GLEASON PARTS AND COUNTABLY GENERATED CLOSED IDEALS IN $H^\infty$

KEI JI IZUCHI AND YUKO IZUCHI

Abstract. It is proved that a countably generated closed ideal in $H^\infty$ whose common zero set is contained in the union set of nontrivial Gleason parts of $H^\infty$ is generated by two Carleson-Newman Blaschke products as a closed ideal.

1. Introduction

Let $H^\infty$ be the Banach algebra of bounded analytic functions on the open unit disk $\mathbb{D}$ with the supremum norm $\| \cdot \|_\infty$. We denote by $M(H^\infty)$ the maximal ideal space of $H^\infty$, that is, $M(H^\infty)$ is the family of nonzero multiplicative linear functionals on $H^\infty$ with the weak*-topology. For a subset $E$ of $M(H^\infty)$, we denote by $\overline{E}$ the closure of $E$ in $M(H^\infty)$. We identify a function $f$ in $H^\infty$ with its Gelfand transform $\hat{f}(m) = m(f)$, $m \in M(H^\infty)$, so we think of $f$ as a continuous function on $M(H^\infty)$. For a sequence $\{a_n\}_n$ in $\mathbb{D}$ satisfying $\sum_{n=1}^\infty (1 - |a_n|) < \infty$, we have the Blaschke product $b(z) = \prod_{n=1}^\infty \frac{-\overline{a}_n z - a_n}{|a_n| (1 - \overline{a}_n z)}$, $z \in \mathbb{D}$, where if $a_n = 0$, we consider that $-\overline{a}_n/|a_n| = 1$. We call $\{a_n\}_n$ and $b(z)$ interpolating if for any bounded sequence of complex numbers $\{c_n\}_n$ there exists $f$ in $H^\infty$ such that $f(a_n) = c_n$ for every $n \geq 1$. In [2], Carleson gave a characterization of interpolating sequences. A Blaschke product $B$ is said to be Carleson-Newman if $B = \prod_{j=1}^m b_j$ for finitely many interpolating Blaschke products $b_1, b_2, \cdots, b_m$. In this case, there are many ways to give such a factorization. If $m$ is the minimal number of interpolating Blaschke products, $B$ is said to be a Carleson-Newman Blaschke product of order $m$.

In the study of the structure of $H^\infty$, Carleson-Newman Blaschke products have played an important role (see [3, 5, 8, 11]). For Blaschke products $b_1$ and $b_2$, we write $b_1 \lessdot b_2$ if $b_1$ is a subproduct of $b_2$.

For $x, y \in M(H^\infty)$, the pseudo-hyperbolic distance is defined by

$$\rho(x, y) = \sup \{ |f(x)| : f(y) = 0, f \in H^\infty, \|f\|_\infty \leq 1 \}.$$ 

A subset $E$ of $M(H^\infty)$ is said to be $\rho$-separated if there is $\varepsilon > 0$ such that $\rho(x, y) \geq \varepsilon$ for every $x, y \in E$ with $x \neq y$. The set

$$P(x) = \{ y \in M(H^\infty) : \rho(y, x) < 1 \}$$

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is called the Gleason part of $H^\infty$ containing $x \in M(H^\infty)$. If $P(x) \neq \{x\}$, $P(x)$ is said to be nontrivial. We denote by $G$ the union set of all nontrivial Gleason parts in $M(H^\infty)$. In [2] (see also [3]), Hoffman studied the structure of Gleason parts of $H^\infty$ extensively. For $x \in M(H^\infty)$, he proved that $x \in G$ if and only if there is an interpolating Blaschke product $b$ satisfying $b(x) = 0$. He also proved that for an interpolating Blaschke product $b$, there exists $\varepsilon > 0$ such that $\{|b| < \varepsilon\} \subset G$, where

$$\{|b| < \varepsilon\} = \{x \in M(H^\infty) : |b(x)| < \varepsilon\}.$$

This fact shows that $G$ is an open subset of $M(H^\infty)$, and for a Carleson-Newman Blaschke product $B$ there is $\varepsilon > 0$ such that $\{|B| < \varepsilon\} \subset G$. Hoffman also showed that for a nontrivial Gleason part $P(x)$ of $H^\infty$, there is a one-to-one, onto and continuous map $L_x : \mathbb{D} \to P(x)$ such that $L_x(0) = x$ and $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. For $f \in H^\infty$, we write

$$Z(f) = \{x \in M(H^\infty) : f(x) = 0\}.$$

It is known that if $b$ is an interpolating Blaschke product with zeros $\{z_n\}_n$ in $\mathbb{D}$, then $Z(b) = \{\overline{z_n}\}_n$, $Z(b)$ is $\rho$-separated and homeomorphic to the Stone-Čech compactification of the set of natural numbers, so $Z(b)$ is a totally disconnected set (see [4] [7]). Hence if $B$ is a Carleson-Newman Blaschke product, then $Z(B)$ is also totally disconnected. Let $f \in H^\infty$. For $z \in \mathbb{D}$, we denote by $\text{ord}(f, z)$ the order of zero of $f$ at $z$. For $x \in G \setminus \mathbb{D}$, we define $\text{ord}(f, x) = \text{ord}(f \circ L_x, 0)$. For $x \in M(H^\infty) \setminus G$, we put as usual $\text{ord}(f, x) = \infty$ if $f(x) = 0$ and $\text{ord}(f, x) = 0$ if $f(x) \neq 0$. Clearly, if $b$ is an interpolating Blaschke product, then $\text{ord}(b, x) \leq 1$. If $b$ is a Carleson-Newman Blaschke product of order $m$, then $\text{ord}(b, x) \leq m$ for every $x$.

Let $I$ be a closed ideal in $H^\infty$. We write

$$Z(I) = \bigcap_{f \in I} Z(f)$$

and

$$\text{ord}(I, x) = \inf_{f \in I} \text{ord}(f, x), \quad x \in M(H^\infty).$$

For each $1 \leq j \leq \infty$ and $f \in H^\infty$, we put

$$Z_j(f) = \{x \in M(H^\infty) : \text{ord}(f, x) \geq j\}$$

and

$$Z_j(I) = \{x \in M(H^\infty) : \text{ord}(I, x) \geq j\}.$$ 

It seems very difficult to study ideal theory in $H^\infty$ generally (see [11]). In [4], Gorkin, Mortini and the first author proved the following two theorems for a closed ideal $I$ satisfying $Z(I) \subset G$. In this case, by Theorem 2.3 in [5], $I$ contains a Carleson-Newman Blaschke product, so $\sup_{x \in Z(I)} \text{ord}(I, x) < \infty$ and $Z(I)$ is totally disconnected (see also [14]).

**Theorem A.** Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$. Then $I$ coincides with the set of all $f$ in $H^\infty$ satisfying $\text{ord}(f, x) \geq \text{ord}(I, x)$ for every $x \in Z(I)$.

This shows that if $I_1, I_2$ are closed ideals in $H^\infty$ such that $Z(I_i) \subset G$ for $i = 1, 2$, $Z(I_1) = Z(I_2)$ and $\text{ord}(I_1, x) = \text{ord}(I_2, x)$ for every $x \in Z(I_1)$, then we have $I_1 = I_2$.
Theorem B. Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} \text{ord}(I,x)$. For each $1 \leq j \leq m$, let $U_j$ be an open subset of $M(H^\infty)$ satisfying $Z_j(I) \subset U_j$. Then $I$ is algebraically generated by Carleson-Newman Blaschke products $B$ of order $m$ in $I$ such that $Z_j(B) \subset U_j$ for $1 \leq j \leq m$.

The above two theorems give us a great deal of information about closed ideals $I$ satisfying $Z(I) \subset G$. In [12, 13], the authors studied closed ideals $I$ satisfying $Z(I) \subset G$ extensively.

For a sequence $\{f_n\}_n$ in $H^\infty$, we denote by $I[f_n : n \geq 1]$ the closed ideal in $H^\infty$ generated by functions $f_n, n = 1, 2, \cdots$; that is,

$$I[f_n : n \geq 1] = \bigcup_{n=1}^\infty \sum_{j=1}^n f_j H^\infty,$$

where the bar indicates the closure in $H^\infty$. The closed ideal $I[f_n : n \geq 1]$ is called a countably generated closed ideal in $H^\infty$. In this paper, we study the structure of countably generated closed ideals $I$ satisfying $Z(I) \subset G$. For a closed subset $E$ of $M(H^\infty)$, let $I(E) = \{f \in H^\infty : f(x) = 0, x \in E\}$. Then $I(E)$ is a closed ideal in $H^\infty$ and $E \subset Z(I(E))$. For closed ideals $I_1, I_2, \cdots, I_m$ in $H^\infty$, let $\bigotimes_{i=1}^m I_i$ and $\bigotimes_{i=1}^m I_i$ be the tensor product and the closed tensor product of $I_1, I_2, \cdots, I_m$, respectively. That is, $\bigotimes_{i=1}^m I_i$ is an ideal generated by functions $\prod_{i=1}^m f_i$, where $f_i \in I_i, 1 \leq i \leq m$, and $\bigotimes_{i=1}^m I_i = \bigotimes_{i=1}^m I_i$. In Section 2, we shall prove the following theorem.

**Theorem 1.1.** Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} \text{ord}(I,x)$. Then the following conditions are equivalent.

(i) $I$ is a countably generated closed ideal.

(ii) There are compact $\rho$-separated $G_\delta$-subsets $E_1, E_2, \cdots, E_m$ of $G$ such that $I = \bigotimes_{j=1}^m I(E_j)$.

(iii) There is a Carleson-Newman Blaschke product $B$ of order $m$ in $I$ such that $\text{ord}(B, x) = \text{ord}(I, x)$ for every $x \in Z(I)$, and $Z(I)$ is a $G_\delta$-set.

(iv) There are two Carleson-Newman Blaschke products $B_1, B_2$ in $I$ such that $I = I[B_1, B_2]$.

For a compact $\rho$-separated $G_\delta$-subset $E$ of $G$, there is an interpolating Blaschke product $b$ satisfying $E \subset Z(b)$, and $I(E)$ is a countably generated closed ideal. We shall show in Example 2.14 that there exist compact $\rho$-separated $G_\delta$-subsets $E_1$ and $E_2$ of $G$ such that $I(E_1) \cap I(E_2)$ is not countably generated. If $I$ is a countably generated closed ideal in $H^\infty$, then by Theorem 1.1, $Z_j(I)$ is a $G_\delta$-set for every $1 \leq j \leq \infty$. But if $I$ is the closed ideal given in Example 2.14, then $Z_2(I)$ is not a $G_\delta$-set.

2. Countably generated closed ideals

To prove Theorem 1.1, we need some lemmas. For a sequence $\{f_n\}_n$ in $H^\infty$ and $1 \leq j \leq \infty$, it is not difficult to show that

$$Z_j(I[f_n : n \geq 1]) = \bigcap_{n=1}^\infty Z_j(f_n)$$

and

$$\text{ord}(I[f_n : n \geq 1], x) = \inf_{n \geq 1} \text{ord}(f_n, x), \quad x \in Z(I[f_n : n \geq 1]).$$
Lemma 2.1. Let $B$ be a Carleson-Newman Blaschke product. Then $Z_j(B)$ is a closed $G_\delta$-set for every $1 \leq j < \infty$.

Proof. Let $B = \prod_{i=1}^{k} b_i$, where $b_i$ is an interpolating Blaschke product for every $1 \leq i \leq k$. Since $\text{ord}(b_i, x) \leq 1$ for $x \in M(H^\infty)$, we have that $Z_j(B) = \emptyset$ for $j > k$. Suppose that $1 \leq j \leq k$. Put $E_i = Z(b_i)$. Then $E_i$ is a closed $G_\delta$-set. We have

$$Z_j(B) = \bigcup \left\{ \bigcap_{l=1}^{j} E_{i_l} : 1 \leq i_1 < i_2 < \cdots < i_j \leq k \right\}.$$ 

Therefore $Z_j(B)$ is a closed $G_\delta$-set. \hfill \square

Lemma 2.2. If $f \in H^\infty$ and $f \neq 0$, then $Z_j(f)$ is a closed $G_\delta$-set for every $1 \leq j \leq \infty$.

Proof. Let $f = B h$, where $B$ is a Blaschke product and $h \in H^\infty$ satisfying $|h| > 0$ on $\mathbb{D}$. Then $Z_\infty(h) = Z(h)$ and $Z_\infty(h)$ is a closed $G_\delta$-set. By Corollary 3.1 in [9], $Z_\infty(B)$ is a closed $G_\delta$-set. Then $Z_\infty(f) = Z_\infty(B) \cup Z_\infty(h)$ is a closed $G_\delta$-set. We have

$$Z(f) \setminus Z_\infty(f) = (Z(B) \cup Z(h)) \setminus Z_\infty(f)$$

$$= (Z(B) \cup Z_\infty(h)) \setminus Z_\infty(f) = Z(B) \setminus Z_\infty(f).$$

By Lemma 4.6 in [9], $Z(B) \setminus Z_\infty(f)$ is a totally disconnected set. Hence there is a sequence of open and closed subsets $\{E_n\}_n$ of $Z(B)$ such that $Z(B) \setminus Z_\infty(f) = \bigcup_{n=1}^{\infty} E_n$ and $E_n \cap E_k = \emptyset$ for $n \neq k$. Let $b_n$ be the subproduct of $B$ with zeros $Z(B) \cap E_n \cap \mathbb{D}$ counting multiplicities. Since $Z(B) \cap \mathbb{D} \subset Z(B) \setminus Z_\infty(f)$, we have $B = \prod_{n=1}^{\infty} b_n$ and $Z(b_n) = E_n$ for every $n \geq 1$. We note that $b_n$ is a Carleson-Newman Blaschke product. For each $1 \leq j < \infty$, we have

$$Z_j(f) = Z_\infty(f) \cup \bigcup_{n=1}^{\infty} Z_j(b_n).$$

By Lemma 2.1 $Z_j(b_n)$ is a closed $G_\delta$-set; so is $Z_j(f)$. \hfill \square

Lemma 2.3. Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} \text{ord}(I, x)$. Then $I$ is a countably generated closed ideal if and only if $Z_j(I)$ is a closed $G_\delta$-set for every $1 \leq j \leq m$. In this case, $I$ is generated by countably many Carleson-Newman Blaschke products.

Proof. Suppose that $I = I[f_n : n \geq 1]$ for a sequence $\{f_n\}_n$ in $H^\infty$. For each $1 \leq j \leq m$, we have $Z_j(I) = \bigcap_{n=1}^{\infty} Z_j(f_n)$. By Lemma 2.2 $Z_j(I)$ is a closed $G_\delta$-set.

Suppose that $Z_j(I)$ is a closed $G_\delta$-set for every $1 \leq j \leq m$. For each $1 \leq j \leq m$, let $\{U_{j,n}\}_n$ be a sequence of open subsets of $G$ such that $Z_j(I) = \bigcap_{n=1}^{\infty} U_{j,n}$. By Theorem B, there is a sequence of Carleson-Newman Blaschke products $\{\varphi_n\}_n$ in $I$ such that $Z_j(\varphi_n) \subset U_{j,n}$ for every $1 \leq j \leq m$ and $n \geq 1$. Let $J = I[\varphi_n : n \geq 1]$. Then $J \subset I$ and $Z(I) \subset Z(J)$. We have $Z(J) \subset Z(\varphi_n) \subset U_{1,n}$ for every $n \geq 1$. Then $Z(J) \subset \bigcap_{n=1}^{\infty} U_{1,n} = Z_1(I) = Z(I)$. Hence $Z(J) = Z(I)$.

Let $x \in Z(I)$ and $\ell = \text{ord}(I, x)$. Since $\varphi_n \in I$, $\ell \leq \text{ord}(\varphi_n, x)$ for every $n \geq 1$. Since $x \notin Z_{\ell+1}(I)$, there is a positive integer $k$ such that $x \notin U_{\ell+1,k}$. Hence
We get
\[ \ell = \text{ord}(J, x) \leq \text{ord}(I, x) \leq \text{ord}(\varphi_k, x) = \ell. \]
Thus we get \( \text{ord}(J, x) = \text{ord}(I, x) \) for every \( x \in Z(I) \). By Theorem A, we have \( J = I \).

The following lemma follows from Theorem 3.1 in [10].

Lemma 2.4. Let \( E \) be a compact \( \rho \)-separated subset of \( G \) and \( U \) be an open subset of \( M(H^\infty) \) satisfying \( E \subset U \). Then there exists an interpolating Blaschke product \( b \) such that \( E \subset Z(b) \subset U \).

Lemma 2.5. Let \( E \) be a compact \( \rho \)-separated \( G_\delta \)-subset of \( G \). Then \( I(E) \) is a countably generated closed ideal in \( H^\infty \), \( E \) is a totally disconnected set, \( Z(I(E)) = E \) and \( \text{ord}(I(E), x) = 1 \) for every \( x \in E \).

Proof. By Lemma 2.4, there is an interpolating Blaschke product \( b \) such that \( E \subset Z(b) \subset G \). Hence \( \text{ord}(I(E), x) = 1 \) for every \( x \in E \). Since \( Z(b) \) is a totally disconnected set, so is \( E \). Let \( \{U_n\}_n \) be a sequence of open subsets of \( G \) satisfying \( E = \bigcap_{n=1}^{\infty} U_n \) and \( Z(b) \cap U_n \) be an open and closed subset of \( Z(b) \) for every \( n \geq 1 \). Let \( b_n \) be the subproduct of \( b \) with zeros \( Z(b) \cap U_n \cap \mathbb{D} \). Then \( E \subset Z(b_n) \subset U_n \). Let \( J = I[b_n : n \geq 1] \). Then we have \( J \subset I(E) \) and
\[ E \subset Z(I(E)) \subset Z(J) \subset \bigcap_{n=1}^{\infty} U_n = E. \]
Hence \( Z(I(E)) = Z(J) = E \). We have \( \text{ord}(J, x) = 1 \) for every \( x \in E \). By Theorem A, we get \( J = I(E) \).

The following lemma follows from the definition of a closed tensor product.

Lemma 2.6. Let \( I_1, I_2, \cdots, I_m \) be countably generated closed ideals in \( H^\infty \). Then
\( \bigotimes_{j=1}^{m} I_j \) is a countably generated closed ideal, \( Z(\bigotimes_{j=1}^{m} I_j) = \bigcup_{j=1}^{m} Z(I_j) \) and \( \text{ord}(\bigotimes_{j=1}^{m} I_j, x) = \sum_{j=1}^{m} \text{ord}(I_j, x) \) for every \( x \in Z(\bigotimes_{j=1}^{m} I_j) \).

For closed ideals \( I_1, I_2, \cdots, I_m \) in \( H^\infty \) satisfying \( Z(I_j) \subset G \) for every \( 1 \leq j \leq m \), in [13] Corollary 9.15 [the authors] proved that \( \bigotimes_{j=1}^{m} I_j = \bigotimes_{j=1}^{m} I_j \).

Lemma 2.7. Let \( I \) be a closed ideal in \( H^\infty \) satisfying \( Z(I) \subset G \) and \( x \in Z(I) \). Let \( B \) be a Carleson-Newman Blaschke product in \( I \) and \( W \) be an open subset of \( M(H^\infty) \) satisfying \( x \in U \subset G \cap W \) and \( Z(I) \cap U \) is an open and closed subset of \( Z(I) \), and there is a Carleson-Newman Blaschke product \( \varphi \) of order \( \text{ord}(I, x) \) such that \( Z(\varphi) \subset U \), \( \varphi \prec B \) and \( \text{ord}(I, y) \leq \text{ord}(\varphi, y) \leq \text{ord}(I, x) \) for every \( y \in Z(I) \cap U \).

Proof. Since \( Z(I) \) is a totally disconnected set (see [4] Theorem 2.2), we may take a sufficiently small open subset \( U \) of \( M(H^\infty) \) such that \( x \in U \subset G \cap W \) and \( Z(I) \cap U \) is an open and closed subset of \( Z(I) \). Since \( \text{ord}(I, y) \) is upper semicontinuous in \( y \in Z(I) \) (see [4] Lemma 1.2), we may assume that \( \text{ord}(I, y) \leq \text{ord}(I, x) \) for every \( y \in Z(I) \cap U \). Let
\[ I_U = \{ f \in H^\infty : \text{ord}(f, y) \geq \text{ord}(I, y), y \in Z(I) \cap U \}. \]
Then by Theorem A, \( I_U \) is a closed ideal in \( H^\infty \), \( I \subset I_U \), \( Z(I_U) = Z(I) \cap U \) and \( \text{ord}(I_U, y) = \text{ord}(I, y) \) for every \( y \in Z(I) \cap U \). By [13] Proposition 8.9, there is a
Carleson-Newman Blaschke product \( \varphi \) of order \( \text{ord}(I, x) \) in \( I_U \) such that \( Z(\varphi) \subset U \), \( \varphi < B \) and \( \text{ord}(\varphi, x) = \text{ord}(I_U, x) \). For each \( y \in Z(I) \cap U \), we have

\[
\text{ord}(I, y) = \text{ord}(I_U, y) \leq \text{ord}(\varphi, y) \leq \text{ord}(I, x).
\]

\[\square\]

**Lemma 2.8.** Let \( I \) be a closed ideal in \( H^\infty \) satisfying \( Z(I) \subset G \) and \( m = \sup_{x \in Z(I)} \text{ord}(I, x) \). Let \( W_1, W_2, \ldots, W_m \) be open subsets of \( M(H^\infty) \) such that \( Z_j(I) \subset W_j \) for every \( 1 \leq j \leq m \) and \( W_m \subset W_{m-1} \subset \cdots \subset W_1 \). Let \( B \) be a Carleson-Newman Blaschke product in \( I \). Then there is a Carleson-Newman Blaschke product \( b \) such that \( b \in I \), \( b < B \) and \( \text{ord}(b, y) \leq j \) for every \( y \in Z(I) \cap (W_j \setminus W_{j+1}) \) and \( 1 \leq j \leq m \), where \( W_{m+1} = \emptyset \).

**Proof.** For each \( x \in Z(I) \), since \( Z(I) \subset \bigcup_{j=1}^m (W_j \setminus W_{j+1}) \) there exists \( 1 \leq j \leq m \) such that \( x \in W_j \setminus W_{j+1} \). Then \( \text{ord}(I, x) \leq j \). By Lemma 2.7, there is an open subset \( U_x \) of \( M(H^\infty) \) satisfying that \( x \in U_x \subset G \cap W_j \) and \( Z(I) \cap U_x \) is an open and closed subset of \( Z(I) \), and there is a Carleson-Newman Blaschke product \( \varphi_x \) of order \( \text{ord}(I, x) \) such that \( Z(\varphi_x) \subset U_x \), \( \varphi_x < B \) and \( \text{ord}(I, y) \leq \text{ord}(\varphi_x, y) \leq \text{ord}(I, x) \) for every \( y \in Z(I) \cap U_x \).

Since \( Z(I) \) is a compact set, there is a finite set \( \{x_1, x_2, \ldots, x_s\} \) in \( Z(I) \) such that \( Z(I) \subset \bigcup_{i=1}^s U_{x_i} \). Let

\[
E_1 = Z(I) \cap U_{x_1}, \quad E_2 = (Z(I) \cap U_{x_2}) \setminus (Z(I) \cap U_{x_1}),
\]

\[
\ldots, \quad E_s = (Z(I) \cap U_{x_s}) \setminus \bigcup_{i=1}^{s-1} (Z(I) \cap U_{x_i}).
\]

Then \( E_i \) is an open and closed subset of \( Z(I) \), \( E_i \cap E_j = \emptyset \) for \( i \neq j \) and \( \bigcup_{i=1}^s E_i = Z(I) \). It may be that \( x_i \notin E_i \) for some \( 1 \leq i \leq s \). We may take open subsets \( V_1, V_2, \ldots, V_s \) of \( M(H^\infty) \) satisfying that \( E_i \subset V_i \subset U_{x_i} \) and \( \overline{V_i} \cap \overline{V_j} = \emptyset \) for \( i \neq j \).

Let \( \psi_i \) be the Blaschke subproduct of \( \varphi_{x_i} \) with zeros \( Z(\varphi_{x_i}) \cap V_i \cap \mathbb{D} \) counting multiplicities. Then \( Z(\psi_i) \cap Z(\psi_j) = \emptyset \) for \( i \neq j \) and \( \text{ord}(\psi_i, y) = \text{ord}(\varphi_{x_i}, y) \) for every \( y \in E_i \) and \( 1 \leq i \leq s \). Let \( b = \prod_{i=1}^s \psi_i \). Then \( b < B \).

Let \( y \in Z(I) \). Then there is the unique \( 1 \leq j \leq m \) such that \( y \in W_j \setminus W_{j+1} \). Also there is the unique \( 1 \leq i \leq s \) such that \( y \in E_i \). So we have

\[
\text{ord}(b, y) = \text{ord}(\psi_i, y) = \text{ord}(\varphi_{x_i}, y) \leq \text{ord}(I, x_i).
\]

Here we have two cases.

**Case 1.** Suppose that \( x_i \in W_j \setminus W_{j+1} \). Then we have

\[
\text{ord}(I, y) \leq \text{ord}(\varphi_{x_i}, y) \leq \text{ord}(I, x_i) \leq j.
\]

Hence \( \text{ord}(I, y) \leq \text{ord}(b, y) \leq j \).

**Case 2.** Suppose that \( x_i \in W_k \setminus W_{k+1} \) for some \( k \neq j \). If \( k < j \), then \( \text{ord}(I, x_i) \leq k < j \). Hence

\[
\text{ord}(I, y) \leq \text{ord}(\varphi_{x_i}, y) = \text{ord}(b, y) < j.
\]

If \( k > j \), then \( y \in U_{x_i} \subset W_k \). Since \( y \notin W_{j+1} \) and \( W_k \subset W_{j+1} \), we have \( y \notin W_k \). This is a contradiction.

By the above two cases, we have \( \text{ord}(I, y) \leq \text{ord}(b, y) \leq j \) for every \( y \in Z(I) \cap (W_j \setminus W_{j+1}) \). By Theorem A, we have \( b \in I \). Thus we get the assertion. \[\square\]
Lemma 2.9. Let $I$ be a countably generated closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} \ord(I, x)$. Let $B$ be a Carleson-Newman Blaschke product in $I$. Then there is a sequence of Carleson-Newman Blaschke products $\{b_n\}_n$ such that $b_1 < B, b_{n+1} < b_n, b_n \in I$ for every $n \geq 1$ and for each $x \in Z(I)$ there is a positive integer $n$ satisfying $\ord(I, x) = \ord(b_n, x)$.

Proof. By Lemma 2.3 $Z_j(I)$ is a closed $G_\delta$-set for every $1 \leq j \leq m$. For each $1 \leq j \leq m$, take a sequence of open subsets $\{W_{j,n}\}_n$ of $M(H^\infty)$ such that $\bigcap_{n=1}^\infty W_{j,n} = Z_j(I)$ and $W_{j,n+1} \subset W_{j,n}$ for every $n \geq 1$. Further we may assume that $W_{j+1,n} \subset W_{j,n}$ for every $1 \leq j \leq m$ and $n \geq 1$, where $W_{m+1,n} = \emptyset$ for every $n \geq 1$. By Lemma 2.8, there is a Carleson-Newman Blaschke product $b_1$ such that $b_1 < B$ and $\ord(b_1, y) \leq j$ for every $y \in Z(I) \cap (W_{j,1} \setminus W_{j+1,1})$ and $1 \leq j \leq m$. By Lemma 2.8 again, there is a Carleson-Newman Blaschke product $b_2$ such that $b_2 < b_1$ and $\ord(b_2, y) \leq j$ for every $y \in Z(I) \cap (W_{j,2} \setminus W_{j+1,2})$ and $1 \leq j \leq m$. Inductively we may get a sequence of Carleson-Newman Blaschke products $\{b_n\}_n$ such that $b_n \in I, b_{n+1} < b_n$ and $\ord(b_n, y) \leq j$ for every $y \in Z(I) \cap (W_{j,n} \setminus W_{j+1,n})$ and $1 \leq j \leq m$.

Let $x \in Z(I)$ and $t = \ord(I, x)$. We consider two cases separately.

Case 1. Suppose that $t < m$. Then $x \notin Z_{t+1}(I)$ and there is a positive integer $k$ such that $x \in Z(I) \cap (W_{t,k} \setminus W_{t+1,k})$. Hence $\ord(b_k, x) \leq t$. Since $b_k \in I$, we have $t = \ord(I, x) \leq \ord(b_k, x) \leq t$. Thus we get $\ord(I, x) = \ord(b_k, x)$.

Case 2. Suppose that $t = m$, that is, $\ord(I, x) = m$. Then $x \in Z(I) \cap (W_{m,n} \setminus W_{m+1,n})$ for every $n \geq 1$. Hence $\ord(b_n, x) = m$. Since $b_n \in I$, we have $m \leq \ord(b_n, x)$. Thus we get $\ord(I, x) = \ord(b_n, x)$ for every $n \geq 1$.

The following is due to Hoffman [7].

Lemma 2.10. For any interpolating Blaschke product $b$ with zeros $\{z_n\}_n$ in $\mathbb{D}$, there exists a positive number $\lambda(b)$ such that a sequence $\{w_n\}_n$ in $\mathbb{D}$ satisfying $\rho(w_n, z_n) < \lambda(b)$ is an interpolating sequence.

Lemma 2.11. Let $I$ be a closed ideal in $H^\infty$ and $Z(I) \subset G$. Let $B$ be a Carleson-Newman Blaschke product in $I$. Then there is a Carleson-Newman Blaschke product $b$ in $I$ satisfying the following conditions.

(i) $\ord(b, x) = \ord(B, x)$ for every $x \in Z(I) \setminus \mathbb{D}$.

(ii) $\ord(b, z) = \ord(I, z)$ for every $z \in Z(I) \cap \mathbb{D}$.

(iii) $\ord(b, z) = 1$ for every $z \in (Z(b) \setminus Z(I)) \cap \mathbb{D}$.

Proof. Let $\varphi_1, \varphi_2, \ldots, \varphi_m$ be interpolating Blaschke products satisfying $B = \prod_{j=1}^m \varphi_j$. Let $\lambda = \min_{1 \leq j \leq m} \lambda(\varphi_j)$. Then $\lambda > 0$. Let $\{z_n\}_n = Z(B) \cap \mathbb{D}$ and $k_n = \ord(B, z_n)$. Then $\sup_{n \geq 1} k_n < \infty$. Let $\{\varepsilon_n\}_n$ be a sequence of numbers with $0 < \varepsilon_n < \lambda$ such that $\varepsilon_n \to 0$ as $n \to \infty$. We shall move the zeros of $B$ a little. Let $n$ be a positive integer. If $z_n \notin Z(I)$, then take $\{w_{n,1}, w_{n,2}, \ldots, w_{n,k_n}\}$ in $\mathbb{D}$ such that $\rho(w_{n,i}, z_n) < \varepsilon_n, w_{n,i} \neq w_{n,j}$ for $i \neq j$ and $\{w_{n,1}, w_{n,2}, \ldots, w_{n,k_n}\} \cap \{z_n\}_n = \emptyset$.

If $z_n \in Z(I)$, put $\ell_n = \ord(I, z_n)$. Then take $\{w_{n,1}, w_{n,2}, \ldots, w_{n,k_n}\}$ in $\mathbb{D}$ as the following: $\rho(w_{n,i}, z_n) < \varepsilon_n$ for every $1 \leq i \leq \ell_n, w_{n,1} = w_{n,2} = \cdots = w_{n,\ell_n} = z_n, w_{n,i} \neq w_{n,j}$ for every $\ell_n \leq i < j \leq k_n$ and $\{w_{n,i} : \ell_n + 1 \leq i \leq k_n\} \cap \{z_n\}_n = \emptyset$. 

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Further, we may assume that
\[ \{w_{n,1}, w_{n,2}, \ldots, w_{n,k_n}\} \cap \{w_{j,1}, w_{j,2}, \ldots, w_{j,k_j}\} = \emptyset \]
for every \( n \neq j \) and
\[ \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} (1 - |w_{n,i}|) < \infty. \]

Let \( b \) be the Blaschke product with zeros \( \{w_{n,i}\}_{n,i} \) counting multiplicities. By Lemma 2.10, \( b \) is a Carleson-Newman Blaschke product. We have \( \text{ord}(b, x) = \text{ord}(B, x) \) for every \( x \in Z(I) \setminus \mathbb{D} \). It is easy to see that \( b \) satisfies (ii) and (iii). Since \( \text{ord}(I, x) \leq \text{ord}(b, x) \) for every \( x \in Z(I) \), by Theorem A we have \( b \in I \). \( \square \)

**Lemma 2.12.** Let \( B \) be a Carleson-Newman Blaschke product and \( \{z_n\}_n \) be an interpolating sequence in \( \mathbb{D} \). If \( 0 < \varepsilon < 1 \), then
\[ \inf_n \sup \{|B(z)| : z \in \mathbb{D}, \rho(z, z_n) < \varepsilon\} > 0. \]

**Proof.** To prove the assertion, suppose not. Then there exists a subsequence \( \{n_j\}_j \) such that
\[ \lim_{j \to \infty} \sup \{|B(z)| : z \in \mathbb{D}, \rho(z, z_{n_j}) < \varepsilon\} = 0. \]

Let \( x \) be a cluster point of \( \{z_{n_j}\}_j \) in \( M(H^\infty) \). By Hoffman’s work [7], it is easy to see that \( B \equiv 0 \) on \( P(x) \), the Gleason part of \( x \). By our assumption, \( B \not\equiv 0 \) on \( P(x) \), and this is a contradiction. \( \square \)

**Lemma 2.13.** Let \( B \) be a Carleson-Newman Blaschke product and \( b \) be an interpolating Blaschke product. Let \( E \) be a closed \( G_\delta \)-subset of \( Z(b) \). Then there is an interpolating Blaschke product \( \varphi \) such that \( E \subset Z(\varphi) \) and \( Z(B) \cap E = Z(B) \cap Z(\varphi) \).

**Proof.** If \( Z(B) \cap E = Z(B) \cap Z(b) \), then put \( \varphi = b \). Then we get the assertion. So we assume that \( Z(B) \cap E \subsetneq Z(B) \cap Z(b) \). By the assumptions, there is a sequence of closed subsets \( \{K_n\}_n \) of \( Z(b) \) such that
\[ (Z(B) \cap Z(b)) \setminus E = \bigcup_{n=1}^{\infty} K_n \]
and \( K_n \cap K_k = \emptyset \) for \( n \neq k \). We note that
\[ \bigcup_{n=1}^{\infty} K_n \setminus \bigcup_{n=1}^{\infty} K_n \subset E. \]

Take a sequence of open subsets \( \{U_n\}_n \) of \( M(H^\infty) \) such that \( K_n \subset U_n \), \( \overline{U_n} \cap \overline{U_k} = \emptyset \) for \( n \neq k \), \( E \cap \overline{U_n} = \emptyset \) and \( Z(b) \cap U_n \) is an open and closed subset of \( Z(b) \) for every \( n \geq 1 \). Let \( b_{n,\ell} \) be the subproduct of \( b \) with zeros \( \{z_{n,\ell}\}_\ell : Z(b) \cap U_n \cap \mathbb{D} \). Then \( K_n \subset Z(b_{n,\ell}) \), \( E \cap Z(b_{n,\ell}) = \emptyset \) for every \( n \geq 1 \) and \( b = \prod_{n=0}^{\infty} b_n \) for some interpolating Blaschke product \( b_0 \). We note that
\[ (Z(B) \cap Z(b)) \setminus \bigcup_{n=1}^{\infty} Z(b_{n,\ell}) \subset E. \]

Let \( \{\varepsilon_n\}_n \) be a sequence of numbers such that \( 0 < \varepsilon_n < \lambda(b) \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \). By Lemma 2.12 there is a sequence of positive numbers \( \{\delta_n\}_n \) such that
\[ \sup \{|B(z)| : z \in \mathbb{D}, \rho(z, z_{n,\ell}) < \varepsilon_n\} > \delta_n \]
for every $\ell \geq 1$. For each $\ell \geq 1$, take $w_{n,\ell} \in \mathbb{D}$ satisfying $\rho(w_{n,\ell}, z_{n,\ell}) < \varepsilon_n$ and $|B(w_{n,\ell})| > \delta_n$. By Lemma 2.10, $\{w_{n,\ell}\}_\ell$ is an interpolating sequence for every $n \geq 1$. For each $n \geq 1$, let $\varphi_n$ be the interpolating Blaschke product with zeros $\{w_{n,\ell}\}_\ell$. Then $Z(B) \cap Z(\varphi_n) = \emptyset$ and $E \cap Z(\varphi_n) = \emptyset$ for every $n \geq 1$. Since

$$\sup_{\ell \geq 1} \rho(w_{n,\ell}, z_{n,\ell}) \leq \varepsilon_n \to 0 \quad (n \to \infty),$$

we have

$$Z\left(\prod_{n=1}^{\infty} b_n \right) \setminus \bigcup_{n=1}^{\infty} Z(b_n) = Z\left(\prod_{n=1}^{\infty} \varphi_n \right) \setminus \bigcup_{n=1}^{\infty} Z(\varphi_n).$$

Put $\varphi = b_0 \prod_{n=1}^{\infty} \varphi_n$. Since

$$\sup_{n,\ell \geq 1} \rho(w_{n,\ell}, z_{n,\ell}) < \lambda(b),$$

by Lemma 2.10, $\varphi$ is an interpolating Blaschke product. Since $E \subset Z(b)$ and $E \cap Z(b_n) = \emptyset$ for every $n \geq 1$, we have

$$E \subset Z(b) \setminus \bigcup_{n=1}^{\infty} Z(b_n)$$

$$= \left( Z(b_0) \cup Z\left(\prod_{n=1}^{\infty} b_n \right) \right) \setminus \bigcup_{n=1}^{\infty} Z(b_n)$$

$$= \left( Z(b_0) \setminus \bigcup_{n=1}^{\infty} Z(b_n) \right) \cup \left( Z\left(\prod_{n=1}^{\infty} \varphi_n \right) \setminus \bigcup_{n=1}^{\infty} Z(\varphi_n) \right)$$

$$= Z(\varphi) \setminus \bigcup_{n=1}^{\infty} Z(\varphi_n).$$

Hence $E \subset Z(\varphi)$. Since $Z(B) \cap Z(\varphi_n) = \emptyset$ for every $n \geq 1$, we have

$$Z(B) \cap E \subset Z(B) \cap Z(\varphi) \subset Z(B) \cap \left( Z(\varphi) \setminus \bigcup_{n=1}^{\infty} Z(\varphi_n) \right)$$

$$= (Z(B) \cap Z(b)) \setminus \bigcup_{n=1}^{\infty} Z(b_n) \subset Z(B) \cap E.$$

Hence we get $Z(B) \cap E = Z(B) \cap Z(\varphi)$. \hfill $\square$

**Proof of Theorem 1.1** (i) $\Rightarrow$ (ii) By Theorem B, there is a Carleson-Newman Blaschke product $b_1$ of order $m$ in $I$. By Lemma 2.11, we may assume that $\text{ord}(b_1, z) = \text{ord}(I, z)$ for every $z \in Z(I) \cap \mathbb{D}$ and $\text{ord}(b_1, z) = 1$ for every $z \in (Z(b_1) \setminus Z(I)) \cap \mathbb{D}$. By Lemma 2.9, there is a sequence of Carleson-Newman Blaschke products $\{b_n\}_n$ such that $b_n \in I$, $b_{n+1} < b_n$ for every $n \geq 1$, and for each $x \in Z(I)$ there is a positive integer $n$ satisfying $\text{ord}(b_n, x) = \text{ord}(I, x)$.

Since the order of $b_1$ is equal to $m$, there are interpolating Blaschke products $\varphi_{1,1}, \varphi_{2,1}, \ldots, \varphi_{m,1}$ such that $b_1 = \prod_{j=1}^{m} \varphi_{j,1}$. Since $b_n \in I$ and $b_{n+1} < b_n$ for every $n \geq 1$, we have $\text{ord}(b_n, z) = \text{ord}(I, z)$ for $z \in Z(I) \cap \mathbb{D}$ and $\text{ord}(b_n, z) = 1$ for $z \in (Z(b_n) \setminus Z(I)) \cap \mathbb{D}$. Then there are the unique interpolating Blaschke products $\varphi_{1,n}, \varphi_{2,n}, \ldots, \varphi_{m,n}$ such that $b_n = \prod_{j=1}^{m} \varphi_{j,n}$ and $\varphi_{j,n+1} < \varphi_{j,n}$ for every $1 \leq j \leq m$. We note that if $z \in Z(I) \cap \mathbb{D}$ and $\varphi_{j,1}(z) = 0$, then $\varphi_{j,n}(z) = 0$ for every $n \geq 1$. 

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For each \(1 \leq j \leq m\), let
\[
E_j = Z(I) \cap \bigcap_{n=1}^{\infty} Z(\varphi_{j,n}).
\]
By Lemma 2.3, \(E_j\) is a compact \(G_\delta\)-set. Since \(\varphi_{j,n}\) is an interpolating Blaschke product, \(E_j\) is a \(\rho\)-separated set. Since \(b_n \in I\),
\[
Z(I) \subset Z(b_n) = \bigcup_{j=1}^{m} Z(\varphi_{j,n}),
\]
so
\[
Z(I) = \bigcup_{j=1}^{m} (Z(I) \cap Z(\varphi_{j,n})).
\]
We have
\[
\bigcup_{j=1}^{m} E_j \subset \bigcup_{j=1}^{m} (Z(I) \cap Z(\varphi_{j,n})) = Z(I).
\]
Suppose that \(\bigcup_{j=1}^{m} E_j \subsetneq Z(I)\) and \(y \in Z(I) \setminus \bigcup_{j=1}^{m} E_j\). For each \(1 \leq j \leq m\), since \(y \notin E_j\) there is a positive integer \(n_j\) such that \(y \notin Z(I) \cap Z(\varphi_{j,n_j})\). Let \(n = \min_{1 \leq j \leq m} n_j\). Then
\[
Z(I) \cap Z(\varphi_{j,n}) \subset Z(I) \cap Z(\varphi_{j,n_j}).
\]
Hence
\[
y \notin \bigcup_{j=1}^{m} (Z(I) \cap Z(\varphi_{j,n})) = Z(I).
\]
But this is a contradiction. Thus we get
\[
Z(I) = \bigcup_{j=1}^{m} E_j.
\]
Let \(x \in Z(I)\). Then there is a positive integer \(n_1\) such that \(\text{ord}(b_{n_1}, x) = \text{ord}(I, x)\). We write \(\ell = \text{ord}(I, x)\). Then there are positive integers \(j_1, j_2, \ldots, j_\ell\) such that
\[
\text{ord}\left(\prod_{i=1}^{\ell} \varphi_{j_i,n_1}, x\right) = \ell \quad \text{and} \quad \text{ord}\left(b_{n_1}/\prod_{i=1}^{\ell} \varphi_{j_i,n_1}, x\right) = 0.
\]
Since \(b_n \in I\) and \(b_n \prec b_{n_1}\) for every \(n \geq n_1\), \(\text{ord}(b_n, x) = \ell\) and \(\varphi_{j_i,n}(x) = 0\) for every \(1 \leq i \leq \ell\) and \(n \geq n_1\). Thus for any \(n \geq n_1\) we have
\[
\text{ord}(I, x) = \text{ord}(b_n, x) = \text{ord}\left(\prod_{j=1}^{m} \varphi_{j,n}, x\right)
\]
\[
= \sum_{i=1}^{\ell} \text{ord}(\varphi_{j_i,n}, x) = \#\{j : x \in E_j, 1 \leq j \leq m\},
\]
where \(\#A\) denotes the number of elements in a set \(A\). Let
\[
J = \bigotimes_{j=1}^{m} I(E_j).
\]
By Lemma 2.5, we have \( ord(I(E_j), x) = 1 \) for every \( x \in E_j \) and \( Z(I(E_j)) = E_j \) for every \( 1 \leq j \leq m \). Hence by Lemma 2.6 \( Z(J) = \bigcup_{j=1}^{m} E_j = Z(I) \) and
\[
ord(J, x) = \sum_{j=1}^{m} ord(I(E_j), x) = \# \{ j : x \in E_j, 1 \leq j \leq m \}
\]
for every \( x \in Z(I) \). By Theorem A, we have \( I = J = \bigodot_{j=1}^{m} I(E_j) \).

(ii) \( \Rightarrow \) (iii) Suppose that condition (ii) holds. By Lemma 2.6, \( Z(I) = \bigcup_{j=1}^{m} Z(E_j) \) for every \( 1 \leq j \leq m \), so \( Z(I) \) is a \( G_\delta \)-set. By Lemma 2.4 for each \( 1 \leq j \leq m \) there is an interpolating Blaschke product \( \varphi_j \) such that \( E_j \subset Z(\varphi_j) \). Let \( \Phi = \prod_{j=1}^{m} \varphi_j \). By Lemma 2.13 for each \( 1 \leq j \leq m \) there exists an interpolating Blaschke product \( b_j \) such that \( E_j \subset Z(b_j) \) and \( Z(\Phi) \cap Z(b_j) = Z(\Phi) \cap E_j = E_j \). We note that \( Z(I) \subset Z(\Phi) \). Let \( B = \prod_{j=1}^{m} b_j \). Then for any \( x \in Z(I) \), we have
\[
ord(B, x) = \ord(\prod_{j=1}^{m} b_j, x) = \sum_{j=1}^{m} \ord(b_j, x) = \# \{ j : x \in E_j, 1 \leq j \leq m \}.
\]
By Lemmas 2.5 and 2.6 we have
\[
ord(I, x) = \ord(\bigodot_{j=1}^{m} I(E_j), x) = \sum_{j=1}^{m} \ord(I(E_j), x) = \# \{ j : x \in E_j, 1 \leq j \leq m \}.
\]
Thus we get \( \ord(B, x) = \ord(I, x) \) for every \( x \in Z(I) \). By Theorem A, we have \( B \in I \).

(iii) \( \Rightarrow \) (iv) Suppose that condition (iii) holds. Let \( B_1 \) be a Carleson-Newman Blaschke product of order \( m \) in \( I \) satisfying \( \ord(B_1, x) = \ord(I, x) \) for every \( x \in Z(I) \). Let \( \varphi_1, \varphi_2, \ldots, \varphi_m \) be interpolating Blaschke products satisfying \( B_1 = \prod_{j=1}^{m} \varphi_j \). For each \( 1 \leq j \leq m \), let \( E_j = Z(I) \cap Z(\varphi_j) \). Since \( Z(I) \) is a \( G_\delta \)-set, \( E_j \) is a closed \( G_\delta \)-set. By Lemma 2.13 there is an interpolating Blaschke product \( b_j \) such that \( Z(B_1) \cap Z(b_j) = E_j \). Let \( B_2 = \prod_{j=1}^{m} b_j \). For any \( x \in Z(I) \), we have
\[
ord(B_2, x) = \sum_{j=1}^{m} \ord(b_j, x) = \# \{ j : x \in E_j, 1 \leq j \leq m \} = \ord(B_1, x) \geq \ord(I, x).
\]
By Theorem A, we have \( B_2 \in I \). We also have
\[
Z(B_1) \cap Z(B_2) = Z(B_1) \cap \bigcup_{j=1}^{m} Z(b_j) = \bigcup_{j=1}^{m} E_j = Z(I) \cap Z(B_1) = Z(I).
\]
Let \( J = I[B_1, B_2] \). Then \( Z(J) = Z(I) \) and \( \ord(J, x) = \ord(I, x) \) for every \( x \in Z(I) \). By Theorem A again, we have \( J = I \).

(iv) \( \Rightarrow \) (i) is trivial. \( \square \)
In the following example, we shall show that there exist compact \( \rho \)-separated \( G_\delta \)-subsets \( E_1 \) and \( E_2 \) of \( G \) such that the ideal \( I(E_1) \cap I(E_2) \) is not countably generated.

**Example 2.14.** Let \( \{ \theta_k \}_k \) be a sequence of numbers such that \( 0 < \theta_{k+1} < \theta_k < 1 \) and \( \theta_k \to 0 \) as \( k \to \infty \). It is known that there is an interpolating Blaschke product \( B_1 \) with zeros \( \{ z_n \}_n \) in \( D \) such that

\[
\overline{\{ z_n \}_n}^c \setminus \{ z_n \}_n = \{ e^{i\theta_k} : k \geq 1 \} \cup \{ 1 \},
\]

where \( \overline{\{ z_n \}_n}^c \) is the closure of \( \{ z_n \}_n \) in \( \mathbb{C} \). Let \( \mathbb{N} \) be the set of positive integers. We may divide \( \mathbb{N} \) as \( \mathbb{N} = \bigcup_{k=1}^{\infty} N_k \) such that \( N_k \cap N_j = \emptyset \) for \( k \neq j \) and

\[
\{ z_n : n \in N_k \}^c \setminus \{ z_n : n \in N_k \} = \{ e^{i\theta_k} \}, \quad k \in \mathbb{N}.
\]

Let \( b_k \) be the subproduct of \( B_1 \) with zeros \( \{ z_n : n \in N_k \} \). Then \( B_1 = \prod_{k=1}^{\infty} b_k \).

Let \( \{ \varepsilon_k \}_k \) be a sequence of numbers such that \( 0 < \varepsilon_k < 1 \) and \( \varepsilon_k \to 0 \) as \( k \to \infty \). Let \( q_k(z) = (b_k(z) - \varepsilon_k)/(1 - \varepsilon_k b_k(z)) \). Taking smaller \( \varepsilon_k \), we may assume that \( B_2 := \prod_{k=1}^{\infty} q_k \) is an interpolating Blaschke product and

\[
\left( \bigcup_{k=1}^{\infty} Z(b_k) \right) \cap \left( \bigcup_{k=1}^{\infty} Z(q_k) \right) = \emptyset.
\]

Let

\[
E_1 = Z(B_1) \setminus D \quad \text{and} \quad E_2 = Z(B_2) \setminus D.
\]

Then \( E_1, E_2 \) are compact \( \rho \)-separated \( G_\delta \)-subsets of \( G \),

\[
E_1 = \left( \bigcup_{k=1}^{\infty} (Z(b_k) \setminus D) \right) \cup \left( E_1 \setminus \bigcup_{k=1}^{\infty} Z(b_k) \right)
\]

and

\[
E_2 = \left( \bigcup_{k=1}^{\infty} (Z(q_k) \setminus D) \right) \cup \left( E_2 \setminus \bigcup_{k=1}^{\infty} Z(q_k) \right).
\]

By Lemma 2.3, \( I(E_1) \) and \( I(E_2) \) are countably generated closed ideals in \( H^\infty \). Let \( I = I(E_1) \cap I(E_2) \). Then \( I = I(E_1 \cup E_2) \). By the construction, we may check that

\[
E_1 \setminus \bigcup_{k=1}^{\infty} Z(b_k) = E_2 \setminus \bigcup_{k=1}^{\infty} Z(q_k)
\]

and

\[
\bigcup_{k=1}^{\infty} (Z(b_k) \setminus D) \setminus \bigcup_{k=1}^{\infty} Z(b_k) = \bigcup_{k=1}^{\infty} (Z(q_k) \setminus D) \setminus \bigcup_{k=1}^{\infty} Z(q_k)
\]

\[
\subset E_1 \setminus \bigcup_{k=1}^{\infty} Z(b_k).
\]

Let \( \Omega \) be the set of all subproducts \( q \) of \( B_2 \) satisfying

\[
\bigcup_{k=1}^{\infty} (Z(q_k) \setminus D) \subset Z(q).
\]
COUNTABLY GENERATED CLOSED IDEALS IN $H^\infty$

Then we have $B_1q \in I$ for every $q \in \Omega$ and

$$\bigcap_{q \in \Omega} Z(q) = \bigcup_{k=1}^{\infty} (Z(q_k) \setminus \mathbb{D}).$$

By this fact, we have

$$Z_2(I) = \bigcup_{k=1}^{\infty} (Z(b_k) \setminus \mathbb{D}) \setminus \bigcup_{k=1}^{\infty} Z(b_k) = \bigcup_{k=1}^{\infty} Z(b_k) \setminus \bigcup_{k=1}^{\infty} Z(b_k),$$

and $Z_2(I)$ is not a $G_\delta$-set (see Example 2.9 in [12]). By Lemma 2.3 $I$ is not countably generated. We note that $I = \overline{I(E_1) \otimes I(E_2 \setminus E_1)}$.

**References**


