GLEASON PARTS AND COUNTABLY GENERATED CLOSED IDEALS IN $H^\infty$

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Abstract. It is proved that a countably generated closed ideal in $H^\infty$ whose common zero set is contained in the union set of nontrivial Gleason parts of $H^\infty$ is generated by two Carleson-Newman Blaschke products as a closed ideal.

1. Introduction

Let $H^\infty$ be the Banach algebra of bounded analytic functions on the open unit disk $\mathbb{D}$ with the supremum norm $\| \cdot \|_\infty$. We denote by $M(H^\infty)$ the maximal ideal space of $H^\infty$, that is, $M(H^\infty)$ is the family of nonzero multiplicative linear functionals on $H^\infty$ with the weak*-topology. For a subset $E$ of $M(H^\infty)$, we denote by $\overline{E}$ the closure of $E$ in $M(H^\infty)$. We identify a function $f$ in $H^\infty$ with its Gelfand transform $\hat{f}(m) = m(f)$, $m \in M(H^\infty)$, so we think of $f$ as a continuous function on $M(H^\infty)$. For a sequence $\{a_n\}_n$ in $\mathbb{D}$ satisfying $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$, we have the Blaschke product

$$b(z) = \prod_{n=1}^{\infty} \frac{-\overline{a}_n z - a_n}{|a_n| - \overline{a}_n z}, \quad z \in \mathbb{D},$$

where if $a_n = 0$, we consider that $-\overline{a}_n/|a_n| = 1$. We call $\{a_n\}_n$ and $b(z)$ interpolating if for any bounded sequence of complex numbers $\{c_n\}_n$ there exists $f$ in $H^\infty$ such that $f(a_n) = c_n$ for every $n \geq 1$. In [2], Carleson gave a characterization of interpolating sequences. A Blaschke product $B$ is said to be Carleson-Newman if $B = \prod_{j=1}^{m} b_j$ for finitely many interpolating Blaschke products $b_1, b_2, \ldots, b_m$. In this case, there are many ways to give such a factorization. If $m$ is the minimal number of interpolating Blaschke products, $B$ is said to be a Carleson-Newman Blaschke product of order $m$. In the study of the structure of $H^\infty$, Carleson-Newman Blaschke products have played an important role (see [3, 5, 8, 11]). For Blaschke products $b_1$ and $b_2$, we write $b_1 \prec b_2$ if $b_1$ is a subproduct of $b_2$.

For $x, y \in M(H^\infty)$, the pseudo-hyperbolic distance is defined by

$$\rho(x, y) = \sup \{ \| f(x) \| : f(y) = 0, f \in H^\infty, \| f \|_\infty \leq 1 \}.$$

A subset $E$ of $M(H^\infty)$ is said to be $\rho$-separated if there is $\varepsilon > 0$ such that $\rho(x, y) \geq \varepsilon$ for every $x, y \in E$ with $x \neq y$. The set

$$P(x) = \{ y \in M(H^\infty) : \rho(y, x) < 1 \}$$

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is called the Gleason part of $H^\infty$ containing $x \in M(H^\infty)$. If $P(x) \neq \{x\}$, $P(x)$ is said to be nontrivial. We denote by $G$ the union set of all nontrivial Gleason parts in $M(H^\infty)$. In [2] (see also [3]), Hoffman studied the structure of Gleason parts of $H^\infty$ extensively. For $x \in M(H^\infty)$, he proved that $x \in G$ if and only if there is an interpolating Blaschke product $b$ satisfying $b(x) = 0$. He also proved that for an interpolating Blaschke product $b$, there exists $\varepsilon > 0$ such that $\{|b| < \varepsilon\} \subset G$, where

$$\{|b| < \varepsilon\} = \{x \in M(H^\infty) : |b(x)| < \varepsilon\}.$$ 

This fact shows that $G$ is an open subset of $M(H^\infty)$, and for a Carleson-Newman Blaschke product $B$ there is $\varepsilon > 0$ such that $\{|B| < \varepsilon\} \subset G$. Hoffman also showed that for a nontrivial Gleason part $P(x)$ of $H^\infty$, there is a one-to-one, onto and continuous map $L_x : \mathbb{D} \to P(x)$ such that $L_x(0) = x$ and $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. For $f \in H^\infty$, we write

$$Z(f) = \{x \in M(H^\infty) : f(x) = 0\}.$$ 

It is known that if $b$ is an interpolating Blaschke product with zeros $\{z_n\}_n$ in $\mathbb{D}$, then $Z(b) = \overline{\{z_n\}_n}$, $Z(b)$ is $\rho$-separated and homeomorphic to the Stone-\v{C}ech compactification of the set of natural numbers, so $Z(b)$ is a totally disconnected set (see [8], [7]). Hence if $B$ is a Carleson-Newman Blaschke product, then $Z(B)$ is also totally disconnected. Let $f \in H^\infty$. For $z \in \mathbb{D}$, we denote by $ord(f, z)$ the order of zero of $f$ at $z$. For $x \in G \setminus \mathbb{D}$, we define $ord(f, x) = ord(f \circ L_x, 0)$. For $x \in M(H^\infty) \setminus G$, we put as usual $ord(f, x) = \infty$ if $f(x) = 0$ and $ord(f, x) = 0$ if $f(x) \neq 0$. Clearly, if $b$ is an interpolating Blaschke product, then $ord(b, x) \leq 1$. If $b$ is a Carleson-Newman Blaschke product of order $m$, then $ord(b, x) \leq m$ for every $x$.

Let $I$ be a closed ideal in $H^\infty$. We write

$$Z(I) = \bigcap_{f \in I} Z(f)$$

and

$$ord(I, x) = \inf_{f \in I} ord(f, x), \quad x \in M(H^\infty).$$

For each $1 \leq j \leq \infty$ and $f \in H^\infty$, we put

$$Z_j(f) = \{x \in M(H^\infty) : ord(f, x) \geq j\}$$

and

$$Z_j(I) = \{x \in M(H^\infty) : ord(I, x) \geq j\}.$$ 

It seems very difficult to study ideal theory in $H^\infty$ generally (see [1]). In [4], Gorkin, Mortini and the first author proved the following two theorems for a closed ideal $I$ satisfying $Z(I) \subset G$. In this case, by Theorem 2.3 in [5], $I$ contains a Carleson-Newman Blaschke product, so $\sup_{x \in Z(I)} ord(I, x) < \infty$ and $Z(I)$ is totally disconnected (see also [14]).

**Theorem A.** Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$. Then $I$ coincides with the set of all $f$ in $H^\infty$ satisfying $ord(f, x) \geq ord(I, x)$ for every $x \in Z(I)$.

This shows that if $I_1, I_2$ are closed ideals in $H^\infty$ such that $Z(I_i) \subset G$ for $i = 1, 2$, $Z(I_1) = Z(I_2)$ and $ord(I_1, x) = ord(I_2, x)$ for every $x \in Z(I_1)$, then we have $I_1 = I_2$. 

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Theorem B. Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} \text{ord}(I, x)$. For each $1 \leq j \leq m$, let $U_j$ be an open subset of $M(H^\infty)$ satisfying $Z_j(I) \subset U_j$. Then $I$ is algebraically generated by Carleson-Newman Blaschke products $B$ of order $m$ in $I$ such that $Z_j(B) \subset U_j$ for $1 \leq j \leq m$.

The above two theorems give us a great deal of information about closed ideals $I$ satisfying $Z(I) \subset G$. In [12, 13], the authors studied closed ideals $I$ satisfying $Z(I) \subset G$ extensively.

For a sequence $\{f_n\}_n$ in $H^\infty$, we denote by $I[f_n : n \geq 1]$ the closed ideal in $H^\infty$ generated by functions $f_n, n = 1, 2, \cdots$; that is,

$$I[f_n : n \geq 1] = \bigcup_{n=1}^\infty \sum_{j=1}^n f_j H^\infty,$$

where the bar indicates the closure in $H^\infty$. The closed ideal $I[f_n : n \geq 1]$ is called a countably generated closed ideal in $H^\infty$. In this paper, we study the structure of countably generated closed ideals $I$ satisfying $Z(I) \subset G$. For a closed subset $E$ of $M(H^\infty)$, let $I(E) = \{ f \in H^\infty : f(x) = 0, x \in E \}$. Then $I(E)$ is a closed ideal in $H^\infty$ and $E \subset Z(I(E))$. For closed ideals $I_1, I_2, \cdots, I_m$ in $H^\infty$, let $\bigotimes_{i=1}^m I_i$ and $\bigotimes_{i=1}^m I_i$ be the tensor product and the closed tensor product of $I_1, I_2, \cdots, I_m$, respectively. That is, $\bigotimes_{i=1}^m I_i$ is an ideal generated by functions $\prod_{i=1}^m f_i$, where $f_i \in I_i, 1 \leq i \leq m$, and $\bigotimes_{i=1}^m I_i = \bigotimes_{i=1}^m I_i$. In Section 2, we shall prove the following theorem.

Theorem 1.1. Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} \text{ord}(I, x)$. Then the following conditions are equivalent.

(i) $I$ is a countably generated closed ideal.

(ii) There are compact $\varrho$-separated $G_\delta$-subsets $E_1, E_2, \cdots, E_m$ of $G$ such that $I = \bigotimes_{j=1}^m I(E_j)$.

(iii) There is a Carleson-Newman Blaschke product $B$ of order $m$ in $I$ such that $\text{ord}(B, x) = \text{ord}(I, x)$ for every $x \in Z(I)$, and $Z(I)$ is a $G_\delta$-set.

(iv) There are two Carleson-Newman Blaschke products $B_1, B_2$ in $I$ such that $I = I[B_1, B_2]$.

For a compact $\varrho$-separated $G_\delta$-subset $E$ of $G$, there is an interpolating Blaschke product $b$ satisfying $E \subset Z(b)$, and $I(E)$ is a countably generated closed ideal. We shall show in Example 2.14 that there exist compact $\varrho$-separated $G_\delta$-subsets $E_1$ and $E_2$ of $G$ such that $I(E_1) \cap I(E_2)$ is not countably generated. If $I$ is a countably generated closed ideal in $H^\infty$, then by Theorem 1.1, $Z_j(I)$ is a $G_\delta$-set for every $1 \leq j \leq \infty$. But if $I$ is the closed ideal given in Example 2.14, then $Z_2(I)$ is not a $G_\delta$-set.

2. Countably generated closed ideals

To prove Theorem 1.1 we need some lemmas. For a sequence $\{f_n\}_n$ in $H^\infty$ and $1 \leq j \leq \infty$, it is not difficult to show that

$$Z_j(I[f_n : n \geq 1]) = \bigcap_{n=1}^\infty Z_j(f_n)$$

and

$$\text{ord}(I[f_n : n \geq 1], x) = \inf_{n \geq 1} \text{ord}(f_n, x), \quad x \in Z(I[f_n : n \geq 1]).$$
Lemma 2.1. Let $B$ be a Carleson-Newman Blaschke product. Then $Z_j(B)$ is a closed $G_\delta$-set for every $1 \leq j < \infty$.

Proof. Let $B = \prod_{i=1}^k b_i$, where $b_i$ is an interpolating Blaschke product for every $1 \leq i \leq k$. Since $\text{ord}(b_i, x) \leq 1$ for $x \in M(H^\infty)$, we have that $Z_j(B) = \emptyset$ for $j > k$. Suppose that $1 \leq j \leq k$. Put $E_i = Z(b_i)$. Then $E_i$ is a closed $G_\delta$-set. We have

$$Z_j(B) = \bigcup \left\{ \bigcap_{\ell=1}^j E_{i_{\ell}} : 1 \leq i_1 < i_2 < \cdots < i_j \leq k \right\}.$$ 

Therefore $Z_j(B)$ is a closed $G_\delta$-set. \hfill \square

Lemma 2.2. If $f \in H^\infty$ and $f \neq 0$, then $Z_j(f)$ is a closed $G_\delta$-set for every $1 \leq j \leq \infty$.

Proof. Let $f = Bh$, where $B$ is a Blaschke product and $h \in H^\infty$ satisfying $|h| > 0$ on $\mathbb{D}$. Then $Z_\infty(h) = Z(h)$ and $Z_\infty(h)$ is a closed $G_\delta$-set. By Corollary 3.1 in [9], $Z_\infty(B)$ is a closed $G_\delta$-set. Then $Z_\infty(f) = Z_\infty(B) \cup Z_\infty(h)$ is a closed $G_\delta$-set. We have

$$Z(f) \setminus Z_\infty(f) = (Z(B) \cup Z(h)) \setminus Z_\infty(f) = (Z(B) \cup Z_\infty(h)) \setminus Z_\infty(f) = Z(B) \setminus Z_\infty(f).$$

By Lemma 4.6 in [9], $Z(B) \setminus Z_\infty(f)$ is a totally disconnected set. Hence there is a sequence of open and closed subsets $\{E_n\}_n$ of $Z(B)$ such that $Z(B) \setminus Z_\infty(f) = \bigcup_{n=1}^\infty E_n$ and $E_n \cap E_k = \emptyset$ for $n \neq k$. Let $b_n$ be the subproduct of $B$ with zeros $Z(B) \cap E_n \cap \mathbb{D}$ counting multiplicities. Since $Z(B) \cap \mathbb{D} \subset Z(B) \setminus Z_\infty(f)$, we have $B = \prod_{n=1}^\infty b_n$ and $Z(b_n) = E_n$ for every $n \geq 1$. We note that $b_n$ is a Carleson-Newman Blaschke product. For each $1 \leq j < \infty$, we have

$$Z_j(f) = Z_\infty(f) \cup \bigcup_{n=1}^\infty Z_j(b_n).$$

By Lemma 2.1 $Z_j(b_n)$ is a closed $G_\delta$-set; so is $Z_j(f)$. \hfill \square

Lemma 2.3. Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} \text{ord}(I, x)$. Then $I$ is a countably generated closed ideal if and only if $Z_j(I)$ is a closed $G_\delta$-set for every $1 \leq j \leq m$. In this case, $I$ is generated by countably many Carleson-Newman Blaschke products.

Proof. Suppose that $I = I[f_n : n \geq 1]$ for a sequence $\{f_n\}_n$ in $H^\infty$. For each $1 \leq j \leq m$, we have $Z_j(I) = \bigcap_{n=1}^\infty Z_j(f_n)$. By Lemma 2.2, $Z_j(I)$ is a closed $G_\delta$-set.

Suppose that $Z_j(I)$ is a closed $G_\delta$-set for every $1 \leq j \leq m$. For each $1 \leq j \leq m$, let $\{U_{j,n}\}_n$ be a sequence of open subsets of $G$ such that $Z_j(I) = \bigcap_{n=1}^\infty U_{j,n}$. By Theorem B, there is a sequence of Carleson-Newman Blaschke products $\{\varphi_n\}_n$ in $I$ such that $Z_j(\varphi_n) \subset U_{j,n}$ for every $1 \leq j \leq m$ and $n \geq 1$. Let $J = I[\varphi_n : n \geq 1]$. Then $J \subset I$ and $Z(I) \subset Z(J)$. We have $Z(J) \subset Z(\varphi_n) \subset U_{1,n}$ for every $n \geq 1$. Then $Z(J) \subset \bigcap_{n=1}^\infty U_{1,n} = Z_1(I) = Z(I)$. Hence $Z(J) = Z(I)$.

Let $x \in Z(I)$ and $\ell = \text{ord}(I, x)$. Since $\varphi_n \in I$, $\ell \leq \text{ord}(\varphi_n, x)$ for every $n \geq 1$. Since $x \notin Z_{\ell+1}(I)$, there is a positive integer $k$ such that $x \notin U_{\ell+1,k}$. Hence
$\ell \leq \text{ord}(\varphi_k, x) \leq \ell$. Therefore

$$\ell = \text{ord}(I, x) \leq \text{ord}(J, x) \leq \text{ord}(\varphi_k, x) = \ell.$$ 

Thus we get $\text{ord}(J, x) = \text{ord}(I, x)$ for every $x \in Z(I)$. By Theorem A, we have $J = I$. □

The following lemma follows from Theorem 3.1 in [10].

**Lemma 2.4.** Let $E$ be a compact $\rho$-separated subset of $G$ and $U$ be an open subset of $M(H^\infty)$ satisfying $E \subset U$. Then there exists an interpolating Blaschke product $b$ such that $E \subset Z(b) \subset U$.

**Lemma 2.5.** Let $E$ be a compact $\rho$-separated $G_\delta$-subset of $G$. Then $I(E)$ is a countably generated closed ideal in $H^\infty$, $E$ is a totally disconnected set, $Z(I(E)) = E$ and $\text{ord}(I(E), x) = 1$ for every $x \in E$.

**Proof.** By Lemma 2.4, there is an interpolating Blaschke product $b$ such that $E \subset Z(b) \subset G$. Hence $\text{ord}(I(E), x) = 1$ for every $x \in E$. Since $Z(b)$ is a totally disconnected set, so is $E$. Let $\{U_n\}_n$ be a sequence of open subsets of $G$ satisfying $E = \bigcap_{n=1}^{\infty} U_n$ and $Z(b) \cap U_n$ be an open and closed subset of $Z(b)$ for every $n \geq 1$. Let $b_n$ be the subproduct of $b$ with zeros $Z(b) \cap U_n \cap \mathbb{D}$. Then $E \subset Z(b_n) \subset U_n$. Let $J = I[b_n : n \geq 1]$. Then we have $J \subset I(E)$ and

$$E \subset Z(I(E)) \subset Z(J) \subset \bigcap_{n=1}^{\infty} U_n = E.$$ 

Hence $Z(I(E)) = Z(J) = E$. We have $\text{ord}(J, x) = 1$ for every $x \in E$. By Theorem A, we get $J = I(E)$. □

The following lemma follows from the definition of a closed tensor product.

**Lemma 2.6.** Let $I_1, I_2, \ldots, I_m$ be countably generated closed ideals in $H^\infty$. Then $\bigotimes_{j=1}^{m} I_j$ is a countably generated closed ideal, $Z(\bigotimes_{j=1}^{m} I_j) = \bigcup_{j=1}^{m} Z(I_j)$ and $\text{ord}(\bigotimes_{j=1}^{m} I_j, x) = \sum_{j=1}^{m} \text{ord}(I_j, x)$ for every $x \in Z(\bigotimes_{j=1}^{m} I_j)$.

For closed ideals $I_1, I_2, \ldots, I_m$ in $H^\infty$ satisfying $Z(I_j) \subset G$ for every $1 \leq j \leq m$, in [13] Corollary 9.15] the authors proved that $\bigotimes_{j=1}^{m} I_j = \bigotimes_{j=1}^{m} I_j$.

**Lemma 2.7.** Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $x \in Z(I)$. Let $B$ be a Carleson-Newman Blaschke product in $I$ and $W$ be an open subset of $M(H^\infty)$ satisfying $x \in W$. Then there is an open subset $U$ of $M(H^\infty)$ such that $x \in U \subset G \cap W$ and $Z(I) \cap U$ is an open and closed subset of $Z(I)$, and there is a Carleson-Newman Blaschke product $\varphi$ of order $\text{ord}(I, x)$ such that $Z(\varphi) \subset U$, $\varphi \prec B$ and $\text{ord}(I, y) \leq \text{ord}(\varphi, y) \leq \text{ord}(I, x)$ for every $y \in Z(I) \cap U$.

**Proof.** Since $Z(I)$ is a totally disconnected set (see [4] Theorem 2.2), we may take a sufficiently small open subset $U$ of $M(H^\infty)$ such that $x \in U \subset G \cap W$ and $Z(I) \cap U$ is an open and closed subset of $Z(I)$. Since $\text{ord}(I, y)$ is upper semicontinuous in $y \in Z(I)$ (see [4] Lemma 1.2]), we may assume that $\text{ord}(I, y) \leq \text{ord}(I, x)$ for every $y \in Z(I) \cap U$. Let $I_U = \{ f \in H^\infty : \text{ord}(f, y) \geq \text{ord}(I, y), y \in Z(I) \cap U \}$.

Then by Theorem A, $I_U$ is a closed ideal in $H^\infty$, $I \subset I_U$, $Z(I_U) = Z(I) \cap U$ and $\text{ord}(I_U, y) = \text{ord}(I, y)$ for every $y \in Z(I) \cap U$. By [13] Proposition 8.9], there is a
Carleson-Newman Blaschke product $\varphi$ of order $ord(I, x)$ in $I_U$ such that $Z(\varphi) \subset U$, $\varphi < B$ and $ord(\varphi, x) = ord(I_U, x)$. For each $y \in Z(I) \cap U$, we have

$$ord(I, y) = ord(I_U, y) \leq ord(\varphi, y) \leq ord(I, x).$$

\[ \square \]

Lemma 2.8. Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} ord(I, x)$. Let $W_1, W_2, \cdots, W_m$ be open subsets of $M(H^\infty)$ such that $Z_j(I) \subset W_j$ for every $1 \leq j \leq m$ and $W_m \subset W_{m-1} \subset \cdots \subset W_1$. Let $B$ be a Carleson-Newman Blaschke product in $I$. Then there is a Carleson-Newman Blaschke product $b$ such that $b \in I$, $b < B$ and $ord(b, y) \leq j$ for every $y \in Z(I) \cap (W_j \setminus W_{j+1})$ and $1 \leq j \leq m$, where $W_{m+1} = \emptyset$.

Proof. For each $x \in Z(I)$, since $Z(I) \subset \bigcup_{j=1}^m (W_j \setminus W_{j+1})$ there exists $1 \leq j \leq m$ such that $x \in W_j \setminus W_{j+1}$. Then $ord(I, x) \leq j$. By Lemma 2.7 there is an open subset $U_x$ of $M(H^\infty)$ satisfying that $x \in U_x \subset G \cap W_j$ and $Z(I) \cap U_x$ is an open and closed subset of $Z(I)$, and there is a Carleson-Newman Blaschke product $\varphi_x$ of order $ord(I, x)$ such that $Z(\varphi_x) \subset U_x$, $\varphi_x < B$ and $ord(I, y) \leq ord(\varphi_x, y) \leq ord(I, x)$ for every $y \in Z(I) \cap U_x$.

Since $Z(I)$ is a compact set, there is a finite set $\{x_1, x_2, \cdots, x_s\}$ in $Z(I)$ such that $Z(I) \subset \bigcup_{i=1}^s U_{x_i}$. Let

$$E_1 = Z(I) \cap U_{x_1}, \quad E_2 = (Z(I) \cap U_{x_2}) \setminus (Z(I) \cap U_{x_1}),$$

$$\cdots,$$

$$E_s = (Z(I) \cap U_{x_s}) \setminus \bigcup_{i=1}^{s-1} (Z(I) \cap U_{x_i}).$$

Then $E_i$ is an open and closed subset of $Z(I)$, $E_i \cap E_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^s E_i = Z(I)$. It may be that $x_i \notin E_i$ for some $1 \leq i \leq s$. We may take open subsets $V_1, V_2, \cdots, V_s$ of $M(H^\infty)$ satisfying that $E_i \subset V_i \subset U_x$, and $V_i \cap \overline{V}_j = \emptyset$ for $i \neq j$. Let $\psi_i$ be the Blaschke subproduct of $\varphi_x$ with zeros $Z(\varphi_x) \cap V_i \cap \mathbb{D}$ counting multiplicities. Then $Z(\psi_i) \cap Z(\psi_j) = \emptyset$ for $i \neq j$ and $ord(\psi_i, y) = ord(\varphi_x, y)$ for every $y \in E_i$ and $1 \leq i \leq s$. Let $b = \prod_{i=1}^s \psi_i$. Then $b < B$.

Let $y \in Z(I)$. Then there is the unique $1 \leq j \leq m$ such that $y \in W_j \setminus W_{j+1}$. Also there is the unique $1 \leq i \leq s$ such that $y \in E_i$. So we have

$$ord(b, y) = ord(\psi_i, y) = ord(\varphi_x, y) \leq ord(I, x_i).$$

Here we have two cases.

Case 1. Suppose that $x_i \in W_j \setminus W_{j+1}$. Then we have

$$ord(I, y) \leq ord(\varphi_x, y) \leq ord(I, x_i) \leq j.$$  

Hence $ord(I, y) \leq ord(b, y) \leq j$.

Case 2. Suppose that $x_i \in W_k \setminus W_{k+1}$ for some $k \neq j$. If $k < j$, then $ord(I, x_i) \leq k < j$. Hence

$$ord(I, y) \leq ord(\varphi_x, y) = ord(b, y) < j.$$  

If $k > j$, then $y \in U_{x_i} \subset W_k$. Since $y \notin W_{j+1}$ and $W_k \subset W_{j+1}$, we have $y \notin W_k$. This is a contradiction.

By the above two cases, we have $ord(I, y) \leq ord(b, y) \leq j$ for every $y \in Z(I) \cap (W_j \setminus W_{j+1})$. By Theorem A, we have $b \in I$. Thus we get the assertion. \[ \square \]
Lemma 2.9. Let $I$ be a countably generated closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} \ord(I, x)$. Let $B$ be a Carleson-Newman Blaschke product in $I$. Then there is a sequence of Carleson-Newman Blaschke products $\{b_n\}_n$ such that $b_1 < B$, $b_{n+1} < b_n$, $b_n \in I$ for every $n \geq 1$ and for each $x \in Z(I)$ there is a positive integer $n$ satisfying $\ord(I, x) = \ord(b_n, x)$.

Proof. By Lemma 2.3, $Z_j(I)$ is a closed $G_\delta$-set for every $1 \leq j \leq m$. For each $1 \leq j \leq m$, take a sequence of open subsets $\{W_{j,n}\}_n$ of $M(H^\infty)$ such that $\bigcap_{n=1}^\infty W_{j,n} = Z_j(I)$ and $W_{j,n+1} \subset W_{j,n}$ for every $n \geq 1$. Further we may assume that $W_{j+1,n} \subset W_{j,n}$ for every $1 \leq j \leq m$ and $n \geq 1$, where $W_{m+1,n} = \emptyset$ for every $n \geq 1$. By Lemma 2.8, there is a Carleson-Newman Blaschke product $b_1$ such that $b_1 < B$ and

$$\ord(b_1, y) \leq j$$

for every $y \in Z(I) \cap (W_{j,1} \setminus W_{j+1,1}$) and $1 \leq j \leq m$. By Lemma 2.8, again, there is a Carleson-Newman Blaschke product $b_2$ such that $b_2 < b_1$ and $\ord(b_2, y) \leq j$ for every $y \in Z(I) \cap (W_{j,2} \setminus W_{j+1,2})$ and $1 \leq j \leq m$. Inductively we may get a sequence of Carleson-Newman Blaschke products $\{b_n\}_n$ such that $b_n \in I$, $b_{n+1} < b_n$ and $\ord(b_n, y) \leq j$ for every $y \in Z(I) \cap (W_{j,n} \setminus W_{j+1,n})$ and $1 \leq j \leq m$.

Let $x \in Z(I)$ and $t = \ord(I, x)$. We consider two cases separately.

Case 1. Suppose that $t < m$. Then $x \notin Z_{t+1}(I)$ and there is a positive integer $k$ such that $x \in Z(I) \cap (W_{t,k} \setminus W_{t+1,k})$. Hence $\ord(b_k, x) \leq t$. Since $b_k \in I$, we have $t = \ord(I, x) \leq \ord(b_k, x) \leq t$. Thus we get $\ord(I, x) = \ord(b_k, x)$.

Case 2. Suppose that $t = m$, that is, $\ord(I, x) = m$. Then $x \in Z(I) \cap (W_{m,n} \setminus W_{m,n+1})$ for every $n \geq 1$. Hence $\ord(b_n, x) \leq m$. Since $b_n \in I$, we have $m \leq \ord(b_n, x)$. Thus we get $\ord(I, x) = \ord(b_n, x)$ for every $n \geq 1$.

The following is due to Hoffman [7].

Lemma 2.10. For any interpolating Blaschke product $b$ with zeros $\{z_n\}_n$ in $\mathbb{D}$, there exists a positive number $\lambda(b)$ such that a sequence $\{w_n\}_n$ in $\mathbb{D}$ satisfying $\rho(w_n, z_n) < \lambda(b)$ is an interpolating sequence.

Lemma 2.11. Let $I$ be a closed ideal in $H^\infty$ and $Z(I) \subset G$. Let $B$ be a Carleson-Newman Blaschke product in $I$. Then there is a Carleson-Newman Blaschke product $b$ in $I$ satisfying the following conditions.

(i) $\ord(b, x) = \ord(B, x)$ for every $x \in Z(I) \setminus \mathbb{D}$.
(ii) $\ord(b, z) = \ord(I, z)$ for every $z \in Z(I) \cap \mathbb{D}$.
(iii) $\ord(b, z) = 1$ for every $z \in (Z(B) \setminus Z(I)) \cap \mathbb{D}$.

Proof. Let $\varphi_1, \varphi_2, \ldots, \varphi_m$ be interpolating Blaschke products satisfying $B = \prod_{j=1}^m \varphi_j$. Let $\lambda = \min_{1 \leq j \leq m} \lambda(\varphi_j)$. Then $\lambda > 0$. Let $\{z_n\}_n = Z(B) \cap \mathbb{D}$ and $k_n = \ord(B, z_n)$. Then $\sup_{n \geq 1} k_n < \infty$. Let $\{\varepsilon_n\}_n$ be a sequence of numbers with $0 < \varepsilon_n < \lambda$ such that $\varepsilon_n \to 0$ as $n \to \infty$. We shall move the zeros of $B$ a little. Let $n$ be a positive integer. If $z_n \notin Z(I)$, then take $\{w_{n,1}, w_{n,2}, \ldots, w_{n,k_n}\}$ in $\mathbb{D}$ such that $\rho(w_{n,i}, z_n) < \varepsilon_n$, $w_{n,i} \neq w_{n,j}$ for $i \neq j$ and

$$\{w_{n,1}, w_{n,2}, \ldots, w_{n,k_n}\} \cap \{z_n\}_n = \emptyset.$$
Further, we may assume that
\[ \{w_{n,1}, w_{n,2}, \ldots, w_{n,k_n}\} \cap \{w_{j,1}, w_{j,2}, \ldots, w_{j,k_j}\} = \emptyset \]
for every \( n \neq j \) and
\[ \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} (1 - |w_{n,i}|) < \infty. \]

Let \( b \) be the Blaschke product with zeros \( \{w_{n,i}\}_{n,i} \) counting multiplicities. By Lemma 2.10, \( b \) is a Carleson-Newman Blaschke product. We have \( \text{ord}(b, x) = \text{ord}(B, x) \) for every \( x \in Z(I) \setminus \mathbb{D} \). It is easy to see that \( b \) satisfies (ii) and (iii). Since \( \text{ord}(I, x) \leq \text{ord}(b, x) \) for every \( x \in Z(I) \), by Theorem A we have \( b \in I \).

**Lemma 2.12.** Let \( B \) be a Carleson-Newman Blaschke product and \( \{z_n\}_n \) be an interpolating sequence in \( \mathbb{D} \). If \( 0 < \varepsilon < 1 \), then
\[ \inf_n \sup_z \{|B(z)| : z \in \mathbb{D}, \rho(z, z_n) < \varepsilon \} > 0. \]

**Proof.** To prove the assertion, suppose not. Then there exists a subsequence \( \{n_j\}_j \) such that
\[ \lim_{j \to \infty} \sup_{z} \{|B(z)| : z \in \mathbb{D}, \rho(z, z_{n_j}) < \varepsilon \} = 0. \]
Let \( x \) be a cluster point of \( \{z_{n_j}\}_j \) in \( M(H^\infty) \). By Hoffman’s work [7], it is easy to see that \( B \equiv 0 \) on \( P(x) \), the Gleason part of \( x \). By our assumption, \( B \not\equiv 0 \) on \( P(x) \), and this is a contradiction. \( \square \)

**Lemma 2.13.** Let \( B \) be a Carleson-Newman Blaschke product and \( b \) be an interpolating Blaschke product. Let \( E \) be a closed \( G_{\delta} \)-subset of \( Z(b) \). Then there is an interpolating Blaschke product \( \varphi \) such that \( E \subset Z(\varphi) \) and \( Z(B) \cap E = Z(B) \cap Z(\varphi) \).

**Proof.** If \( Z(B) \cap E = Z(B) \cap Z(b) \), then put \( \varphi = b \). Then we get the assertion. So we assume that \( Z(B) \cap E \subsetneq Z(B) \cap Z(b) \). By the assumptions, there is a sequence of closed subsets \( \{K_n\}_n \) of \( Z(b) \) such that
\[ (Z(B) \cap Z(b)) \setminus E = \bigcup_{n=1}^{\infty} K_n \]
and \( K_n \cap K_k = \emptyset \) for \( n \neq k \). We note that
\[ \bigcup_{n=1}^{\infty} K_n \setminus \bigcup_{n=1}^{\infty} K_n \subset E. \]
Take a sequence of open subsets \( \{U_n\}_n \) of \( M(H^\infty) \) such that \( K_n \subset U_n \), \( \overline{U_n} \cap \overline{U_k} = \emptyset \) for \( n \neq k \), \( E \cap \overline{U_n} = \emptyset \) and \( Z(b) \cap \overline{U_n} \) is an open and closed subset of \( Z(b) \) for every \( n \geq 1 \). Let \( b_n \) be the subproduct of \( b \) with zeros \( \{z_{n,\ell}\}_\ell := Z(b) \cap U_n \cap \mathbb{D} \). Then \( K_n \subset Z(b_n) \), \( E \cap Z(b_n) = \emptyset \) for every \( n \geq 1 \) and \( b = \prod_{n=0}^{\infty} b_n \) for some interpolating Blaschke product \( b_0 \). We note that
\[ (Z(B) \cap Z(b)) \setminus \bigcup_{n=1}^{\infty} Z(b_n) \subset E. \]

Let \( \{\varepsilon_n\}_n \) be a sequence of numbers such that \( 0 < \varepsilon_n < \lambda(b) \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \). By Lemma 2.12, there is a sequence of positive numbers \( \{\delta_n\}_n \) such that
\[ \sup \{|B(z)| : z \in \mathbb{D}, \rho(z, z_{n,\ell}) < \varepsilon_n \} > \delta_n \]
for every \( \ell \geq 1 \). For each \( \ell \geq 1 \), take \( w_{n,\ell} \in \mathbb{D} \) satisfying \( \rho(w_{n,\ell}, z_{n,\ell}) < \varepsilon_n \) and \( |B(w_{n,\ell})| > \delta_n \). By Lemma 2.10 \( \{w_{n,\ell}\}_\ell \) is an interpolating sequence for every \( n \geq 1 \). For each \( n \geq 1 \), let \( \varphi_n \) be the interpolating Blaschke product with zeros \( \{w_{n,\ell}\}_\ell \). Then \( Z(B) \cap Z(\varphi_n) = \emptyset \) and \( E \cap Z(\varphi_n) = \emptyset \) for every \( n \geq 1 \). Since
\[
\sup_{\ell \geq 1} \rho(w_{n,\ell}, z_{n,\ell}) \leq \varepsilon_n \to 0 \quad (n \to \infty),
\]
we have
\[
Z\left( \prod_{n=1}^{\infty} b_n \right) \setminus \bigcup_{n=1}^{\infty} Z(b_n) = Z\left( \prod_{n=1}^{\infty} \varphi_n \right) \setminus \bigcup_{n=1}^{\infty} Z(\varphi_n).
\]
Put \( \varphi = b_0 \prod_{n=1}^{\infty} \varphi_n \). Since
\[
\sup_{n,\ell \geq 1} \rho(w_{n,\ell}, z_{n,\ell}) < \lambda(b),
\]
by Lemma 2.10 \( \varphi \) is an interpolating Blaschke product. Since \( E \subset Z(b) \) and \( E \cap Z(b_n) = \emptyset \) for every \( n \geq 1 \), we have
\[
E \subset Z(b) \setminus \bigcup_{n=1}^{\infty} Z(b_n)
\quad = \quad \left( Z(b_0) \cup Z\left( \prod_{n=1}^{\infty} b_n \right) \right) \setminus \bigcup_{n=1}^{\infty} Z(b_n)
\quad = \quad \left( Z(b_0) \setminus \bigcup_{n=1}^{\infty} Z(b_n) \right) \cup \left( Z\left( \prod_{n=1}^{\infty} \varphi_n \right) \setminus \bigcup_{n=1}^{\infty} Z(\varphi_n) \right)
\quad = \quad Z(\varphi) \setminus \bigcup_{n=1}^{\infty} Z(\varphi_n).
\]
Hence \( E \subset Z(\varphi) \). Since \( Z(B) \cap Z(\varphi_n) = \emptyset \) for every \( n \geq 1 \), we have
\[
Z(B) \cap E \subset Z(B) \cap Z(\varphi) \subset Z(B) \cap \left( Z(\varphi) \setminus \bigcup_{n=1}^{\infty} Z(\varphi_n) \right)
\quad = \quad (Z(B) \cap Z(b)) \setminus \bigcup_{n=1}^{\infty} Z(b_n) \subset Z(B) \cap E.
\]
Hence we get \( Z(B) \cap E = Z(B) \cap Z(\varphi) \).
\[\square\]

**Proof of Theorem 1.1** (i) \( \Rightarrow \) (ii) By Theorem B, there is a Carleson-Newman Blaschke product \( b_1 \) of order \( m \) in \( I \). By Lemma 2.11, we may assume that \( \text{ord}(b_1, z) = \text{ord}(I, z) \) for every \( z \in Z(I) \cap \mathbb{D} \) and \( \text{ord}(b_{n+1}, z) = 1 \) for every \( z \in (Z(b_1) \setminus Z(I)) \cap \mathbb{D} \). By Lemma 2.9 there is a sequence of Carleson-Newman Blaschke products \( \{b_n\}_n \) such that \( b_n \in I \), \( b_{n+1} < b_n \) for every \( n \geq 1 \), and for each \( x \in Z(I) \) there is a positive integer \( n \) satisfying \( \text{ord}(b_n, x) = \text{ord}(I, x) \).

Since the order of \( b_1 \) is equal to \( m \), there are interpolating Blaschke products \( \varphi_{1,1}, \varphi_{2,1}, \ldots, \varphi_{m,1} \) such that \( b_1 = \prod_{j=1}^{m} \varphi_{j,1} \). Since \( b_n \in I \) and \( b_{n+1} < b_n \) for every \( n \geq 1 \), we have \( \text{ord}(b_n, z) = \text{ord}(I, z) \) for \( z \in Z(I) \cap \mathbb{D} \) and \( \text{ord}(b_n, z) = 1 \) for \( z \in (Z(b_n) \setminus Z(I)) \cap \mathbb{D} \). Then there are the unique interpolating Blaschke products \( \varphi_{1,n}, \varphi_{2,n}, \ldots, \varphi_{m,n} \) such that \( b_n = \prod_{j=1}^{m} \varphi_{j,n} \) and \( \varphi_{j,n+1} < \varphi_{j,n} \) for every \( 1 \leq j \leq m \). We note that if \( z \in Z(I) \cap \mathbb{D} \) and \( \varphi_{j,1}(z) = 0 \), then \( \varphi_{j,n}(z) = 0 \) for every \( n \geq 1 \).
For each $1 \leq j \leq m$, let
\[ E_j = Z(I) \cap \bigcap_{n=1}^{\infty} Z(\varphi_{j,n}). \]

By Lemma 2.3, $E_j$ is a compact $G_\delta$-set. Since $\varphi_{j,n}$ is an interpolating Blaschke product, $E_j$ is a $\rho$-separated set. Since $b_n \in I$,
\[ Z(I) \subset Z(b_n) = \bigcup_{j=1}^{m} Z(\varphi_{j,n}), \]
so
\[ Z(I) = \bigcup_{j=1}^{m} (Z(I) \cap Z(\varphi_{j,n})). \]

We have
\[ \bigcup_{j=1}^{m} E_j \subset \bigcup_{j=1}^{m} (Z(I) \cap Z(\varphi_{j,n})) = Z(I). \]

Suppose that $\bigcup_{j=1}^{m} E_j \subsetneq Z(I)$ and $y \in Z(I) \setminus \bigcup_{j=1}^{m} E_j$. For each $1 \leq j \leq m$, since $y \notin E_j$ there is a positive integer $n_j$ such that $y \notin Z(I) \cap Z(\varphi_{j,n_j})$. Let $n = \min_{1 \leq j \leq m} n_j$. Then
\[ Z(I) \cap Z(\varphi_{j,n}) \subset Z(I) \cap Z(\varphi_{j,n_j}). \]

Hence
\[ y \notin \bigcup_{j=1}^{m} (Z(I) \cap Z(\varphi_{j,n})) = Z(I). \]

But this is a contradiction. Thus we get
\[ Z(I) = \bigcup_{j=1}^{m} E_j. \]

Let $x \in Z(I)$. Then there is a positive integer $n_1$ such that $ord(b_{n_1}, x) = ord(I, x)$. We write $\ell = ord(I, x)$. Then there are positive integers $j_1, j_2, \cdots, j_\ell$ such that
\[ ord\left( \prod_{i=1}^{\ell} \varphi_{j_i,n_1}, x \right) = \ell \quad \text{and} \quad ord\left( b_{n_1} / \prod_{i=1}^{\ell} \varphi_{j_i,n_1}, x \right) = 0. \]

Since $b_n \in I$ and $b_n \prec b_{n_1}$ for every $n \geq n_1$, $ord(b_n, x) = \ell$ and $\varphi_{j_i,n}(x) = 0$ for every $1 \leq i \leq \ell$ and $n \geq n_1$. Thus for any $n \geq n_1$ we have
\[ ord(I, x) = ord(b_n, x) = ord\left( \prod_{j=1}^{m} \varphi_{j,n}, x \right) \]
\[ = \sum_{i=1}^{\ell} ord(\varphi_{j_i,n}, x) = \# \{ j : x \in E_j, 1 \leq j \leq m \}, \]
where $\#A$ denotes the number of elements in a set $A$. Let
\[ J = \prod_{j=1}^{m} I(E_j). \]
By Lemma \ref{2.5}, we have \( \text{ord}(I(E_j), x) = 1 \) for every \( x \in E_j \) and \( Z(I(E_j)) = E_j \) for every \( 1 \leq j \leq m \). Hence by Lemma \ref{2.6} \( Z(J) = \bigcup_{j=1}^m E_j = Z(I) \) and

\[
\text{ord}(J, x) = \sum_{j=1}^m \text{ord}(I(E_j), x) = \# \{ j : x \in E_j, 1 \leq j \leq m \}
\]

for every \( x \in Z(I) \). By Theorem A, we have \( I = J = \bigotimes_{j=1}^m I(E_j) \).

(ii) \( \Rightarrow \) (iii) Suppose that condition (ii) holds. By Lemma \ref{2.6},

\[
Z(I) = \bigcup_{j=1}^m Z(I(E_j)) = \bigcup_{j=1}^m E_j,
\]

so \( Z(I) \) is a \( G_\delta \)-set. By Lemma \ref{2.4}, for each \( 1 \leq j \leq m \) there is an interpolating Blaschke product \( \varphi_j \) such that \( E_j \subset Z(\varphi_j) \). Let \( \Phi = \prod_{j=1}^m \varphi_j \). By Lemma \ref{2.13} for each \( 1 \leq j \leq m \) there exists an interpolating Blaschke product \( b_j \) such that \( E_j \subset Z(b_j) \) and \( Z(\Phi) \cap Z(b_j) = Z(\Phi) \cap E_j = E_j \). We note that \( Z(I) \subset Z(\Phi) \). Let \( B = \prod_{j=1}^m b_j \). Then for any \( x \in Z(I) \), we have

\[
\text{ord}(B, x) = \text{ord}\left( \prod_{j=1}^m b_j, x \right) = \sum_{j=1}^m \text{ord}(b_j, x) = \# \{ j : x \in E_j, 1 \leq j \leq m \}.
\]

By Lemmas \ref{2.5} and \ref{2.6}, we have

\[
\text{ord}(I, x) = \text{ord}\left( \bigotimes_{j=1}^m I(E_j), x \right) = \sum_{j=1}^m \text{ord}(I(E_j), x) = \# \{ j : x \in E_j, 1 \leq j \leq m \}.
\]

Thus we get \( \text{ord}(B, x) = \text{ord}(I, x) \) for every \( x \in Z(I) \). By Theorem A, we have \( B \in I \).

(iii) \( \Rightarrow \) (iv) Suppose that condition (iii) holds. Let \( B_1 \) be a Carleson-Newman Blaschke product of order \( m \) in \( I \) satisfying \( \text{ord}(B_1, x) = \text{ord}(I, x) \) for every \( x \in Z(I) \). Let \( \varphi_1, \varphi_2, \ldots, \varphi_m \) be interpolating Blaschke products satisfying \( B_1 = \prod_{j=1}^m \varphi_j \). For each \( 1 \leq j \leq m \), let \( E_j = Z(I) \cap Z(\varphi_j) \). Since \( Z(I) \) is a \( G_\delta \)-set, \( E_j \) is a closed \( G_\delta \)-set. By Lemma \ref{2.13}, there is an interpolating Blaschke product \( b_j \) such that \( Z(B_1) \cap Z(b_j) = E_j \). Let \( B_2 = \prod_{j=1}^m b_j \). For any \( x \in Z(I) \), we have

\[
\text{ord}(B_2, x) = \sum_{j=1}^m \text{ord}(b_j, x) = \# \{ j : x \in E_j, 1 \leq j \leq m \} = \text{ord}(B_1, x) \geq \text{ord}(I, x).
\]

By Theorem A, we have \( B_2 \in I \). We also have

\[
Z(B_1) \cap Z(B_2) = Z(B_1) \cap \bigcup_{j=1}^m Z(b_j) = \bigcup_{j=1}^m E_j = Z(I) \cap Z(B_1) = Z(I).
\]

Let \( J = I[B_1, B_2] \). Then \( Z(J) = Z(I) \) and \( \text{ord}(J, x) = \text{ord}(I, x) \) for every \( x \in Z(I) \).

By Theorem A again, we have \( J = I \).

(iv) \( \Rightarrow \) (i) is trivial. \hfill \square
In the following example, we shall show that there exist compact \( \rho \)-separated \( G_\delta \)-subsets \( E_1 \) and \( E_2 \) of \( G \) such that the ideal \( I(E_1) \cap I(E_2) \) is not countably generated.

**Example 2.14.** Let \( \{\theta_k\}_k \) be a sequence of numbers such that \( 0 < \theta_{k+1} < \theta_k < 1 \) and \( \theta_k \to 0 \) as \( k \to \infty \). It is known that there is an interpolating Blaschke product \( B_1 \) with zeros \( \{z_n\}_n \) in \( \mathbb{D} \) such that

\[
\left\{ z_n \right\}_n^c \setminus \left\{ z_n \right\}_n = \{ e^{i\theta_k} : k \geq 1 \} \cup \{1\},
\]

where \( \left\{ z_n \right\}_n^c \) is the closure of \( \{z_n\}_n \) in \( \mathbb{C} \). Let \( \mathbb{N} \) be the set of positive integers. We may divide \( \mathbb{N} \) as \( \mathbb{N} = \bigcup_{k=1}^{\infty} N_k \) such that \( N_k \cap N_j = \emptyset \) for \( k \neq j \) and

\[
\left\{ z_n : n \in N_k \right\}_n^c \setminus \left\{ z_n : n \in N_k \right\} = \{ e^{i\theta_k} \}, \quad k \in \mathbb{N}.
\]

Let \( b_k \) be the subproduct of \( B_1 \) with zeros \( \{z_n : n \in N_k\} \). Then \( B_1 = \prod_{k=1}^{\infty} b_k \).

Let \( \{\varepsilon_k\}_k \) be a sequence of numbers such that \( 0 < \varepsilon_k < 1 \) and \( \varepsilon_k \to 0 \) as \( k \to \infty \).

Let \( q_k(z) = (b_k(z) - \varepsilon_k)/(1 - \varepsilon_kb_k(z)) \).

Taking smaller \( \varepsilon_k \), we may assume that \( B_2 := \prod_{k=1}^{\infty} q_k \) is an interpolating Blaschke product and

\[
\left( \bigcup_{k=1}^{\infty} Z(b_k) \right) \cap \left( \bigcup_{k=1}^{\infty} Z(q_k) \right) = \emptyset.
\]

Let

\[
E_1 = Z(B_1) \setminus \mathbb{D} \quad \text{and} \quad E_2 = Z(B_2) \setminus \mathbb{D}.
\]

Then \( E_1, E_2 \) are compact \( \rho \)-separated \( G_\delta \)-subsets of \( G \),

\[
E_1 = \left( \bigcup_{k=1}^{\infty} (Z(b_k) \setminus \mathbb{D}) \right) \cup \left( E_1 \setminus \bigcup_{k=1}^{\infty} Z(b_k) \right)
\]

and

\[
E_2 = \left( \bigcup_{k=1}^{\infty} (Z(q_k) \setminus \mathbb{D}) \right) \cup \left( E_2 \setminus \bigcup_{k=1}^{\infty} Z(q_k) \right).
\]

By Lemma 2.3, \( I(E_1) \) and \( I(E_2) \) are countably generated closed ideals in \( H^\infty \). Let \( I = I(E_1) \cap I(E_2) \). Then \( I = I(E_1 \cup E_2) \). By the construction, we may check that

\[
E_1 \setminus \bigcup_{k=1}^{\infty} Z(b_k) = E_2 \setminus \bigcup_{k=1}^{\infty} Z(q_k)
\]

and

\[
\bigcup_{k=1}^{\infty} (Z(b_k) \setminus \mathbb{D}) \setminus \bigcup_{k=1}^{\infty} Z(b_k) = \bigcup_{k=1}^{\infty} (Z(q_k) \setminus \mathbb{D}) \setminus \bigcup_{k=1}^{\infty} Z(q_k)
\]

\[
\subset E_1 \setminus \bigcup_{k=1}^{\infty} Z(b_k).
\]

Let \( \Omega \) be the set of all subproducts \( q \) of \( B_2 \) satisfying

\[
\bigcup_{k=1}^{\infty} (Z(q_k) \setminus \mathbb{D}) \subset Z(q).
\]
Then we have $B_1q \in I$ for every $q \in \Omega$ and

$$\bigcap_{q \in \Omega} Z(q) = \bigcup_{k=1}^{\infty} (Z(q_k) \setminus \mathbb{D}).$$

By this fact, we have

$$Z_2(I) = \bigcup_{k=1}^{\infty} (Z(b_k) \setminus \mathbb{D}) \setminus \bigcup_{k=1}^{\infty} Z(b_k) = \bigcup_{k=1}^{\infty} Z(b_k) \setminus \bigcup_{k=1}^{\infty} Z(b_k),$$

and $Z_2(I)$ is not a $G_\delta$-set (see Example 2.9 in [12]). By Lemma 2.3, $I$ is not countably generated. We note that $I = I(E_1) \otimes I(E_2 \setminus E_1)$. \hfill \Box

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