - n_i) \leq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$ . Then the periodic point  $z_*$  satisfies that its period is  $p_* = \sum_{j=1}^k (n_j + m_j + p_{*j})$ ,

$$z_* \in \bigcap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta_* e^{-2|j|\eta}),$$

and for  $2 \leq i \leq k$ ,

$$f^{\sum_{j=1}^{i-1} (n_j + p_{*j}) + \sum_{j=2}^i m_j}(z_*) = f^{\sum_{j=1}^{i-1} (t_{l_2(x_j)+s_2(x_j)} + N_j) - \sum_{j=2}^i t_{l_1(x_j)+s_1(x_j)}}(z_*) \in \bigcap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta_* e^{-2|j|\eta}). \quad \square$$

**Proposition 5.2.** *Let  $f$  be a  $C^{1+\alpha}$  ( $\alpha > 0$ ) diffeomorphism. Then for any small  $0 < \sigma < 1$ , any small  $\eta > 0$ , any  $\eta$ -slowing varying positive function  $q$  (that is,  $q(f^{\pm 1}(x)) \leq e^\eta q(x)$ ) and any  $\theta > 0$ , there is a subset  $\Lambda_\sigma$  with  $\mu(\Lambda_\sigma) > 1 - \sigma$  such that for every  $x \in \Lambda_\sigma$ , any integer  $m, n$ , there exists  $K_* := K_*(\eta, \theta, x, m, n)$  satisfying*

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K_*(\eta, \theta, x, m, n)}{m + n} = 0$$

and so that the following holds. Given points  $x_1, x_2, \dots, x_k$  in  $\Lambda_\sigma$  and positive integers  $m_1, \dots, m_k, n_1, \dots, n_k$ , there is  $p_{*i} \geq 0$  with

$$\sum_{i=1}^k p_{*i} \leq \sum_{i=1}^k K_*(\eta, \theta, x_i, m_i, n_i)$$

(in particular, if  $\mu$  is mixing, for any  $t \geq \sum_{i=1}^k K_*(\eta, \theta, x_i, m_i, n_i)$  there is  $p_{*i}$  with  $\sum_{i=1}^k p_{*i} = t$ ) and a periodic point  $z_*$  with period  $p_* = \sum_{j=1}^k (n_j + m_j + p_{*j})$  such that

$$z_* \in \tilde{B}_{m_1}^{n_1}(x_1, \theta) := \bigcap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta q(f^j(x_1))^{-2}),$$

and for  $2 \leq i \leq k$ ,

$$f^{\sum_{j=1}^{i-1} (n_j + p_{*j}) + \sum_{j=2}^i m_j} (z_*) \in \tilde{B}_{m_i}^{n_i}(x_i, \theta) := \bigcap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta q(f^j(x_i))^{-2}).$$

This property is also valid for one side case.

*Proof.* Recall  $\varepsilon$  to be the number that appeared in the definition of a Pesin set. Let  $0 < \eta \leq \varepsilon/2$  and  $q$  be an  $\eta$ -slowing varying positive function. Recall  $\Lambda_{\frac{\sigma}{2}}^*$  to be the set introduced in the proof of Proposition 5.1 for  $\frac{\sigma}{2}$ . Let  $\theta_* > 0$  be small enough such that we can take  $\Lambda_\sigma \subseteq \Lambda_{\frac{\sigma}{2}}^*$  with  $\mu(\Lambda_\sigma) > 1 - \sigma$  and every point  $x \in \Lambda_\sigma$  satisfies  $\theta q^{-2}(x) \geq \theta_*$ .

Let  $x \in \Lambda_\sigma \subseteq \Lambda_{\frac{\sigma}{2}}^*$ ,  $m, n$  be two positive integers. Let  $K(\eta, \theta_*, x, m, n)$  be the number as in Proposition 5.1. Then if we can take

$$K_*(\eta, \theta, x, m, n) := K(\eta, \theta_*, x, m, n),$$

thus this  $K_*(\eta, \theta, x, m, n)$  also satisfies

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K_*(\eta, \theta, x, m, n)}{m + n} = 0.$$

Re-denote the  $l_1, s_1$  and  $l_2, s_2$  with respect to  $x$  in the proof of Theorem 1.4 by  $l_1(x), s_1(x)$  and  $l_2(x), s_2(x)$ .

Given points  $x_1, x_2, \dots, x_k$  in  $\Lambda_\sigma \subseteq \Lambda_{\frac{\sigma}{2}}^*$  and positive integers  $m_1, \dots, m_k, n_1, \dots, n_k$ , by Proposition 5.1, there is  $p_{*i} \geq 0$  with

$$\sum_{i=1}^k p_{*i} \leq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$$

(in particular, if  $\mu$  is mixing, for any  $t \geq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$  there is  $p_{*i}$  with  $\sum_{i=1}^k p_{*i} = t$ ), and there is a periodic point  $z_*$  with period  $p_* = \sum_{j=1}^k (n_j + m_j + p_{*j})$ ,

$$z_* \in \bigcap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta_* e^{-2|j|\eta}),$$

and for  $2 \leq i \leq k$ ,

$$f^{\sum_{j=1}^{i-1} (n_j + p_{*j}) + \sum_{j=2}^i m_j} (z_*) \in \bigcap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta_* e^{-2|j|\eta}).$$

Since  $x_i \in \Lambda_\sigma$ , then  $\theta q^{-2}(x_i) \geq \theta_*$ . Using  $q(f^j(x_i)) \leq e^{|j|\eta} q(x_i)$ , we have

$$\begin{aligned} z_* &\in \bigcap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta_* e^{-2|j|\eta}) \subseteq \bigcap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta q^{-2}(x_1) e^{-2|j|\eta}) \\ &\subseteq \bigcap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta q^{-2}(f^j(x_1))), \end{aligned}$$

and similarly for  $2 \leq i \leq k$ ,

$$f^{\sum_{j=1}^{i-1} (n_j + p_{*j}) + \sum_{j=2}^i m_j} (z_*) \in \bigcap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta q^{-2}(f^j(x_i))).$$

Now we start to prove Theorem 1.10.

*Proof of Theorem 1.10.* We use Proposition 5.2 to give a proof. Let  $\Lambda_\sigma$  be a fixed positive  $\mu$ -measure set from Proposition 5.2. Note that  $\bigcup_{j \geq 0} f^j(\Lambda_\sigma)$  is a full  $\mu$ -measure set by the ergodicity of  $\mu$ . We consider  $x \in \bigcup_{j \geq 0} f^j(\Lambda_\sigma)$ . Clearly we can take a finite (and fixed) number  $t(x) \geq 0$  (which can be chosen the first time) such that  $x \in f^{t(x)}(\Lambda_\sigma)$ , and thus  $f^{-t(x)}(x) \in \Lambda_\sigma$ . Let  $K_*(\eta, \theta, f^{-t(x)}(x), m, n + t(x))$  be the number as in Proposition 5.2 and define

$$K(\eta, \theta, x, m, n) := K_*(\eta, \theta, f^{-t(x)}(x), m, n + t(x)) + t(x).$$

Then  $K(\eta, \theta, x, m, n)$  satisfies

$$\begin{aligned} & \lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} \\ &= \lim_{\eta \rightarrow 0} \limsup_{m, n + t(x) \rightarrow +\infty} \frac{K_*(\eta, \theta, f^{-t(x)}(x), m, n + t(x)) + t(x)}{m + n + t(x)} = 0. \end{aligned}$$

Given  $x_1, \dots, x_k \in \bigcup_{j \geq 0} f^j(\Lambda_\sigma)$  and positive integers  $m_1, m_2, \dots, m_k, n_1, \dots, n_k$  large enough, we consider points  $f^{-t(x_1)}(x_1), \dots, f^{-t(x_k)}(x_k) \in \Lambda_\sigma$  and positive integers

$$m_1, m_2, \dots, m_k, n_1 + t(x_1), \dots, n_k + t(x_k).$$

By Proposition 5.2 there is  $p_{*i} \geq 0$  with

$$\sum_{i=1}^k p_{*i} \leq \sum_{i=1}^k K_*(\eta, \theta, f^{-t(x_i)}(x_i), m_i, n_i + t(x_i))$$

(in particular, if  $\mu$  is mixing, for any  $t \geq \sum_{i=1}^k K_*(\eta, \theta, f^{-t(x_i)}(x_i), m_i, n_i + t(x_i))$  there is  $p_{*i}$  with  $\sum_{i=1}^k p_{*i} = t$ ) and a periodic point  $z_*$  with period

$$p_* = \sum_{j=1}^k (m_j + n_j + t(x_j) + p_{*j})$$

such that

$$z_* \in \tilde{B}_{m_1}^{n_1 + t(x_1)}(f^{-t(x_1)}(x_1), \theta) := \bigcap_{j=-m_1}^{n_1 + t(x_1)} f^{-j} B(f^{j-t(x_1)}(x_1), \theta q (f^{j-t(x_1)}(x_1))^{-2}),$$

and for  $2 \leq i \leq k$ ,

$$\begin{aligned} & f^{\sum_{j=1}^{i-1} (n_j + t(x_j) + p_{*j}) + \sum_{j=2}^i m_j} (z_*) \in \tilde{B}_{m_i}^{n_i + t(x_i)}(f^{-t(x_i)}(x_i), \theta) \\ &= \bigcap_{j=-m_j}^{n_i + t(x_i)} f^{-j} B(f^{j-t(x_i)}(x_i), \theta q (f^{j-t(x_i)}(x_i))^{-2}). \end{aligned}$$

Let  $p_i = p_{*i} + t(x_{i+1})$ ,  $p = p_*$ , where  $x_{k+1} = x_1$ . Then

$$\begin{aligned} \sum_{i=1}^k p_i &\leq \sum_{i=1}^k K_*(\eta, \theta, f^{-t(x_i)}(x_i), m_i, n_i + t(x_i)) + \sum_{i=1}^k t(x_{i+1}) \\ &= \sum_{i=1}^k K(\eta, \theta, x_i, m_i, n_i + t(x_i)). \end{aligned}$$

Let  $z = f^{t(x_1)}(z_*)$ ; then  $z$  is the needed periodic point. More precisely,

$$\begin{aligned} z &= f^{t(x_1)}(z_*) \in f^{t(x_1)}\tilde{B}_{m_1}^{n_1+t(x_1)}(f^{-t(x_1)}(x_1), \theta) \\ &= f^{t(x_1)} \bigcap_{j=-m_1}^{n_1+t(x_1)} f^{-j} B(f^{j-t(x_1)}(x_1), \theta q(f^{j-t(x_1)}(x_1))^{-2}) \\ &= \bigcap_{j=-m_1-t(x_1)}^{n_1} f^{-j} B(f^j(x_1), \theta q(f^j(x_1))^{-2}) \subseteq \tilde{B}_{m_1}^{n_1}(x_1, \theta), \end{aligned}$$

and similarly for  $2 \leq i \leq k$ , we have

$$\begin{aligned} f^{\sum_{j=1}^{i-1}(n_j+p_j)+\sum_{j=2}^i m_j}(z) &= f^{\sum_{j=1}^{i-1}(n_j+p_j)+\sum_{j=2}^i m_j+t(x_1)}(z_*) \\ &= f^{t(x_i)} \circ f^{\sum_{j=1}^{i-1}(n_j+t(x_j)+p_{*j})+\sum_{j=2}^i m_j}(z_*) \\ &\in f^{t(x_i)}\tilde{B}_{m_i}^{n_i+t(x_i)}(f^{-t(x_i)}(x_i), \theta) \\ &= f^{t(x_i)} \bigcap_{j=-m_i}^{n_i+t(x_i)} f^{-j} B(f^{j-t(x_i)}(x_i), \theta q(f^{j-t(x_i)}(x_i))^{-2}) \\ &= \bigcap_{j=-m_i-t(x_i)}^{n_i} f^{-j} B(f^j(x_i), \theta q(f^j(x_i))^{-2}) \subseteq \tilde{B}_{m_i}^{n_i}(x_i, \theta). \quad \square \end{aligned}$$

*Remark 5.3.* From the above discussion, in fact one also has a precise description of  $p_{*i}$ :

$$p_{*i} \leq K_*(\eta, \theta, x_i, m, n) + K_*(\eta, \theta, x_{i+1}, m, n),$$

and then

$$p_i \leq K(\eta, \theta, x_i, m, n) + K(\eta, \theta, x_{i+1}, m, n).$$

In other words,  $p_{*i}, p_i$  only depends on the point  $x_i$  and the next point  $x_{i+1}$ .

### 6. $C^1$ PESIN THEORY

To prove Theorem 1.12 we need the exponentially shadowing lemma in [28] ( $C^1$  Pesin Theory). Before that we introduce some notions. Given  $x \in M$  and  $n \in \mathbb{N}$ , let

$$\{x, n\} := \{f^j(x) \mid j = 0, 1, \dots, n\}.$$

In other words,  $\{x, n\}$  represents the orbit segment from  $x$  to  $f^n(x)$  with length  $n$ . For a sequence of points  $\{x_i\}_{i=-\infty}^{+\infty}$  in  $M$  and a sequence of positive integers  $\{n_i\}_{i=-\infty}^{+\infty}$ , we call  $\{x_i, n_i\}_{i=-\infty}^{+\infty}$  a  $\delta$ -pseudo-orbit if  $d(f^{n_i}(x_i), x_{i+1}) < \delta$  for all  $i$ . Given  $\varepsilon > 0$  and  $\tau > 0$ , we call a point  $x \in M$  an (exponentially)  $(\tau, \varepsilon)$ -shadowing point for a pseudo-orbit  $\{x_i, n_i\}_{i=-\infty}^{+\infty}$  if

$$d(f^{c_i+j}(x), f^j(x_i)) < \tau \cdot e^{-\min\{j, n_i-j\}\varepsilon},$$

$\forall j = 0, 1, 2, \dots, n_i$  and  $\forall i \in \mathbb{Z}$ , where  $c_i$  is defined as

$$(6.7) \quad c_i = \begin{cases} 0, & \text{for } i = 0, \\ \sum_{j=0}^{i-1} n_j, & \text{for } i > 0, \\ -\sum_{j=i}^{-1} n_j, & \text{for } i < 0. \end{cases}$$

**Lemma 6.1.** *Let us assume same conditions as in Theorem 1.12. Then for each  $\sigma > 0$ , there exist a compact set  $\Lambda_\sigma \subseteq M$ ,  $\varepsilon_\sigma > 0$  and  $T_\sigma \in \mathbb{N}$  such that  $\mu(\Lambda_\sigma) > 1 - \sigma$  and the following (exponentially) shadowing lemma holds. For all  $\tau > 0$ , there exists  $\delta = \delta(\sigma, \tau) > 0$  such that if a  $\delta$ -pseudo-orbit  $\{x_i, n_i\}_{i=-\infty}^{+\infty}$  satisfies  $n_i \geq T_\sigma$  and  $x_i, f^{n_i}(x_i) \in \Lambda_\sigma$  for all  $i$ , then there exists an (exponentially)  $(\tau, \varepsilon_\sigma)$ -shadowing point  $x \in M$  for  $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ . If further  $\{x_i, n_i\}_{i=-\infty}^{+\infty}$  is periodic, i.e., there exists an integer  $m > 0$  such that  $x_{i+m} = x_i$  and  $n_{i+m} = n_i$  for all  $i$ , then the shadowing point  $x$  can be chosen to be periodic.*

Before ending this section we explain and introduce more about the above lemma of  $C^1$  Pesin Theory, providing an example as to why the requirement that the system of class  $C^1$  is paired with the requirement that the Oseledec splitting is dominated. First, the result of the shadowing lemma in [26] by Sun and the second author, motivated by Liao’s closing lemma [17] and its generalized shadowing lemma by Gan in [8] for quasi-hyperbolic orbit segments in the topological sense, was realized for almost every point in the statistical sense. For the convenience of the readers, here we state the shadowing lemma of [8]. Before that we need to recall a notion of Liao’s quasi-hyperbolic orbit segment [8, 17].

**Definition 6.2.** Fix arbitrarily two constants  $\zeta > 0$  and  $e \in \mathbb{Z}^+$  and consider an orbit segment

$$\{x, n\} := \{f^i(x) \mid i = 0, 1, 2, \dots, n\},$$

where  $x \in M$  and  $n \in \mathbb{N}$ . We call  $\{x, n\}$  a  $(\zeta, e, q)$ -quasi-hyperbolic orbit segment with respect to a splitting

$$T_x M = E_x^s \oplus E_x^u,$$

if  $\dim E_x^s = q$  and there is a partition

$$0 = t_0 < t_1 < \dots < t_m = n \quad (m \geq 1)$$

such that  $t_k - t_{k-1} \leq e$  and

- (1)  $\frac{1}{t_k} \sum_{j=1}^k \log \|Df^{t_j - t_{j-1}}|_{f^{t_{j-1}}(E_x^s)}\| \leq -\zeta,$
- (2)  $\frac{1}{t_m - t_{k-1}} \sum_{j=k}^m \log m(Df^{t_j - t_{j-1}}|_{f^{t_{j-1}}(E_x^u)}) \geq \zeta,$
- (3)  $\frac{1}{t_k - t_{k-1}} \log \frac{\|Df^{t_k - t_{k-1}}|_{f^{t_{k-1}}(E_x^s)}\|}{m(Df^{t_k - t_{k-1}}|_{f^{t_{k-1}}(E_x^u)})} \leq -2\zeta, \quad k = 1, 2, \dots, m.$

**Theorem 6.3** ([8]). *For any  $\zeta > 0$ ,  $e \in \mathbb{Z}^+$ , there exist  $L > 0$ ,  $d_0 > 0$  with the following property for any  $d \in (0, d_0]$ . If for a  $(\zeta, e)$ -quasi-hyperbolic  $d$ -pseudo-orbit  $\{x_i, n_i\}_{i=-\infty}^{+\infty}$  with respect to  $Df$ -invariant splittings  $T_{x_i} M = E^s(x_i) \oplus E^u(x_i)$  one has  $Df^{n_i}(E^\xi(x_i)) \cap T^\# M \subseteq U(E^\xi(x_{i+1}) \cap T^\# M, d)$  ( $\xi = s, u$ ) for all  $i$ , then there exists an  $Ld$ -shadowing point  $x \in M$  for  $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ . Moreover, if  $\{x_i, n_i\}_{i=-\infty}^{+\infty}$  is periodic, i.e., there exists an  $m > 0$  such that  $x_{i+m} = x_i$  and  $n_{i+m} = n_i$  for all  $i$ , then the shadowing point  $x$  can be chosen to be periodic with period  $c_m$ , where  $c_i$  is the same as in (6.7).*

Secondly, motivated by the existence of the Fake Stable Manifold Theorem for dominated splitting in [10], the second author used a fake stable manifold to realize the truth stable manifold and then generalized the shadowing to be exponentially shadowing [28]. There are also many applications. In [26] they used the shadowing lemma to generalize the classical result of Sigmund [25] that periodic measures are dense in invariant measures. Also in [28], the second author realized some results about the existence of horseshoe, the closure of an unstable manifold of periodic

points satisfying positive measure and such periodic points dense in the support of the given hyperbolic measure, from whose proof the stable manifold theorem and shadowing are both important, and furthermore the second author generalized the Livsic Theorem [5], from whose proof the exponentially shadowing played a crucial role.

7. PROOF OF THEOREM 1.12

Here we point out that the result of Lemma 6.1 is weaker than the statements of Katok’s shadowing lemma since it holds only for the pseudo-orbit whose beginning and ending points are in the same Pesin block. However, Lemma 6.1 is enough to prove Theorem 1.12, since it also can deduce all propositions in Section 5. In other words, every ergodic measure of a homeomorphism with the property stated as in Lemma 6.1 has (generalized) non-uniform specification. Here we only give a proof of non-uniform specification (Definition 1.3).

Since the given hyperbolic measure  $\mu$  is ergodic, the number  $\varepsilon_\sigma$  in Lemma 6.1 can be chosen independent of  $\sigma$  from Remark 1.4 in [28], and thus we can take a fixed number  $\varepsilon$ . In other words, for each  $\sigma > 0$  there exist a compact set  $\Lambda_\sigma \subseteq M$  and  $T_\sigma \in \mathbb{N}$  such that  $\mu(\Lambda_\sigma) > 1 - \sigma$ , and the (exponentially) shadowing lemma holds as in Lemma 6.1 for  $\varepsilon_\sigma \equiv \varepsilon$ .

Set  $\tilde{\Lambda}_\sigma = \text{supp}(\mu|_{\Lambda_\sigma})$  and  $\tilde{\Lambda} = \bigcup_{\sigma>0} \tilde{\Lambda}_\sigma$ . Clearly,  $\tilde{\Lambda}$  is of  $\mu$ -full measure. Let  $\Delta_\sigma \subseteq \tilde{\Lambda}_\sigma$  be the set of all points whose recurrence times are well defined for  $\Gamma = \tilde{\Lambda}_\sigma$  and satisfy

$$(7.8) \quad \lim_{i,j \rightarrow +\infty} \frac{t_{i+1}(x) - t_{-j-1}(x)}{t_i(x) - t_{-j}(x)} = 1.$$

By the Poincaré Recurrence Theorem and Proposition 3.4,  $\mu(\Delta_\sigma) = \mu(\tilde{\Lambda}_\sigma)$ , and thus  $\bigcup_{\sigma>0} \Delta_\sigma$  is a set with full measure. So we only need to prove that for every fixed  $\Delta_\sigma$  with positive measure, all points in  $\Delta_\sigma$  satisfy the conditions of non-uniform specification property. Take and fix a point  $x \in \Delta_\sigma$ .

Let  $0 < \eta \leq \varepsilon/2$  and  $q$  be an  $\eta$ -slowing varying positive function,  $\theta > 0$  and let  $m, n$  be two positive integers. We may assume that  $m, n \geq T_\sigma$  (otherwise, consider  $m' = m + T_\sigma, n' = n + T_\sigma$ ). Let  $\tau = \theta q^{-2}(x) > 0$ . Then for this  $\tau$  there exists  $\delta = \delta(\tau, \sigma) > 0$  satisfying Lemma 6.1.

Take and fix for  $\tilde{\Lambda}_\sigma$  a finite cover  $\alpha_\sigma = \{V_1, V_2, \dots, V_{r_\sigma}\}$  by non-empty open balls  $V_i$  in  $M$  such that  $\text{diam}(U_i) < \delta$  and  $\mu(U_i) > 0$ , where  $U_i = V_i \cap \tilde{\Lambda}_\sigma, i = 1, 2, \dots, r_\sigma$ . Since  $\mu$  is  $f$ -ergodic, by the Birkhoff Ergodic Theorem we have

$$(7.9) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{h=0}^{n-1} \mu(f^{-h}(U_i) \cap U_j) = \mu(U_i)\mu(U_j) > 0.$$

Then take

$$X_{i,j} = \min\{h \in \mathbb{N} \mid h \geq \mathbf{T}_\sigma, \mu(f^{-h}(U_i) \cap U_j) > 0\}.$$

By (7.9),  $\mathbf{T}_\sigma \leq X_{i,j} < +\infty$ . Let

$$M_\sigma = \max_{1 \leq i,j \leq r_\sigma} X_{i,j}.$$

Note that  $M_\sigma$  is dependent on  $\sigma, \theta$  and the  $\eta$ -slowing varying positive function  $q$ , but independent of  $m, n$ . So

$$(7.10) \quad \frac{M_\sigma}{m}, \frac{M_\sigma}{n} \rightarrow 0$$

as  $m, n \rightarrow \infty$ , respectively.

Take positive integers  $l_1$  and  $l_2$  such that

$$(7.11) \quad t_{-l_1} < -m \leq t_{-l_1+1} \quad \text{and} \quad t_{l_2} > n \geq t_{l_2-1}.$$

Take positive integers  $s_1$  and  $s_2$  such that

$$(7.12) \quad t_{-l_1-s_1} \leq (1 + \frac{2\eta}{\varepsilon})t_{-l_1} < t_{-l_1-s_1+1} \quad \text{and} \quad t_{l_2+s_2} \geq (1 + \frac{2\eta}{\varepsilon})t_{l_2} > t_{l_2+s_2-1}.$$

Take  $K := K(\eta, \theta, x, m, n) = t_{l_2+s_2} - t_{-l_1-s_1} + M_\sigma - m - n$ . The calculation of

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} = 0$$

is similar to the proof of Theorem 1.4 by using (7.11), (7.10), (7.8) and (7.12). We omit the details.

Since  $f^{t_{-l_1-s_1}}(x), f^{t_{l_2+s_2}}(x) \in \tilde{\Lambda}_\sigma$ , we can take  $U_i$  and  $U_j$  such that

$$f^{t_{-l_1-s_1}}(x) \in U_i, f^{t_{l_2+s_2}}(x) \in U_j.$$

By (7.9), there exist  $y \in U_j$  and  $\mathbf{T}_\sigma \leq N \leq M_\sigma$  such that  $f^N(y) \in U_i$ .

Note that

$$\begin{aligned} f^{t_{-l_1-s_1}}(x), x, f^{t_{l_2+s_2}}(x), y, f^N(y) &\in \tilde{\Lambda}_k \subseteq \Lambda_k, \\ -t_{-l_1-s_1} \geq m \geq T_\sigma, t_{l_2+s_2} \geq n \geq T_\sigma, N \geq T_\sigma \end{aligned}$$

and

$$d(f^{t_{l_2+s_2}}(x), y) < \delta, d(f^{t_{-l_1-s_1}}(x), f^N(y)) < \delta.$$

So if we repeat the orbit segments of

$$\{f^{t_{-l_1-s_1}}(x), -t_{-l_1-s_1}\}, \{x, t_{l_2+s_2}\}, \{y, N\}$$

infinitely many times, then we get a periodic  $\delta$ -pseudo-orbit. Then by Lemma 6.1 there exists a periodic point  $z$  with period  $p = t_{l_2+s_2} - t_{-l_1-s_1} + N$  such that

$$d(f^j(x), f^j(z)) < \tau \cdot e^{-\min\{j, t_{l_2+s_2}-j\}\varepsilon},$$

$\forall j = 0, 1, 2, \dots, t_{l_2+s_2}$  and

$$d(f^j(x), f^j(z)) < \tau \cdot e^{-\min\{-j, -t_{-l_1-s_1}+j\}\varepsilon},$$

$\forall j = t_{-l_1-s_1}, \dots, -2, -1, 0$ .

Now we start to verify the conditions of non-uniform specification property. Clearly we have  $p \leq m + n + K$  since  $N \leq M_\sigma$ , and thus we only need to show

$$z \in \tilde{B}_m^n(x, \theta) := \bigcap_{i=-m}^n f^{-i} B(f^k(x), \theta q(f^k(x))^{-2}).$$

First, we consider  $0 \leq i \leq t_{l_2} - 1$  and calculate  $d(f^i(x), f^i(z))$ . More precisely,

$$\begin{aligned} & d(f^i(x), f^i(z)) \\ & \leq \max\{\tau e^{-i\varepsilon}, \tau e^{-(t_{l_2}+s_2-i)\varepsilon}\} \\ & \leq \max\{\tau e^{-i\varepsilon}, \tau e^{-(t_{l_2}+s_2-t_{l_2})\varepsilon}\} \quad (\text{using } i \leq t_{l_2}) \\ & \leq \max\{\tau e^{-i\varepsilon}, \tau e^{-2t_{l_2}\eta}\} \quad (\text{using (7.12)}) \\ & \leq \max\{\tau e^{-2i\eta}, \tau e^{-2i\eta}\} \quad (\text{using } 2\eta < \varepsilon \text{ and } i \leq t_{l_2}) \\ & = \tau e^{-2i\eta} \\ & = \theta q^{-2}(x) e^{-2i\eta} \quad (\text{by the choice of } \tau) \\ & \leq \theta q(f^i(x))^{-2} \quad (\text{using } q(f^i(x)) \leq q(x) e^{i\eta}). \end{aligned}$$

Secondly we can follow a similar method to show that for  $t_{-l_1} + 1 \leq i \leq 0$ ,

$$d(f^i(x), f^i(z)) \leq \theta q(f^i(x))^{-2}.$$

Notice that  $t_{-l_1} < -m$  and  $n < t_{l_2}$ , and thus  $z \in \tilde{B}_m^n(x, \theta)$ . □

*Remark 7.1.* From the proofs of Theorem 1.12 and Theorem 1.4, we point out that for an invariant measure of a homeomorphism  $f : X \rightarrow X$  on a compact metric space, a sufficient condition of non-uniform specification for  $\mu$  is that: There exists  $\varepsilon > 0$  such that for any  $\sigma > 0$ , there is a subset  $\Gamma_\sigma$  with  $\mu$  positive measure larger than  $1 - \sigma$  such that for any  $\tau > 0$  and any integer  $m, n$ , if  $f^{-m}(x), x, f^n(x) \in \Gamma_\sigma$ , there exists  $L := L(\sigma, \tau, x, m, n)$  such that:

- (i) there exists a periodic point  $z$  with period  $p \leq m + n + L$  such that

$$d(f^j(x), f^j(z)) < \tau \cdot e^{-\min\{j, n-j\}\varepsilon},$$

$\forall j = 0, 1, 2, \dots, n$  and

$$d(f^j(x), f^j(z)) < \tau \cdot e^{-\min\{-j, m+j\}\varepsilon},$$

$\forall j = -m, \dots, -2, -1, 0$ ;

- (ii) the dependence of  $L$  on  $m, n$  satisfies

$$\limsup_{m, n \rightarrow +\infty} \frac{L(\sigma, \tau, x, m, n)}{m + n} = 0.$$

In particular, exponential shadowing is such a sufficient condition. Note that from the proofs of Theorem 1.12 and Theorem 1.4,  $L$  is the number  $M_k$  or  $M_\sigma$  independent of  $m, n$ . We can also give a similar sufficient condition with several orbit segments for generalized non-uniform specification (we omit the details). There are also some other related papers [30, 7, 11, 5, 28] where exponential shadowing plays an important role. Exponential closing (which is the particular exponential shadowing for one pseudo-orbit segment) has played a crucial role in [30, 7, 11] to prove that Lyapunov exponents of ergodic measures can be approximated by ones of periodic measures. It has also been used in [5, 28, 11] to calculate Hölder functions or Hölder cocycles to get some convergence properties for proving the corresponding Livšic Theorem.

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