

ON UNIPOTENT ALGEBRAIC G -GROUPS AND 1-COHOMOLOGY

DAVID I. STEWART

ABSTRACT. In this paper we consider non-abelian 1-cohomology for groups with coefficients in other groups. We prove versions of the ‘five lemma’ arising from this situation. We go on to show that a connected unipotent algebraic group Q acted on morphically by a connected algebraic group G admits a filtration with successive quotients having the structure of G -modules. From these results we deduce extensions to results due to Cline, Parshall, Scott and van der Kallen. First, if G is a connected, reductive algebraic group with Borel subgroup B and Q a unipotent algebraic G -group, we show the restriction map $H^1(G, Q) \rightarrow H^1(B, Q)$ is an isomorphism. We also show that this situation admits a notion of rational stability and generic cohomology. We use these results to obtain corollaries about complete reducibility and subgroup structure.

1. INTRODUCTION

The low degree cohomology of a group G with coefficients in another group Q on which G acts gives rise to important invariants in group theory. For instance, in degree 0, $H^0(G, Q)$ is the group of fixed points of the action of G on Q , $H^1(G, Q)$ is the set of conjugacy classes of complements to Q in the semidirect product GQ , and $H^2(G, Q)$ (where defined) measures the number of non-equivalent group extensions

$$1 \rightarrow Q \rightarrow E \rightarrow G \rightarrow 1.$$

In this paper we consider the situation where Q is, in general, a non-abelian G -group. In such a situation $H^1(G, Q)$ is just a pointed set (in contrast to the situation where $Q = V$ is a representation for G when $H^1(G, V)$ is a vector space). One needs to work harder to employ the usual tools of cohomology since exact sequences do not, unfortunately, say very much about pointed sets. For instance, if $1 \rightarrow A \rightarrow B$ is exact, the map $A \rightarrow B$ need not be an injection, merely that anything mapping to the distinguished element in B should have been the distinguished element in A . However one can usually rescue the situation for non-abelian 1-cohomology by employing ‘twisting’, a technique familiar in Galois cohomology.

Before we describe our results, let us say at the outset that we will consider in detail the situation where G is a connected (possibly reductive) algebraic group over an algebraically closed field k , and Q is a unipotent algebraic G -group. However up until the end of §3.1 all our results apply equally in the category of abstract groups. For clarity we give our results up to this point in the abstract setting and then in §3.2 indicate the appropriate modifications to use these results in the rational (algebraic) situation; the proofs are the same, *mutatis mutandis*.

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Letting G first be an abstract group and Q an abstract G -group, we give in §2.1 the well-known definitions of the 0th, 1st and 2nd cohomology groups of G with coefficients in Q in terms of cocycles (where these definitions exist) as well as the notion of twisting an action by a cocycle. When Q fits into a short exact sequence

$$1 \rightarrow R \rightarrow Q \rightarrow S \rightarrow 1$$

and where the image of R in Q is contained in the centre $Z(Q)$ of Q , there is an exact sequence of non-abelian cohomology (Proposition 2.2.1(i)). This exact sequence fits into various commutative diagrams (Proposition 2.2.1(ii)–(x)). In particular we consider the restriction of this exact sequence to a subgroup B of G and show how conditions on the restriction maps of cohomology $H^n(G, R) \rightarrow H^n(B, R)$ and $H^n(G, S) \rightarrow H^n(B, S)$ for $0 \leq n \leq 2$ can be used to deduce properties of the restriction $H^1(G, Q) \rightarrow H^1(B, Q)$ (Theorems 3.1.1 and 3.1.2). For abelian Q this would come down to a simple application of the equally simple ‘five lemma’ (see e.g. [Rot09, 2.72]), but in the non-abelian case, we must use the technique of twisting.

We apply this in the special situation where G is a connected algebraic group acting (morphically) on a connected unipotent algebraic group Q . We first show that Q has a central filtration by G -modules (Theorem 3.3.5). This filtration allows us to go further than the results obtained in [Ric82, §6] where this general set-up was first considered; in particular, one can bring second cohomology into play.¹

Perhaps surprisingly, it turns out that unipotent algebraic groups are just as good as vector spaces as far as low-degree cohomology is concerned. By way of demonstration, we can use our work to generalise several of the results in the paper [CPSvdK77]. Consider the case where G is connected and reductive, and let P be a parabolic subgroup of G . In [CPSvdK77], it is proved that $H^n(G, V) \cong H^n(P, V)$ for any finite dimensional rational G -module V . We use our filtration to extend this result by applying our Theorem 3.1.1 inductively. We show that $H^1(G, Q) \cong H^1(P, Q)$ for any unipotent algebraic group Q on which G acts morphically.

The main result of [CPSvdK77] is the proof of the existence of rational stability and generic cohomology. Let G be additionally defined and split over the prime subfield \mathbb{F}_p . If $V^{[r]}$ denotes the r th Frobenius twist of V , then it is shown that for each $m \in \mathbb{N}$ and V a finite dimensional module there is some r_0 with $H^m(G, V^{[r]}) \cong H^m(G, V^{[r_0]})$ for any $r \geq r_0$. In addition, there is some s_0 such that for each $s \geq s_0$, restriction $H^m(G, V^{[r_0]}) \rightarrow H^m(G(p^s), V^{[r_0]}) \cong H^m(G(p^s), V)$ is an isomorphism. We extend each of these theorems in the case $m = 1$, replacing V with a unipotent algebraic group (Theorem 3.5.6).

The above has a number of corollaries on subgroup structure related to Serre’s notion of G -complete reducibility. If H is a closed, connected, reductive subgroup of a parabolic subgroup P of G , one can find its image \bar{H} under the map $\pi : P \rightarrow L$ for L a Levi subgroup of P . One need not have H a complement to Q in

¹We remark that after this paper was submitted it came to our attention that George McNinch ([McN13]) has a preprint establishing the existence of such a central filtration, while working in the more general context of connected algebraic groups over arbitrary fields k . In such a situation one can have unipotent radicals of algebraic groups which are not defined over k . McNinch defines a condition **(R)**: the unipotent radical of G is defined over k . He then proves that when **(R)** holds, such a filtration of Q exists whenever Q is split over k . Now, since these conditions hold automatically for $k = \bar{k}$ our result can be recovered from his. Our proofs, however, appear to be substantively different. The one given here is significantly shorter, no doubt owing to the technical difficulties involved in McNinch’s more general result. Note that the problem of whether there exists such a filtration of Q when G is disconnected (e.g. finite) remains open.

$\bar{H}Q$, even though it is when it is considered as an abstract group. Where it is a complement, it corresponds, up to Q -conjugacy, to an element of $H^1(\bar{H}, Q)$. We show in Lemma 3.6.1 in almost all cases that H is a complement, and even when it is not, it still corresponds to an element of $H^1(\bar{H}, Q^{[1]})$. This gives a cohomological characterisation of conjugacy classes of subgroups of parabolics. We can then prove results on G -complete reducibility using this characterisation, via a result of Bate, Martin, Roerhle and Tange. First, let B be a Borel subgroup of H with unipotent radical $U = R_u(B)$. Then H is conjugate to a subgroup of L if and only if B is, if and only if U is (Corollary 3.6.2). Second, for some q which can be explicitly calculated, there is a finite reductive subgroup $H(q)$ of H such that H is conjugate to a subgroup of L if and only if $H(q)$ is (Corollary 3.6.3).

2. PRELIMINARIES

2.1. **Definitions.** We start with some standard material. Let G be a group and let Q be a G -set; that is, a set equipped with a right action of G .

Then we define $H^0(G, Q)$ as the set Q^G of fixed points of G acting on Q , i.e. $H^0(G, Q) = \{q \in Q : q^g = q, \forall g \in G\}$.

If further Q is a G -group, then we may define the 1-cohomology $H^1(G, Q)$ as follows.

First, a 1-cocycle is a map $\gamma : G \rightarrow Q$ satisfying the cocycle condition $\gamma(gh) = \gamma(g)^h \gamma(h)$. The set of all 1-cocycles is denoted by $Z^1(G, Q)$. We say that two 1-cocycles γ, δ are cohomologous and write $\gamma \sim \delta$ if $\delta(g) = q^{-g} \gamma(g) q$ for some element $q \in Q$. Then we define $H^1(G, Q)$ as the set of equivalence classes of $Z^1(G, Q)$ under \sim ; i.e. $H^1(G, Q) = Z^1(G, Q) / \sim$.

There is a distinguished point in $Z^1(G, Q)$ given by the trivial cocycle $\mathbf{1} : G \rightarrow Q$; $\mathbf{1}(g) = 1$. The equivalence class of the trivial cocycle $[\mathbf{1}] \subseteq Z^1(G, Q)$ is the set of coboundaries, and we denote this distinguished point in $H^1(G, Q)$ by $B^1(G, Q)$.

If R is an abelian G -group we can go further and define the second cohomology: define a 2-cocycle to be a map $\gamma : G \times G \rightarrow R$ satisfying the 2-cocycle condition $\gamma(gh, k) \gamma(g, h)^k = \gamma(g, hk) \gamma(g, k)$. The set of all 2-cocycles is denoted $Z^2(G, R)$. We say that two 2-cocycles γ and δ are cohomologous and write $\gamma \sim \delta$ if there is a map (morphism) $\phi : G \rightarrow R$ with $\delta(g, h) = \gamma(g, h) \phi(g)^h \phi(h) \phi(gh)^{-1}$. We then define $H^2(G, R)$ to be the set of equivalence classes of $Z^2(G, R)$ modulo \sim , i.e. $H^2(G, R) := Z^2(G, R) / \sim$. Again there is a distinguished point in $Z^2(G, R)$ given by the trivial 2-cocycle $\mathbf{1}$ and its class in $H^2(G, R)$ is denoted by $B^2(G, R)$.

2.2. **Twisting, sequences and commutative diagrams.** Recall from [Ser94, I.§5.3] the notion of twisting the action of G on a G -group Q by a cocycle in $Z^1(G, Q)$. To wit, fix an arbitrary 1-cocycle $\gamma \in Z^1(G, Q)$ and define $q * g = (q^g) \gamma(g) = \gamma(g)^{-1} q^g \gamma(g)$ for each $q \in Q$.² One checks that this is a new action of G on Q . We denote the new G -group by Q_γ , which coincides with Q as a group.

We give a few facts about the set-up so far.

Proposition 2.2.1. *Let $1 \rightarrow R \rightarrow Q \begin{matrix} \xrightarrow{\pi} \\ \xleftarrow{\sigma} \end{matrix} S \rightarrow 1$ be a short exact sequence of G -groups, with the image of R in Q contained in the centre $Z(Q)$ of Q and with σ*

²More generally, R can be a further right G -set with a right action on Q which commutes with the action of G on Q , i.e. $(q^r)^g = q^{gr^g}$. Then one can define $q * g = q^g \gamma(g)$ for $q \in Q, g \in G$ and $\gamma \in Z^1(G, R)$.

a section of π (σ not necessarily a homomorphism). Additionally, fix $\gamma \in Z^1(G, Q)$ and denote the induced element in $Z^1(G, S)$ also by γ . Then the following hold:

- (i) There is an exact sequence of cohomology

$$1 \rightarrow H^0(G, R) \rightarrow H^0(G, Q) \rightarrow H^0(G, S) \\ \xrightarrow{\delta_G} H^1(G, R) \rightarrow H^1(G, Q) \rightarrow H^1(G, S) \xrightarrow{\Delta_G} H^2(G, R),$$

with $\delta_G(s)(g) = [\sigma(s)^{-g}\sigma(s)]$; $\Delta_G(\gamma)(g, h) = [\sigma(\gamma(g))^h\sigma(\gamma(h))\sigma(\gamma(gh)^{-1})]$.

- (ii) Moreover, the association of the short exact sequence to the longer is functorial: Suppose we have another short exact sequence of G -groups

$$1 \rightarrow R' \rightarrow Q' \begin{array}{c} \xrightarrow{\pi'} \\ \xleftarrow{\sigma'} \end{array} S' \rightarrow 1$$

with the image of R' in $Z(Q')$ and with σ' a section of π' , and we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R & \longrightarrow & Q & \longrightarrow & S & \longrightarrow & 1 \\ & & \zeta \downarrow & & \eta \downarrow & & \theta \downarrow & & \\ 1 & \longrightarrow & R' & \longrightarrow & Q' & \longrightarrow & S' & \longrightarrow & 1, \end{array}$$

where the vertical arrows are G -equivariant homomorphisms. Then there is a commutative diagram

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & R^G & \longrightarrow & Q^G & \longrightarrow & S^G & \longrightarrow & H^1(G, R) & & \\ & & \zeta \downarrow & & \eta \downarrow & & \theta \downarrow & & \zeta \circ - \downarrow & & \\ 1 & \longrightarrow & R'^G & \longrightarrow & Q'^G & \longrightarrow & S'^G & \longrightarrow & H^1(G, R') & & \\ & & & & \longrightarrow & H^1(G, Q) & \longrightarrow & H^1(G, S) & \longrightarrow & H^2(G, R) & \\ & & & & & \eta \circ - \downarrow & & \theta \circ - \downarrow & & \zeta \circ - \downarrow & \\ & & & & & \longrightarrow & H^1(G, Q') & \longrightarrow & H^1(G, S') & \longrightarrow & H^2(G, R') \end{array}$$

of the two sequences arising from (i) above.

- (iii) There is another short exact sequence of G -groups,

$$1 \rightarrow R \rightarrow Q_\gamma \rightarrow S_\gamma \rightarrow 1,$$

with the image of R contained in the centre of Q_γ . Moreover, in the situation of (ii), there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R & \longrightarrow & Q_\gamma & \longrightarrow & S_\gamma & \longrightarrow & 1 \\ & & \zeta \downarrow & & \eta \downarrow & & \theta \downarrow & & \\ 1 & \longrightarrow & R' & \longrightarrow & Q'_{\gamma'} & \longrightarrow & S'_{\gamma'} & \longrightarrow & 1, \end{array}$$

where γ' indicates the cocycle $\eta \circ \gamma$.

- (iv) Thus we get by (i) a new exact sequence of cohomology:

$$1 \rightarrow R^G \rightarrow Q_\gamma^G \rightarrow S_\gamma^G \rightarrow H^1(G, R) \rightarrow H^1(G, Q_\gamma) \rightarrow H^1(G, S_\gamma) \rightarrow H^2(G, R_\gamma),$$

which is also functorial in the manner of (ii).

(v) The map

$$\theta_\gamma : H^1(G, Q_\gamma) \rightarrow H^1(G, Q); [\delta] \mapsto [\delta\gamma],$$

where $\delta\gamma$ denotes the map $g \mapsto \gamma(g)\delta(g)$, is a well-defined bijection, taking the trivial class in $H^1(G, Q_\gamma)$ to the class of γ in $H^1(G, Q)$.

Now let B be a subgroup of G . Then:

(vi) Restriction to B of the exact sequence of cohomology from (i) gives rise to the following commutative diagram, where the rows are exact and the vertical arrows are restrictions:

$$\begin{array}{ccccccc} 1 & \longrightarrow & R^G & \longrightarrow & Q^G & \longrightarrow & S^G \xrightarrow{\delta_G} H^1(G, R) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & R^B & \longrightarrow & Q^B & \longrightarrow & S^B \xrightarrow{\delta_B} H^1(B, R) \\ & & & & \longrightarrow & H^1(G, Q) & \longrightarrow H^1(G, S) \longrightarrow H^2(G, R) \\ & & & & & \downarrow & \downarrow \\ & & & & & \longrightarrow & H^1(B, Q) \longrightarrow H^1(B, S) \longrightarrow H^2(B, R). \end{array}$$

(vii) If $\beta \in Z^1(B, Q)$ denotes the restriction of γ to B , then we have the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & R & \longrightarrow & Q_\gamma & \longrightarrow & S_\gamma \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & R & \longrightarrow & Q_\beta & \longrightarrow & S_\beta \longrightarrow 1, \end{array}$$

where the vertical arrows are restrictions. Moreover,

(viii) restriction from G to B gives rise to the following commutative diagram, where the rows are exact and the vertical arrows are restrictions:

$$\begin{array}{ccccccc} 1 & \longrightarrow & R^G & \longrightarrow & Q_\gamma^G & \longrightarrow & S_\gamma^G \xrightarrow{\delta_G} H^1(G, R) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & R^B & \longrightarrow & Q_\beta^B & \longrightarrow & S_\beta^B \xrightarrow{\delta_B} H^1(B, R) \\ & & & & \longrightarrow & H^1(G, Q_\gamma) & \longrightarrow H^1(G, S_\gamma) \longrightarrow H^2(G, R) \\ & & & & & \downarrow & \downarrow \\ & & & & & \longrightarrow & H^1(B, Q_\beta) \longrightarrow H^1(B, S_\beta) \longrightarrow H^2(B, R). \end{array}$$

(ix) We have the following commutative diagram:

$$\begin{array}{ccc} H^1(G, Q) & \longrightarrow & H^1(G, S) \\ \theta_\gamma \uparrow & & \theta_\beta \uparrow \\ H^1(G, Q_\gamma) & \longrightarrow & H^1(G, S_\beta), \end{array}$$

where the vertical maps are the bijections of (iv) and the horizontal maps are induced by the G -homomorphism $Q \rightarrow S$.

(x) We also have the following commutative diagram:

$$\begin{CD} H^1(G, Q_\gamma) @>\theta_\gamma>> H^1(G, Q) \\ @VVV @VVV \\ H^1(B, Q_\beta) @>\theta_\beta>> H^1(B, Q), \end{CD}$$

where the vertical maps arise from restriction.

Proof. (i) This is [Ser94, I.Prop. 38].

(ii) is a straightforward check. For example, to see that

$$\begin{CD} S^G @>\delta_G>> H^1(G, R) \\ @V\theta VV @VV\eta^\circ V \\ S'^G @>\delta'_G>> H^1(G, R') \end{CD}$$

commutes: on elements we have

$$\begin{CD} qR @>>> [q^{-\bullet}q] \\ @VV\theta V @VVV \\ \theta(qR) = \zeta(q)R' @>>> [\zeta(q)^{-\bullet}\zeta(q)] = \zeta([q^{-\bullet}q]). \end{CD}$$

(iii) is explained in [Ser94, p. 52].

(iv) is clear from (i) and (iii).

(v) is [Ser94, I.Prop. 35 bis].

(vi) is easy to check from the definitions of the maps concerned. For instance, if $qR \in Q/R \cong S$ with $(qR)^g = qR$ for all g , then $(\text{res}_B^G \circ \delta_G)(qR)(b) = [q^{-\bullet}q](b) = q^{-b}q = \delta_B(qR) = \delta_B(\text{res}_B^G qR)$ for all $b \in B$.

(vii) is clear.

(viii) follows from (iv), (vi) and (vii).

(ix) is explained in [Ser94, p. 47].

(x) On elements

$$\begin{CD} [\delta] @>>> [\delta\gamma] \\ @VVV @VVV \\ [\delta|_B] @>>> [\delta|_B\beta] = [\delta|_B\gamma|_B]. \end{CD}$$

□

Putting together the components of the last proposition:

Proposition 2.2.2. (i) *With the hypotheses of Proposition 2.2.1, we have the following commutative partial cuboid:*

$$\begin{array}{ccccc}
 & H^1(G, R) & \longrightarrow & H^1(G, Q) & \longrightarrow & H^1(G, S) , \\
 & \downarrow & & \nearrow & \downarrow & \nearrow \\
 H^1(G, R) & \longrightarrow & H^1(G, Q_\gamma) & \longrightarrow & H^1(G, S_\gamma) & \\
 \downarrow & & \downarrow & & \downarrow & \\
 & H^1(B, R) & \longrightarrow & H^1(B, Q) & \longrightarrow & H^1(B, S) \\
 \downarrow & & \downarrow & & \downarrow & \\
 H^1(B, R) & \longrightarrow & H^1(B, Q_\beta) & \longrightarrow & H^1(B, S_\beta) &
 \end{array}$$

where rightward arrows are part of four exact sequences running through the central vertical square.

(ii) *With the hypotheses of Proposition 2.2.1(ii), we have the following commutative partial cuboid:*

$$\begin{array}{ccccc}
 & H^1(G, R) & \longrightarrow & H^1(G, Q) & \longrightarrow & H^1(G, S) , \\
 & \downarrow & & \nearrow & \downarrow & \nearrow \\
 H^1(G, R) & \longrightarrow & H^1(G, Q_\gamma) & \longrightarrow & H^1(G, S_\gamma) & \\
 \downarrow & & \downarrow & & \downarrow & \\
 & H^1(G, R') & \longrightarrow & H^1(G, Q') & \longrightarrow & H^1(G, S') \\
 \downarrow & & \downarrow & & \downarrow & \\
 H^1(G, R') & \longrightarrow & H^1(G, Q'_{\gamma'}) & \longrightarrow & H^1(G, S'_{\gamma'}) &
 \end{array}$$

Proof. For the first partial cuboid, the front and back faces are subdiagrams of Proposition 2.2.1(vi),(viii); the top and bottom faces commute by (ix); the vertical squares commute by (x).

For the second partial cuboid, the front and back faces commute by Proposition 2.2.1(ii) and (iv). The vertical squares commute by (ix). □

3. MAIN RESULTS

3.1. Versions of the five lemma.

Theorem 3.1.1. *Assume the hypotheses of Proposition 2.2.1 and let B be a subgroup of G . Then in the following subdiagram of Proposition 2.2.1(vi),*

$$\begin{array}{ccccccccc}
 S^G & \xrightarrow{\delta_G} & H^1(G, R) & \xrightarrow{\iota_G} & H^1(G, Q) & \xrightarrow{\pi_G} & H^1(G, S) & \xrightarrow{\Delta_G} & H^2(G, R) \\
 h_1 \downarrow & & h_2 \downarrow & & h_3 \downarrow & & h_4 \downarrow & & h_5 \downarrow \\
 S^B & \xrightarrow{\delta_B} & H^1(B, R) & \xrightarrow{\iota_B} & H^1(B, Q) & \xrightarrow{\pi_B} & H^1(B, S) & \xrightarrow{\Delta_B} & H^2(B, R),
 \end{array}$$

the following hold:

- (i) *If h_2 and h_4 are surjective and h_5 is injective, then h_3 is surjective.*
- (ii) *If h_2 and h_4 are injective and the restriction maps $S_\gamma^G \rightarrow S_\beta^B$ are surjective for any $\gamma \in Z^1(G, S)$ with $\gamma|_B^G = \beta$, then h_3 is injective.*
- (iii) *If the hypotheses of (i) and (ii) hold, then h_3 is an isomorphism.*

Proof. Assume the hypotheses of (i) and take $\gamma \in H^1(B, Q)$. We need to produce a preimage β , say, such that $h_3(\beta) = \gamma$. We begin with the diagram chase used to prove the five lemma for diagrams of abelian groups.

Let $\delta := \pi_B(\gamma)$. As the bottom row is exact, $\Delta_B(\delta) = 1$. As h_4 is surjective, we have a preimage ϵ with $h_4(\epsilon) = \delta$.

Since the last square commutes, $h_5(\Delta_G(\epsilon)) = 1$, and since h_5 is injective, $\Delta_G(\epsilon) = 1$.

Now, the top row is exact, so $\epsilon \in \ker \Delta_G$ and hence $\epsilon \in \text{im } \pi_G$; say, $\epsilon = \pi_G(\eta)$. Let $h_3(\eta) = \theta$.

Since the penultimate square commutes, we have $\pi_B(\theta) = \delta$.

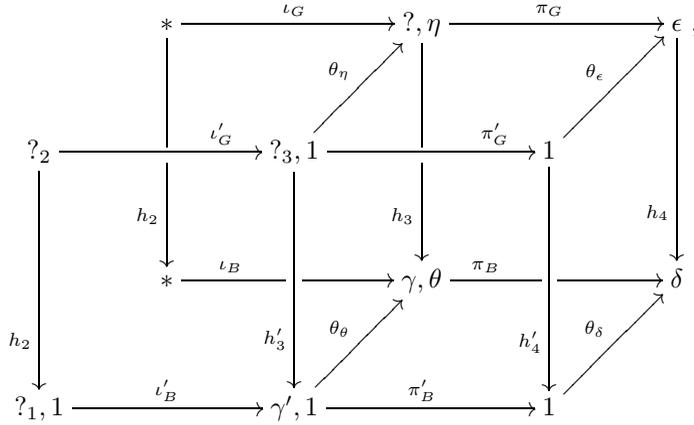
The picture is now as follows:

$$\begin{array}{ccccccccc}
 * & \xrightarrow{\delta_G} & * & \xrightarrow{\iota_G} & ?, \eta & \xrightarrow{\pi_G} & *, \epsilon & \xrightarrow{\Delta_G} & 1 \\
 h_1 \downarrow & & h_2 \downarrow & & h_3 \downarrow & & h_4 \downarrow & & h_5 \downarrow \\
 * & \xrightarrow{\delta_B} & * & \xrightarrow{\iota_B} & \gamma, \theta & \xrightarrow{\pi_B} & \delta & \xrightarrow{\Delta_B} & 1
 \end{array}$$

where we would like to assert the existence of an element replacing ‘?’ and mapping to γ .

In the case of abelian groups, one would continue the proof of the five lemma by taking the difference $\gamma - \theta$, observing that this maps under π_B to 1 and continuing the diagram chase. Since we cannot do this in the case of pointed sets we use twisting to reduce to the case $\theta = 1$. For this we use Proposition 2.2.2(i), replacing

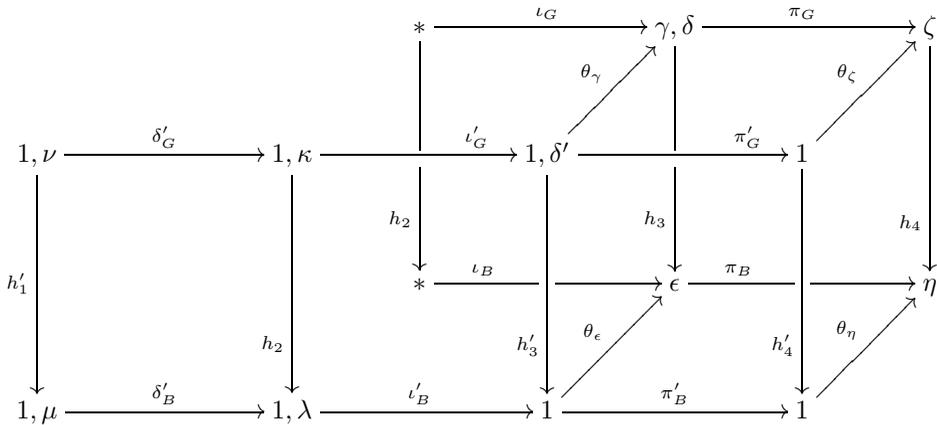
the back face with the middle of the diagram above.



Here we are using the fact that the bijection in Proposition 2.2.1(v) takes the trivial element in $H^1(G, Q_\eta)$ to the element η in $H^1(G, Q)$ and the fact that the partial cuboid is commutative.

Now in the front bottom row, as $\gamma' \in \ker \pi'_B$, we have $\gamma' \in \text{im } \iota'_B$. Thus we may replace $?_1$ with an element κ mapping to γ' . Since h_2 is a surjection, we may replace $?_2$ with an element λ mapping to κ . Then we replace $?_3$ with $\mu = \iota'_G(\lambda)$, and by the fact that the front left square commutes, $h'_3(\mu) = \gamma'$. Finally if we replace $?$ with $\nu := \theta_\eta(\mu)$ the commutativity of the central vertical square gives us our preimage of γ .

For (ii) the picture in the partial cuboid is as follows:



where we have twisted by γ . Here a preimage κ of δ' under ι'_G exists since δ' is in the kernel of π'_G . Similarly μ is a preimage of λ under δ'_B , and ν is a preimage of μ under h'_1 using the fact that that map is surjective.

This shows that $\delta' = \iota'_G(\delta'_G(\nu))$ and hence is equal to 1 since the composition of these two maps is trivial. Thus $\gamma = \delta$ since θ_γ is a bijection.

(iii) is now obvious. □

An identical proof using Proposition 2.2.2(ii) shows:

Theorem 3.1.2. *Assume the hypotheses of Proposition 2.2.1(ii). Then in the following subdiagram of Proposition 2.2.1(ii),*

$$\begin{array}{ccccccccc}
 S^G & \xrightarrow{\delta_G} & H^1(G, R) & \xrightarrow{\iota_G} & H^1(G, Q) & \xrightarrow{\pi_G} & H^1(G, S) & \xrightarrow{\Delta_G} & H^2(G, R) \\
 h_1 \downarrow & & h_2 \downarrow & & h_3 \downarrow & & h_4 \downarrow & & h_5 \downarrow \\
 S'^G & \xrightarrow{\delta_G} & H^1(G, R') & \xrightarrow{\iota_B} & H^1(G, Q') & \xrightarrow{\pi_B} & H^1(G, S') & \xrightarrow{\Delta_B} & H^2(G, R'),
 \end{array}$$

the following hold:

- (i) *If h_2 and h_4 are surjective and h_5 is injective, then h_3 is surjective.*
- (ii) *If h_2 and h_4 are injective and the restriction maps $S_\gamma^G \rightarrow S_{\gamma'}^G$ are surjective for any $\gamma \in Z^1(G, S)$ with $\gamma' = \eta \circ \gamma$, then h_3 is injective.*
- (iii) *If the hypotheses of (i) and (ii) hold, then h_3 is an isomorphism.*

Proposition 3.1.3. *Assume the hypotheses of 2.2.1(ii) and let B be a subgroup of G . Then in each of the following diagrams*

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & R^G & \longrightarrow & Q^G & \longrightarrow & S^G & \xrightarrow{\delta_G} & H^1(G, R) \\
 h_1 \downarrow & & h_2 \downarrow & & h_3 \downarrow & & h_4 \downarrow & & h_5 \downarrow \\
 1 & \longrightarrow & R^B & \longrightarrow & Q^B & \longrightarrow & S^B & \xrightarrow{\delta_B} & H^1(B, R)
 \end{array}$$

and

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & R^G & \longrightarrow & Q^G & \longrightarrow & S^G & \xrightarrow{\delta_G} & H^1(G, R) \\
 h_1 \downarrow & & h_2 \downarrow & & h_3 \downarrow & & h_4 \downarrow & & h_5 \downarrow \\
 1 & \longrightarrow & R'^G & \longrightarrow & Q'^G & \longrightarrow & S'^G & \xrightarrow{\delta_G} & H^1(G, R') :
 \end{array}$$

- (i) *If h_2 and h_4 are surjective and h_5 is injective, then h_3 is surjective.*
- (ii) *If h_2 and h_4 are injective, then h_3 is injective.*
- (iii) *If h_2 and h_4 are isomorphisms and h_5 is an injection, then h_3 is an isomorphism.*

Proof. The usual proof of the five lemma [Rot09, 2.72] goes through in these cases. Where one would take the ‘difference’ of two elements g and h in an abelian group one uses the element gh^{-1} . The proof then works as normal. \square

Corollary 3.1.4. *Suppose Q is a G -group admitting a filtration by G -groups $Q = Q_1 \geq Q_2 \geq Q_3 \geq \dots \geq Q_{n+1} = \{1\}$ with each Q_i normalised by G , $Q_i \triangleleft Q$ and $Q_i/Q_{i+1} \leq Z(Q/Q_{i+1})$.*

(a) *If B is a subgroup of G and for all $1 \leq j \leq n$ we have*

- (i) $H^1(G, Q_j/Q_{j+1}) \cong H^1(B, Q_j/Q_{j+1})$,
- (ii) $H^0(G, Q_j/Q_{j+1}) \twoheadrightarrow H^0(B, Q_j/Q_{j+1})$,
- (iii) $H^2(G, Q_j/Q_{j+1}) \hookrightarrow H^2(B, Q_j/Q_{j+1})$,

then $H^1(G, Q) \cong H^1(B, Q)$.

(b) *Suppose Q' is another G -group admitting another such filtration by G -groups $Q' = Q'_1 \geq Q'_2 \geq Q'_3 \geq \dots \geq Q'_{n+1} = \{1\}$ and that there is a G -homomorphism*

$\rho : Q \rightarrow Q'$ such that $\rho(Q_i) \leq Q'_i$. If for all $1 \leq j \leq n$ we have

- (i) $H^1(G, Q_j/Q_{j+1}) \cong H^1(G, Q'_j/Q'_{j+1})$,
- (ii) $H^0(G, Q_j/Q_{j+1}) \twoheadrightarrow H^0(G, Q'_j/Q'_{j+1})$,
- (iii) $H^2(G, Q_j/Q_{j+1}) \hookrightarrow H^2(G, Q'_j/Q'_{j+1})$,

then $H^1(G, Q) \cong H^1(G, Q')$.

Proof. (a) This is a simple induction using the previous results. First the fact that Q_j is normalised by both Q and G mean that $(Q_\gamma)_j := (Q_j)_\gamma$ gives a filtration of Q_γ . We wish to show that the hypotheses (i) and (ii) give us: (*) For each $\gamma \in H^1(G, Q)$, we have $H^0(G, Q_\gamma/(Q_\gamma)_j) \twoheadrightarrow H^0(B, Q_\gamma/(Q_\gamma)_j)$.

Suppose (*) is known to hold for all G -groups with filtrations up to $n-1$; then we must just check that $Q^G \twoheadrightarrow Q^B$. So set $R = Q_n = (Q_\gamma)_n$, and $S = Q_\gamma/(Q_\gamma)_n$. The hypotheses of Proposition 3.1.3(i) now hold and we have $Q^G \twoheadrightarrow Q^B$ as required, proving (*).

For the conclusion of (a) note that the case $n = 1$ is immediate from the hypotheses. If (a) is known to hold for G -groups admitting a filtration of length up to $n-1$, then set $R = Q_n$ and $S = Q/Q_n$. Then by (*) the hypotheses of Theorem 3.1.1(iii) hold. We conclude $H^1(G, Q) \cong H^1(B, Q)$.

(b) is similar, using Theorem 3.1.2(iii). □

Remark 3.1.5. The results in this section involving B exist in a more general form which we do not need for the purposes of this paper. It is not necessary for B to be a subgroup of G —merely that there is some homomorphism $\rho : B \rightarrow G$. Then, for instance, the ‘restriction maps’ of 1-cohomology become ‘composition with ρ ’.

3.2. Algebraic G -groups. Let G be an algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$. Let Q be another algebraic group defined over k . Then we say Q is an algebraic G -group if Q is a G -variety (i.e. the map $G \times Q \rightarrow Q; (g, q) \mapsto q^g$ is a morphism of varieties) and G acts as group automorphism of Q . In this case, following [Hum75, 8.4] one can form a semidirect product GQ with multiplication defined by $g_1q_1 \cdot g_2q_2 = g_1g_2q_1^{g_2}q_2$.

One checks that the results and proofs of the previous sections apply to the category of algebraic groups provided one is clear on the nature of this category: First, one insists that maps are morphisms of varieties, i.e. continuous in the Zariski topology. Thus by definition $Z^i(G, Q)$ is the set of regular maps $\gamma : \prod_1^i G \rightarrow Q$ satisfying the appropriate cocycle condition. This moreover ensures that G acts regularly on Q_γ .³

With this change note that where we have made cocycle-coboundary definitions for $H^i(G, Q)$, these coincide with the usual rational (Hochschild) cohomology definitions of cohomology for algebraic groups in the case where Q is a (rational) G -module, such as those used in [CPSvdK77]. This is all worked out carefully in [McN10, 4.1]. This means in particular that the results and proofs quoted in Proposition 2.2.1 go over to our situation; this applies especially to results quoted from [Ser94].

Second, by a subgroup we will always mean a closed subgroup; i.e. the image of an injection of algebraic groups. For instance, since a cocycle γ in $Z^1(G, Q)$ is a morphism $\gamma : G \rightarrow Q$, the subgroup $\{g\gamma(g) : g \in G\}$ of the semidirect product

³Of course one still has the original cohomology theory H_{abs}^i defined via abstract cocycles, but we will not be considering this theory when G is an algebraic group.

GQ is now a closed subvariety of GQ . By a complement to Q in GQ , we use the definition in [McN10, 4.3.1]:

Definition 3.2.1. Let $G = H \ltimes Q$ be a semidirect product of algebraic groups as in [Jan03, I.2.6].

A closed subgroup H' of G is a *complement* to Q if it satisfies the following equivalent conditions:

- (i) Multiplication is an isomorphism $H' \ltimes Q \rightarrow G$.
- (ii) $\pi_{H'} : H' \rightarrow H$ is an isomorphism of algebraic groups.
- (iii) Viewed as group-schemes, $H'Q = G$ and $H' \cap Q = \{1\}$.
- (iv) For the groups of k -points, one has $H'(k)Q(k) = G(k)$, $H'(k) \cap Q(k) = \{1\}$ and $\text{Lie}(H') \cap \text{Lie}(Q) = 0$.⁴

One checks that one retains the correspondence between complements and 1-cocycles in this category:

Lemma 3.2.2. *The set of 1-cocycles $Z^1(H, Q)$ is in bijection with the set of complements to Q in HQ . Two cocycles are equivalent if the corresponding complements are conjugate by an element of $H(k)$.*

Remark 3.2.3. This is a non-abelian generalisation from the case where Q is an H -module. All the proofs are exactly the same. See [McN10, 4.5].

Lastly, where we consider extensions of algebraic G -groups we should emphasise the importance of the existence of the section σ in Proposition 2.2.1.

Definition 3.2.4. Let

$$1 \rightarrow R \rightarrow Q \xrightarrow{\pi} S \rightarrow 1$$

be a short exact sequence of algebraic groups. We call Q a *sectioned extension* of R by S if there exists a regular map of varieties $\sigma : Q \rightarrow S$ with $\pi \circ \sigma$ the identity on S .

Note that a sectioned extension is in particular a strongly exact sequence; that is to say, the map $d\pi : \text{Lie } Q \rightarrow \text{Lie } S$ is onto. If Q is a sectioned extension of algebraic groups R and S with R abelian, it follows that Q corresponds to a (regular) 2-cocycle class in $H^2(S, R)$; e.g. [Ser88, VII.4(a)]. (Obviously in the category of abstract groups, all extensions will be ‘sectioned’, since they simply need to be defined set-wise.) We record the obvious fact:

Lemma 3.2.5. *For any algebraic group G and any G -group Q , the extension of G -groups*

$$1 \rightarrow Q^\circ \rightarrow Q \rightarrow Q/Q^\circ \rightarrow 1$$

is sectioned.

Remark 3.2.6. Much of the cohomology theory for algebraic groups predicated on regular maps fails if one considers non-sectioned extensions of algebraic groups. We thank Steve Donkin for the following example. Let $R = \mathbb{F}_p$, $Q = \mathbf{G}_a$ and $S = Q/\mathbb{F}_p$, where \mathbf{G}_a is the 1-dimensional additive group. (Then S is isomorphic to \mathbf{G}_a , too.) Imagine there were a morphism of varieties $\sigma : S \rightarrow Q$ with σ a section of the quotient map $Q \rightarrow S$. Then $q + \mathbb{F}_p = \pi\sigma(q + \mathbb{F}_p) = \sigma(q + \mathbb{F}_p) + \mathbb{F}_p$.

⁴The definition and equivalence of (i) and (ii) above are [McN10, 4.3.1]. (i) is equivalent to (iii) using the remark at the end of [Jan03, I.2.6]. For (iv), $H'(k)Q(k) = G(k) \iff HQ = G$ and by [Jan03, I.7.9(2)], $\text{Lie}(H' \cap Q) = \text{Lie}(H') \cap \text{Lie}(Q)$, so (iii) is equivalent to (iv).

So $\sigma(q + \mathbb{F}_p) - q \in \mathbb{F}_p$. Since S is connected, F_p is discrete and σ is continuous, the image of this map must be constant, hence 0. So $\sigma(q + \mathbb{F}_p)$ is constantly q . If $0 \neq r \in F_p$, then $\sigma(q + r + \mathbb{F}_p) = q = q + r$. So $r = 0$, a contradiction.

Indeed the extension

$$1 \rightarrow R \rightarrow Q \rightarrow S \rightarrow 1$$

cannot correspond to a 2-cocycle $\phi : S \times S \rightarrow R$ since such regular cocycles from a continuous to a discrete group must again be trivial.

Worse, letting $G \cong S$, no such exact sequence of Proposition 2.2.1(i) can exist using rational cohomology. If it did, one would have

$$H^1(G, R) \rightarrow H^1(G, Q) \rightarrow H^1(G, S) \rightarrow H^2(G, R),$$

with $H^i(G, R) = 0$ for $i = 1, 2$, as continuous maps $\gamma : G \rightarrow R$ or $\gamma : G \times G \rightarrow R$ must be constant (and so cocycles are zero maps). Thus we would have $H^1(G, Q) \cong H^1(G, S)$. The latter contains the identity map, but the above argument also shows there can be no preimage, a contradiction.

Happily, sectioned extensions exist in some general settings as in the lemma below. We need some further definitions:

Definition 3.2.7. Let G be a group and Q a G -group. We say that a (finite) filtration

$$Q = Q_1 \geq Q_2 \geq \dots \geq Q_{n+1} = 1$$

of Q with $Q_i \triangleleft Q$ is *central* if $Q_i/Q_{i+1} \leq Z(Q/Q_{i+1})$; we say that it is a *filtration by G -groups* if each Q_i is normalised by G ; we say it is a *filtration by G -modules* if it is a filtration by G -groups and for each i , Q_i/Q_{i+1} has the structure of a (rational, linear) G -module; finally, we say that it is *sectioned* if for each i there is a map $Q/Q_i \rightarrow Q$.

The following is a result of Rosenlicht.

Lemma 3.2.8 (cf. [Spr98, 14.2.6]). *Let Q be a connected unipotent algebraic subgroup of an algebraic group G . Then*

- (i) *there exists a (regular) section to the morphism $G \rightarrow G/Q$ of varieties;*
- (ii) *G is isomorphic (as a variety) to $G/Q \times Q$.*

In particular, any filtration of Q by unipotent subgroups is sectioned.

3.3. Unipotent algebraic G -groups. Here we prove that any connected unipotent algebraic G -group Q admits a (sectioned) central filtration by G -modules. This allows us to use results like Corollary 3.1.4. First we need four technical lemmas.

Lemma 3.3.1. *Let G be an algebraic group defined over k and let M be a closed G -stable subgroup of a finite dimensional simple G -module V . Then $M = V$ or M is finite.*

Proof. Since V is a G -module, there is a G -equivariant isomorphism between V and $T_0(V)$, its tangent space at the identity. Under this map, M is sent to a G -subspace $T_0(M)$ of $T_0(V) \cong V$, which is either 0 or V since V is simple. If $T_0(M) = V$, then $M = V$ by dimensions, or if $T_0(M) = 0$, then M is 0-dimensional, so finite. \square

Recall that a vector group V is an algebraic group isomorphic to a direct product of n copies of \mathbf{G}_a , the additive group of the field.

Lemma 3.3.2. *Let U be a connected unipotent algebraic group over a field k of characteristic $p > 0$, V a vector group over k and let $\pi : U \rightarrow V$ be an isogeny. Then U is a vector group, hence isomorphic to V .*

Proof. If $u \in U$, $\pi(u^p) = p\pi(u) = 0$, so the map $u \mapsto u^p$ is a continuous map from U to the kernel of π . Such a map must be constant as U is connected, thus identically 0. Hence $u^p = 0$. By [Ser88, Proposition 11], U is thus a vector group; dimensions imply that $U \cong V$. □

Lemma 3.3.3. *Let G be a connected algebraic group and let*

$$1 \rightarrow F \rightarrow W \xrightarrow{\pi} V \rightarrow 1$$

be a short exact sequence of algebraic G -groups with W a vector group, V a simple G -module and F a finite group. Then either W has the structure of a trivial G -module or $F = 0$.

In any case, W is isomorphic to V as a G -module.

Proof. We have that $R_u(G)$ acts trivially on simple G -modules, so V inherits the structure of a simple H -module for the reductive group $H = G/R_u(G)$. Take a maximal torus T of H .

If T acts trivially on V , then as V is simple for H , we have $V \cong k$. Hence, by Lemma 3.3.2, $W \cong k$ also, with G acting trivially and so W has the structure of a trivial G -module.

Otherwise, let $v^+ \in V$ be a highest weight vector for H with non-trivial T -weight. Denote by $\langle Gv^+ \rangle$ the closed subgroup of the (additive) group V generated by elements gv^+ . We claim $V = \langle Gv^+ \rangle$. To see this, observe that $Tv^+ = k^\times v^+$, and since $0 \in \langle Gv^+ \rangle$, we have $kv^+ \leq \langle Gv^+ \rangle$. Hence for each $k_i \in k$ and $g_i \in G$, we have $k_i g_i v^+ \in \langle Gv^+ \rangle$. Then if $m = k_1 g_1 + \dots + k_l g_l \in kG$, we have $mv^+ \in \langle Gv^+ \rangle$. Thus $\langle Gv^+ \rangle = kGv^+ = V$. This proves the claim.

Now take $K = (\pi^{-1}(kv^+))^\circ$ and let $0 \neq w^+ \in K$. We have Tw^+ is a connected subset of $\pi^{-1}(kv^+)$, so $\langle Tw^+ \rangle = K$ and K is a closed, connected 1-dimensional subgroup of W , hence $K \cong k, K = Tw^+ \cup \{0\}$ and $\pi(K) = kv^+$. The image of the map $\phi : G \times K \rightarrow W$ by $(g, x) \rightarrow gx$ is the image of an irreducible variety containing the identity of W . Hence by [Hum75, Prop. 7.5], $\langle GK \rangle = \langle GTw^+ \rangle = \langle Gw^+ \rangle$ is a connected subgroup of G whose elements all have the form $g_1 w^+ + \dots + g_n w^+$. Moreover, $\pi \langle Gw^+ \rangle = \langle Gv^+ \rangle$, which is in turn equal to V by the above claim, so we have that $\langle GK \rangle$ is a connected subgroup of W isogenous to V . Thus $\langle Gw^+ \rangle = W$ by dimensions and for any element $w \in W$ we can write $w = g_1 w^+ + \dots + g_n w^+$. So let $f \in F \leq W$ and write $f = g_1 w^+ + \dots + g_n w^+$. The map $\rho : K \rightarrow W$ by $x \mapsto g_1 x + \dots + g_n x$ is the image of a connected 1-dimensional variety, hence is 1-dimensional or constant. Now, for any $\lambda \in k$, we have $g_1 \lambda v^+ + \dots + g_n \lambda v^+ = \lambda \pi(f) = 0$, so that $\pi \rho(K) = 0$. But since π has only finite kernel, the map ρ must have constant image. Since 0 is in the image of ρ , it now follows that $f = 0$. □

Lemma 3.3.4. *Let G be a connected algebraic group and let Q be a connected unipotent algebraic G -group with a (central) filtration $\{Q_i\}$ such that successive quotients have the structure of G -modules or finite groups. Then Q has a (central) filtration by G -modules.*

Proof. Let $m := |\{i : Q_i/Q_{i+1} \text{ is finite}\}|$. Take j to be a minimum such that $A := Q_j/Q_{j+1}$ is finite (note that $j > 1$ or else Q has a disconnected quotient).

Then $C := Q_{j-1}/Q_j$ is a G -module. Letting $B = Q_{j-1}/Q_{j+1}$, we have the following complex induced from the short exact sequence $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$:

$$(*) \quad 1 \rightarrow A \cap B^\circ \xrightarrow{\iota} B^\circ \xrightarrow{\pi} C \rightarrow 1.$$

Clearly ι is an injection; the image of B° under π is a connected subgroup of C of the same dimension as C and hence is equal to C , showing that π is surjective. If $b \in \ker \pi$, then we obviously have $b \in A \cap B^\circ$; thus the complex is a short exact sequence.

Now, B° is a connected unipotent group isogenous to the vector group C , so by Lemma 3.3.2, $B^\circ \cong C$ as a vector group. Thus we may apply Lemma 3.3.3 to the exact sequence $(*)$ to see that B° is a simple G -module. Thus B has a filtration $1 \rightarrow B^\circ \rightarrow B \rightarrow B/B^\circ \rightarrow 1$ with B° a G -module and B/B° finite. Let $\{Q'_i\}$ be the filtration of Q with $Q'_i = Q_i$ for all $i \neq j, j - 1$ and $Q_{j-1} = B/B^\circ, Q_j = B^\circ$. Then $\{Q'_i\}$ is a filtration of Q by G -modules or finite groups with $j - 1$ the minimum value of i such that Q_i/Q_{i+1} is finite. By induction we get a filtration $\{R_i\}$ of Q with Q/R_2 finite. This is the image of a connected group and is hence trivial. Thus $\{R_i\}_{i \geq 2}$ is a (central) filtration of Q by G -modules and finite groups with $m - 1$ successive quotients being finite.

Repeating this process, we get a central filtration Q by G -modules with no finite quotients. □

We are now in a position to prove our second main result.

Theorem 3.3.5. *Let G be a connected algebraic group over an algebraically closed field k and Q a connected unipotent G -group. Then Q has a (sectioned) central filtration by G -modules.*

Proof. Form the semidirect product $H = GQ$ and embed it as a closed subgroup of $GL_n(k)$ in such a way that Q is represented by strictly upper triangular matrices. Write $M_n(k)$ for the algebra of $n \times n$ matrices over k and regard $M_n(k)$ as a G -module via matrix conjugation. Let A be the subalgebra of $M_n(k)$ generated by $Q \leq GL_n(k) \subset M_n(k)$. Then every element in Q , and therefore A , is $\lambda I + x_n$ for x_n nilpotent. Thus A is a local algebra with radical J , say; in fact, J is the ideal given by all the nilpotent elements of A . Now G acts on A by algebra automorphisms; thus G also acts as a group of automorphisms of each power J^i of J since each is a characteristic ideal. Equally, G acts on each of the multiplicative groups $1 + J^i$, again by matrix conjugation. Note that for $j_1 \in J^i$ and $j_2 \in J$ we have $(1 + j_1)(1 + j_2) = (1 + j_2)(1 + j_1)$ modulo J^{i+1} and so $(1 + J^i)/(1 + J^{i+1}) \leq Z((1 + J)/(1 + J^{i+1}))$. Moreover, there is a G -equivariant automorphism from the additive G -group J^i/J^{i+1} to the multiplicative G -group $(1 + J^i)/(1 + J^{i+1})$. So $R = 1 + J$ has a central filtration by G -modules. We may refine this to a filtration $R = R_0 > R_1 > R_2 > \dots > 1$ with consecutive quotients being simple G -modules.

Now, since Q was unipotent, $Q - 1 \subset A$ consists of nilpotent matrices and is therefore contained in J . So $Q \subseteq R = 1 + J$. Thus, letting $Q_i = Q \cap R_i$, we have a filtration of Q with successive quotients being either simple G -modules, or finite G -groups by Lemma 3.3.1. By Lemma 3.3.4, there is now a central filtration of Q by G -modules.

Recall that such a filtration is automatically sectioned by Lemma 3.2.8. □

3.4. Non-abelian cohomology of a group extension. The following is a generalisation (by a small amount) of the five-term exact sequence from the Lyndon-Hochschild-Serre spectral sequence to our non-abelian situation. Here our groups can be algebraic or abstract, bearing in mind the impositions made in §3.2.

Lemma 3.4.1. *Let $1 \rightarrow U \rightarrow G \xrightarrow{\pi} T \rightarrow 1$ be an extension of groups and let Q be a G -group. Then there is an exact sequence*

$$1 \rightarrow H^1(T, Q^U) \xrightarrow{\text{inf}} H^1(G, Q) \xrightarrow{\text{res}} H^1(U, Q)^T.$$

Moreover,

- (i) *if $H^1(T, Q_\gamma^U) = \{1\}$ for all $\gamma \in Z^1(G, Q)$, the map $H^1(G, Q) \xrightarrow{\text{res}} H^1(U, Q)$ is injective;*
- (ii) *if $H^1(U, Q_\gamma)^T = \{1\}$ for all $\gamma \in Z^1(G, Q)$, then $H^1(T, Q^U) \xrightarrow{\text{inf}} H^1(G, Q)$ is an isomorphism of pointed sets.*

Proof. The existence of the exact sequence can be found in [Ser94, I.5.8(a)] and [Ric82, 6.2.3].

Assuming the hypotheses for (i), take δ and γ in $Z^1(G, Q)$ with $\delta(u) = q^{-u}\gamma(u)q$ for each $u \in U$. We wish to show $\delta \sim \gamma$. First, replacing γ by the cocycle $g \mapsto q^{-g}\gamma(g)q$ we have $\delta(u) = \gamma(u)$. Now consider the exact sequence

$$1 \rightarrow H^1(T, Q_\gamma^U) = 1 \xrightarrow{\text{inf}} H^1(G, Q_\gamma) \xrightarrow{\text{res}} H^1(U, Q_\gamma)^T.$$

The image of δ under the map θ_γ^{-1} given in Proposition 2.2.1(v) is now in the kernel of res. Thus $\theta_\gamma^{-1}(\delta) = 1$ by exactness of this sequence. So $\delta = \theta_\gamma(1) = \gamma$.

Statement (ii) is similar. □

Question 3.4.2. If, moreover, Q^U is abelian, can one extend this sequence to the right with the term

$$\rightarrow H^2(T, Q^U)?$$

The lemma has the following consequences for algebraic groups.

Corollary 3.4.3. *Let $B = U \rtimes T$ be a connected solvable algebraic group with unipotent radical U and maximal torus T . Then if Q is a unipotent B -group, the restriction map $H^1(B, Q) \rightarrow H^1(U, Q)$ is an injection.*

Let G be a connected, reductive group with $G_s = G/Z(G)^\circ$ a semisimple quotient. If Q is a unipotent G -group we have $H^1(G, Q) \cong H^1(G_s, Q^{Z(G)^\circ})$.

Proof. We have $H^1(T, Q^U) = 0$, for any unipotent group Q , by [Ric82, Lemma 6.2.6], or an easy application of Proposition 2.2.1(i), Theorem 3.3.5 and the fact that we have $H^1(T, V) = 0$ for all T -modules and the finite T -group Q/Q° (as T is linearly reductive and connected, respectively). The hypotheses of Lemma 3.4.1(i) hold and the result follows.

For the second part, since $Z(G)^\circ$ is a torus, we also have $H^1(Z(G)^\circ, Q) = 0$ for any unipotent algebraic group Q . Thus the statement follows from Lemma 3.4.1(ii). □

3.5. Application: 1-cohomology of reductive G with coefficients in a unipotent group. In this section, G will be a connected, reductive group defined over an algebraically closed field k and B a Borel subgroup of G . Recall that for any parabolic subgroup P of G , there is a short exact sequence

$$1 \rightarrow R_u(P) \rightarrow P \xrightarrow{\pi} L \rightarrow 1$$

where $R_u(P)$ is the unipotent radical of P , and L is a complement to $R_u(P)$ in P , i.e. a Levi subgroup. We have a splitting homomorphism $\iota : L \rightarrow P$ such that $\pi \circ \iota$ is the identity map on L .

In [CPSvdK77], Cline, Parshall, Scott and van der Kallen proved a result which has the following generalisation.

Theorem 3.5.1 ([Jan03, II.4.7]). *Let G be a connected, reductive algebraic group over k and let P be a parabolic subgroup of G . If V is a rational G -module, then $H^n(G, V) \cong H^n(P, V)$ for all $n \geq 0$.*

Using the results of the previous sections we will generalise this result in the case $n = 1$ by replacing V by an arbitrary unipotent G -group Q .

Theorem 3.5.2. *Let G be a connected, reductive algebraic group over an algebraically closed field k and let Q be a unipotent G -group. Let P be a parabolic subgroup of G . Then the restriction map $H^1(G, Q) \rightarrow H^1(P, Q)$ is an isomorphism of pointed sets.*

Proof. We have $H^1(G, Q) \cong H^1(G, Q^\circ)$ and $H^1(P, Q) \cong H^1(P, Q^\circ)$ since G and P are connected. Also, by Theorem 3.3.5 the connected component Q° has a sectioned central filtration by (rational) G -modules. Now if V is a G -module, $H^i(G, V) \cong H^i(P, V)$ by Theorem 3.5.1 for $0 \leq i \leq 2$. We have the hypotheses for Corollary 3.1.4 and so we conclude that $H^1(G, Q) \cong H^1(P, Q)$. \square

Remark 3.5.3. We have a more general result when $n = 0$ is clear: for any affine G -variety X we may choose a G -equivariant embedding of X into a G -module W with fixed points $X^G = X \cap W^G = X \cap W^B = X^B$.

To prove our second generalisation of results in [CPSvdK77], let us recall (and extend) the notion of a Frobenius twist—if G is defined over a field k of characteristic p one has Frobenius morphisms $F_r = F_1^r : G \rightarrow G^{(r)}$ induced by the field automorphism $\sigma_q : k \rightarrow k$ by $x \mapsto x^{p^r}$ (see [Jan03, I.9.2]). Where G is defined over the prime field $k_0 \leq k$ we have $F_r(G) \cong G$. In this case, if Q is any algebraic G -group (for instance a G -module), then one can define a new G -group $Q^{[r]}$ which is identical to Q as a group, but where we twist the G -action by precomposing with F_r . If both Q and the action map $G \times Q \rightarrow Q$ are also defined over k_0 , one has a G -equivariant map of algebraic groups given by $Q \rightarrow Q^{[r]}$; call this the twist map. In the case Q is a G -module, it is also k -semilinear. Denote the fixed points $G(k)^{F_r}$ of the group of k -points $G(k)$ by $G(q)$. If G is connected reductive, $G(q)$ is called a finite reductive group.

The following is an easy improvement of the main result [CPSvdK77, 6.6] to the case where G is connected and reductive. We will only need it in the case where $m = 1$.

Lemma 3.5.4. *Let G be a connected reductive algebraic group defined and split over the prime field k_0 of k with root system Φ . Let V be a rational G -module. Then*

there exist non-negative integers $e_0 = e_0(\Phi, m, V)$ and $f_0 = f_0(\Phi, m, V)$ so that for all $e \geq e_0$ and $f \geq f_0$, the restriction map $H^n(G, V^{[e]}) \rightarrow H^n(G(p^{e+f}), V^{[e]})$ is an isomorphism for $n \leq m$ and an injection for $n = m + 1$.

Proof. Let $Z = Z(G)^\circ$ be the connected centre of G . Set $G_s := G/Z$, a semisimple quotient of G , and set $W = V^Z$, the Z -fixed points of V .

Invoking [CPSvdK77, 6.6], we find integers $e'_0 = e'_0(\Phi, m, W)$ and $f'_0 = f'_0(\Phi, m, W)$ so that for all $e \geq e'_0$ and $f \geq f'_0$, the restriction map

$$H^n(G_s, W^{[e]}) \rightarrow H^n(G_s(p^{e+f}), W^{[e]})$$

is an isomorphism for $n \leq m$ and an injection for $n = m + 1$.

We claim there exists an $e_0 \geq e'_0$ and $f_0 \geq f'_0$ so that for all $e \geq e_0$ and $f \geq f_0$ we have (a) $H^n(G, V^{[e]}) \cong H^n(G_s, W^{[e]})$; and (b) $H^n(G(p^{e+f}), V^{[e]}) \cong H^n(G_s(p^{e+f}), W^{[e]})$ for all $n \geq 0$. Clearly this will finish the proof.

To prove the claim, take $e_0 = e'_0$ and $f_0 \geq f'_0$ so that each Z -weight of V is p^{f_0} -restricted. Then for all $f \geq f_0$ and $q = p^{e+f}$ we have $(V^{[e]})^{Z(q)} = (V^{[e]})^Z$. We also have that $H^n(Z(q), V^{[e]}) = 0 = H^n(Z, V^{[e]})$ for all $n \geq 1$, the former equality since $|Z(q)|$ is coprime to p , the latter as Z is connected.

Now the Lyndon–Hochschild–Serre spectral sequence applied to $Z \triangleleft G$ degenerates to a line, yielding isomorphisms $H^n(G, V^{[e]}) \cong H^n(G/Z, (V^{[e]})^Z) = H^n(G_s, W^{[e]})$ for all $n \geq 0$. This proves (a). For (b), the Lyndon–Hochschild–Serre spectral sequence applied to $Z(q) \triangleleft G(q)$ similarly degenerates, yielding isomorphisms $H^n(G(q), V^{[e]}) \cong H^n(G(q)/Z(q), (V^{[e]})^{Z(q)}) = H^n(G(q)/Z(q), W^{[e]})$ for all $n \geq 0$. Furthermore, as Z is connected, we have by [DM91, 3.13] that $G(q)/Z(q) \cong (G/Z)(q) = G_s(q)$. Thus $H^n(G(q), V^{[e]}) \cong H^n(G_s(q), W^{[e]})$. This proves (b). The claim is proved and this finishes the proof of the lemma. \square

Remark 3.5.5. In the proof of the next theorem, we will need to know that if G, Q and the action map $G \times Q \rightarrow Q$ are defined over k_0 , then the filtration of Q asserted by Theorem 3.3.5 is also defined over k_0 . We outline the embellishments to the proof. Note from [Spr98, 2.3.7(ii)] that the embedding $GQ \hookrightarrow GL_n(k) = GL(V)$ can be taken to be defined over k_0 ; according to an appropriate k_0 -basis of V we can also insist $Q \hookrightarrow U_n(k)$. The remainder goes through fine: for instance $A = (k_0Q) \otimes_{k_0} k$, similarly for J and all of its powers; hence also for the filtration $\{Q_i\}$. As k_0 is perfect, by [Spr98, 12.2.1] quotients are defined over k_0 ; in particular the quotients Q_i/Q_{i+1} . Now the invocation of Lemma 3.3.4 under these hypotheses does yield another filtration over k_0 : B° is defined over k_0 if B is; also, intersections of k_0 -groups are defined over k_0 by [Spr98, 12.1.15]. In conclusion, we see that Q has a central filtration $\{Q_i\}$ with each Q_i Frobenius stable, and Q_i/Q_{i+1} is a G -module.

As in the case of a G -module, one always has an injection $H^1(G, Q) \hookrightarrow H^1(G, Q^{[1]})$, for instance, by looking at the cocycle-coboundary definitions. As in Lemma 3.5.4, repeating this process quickly yields isomorphisms:

Theorem 3.5.6. *Let G be a connected, reductive algebraic group defined over an algebraically closed field k of characteristic $p > 0$ and let Q be a unipotent G -group with G, Q and the action map $G \times Q \rightarrow Q$ defined over the prime field k_0 of k . Then there exists an integer r_0 such that for each $r \geq r_0$, the restriction map $H^1(G, Q^{[1]}) \rightarrow H^1(G(p^r), Q^{[1]})$ is an isomorphism. Moreover this holds with $Q^{[1]}$ replaced by Q unless $p = 2$, G has a component of type C_l with $l \geq 1$ and Q contains the 2l-dimensional ‘natural representation’ $V(\omega_1)$ as a G -composition factor.*

In particular $H^1(G, Q^{[1]}) \cong H^1(G, Q^{[e]})$ for $e \geq 1$. Moreover $H^1(G, Q) \cong H^1(G, Q^{[e]})$ for $e \geq 0$ unless $p = 2$, $G/Z(G)^\circ$ contains a factor of type C_l and Q contains the $2l$ -dimensional G -composition factor $V(\omega_1)$.

Proof. Assume first Q is connected, so we may use Remark 3.5.5 to take a Frobenius stable central filtration $\{Q_j\}$ of Q by G -modules. First assume we are not in the special case: that is where $p = 2$, G contains a factor of type C_l and for some j , Q_j/Q_{j+1} has a G -composition factor isomorphic to $V(\omega_1)$. We will show $H^1(G, Q) \cong H^1(G, Q^{[1]})$.

As in Lemma 3.5.4 one can generalise [CPSvdK77, 7.1] to show that since we are not in the exceptional case, for each $1 \leq j \leq n$, we have $H^1(G, Q_j/Q_{j+1}) \cong H^1(G, (Q_j/Q_{j+1})^{[1]})$. One also has $H^2(G, Q_j/Q_{j+1}) \hookrightarrow H^2(G, (Q_j/Q_{j+1})^{[1]})$, since $H^m(G, V) \rightarrow H^m(G, V^{[1]})$ is always an injection. Finally it is clear that $H^0(G, Q_j/Q_{j+1}) \cong H^0(G, (Q_j/Q_{j+1})^{[1]})$. We have the hypotheses for Corollary 3.1.4(b) and so we conclude that $H^1(G, Q) \cong H^1(G, Q^{[1]})$.

Now, Lemma 3.5.4 asserts the existence of, for each G -module Q_j/Q_{j+1} , an integer r_j such that for $r \geq r_j$ restriction induces isomorphisms $H^i(G, Q_j/Q_{j+1}) \rightarrow H^i(G(p^r), Q_j/Q_{j+1})$ for $i = 0, 1$, and an injection

$$H^2(G, Q_j/Q_{j+1}) \rightarrow H^2(G(p^r), Q_j/Q_{j+1}).$$

Take $r_0 = \max_j r_j$. Then for any $r \geq r_0$, take $B = G(p^r)$; we see that the hypotheses of Corollary 3.1.4(a) hold and we conclude $H^1(G, Q) \cong H^1(G(p^r), Q)$.

Now we deal with the case where Q is not connected; note again that Q and Q/Q° are defined over k_0 . We have $H^1(G, Q) \cong H^1(G, Q^\circ)$, as G is connected.

We claim there is also an isomorphism $H^1(G(q), Q^\circ) \cong H^1(G(q), Q)$ for all sufficiently large q . Clearly this will now finish the proof in the non-exceptional case of the lemma. To prove the claim, from the short exact sequence

$$1 \rightarrow Q^\circ \rightarrow Q \rightarrow Q/Q^\circ \rightarrow 1$$

we deduce that in the exact sequence of Proposition 2.2.1(i) the map $(Q/Q^\circ)^G = Q/Q^\circ \rightarrow H^1(G, Q^\circ)$ is trivial. We have shown that there exists an integer r_0 with $H^1(G, Q^\circ) \rightarrow H^1(G(q), Q^\circ)$ if $q = p^r$ and $r \geq r_0$. Thus using the commutative diagram of Proposition 2.2.1(vi) with $G(q)$ taking the role of B , we see the map $(Q/Q^\circ)^G \rightarrow H^1(G, Q^\circ)$ is also trivial. We now wish to show $H^1(G(q), Q/Q^\circ) = \{1\}$. Since G acts trivially on Q/Q° , so does $G(q)$. As Q/Q° is a p -group, we can take a filtration of it by subgroups isomorphic to \mathbb{F}_p under addition. Now q is already large enough so that $H^1(G(q), k) = 0$ from Lemma 3.5.4. Since $H^1(G(q), \mathbb{F}_p) \rightarrow H^1(G(q), k) = 0$ is clearly an injection, we have $H^1(G(q), Q/Q^\circ) = \{1\}$ by applying the long exact sequence of Proposition 2.2.1(i) inductively through this filtration of Q/Q° . One last application of Proposition 2.2.1(i) applied to $1 \rightarrow Q^\circ \rightarrow Q \rightarrow Q/Q^\circ \rightarrow 1$ yields the exact sequence $1 \rightarrow H^1(G(q), Q^\circ) \rightarrow H^1(G(q), Q) \rightarrow 1$. An easy twisting argument now shows that $H^1(G(q), Q^\circ) \rightarrow H^1(G(q), Q)$ is an isomorphism of pointed sets, thereby proving the claim.

In the exceptional case when $p = 2$, exactly the same proof shows that $H^1(G, Q^{[1]}) \cong H^1(G, Q^{[2]})$, using the exceptional case from [CPSvdK77, 7.1]. \square

Definition 3.5.7. Following [CPSvdK77, 6.7(b)], we denote by $H_{\text{gen}}^1(G, Q)$ the stable value of $H^1(G(p^r), Q)$.

Remark 3.5.8. If one knows the weights of G on a filtration of Q , one can establish an explicit value of r_0 in the theorem above using the proof of Lemma 3.5.4 and the explicit constants given for e_0 and f_0 in [CPSvdK77].

3.6. Subgroup structure. Finally, we give a corollary on subgroup structure. For that we need the following lemma, which pins down when abstract complements in semidirect products of connected, reductive groups with unipotent groups are also complements as algebraic groups.

Lemma 3.6.1 (cf. [LS96, 1.5]). *Let G be a connected, reductive algebraic group over an algebraically closed field k of characteristic p and let Q be a unipotent G -group. Suppose H is a closed, connected, reductive subgroup of the semidirect product GQ and denote by \bar{H} the subgroup of G given by the image of H under the quotient map $\pi : GQ \rightarrow G$. Assume $\bar{H} = G$.*

Then as abstract groups $H(k)$ is a complement to $Q(k)$ in $G(k)Q(k)$; and either (a) H is a complement to Q in GQ ; or (b)

- (i) $p = 2$;
- (ii) SO_{2n+1} is a component of the semisimple group $H/Z(H)^\circ$;
- (iii) the image of this component in $G/Z(G)^\circ$ is isomorphic to Sp_{2n} ; and
- (iv) the natural module for Sp_{2n} appears in a filtration of Q by G -modules.

In case (b), if G and Q and the action map $G \times Q \rightarrow Q$ are defined over the prime field k_0 of k , then H corresponds to a cocycle $\gamma \in Z^1(G, Q^{[1]})$ such that $[\gamma]$ has no preimage in $H^1(G, Q)$ under the inclusion $H^1(G, Q) \rightarrow H^1(G, Q^{[1]})$.

Thus there is a bijection between the set of conjugacy classes of closed, connected, reductive subgroups H of GQ with $\bar{H} = G$ and the set $H^1(G, Q^{[1]})$.

Proof. We first tackle the case where H (and therefore G) is semisimple. Then $H = H_1 \dots H_n$. Let $G_i := \pi(H_i)$. Clearly, $\prod G_i = G$.

Clearly if H is a complement to Q in GQ , then it is an abstract complement, too.

Assume H is not a complement to Q . We still have $\bar{H}Q = HQ$ and so by Definition 3.2.1 we must have either $H(k) \cap Q(k) \neq \{1\}$ or $\text{Lie}(H) \cap \text{Lie}(Q) \neq 0$. Since $H(k) \cap Q(k) \leq Z(H(k)) \cap Q(k)$ the elements in the intersection are both semisimple and unipotent, hence just the identity. It now follows that $H(k)$ is an abstract complement to $Q(k)$ in $G(k)Q(k)$.

Now as $\text{Lie}(H) \cap \text{Lie}(Q) \neq 0$, we see that $\text{Lie}(H)$ contains a nilpotent ideal $I = \text{Lie} H \cap \text{Lie} Q$, and the map of abstract groups $H(k) \rightarrow G(k)$ is an isomorphism. It follows that $p = 2$, H_i is of type B_{n_i} for some i and n_i and I is generated by the short root spaces (see [Hum95, Ch. 0]). Relabel the H_i (and G_i) so that $H = H_1 \dots H_m H_{m+1} \dots H_n$ with $\text{Lie} H_i \cap I \neq 0$ if $1 \leq i \leq m$ and $\text{Lie} H_i \cap I = 0$ otherwise; so H_i is of type B_{n_i} for $1 \leq i \leq m$.

Since for each $1 \leq i \leq m$ the image of $\text{Lie}(H_i)$ is an ideal in $\text{Lie}(G_i)$, we must have G_i of type C_{n_i} . Now H_i must be adjoint, or else I contains the non-nilpotent subalgebra $Z(\text{Lie}(H_i))$, since it contains the Lie algebra of a short root SL_2 -subgroup. So $H_i = SO_{2n_i+1}$ and $G_i = Sp_{2n_i}$, with the map $\pi : H_i \rightarrow G_i$ being the well-known special isogeny. Define $\gamma : H \rightarrow GQ$ by $\gamma(h) = \pi(h)^{-1}h$. Then since $\gamma(h) \in \ker \pi$ we have in fact $\gamma : H \rightarrow Q$, with $\pi(h) = h\gamma(h)^{-1}$.

As H is semisimple, $\text{Lie} H_i \cap \text{Lie} H_j = 0$ for $i \neq j$, and as $H_i(k) \cap H_j(k) \leq Z(H_i)(k) = \{1\}$ for $1 \leq i \leq m$ (as H_i is adjoint), we have $H \cong H_1 \times \dots \times H_m \times (H_{m+1}H_{m+2} \dots H_n)$. Now as $\text{Lie} G_i \cap \text{Lie} G_j = 0$ for $i \neq j$ and as π is

an isomorphism of abstract groups, we must similarly have $G \cong G_1 \times \cdots \times G_m \times (G_{m+1} \cdots G_n)$. Let $J = \prod_{i=m+1}^n H_i$ and let $K = \prod_{i=m+1}^n H_i$. As the map $\pi|_J$ is an isomorphism of abstract groups $J \rightarrow K$ and induces an isomorphism on Lie algebras, $\pi|_J$ is an isomorphism by [Spr98, 5.3.3].

Now we can define a map $\sigma : G \rightarrow H$. Let σ_i be the other well-known special isogeny $Sp_{2n_i} \rightarrow SO_{2n_i+1}$ for $1 \leq i \leq m$, and $\tau : K \rightarrow J$ the inverse of $\pi|_J$. Then we define $\sigma : G \rightarrow H$ by setting $\sigma|_{H_i} = \sigma_i$ for $1 \leq i \leq n$ and $\sigma|_K = \tau$.

Then the composition of maps $G \xrightarrow{\sigma} H \xrightarrow{\pi} G$ induces the first Frobenius endomorphism $F : H_i \rightarrow H_i$ for $1 \leq i \leq m$, and an isomorphism on K . So under this morphism we have $g \mapsto h \mapsto h\gamma(h)^{-1} = F(g)$ for $g \in H_i$. Defining $\delta = \gamma \circ \sigma$ we have $h = F(g)\delta(g)$. Now one checks that δ defines a cocycle in $Z^1(G, Q^{[1]})$: for $g_1, g_2 \in G_i$ with $1 \leq i \leq m$, we have $F(g_1)\delta(g_1)F(g_2)\delta(g_2) = F(g_1)F(g_2)\delta(g_1g_2)$ by the fact that σ_i is a homomorphism, so that $\delta(g_1g_2) = \delta(g_1)^{F(g_2)}\delta(g_2)$, showing that $\delta|_{G_i}$ defines a cocycle in $H^1(G_i, Q^{[1]})$. For $g_1, g_2 \in K$, $\delta|_K$ defines a cocycle in $Z^1(K, Q)$, so that under the injection $Z^1(K, Q) \rightarrow Z^1(K, Q^{[1]})$, $\delta|_K$ defines a cocycle in $Z^1(K, Q^{[1]})$. Hence δ defines a cocycle in $Z^1(G, Q^{[1]})$.

It is clear that if H^i gives rise to δ_i for $i \in \{1, 2\}$, then $H^1 = H^2 \iff \delta_1 = \delta_2$. Moreover if H^1 is conjugate to H^2 it is easy to check that $\delta_1 \sim \delta_2$ in $Z^1(G, Q^{[1]})$.

Certainly $[\delta]$ can have no preimage $[\gamma]$ in $H^1(G, Q)$: if H' were the complement corresponding to γ we would have that $H' \cong G$; then $H = F(H') \cong G$ —a contradiction. This proves (b).

Part (iv) follows from (b) noting that we must be in the exceptional case of Theorem 3.5.6.

The last statement is now clear: a cocycle class in $H^1(G, Q^{[1]})$ is either in the image of $H^1(G, Q)$ or it is not. This concludes the case where H is semisimple.

Assume H is a closed, connected, reductive subgroup of GQ . Since $Z(H)^\circ$ is a torus we may replace H with a conjugate to have $Z(H)^\circ = Z(G)^\circ$. Thus each conjugacy class of subgroups considered has a representative H with $Z := Z(H)^\circ = Z(G)^\circ$. Let H_1 and H_2 be two such representatives. Then H_1 is Q -conjugate to H_2 if and only if H_1/Z is Q^Z -conjugate to H_2/Z in the semidirect product $(G/Z)Q^Z$.

Thus the Q -conjugacy classes of reductive subgroups H with $\bar{H} = G$ is in bijection with the Q^Z -conjugacy classes of semisimple subgroups K of $(G/Z)Q^Z$ with $\bar{K} = G/Z$, where \bar{K} denotes the image of K under the quotient morphism $(G/Z)Q^Z \rightarrow G/Z$. We have already shown that this is in bijection with $H^1(G/Z, (Q^Z)^{[1]})$. Now by Lemma 3.4.1 we have $H^1(G/Z, (Q^Z)^{[1]}) \cong H^1(G, Q^{[1]})$, since $H^1(Z, Q^{[1]}) = \{1\}$. □

We recall Serre’s notion of G -complete reducibility from [Ser98]. A subgroup H of G is said to be G -completely reducible (or G -cr) if whenever H is contained in a parabolic subgroup P of G , it is contained in some Levi subgroup of that parabolic.

Using the above theorem we can show that a closed reductive subgroup H of G is G -cr if and only if whenever H is in a parabolic subgroup P of G , each parabolic subgroup P_1 of H is in some Levi subgroup of that parabolic. In other words

Corollary 3.6.2. *Let H be a closed, connected, reductive subgroup of G contained in a parabolic $P = R_u(P)L$ of G and let P_1 be any parabolic subgroup of H . Then H is G -conjugate to a subgroup of L if and only if P_1 is as well.*

Moreover, if U is the unipotent radical of a Borel subgroup of H , then H is G -conjugate to a subgroup of L if and only if U is as well.

Proof. One direction is trivial, so assume that P_1 is G -conjugate to a subgroup of L .

Since H is reductive, we have as usual that if $\bar{H} := \pi(H)$ denotes the projection of H to L , then $HQ = \bar{H}Q$ and we are in the situation of Lemma 3.6.1. As k is algebraically closed, it is perfect, hence L , Q and the conjugation action $L \times Q \rightarrow Q$ are defined over k_0 ; so we have that H corresponds to some cocycle $\gamma \in Z^1(\bar{H}, R_u(P)^{[1]})$. Now consider $\bar{P}_1 := \pi(P_1) \leq \bar{H}$ and let β denote the restriction $\gamma \downarrow_{\bar{P}_1}$. Then $\beta \in Z^1(\bar{P}_1, R_u(P)^{[1]})$.

The hypothesis that P_1 is G -conjugate to L implies that P_1 is $R_u(P)$ -conjugate to L by [BMRT09, 5.9(ii)]. Thus β must be in the trivial cocycle class in $H^1(\bar{P}_1, R_u(P)^{[1]})$. But by Theorem 3.5.2, $H^1(\bar{P}_1, R_u(P)) \cong H^1(\bar{H}, R_u(P)^{[1]})$ and so γ is in the trivial cocycle class in $H^1(\bar{H}, R_u(P)^{[1]})$. Thus H is $R_u(P)$ -conjugate to a subgroup of L and so clearly it is G -conjugate to a subgroup of L .

Similarly, for the last part of the corollary, if $P_1 = B$ is a Borel subgroup of H , then we have $H^1(\bar{H}, R_u(P)^{[1]}) \cong H^1(\bar{B}, R_u(P)^{[1]})$ and by Corollary 3.4.3 the restriction map to $H^1(\bar{U}, R_u(P)^{[1]})^T$ is injective. Suppose $U = R_u B$ is $R_u(P)$ -conjugate to a subgroup of L . Then if H corresponds to γ and β denotes the restriction of γ to \bar{U} , then the hypothesis implies that β is in the trivial cocycle class. But by the injectivity of the restriction map, γ must have been in the trivial cocycle class and so H is G -conjugate to L . \square

Corollary 3.6.3. *Let H be a closed, connected, reductive subgroup of G contained in a parabolic $P = R_u(P)L$ of G . Then there exists a finite subgroup $H(q)$ of H such that $H(q)$ is conjugate to a subgroup of L if and only if H is also.*

Proof. This works the same as the previous corollary, using the following fact: As L , $R_u(P)$, and the conjugation action of L on $R_u(P)$ are defined over the prime field k_0 , we can invoke Theorem 3.5.6 to obtain an isomorphism $H^1(H(q), R_u(P)) \cong H^1(H, R_u(P))$. \square

Remark 3.6.4. Note that in the above theorem one can use Remark 3.5.8 to get an explicit value of q .

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DEPARTMENT OF MATHEMATICS AND STATISTICS, NEW COLLEGE, OXFORD, OX1 3BN, UNITED KINGDOM