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GALOIS REPRESENTATIONS WITH QUATERNION MULTIPLICATION ASSOCIATED TO NONCONGRUENCE MODULAR FORMS

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Abstract. In this paper we study the compatible family of degree-4 Scholl representations ρ_ℓ associated with a space S of weight $\kappa > 2$ noncongruence cusp forms satisfying Quaternion Multiplication over a biquadratic extension of $\mathbb{Q}$. It is shown that ρ_ℓ is automorphic, that is, its associated L-function has the same Euler factors as the L-function of an automorphic form for GL_4 over $\mathbb{Q}$. Further, it yields a relation between the Fourier coefficients of noncongruence cusp forms in S and those of certain automorphic forms via the three-term Atkin and Swinnerton-Dyer congruences.

Contents

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1. Introduction

To a d-dimensional space $S_\kappa(\Gamma)$ of cusp forms of weight $\kappa > 2$ for a noncongruence subgroup Γ under general assumptions, in [Sch85] Scholl attached a compatible family of $2d$-dimensional ℓ-adic representations ρ_ℓ of $G := G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Due to the motivic nature of Scholl's construction, one expects these representations to be *automorphic* in the sense that they are related to automorphic forms as predicted by Langlands. For general Scholl representations, this is too much to hope for at present, owing to the ineffective Hecke operators on noncongruence modular forms and the currently available modularity techniques; but it is achievable if the space admits extra symmetries, as shown in [LLY05, ALL08, Lon08]. In each of these examples, the automorphy was established using the Faltings-Serre modularity technique, which becomes inefficient for the general situation. On the other hand, tremendous progress in modularity has been made in recent years. In this paper, we prove the automorphy of degree-4 Scholl representations which admit quaternion multiplication by using modern technology. Moreover, the Atkin and Swinnerton-Dyer congruences for forms in the underlying space are also established.

Now we outline our method and main results. The recent settlement of Serre's conjecture over $\mathbb{Q}$ (cf. Theorem 2.1.2) by Khare, Wintenberger and Kisin ([KW09], [Kis09a]) is useful for establishing the automorphy of 2-dimensional representations of G coming from geometry. In our situation, the Scholl representations ρ_ℓ were constructed from geometry, as a variation of Deligne's construction of ℓ-adic Galois representations for congruence cusp forms. They have Hodge-Tate weights 0 and $1-\kappa$, each of multiplicity d, and all eigenvalues of the characteristic polynomial of a geometric Frobenius element are algebraic integers with the same complex absolute value. When $d = 1$, by appealing to the now established Serre's conjecture, one concludes that ρ_ℓ arises from a congruence newform. When $d > 1$, all existing potential automorphy criteria, such as [BLTDR10], assume the regularity condition that the Hodge-Tate weights are distinct, hence they cannot be applied to Scholl representations directly. In this paper we consider the case $d = 2$ and that the representation ρ_ℓ has Quaternion Multiplication (QM) over a field K. We prove that when K is quadratic over $\mathbb{Q}$, after extending the scalar field, ρ_ℓ decomposes into the sum of two modular representations (Theorem 4.2.3). In the more interesting case that K is biquadratic over $\mathbb{Q}$, we show that ρ_ℓ is a tensor product of two 2-dimensional projective representations $\tilde{\rho}$ and $\tilde{\gamma}$ of G, in which $\tilde{\gamma}$ has finite image. By using Tate's vanishing theorem $\mathrm{H}^2(G, \mathbb{C}^\times) = 0$, we lift $\tilde{\gamma}$ to an (ordinary) representation γ of G with finite image. Consequently, $\tilde{\rho}$ can be lifted to a representation η of G such that $\rho_\ell = \eta \otimes \gamma$ is a tensor product of two degree-2 representations of G. Of these γ is induced from a finite character of the absolute Galois group G_F of a quadratic field F contained in K, and hence is automorphic. Moreover, it is odd if K is not totally real, and even otherwise (Theorem 3.2.1 and Proposition 3.2.3). The other component η is shown to be modular, arising from a weight κ newform of a congruence subgroup (Theorem 4.2.4). In fact, for each quadratic field F contained in K, ρ_ℓ is induced from a degree-2 representation of G_F which comes from an automorphic form h_F of $\mathrm{GL}_2(\mathbb{A}_{\mathrm{F}})$ (Theorem 4.3.1). Consequently, for K biquadratic over $\mathbb{Q}$, the automorphy of ρ_ℓ can be seen in many ways: in addition to what is described above, it also corresponds to an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, which also implies that it comes from an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ by a result of Ramakrishnan [Ram00]

(Remark 4.2.5). This approach can be extended to handle symmetries in a more general context, which will be dealt with in our future work.

The automorphy of Scholl representations is useful for understanding arithmetic properties of noncongruence modular forms. One application is the following intriguing link between the coefficients of noncongruence cusp forms and congruence automorphic forms. As alluded to above, Hecke operators act ineffectively on genuine noncongruence modular forms. Based on their empirical data, Atkin and Swinnerton-Dyer predicted in [ASD71] that, under some general assumptions, the space of noncongruence cusp forms $S_\kappa(\Gamma)$ should still possess p-adic Hecke-like eigenbasis for almost all primes p in the sense that the Fourier coefficients $a(n)$ of each basis function are p-adically integral and satisfy 3-term congruences, called Atkin and Swinnerton-Dyer (ASD) congruences, of the form

$$a(np) - A(p)a(n) + B(p)a(n/p) \equiv 0 \mod p^{(\kappa-1)(1+ord_p n)} \qquad \forall n \geq 1, \tag{1.1}$$

for some algebraic integers $A(p)$ and $B(p)$. (See §4.3 for more details.) In general, the p-adic Hecke-like eigenbasis varies with p. Scholl's paper [Sch85] laid the groundwork on ASD congruences. In particular, the product of $x^2 - A(p)x + B(p)$ is expected to be the characteristic polynomial $H_p(x)$ of the action of the Frobenius at p under Scholl representations. When the Scholl representations attached to $S_\kappa(\Gamma)$ are 4-dimensional and admit QM over a biquadratic field K, more can be said about the degree 4 polynomial $H_p(x)$. Namely, there is a quadratic subfield F of K in which p splits into two places $\mathfrak{p}_\pm$ and $H_p(x)$ is the product of two polynomials $x^2 - A_\pm(p)x + B(p)$ arising from the two Euler factors $(1 - A_\pm(p)p^{-s} + B(p)p^{-2s})^{-1}$ at $\mathfrak{p}_\pm$ of the L-function attached to the aforementioned automorphic form h_F for $\mathrm{GL}_2(\mathbb{A}_F)$. (If p splits in more than one F, the factors remain the same.) We show that for almost all primes p, $S_\kappa(\Gamma)$ possesses a p-adically integral basis $f_{\pm,p}$, depending only on the congruence class of p modulo the discriminant of K, such that the Fourier coefficients of $f_{\pm,p}$ satisfy 3-term ASD congruences (1.1) with $(A(p), B(p)) = (A_\pm(p), B(p))$ (Theorem 4.3.2).

In addition to establishing congruence relations with Fourier coefficients of automorphic forms as explained above (see also [ALL08, Thm. 15] and [LLY05, Thm. 1.4]), the automorphy of Scholl representations in conjunction with the ASD congruences is used to establish the unbounded denominator conjecture for the case $d = 1$ (cf. [LL10]).

This paper is organized in the following manner. In §2, we obtain a simpler modularity criterion for 2-dimensional representations of G. §3 is devoted to the study of 4-dimensional Galois representations with QM over a biquadratic field. In §4, we prove the automorphy of Scholl representations of G for the cases $d = 1$ and $d = 2$ with QM, as well as the implications to Atkin and Swinnerton-Dyer congruences. The main results are recorded in Theorems 4.1.1, 4.2.4, 4.3.1 and 4.3.2. To illustrate our main results and methods, explicit examples are exhibited in §5. We recast the known results on automorphy and ASD congruences obtained in [LLY05, ALL08, Lon08] in the framework of QM. A new example of Scholl representations admitting QM over a biquadratic field is also given. The novelty of this example is the role played by the Atkin-Lehner involution. The automorphy of Galois representations and the conjectural ASD congruences are established.

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1.1. **Notation.** Given a number field F, denote by $\mathcal{O}_F$ its ring of integers and G_F the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. For any finite prime $\mathfrak{l}$ of $\mathcal{O}_F$ dividing the rational prime ℓ, write $F_\mathfrak{l}$ for the completion of F at $\mathfrak{l}$ and $k_\mathfrak{l} := \mathcal{O}_F/\mathfrak{l}$ as its residue field. Its Galois group $G_{F_\mathfrak{l}} := \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/F_\mathfrak{l})$ is identified with a decomposition group of $\mathfrak{l}$ in G_F. Write $I_{F_\mathfrak{l}} \subset G_{F_\mathfrak{l}}$ for the inertia subgroup over $\mathfrak{l}$ and $\mathrm{Fr}_\mathfrak{l}$ the *arithmetic* Frobenius over $\mathfrak{l}$, which is the usual topological generator of $G_{F_\mathfrak{l}}/I_{F_\mathfrak{l}} \simeq \mathrm{Gal}(\bar{k}_\mathfrak{l}/k_\mathfrak{l})$ sending $x \in \bar{k}_\mathfrak{l}$ to $x^{|k_\mathfrak{l}|}$. The *geometric* Frobenius, which is $\mathrm{Fr}_\mathfrak{l}^{-1}$, is denoted by $\mathrm{Frob}_\mathfrak{l}$.

We reserve G for $G_\mathbb{Q}$, G_ℓ for $G_{\mathbb{Q}_\ell}$ and I_ℓ for $I_{\mathbb{Q}_\ell}$. For each ℓ we fix an embedding $\iota_\ell : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. By an ℓ-adic Galois representation over E we mean a continuous representation $\rho : H \to \mathrm{Aut}_E(V)$ where H is a subgroup of G and V a finite-dimensional vector space over E, a finite extension of $\mathbb{Q}_\ell$. The reference to the ℓ-adic field E will be dropped when it is not important. We refer ρ to V if no confusions arise. Denote by ρ^{ss} or V^{ss} the semi-simplification of ρ. Let ρ be a Galois representation of H as above. If H is compact, then there exists an $\mathcal{O}_E$-lattice T in V invariant under $\rho(H)$. Let $\mathfrak{m}_E$ be the maximal ideal of $\mathcal{O}_E$. We get a *residual representation* $\bar{\rho}$ of H on the vector space $T/\mathfrak{m}_E T$ over the residue field $k := \mathcal{O}_E/\mathfrak{m}_E$. Although $\bar{\rho}$ depends on the choice of T, its semi-simplification $\bar{\rho}^{\mathrm{ss}}$ does not. Two continuous linear representations ρ_i, $i = 1, 2$, of the topological group H acting on finite-dimensional E_i-vector spaces V_i, where E_1 and E_2 are contained in a finite common extension E, are said to be *equivalent*, denoted $\rho_1 \sim \rho_2$, if $\rho_1 \otimes_{E_1} E \simeq \rho_2 \otimes_{E_2} E$ as representations of H.

2. Modularity of degree-2 ℓ-adic Galois representations

In this section, we derive from known modularity results for 2-dimensional ℓ-adic Galois representations of G a useful modularity criterion for applications in later sections.

2.1. **Modularity of 2-dimensional ℓ-adic Galois representations.** Let $\kappa \geq 2$ and $N \geq 1$ be integers and $S_\kappa(\Gamma_1(N), \mathbb{C})$ be the space of cusp forms of weight κ and level N. Suppose that $f = \sum_{i=1}^{\infty} a_n q^n$ is a newform normalized with $a_1 = 1$. The following is a classical result proved by Deligne ([De]) when $\kappa > 1$, and Deligne and Serre when $\kappa = 1$ ([DS75]).

Theorem 2.1.1. *For a newform f as above, the field of coefficients*

$$E_f = \mathbb{Q}(a_n, n \geq 1) \subset \mathbb{C}$$

is a number field. Moreover, for any prime $\lambda \mid \ell$ of E_f, there exists a continuous representation

$$\rho_{f,\lambda} : G \ \to \ GL_2(E_{f,\lambda}),$$

where $E_{f,\lambda}$ is the completion of E_f at λ, such that

1. *$\rho_{f,\lambda}$ is odd and absolutely irreducible.*
2. *For any $p \nmid N\ell$, $\rho_{f,\lambda}$ is unramified at p and $\mathrm{Tr}(\rho_{f,\lambda}(\mathrm{Fr}_p)) = a_p$.*
3. *If $\ell \nmid N$, then $\rho_{f,\lambda}|_{G_\ell}$ is crystalline with Hodge-Tate weights $\{0, k-1\}$.*

In what follows, an ℓ-adic representation ρ of G is said to be *modular* if there exists a modular form $f \in S_\kappa(\Gamma_1(N), \mathbb{C})$ and a prime $\lambda \mid \ell$ of E_f such that $\rho_{f,\lambda} \sim \rho$ or its dual $\rho^\vee$.

Recall that for any prime p, $G_p \subset G$ denotes a decomposition group at p and $I_p \subset G_p$ is the inertia subgroup. Let $\bar{\rho} : G \to \mathrm{GL}_2(k)$ be a representation over a finite field k of characteristic ℓ. Call $\bar{\rho}$ *modular* if there exists a modular form $f \in S_\kappa(\Gamma_1(N), \mathbb{C})$ and a prime $\lambda \mid \ell$ of E_f such that the residual representation $\bar{\rho}_{f,\lambda}$ is equivalent to $\bar{\rho}$.

The following has been conjectured by Serre and proved by Khare, Wintenberger and Kisin ([KW09, Kis09a]). We refer to [Kis05] for a nice review of Serre's conjecture.

Theorem 2.1.2 (Serre's conjecture)**.** *Any odd and absolutely irreducible representation $\bar{\rho} : G \to \mathrm{GL}_2(k)$ is modular.*

Remark 2.1.3. The precise Serre's conjecture also predicts the (minimal) weight and the level of the modular form f, which are not needed here.

The aim of this subsection is to reprove the following result.

Theorem 2.1.4 ([DM03], [Die08])**.** *Let $E/\mathbb{Q}_\ell$ be a finite extension. Suppose that $\rho : G \to \mathrm{GL}_2(\mathcal{O}_E)$ is an ℓ-adic Galois representation such that*

(1) *ρ is odd and absolutely irreducible;*
(2) *ρ is unramified at almost all primes;*
(3) *$\rho|_{G_\ell}$ is crystalline with Hodge-Tate weights $\{0, r\}$ such that $1 \le r \le \ell - 2$ and $\ell + 1 \nmid 2r$.*

Then ρ is modular.

This theorem was essentially proved in [DM03], as explained in [Die08]. We sketch the proof below because some ingredients will be used later. We remark that the above modularity lifting theorem is an easy version (compared to that of [Kis09b]), however it is enough for the applications in this paper.

The proof will use the discussion of local representation at ℓ and several modularity lifting theorems. First we discuss the local representation $\rho|_{G_\ell}$ and its reduction. For $i \ge 1$, the map

$$\omega_i : I_\ell \to \mathbb{F}_{\ell^i}^\times \text{ defined by } g \mapsto \frac{g(\sqrt[\ell^i-1]{\ell})}{\sqrt[\ell^i-1]{\ell}} \mod \ell$$

is the fundamental character of level i. Note that $\omega_1 = \epsilon_\ell \mod \ell$, where ϵ_ℓ is the ℓ-adic cyclotomic character.

Since $\rho|_{G_\ell}$ is crystalline with Hodge-Tate weights $\{0, r\}$ and $r \le \ell - 2$, there are two possibilities:

Type I: $\rho|_{G_\ell}$ is absolutely reducible. In this case, $\rho|_{I_\ell} \sim \begin{pmatrix} \epsilon_\ell^r & * \\ 0 & 1 \end{pmatrix}$ and $(\bar{\rho} \otimes_k \bar{k})|_{I_\ell} \simeq \begin{pmatrix} \omega_1^r & * \\ 0 & 1 \end{pmatrix}$.

Type II: $\rho|_{G_\ell}$ is absolutely irreducible. In this case, $\bar{\rho} \otimes_k \bar{k}|_{I_\ell} \simeq \begin{pmatrix} \omega_2^r & 0 \\ 0 & \omega_2^{\ell r} \end{pmatrix}$. In particular, $(\bar{\rho} \otimes_k \bar{k})|_{G_\ell}$ is irreducible because Fr_ℓ will "swap" ω_2 and ω_2^ℓ. (See the proof of case (3) for the precise statement.)

The reader is referred to §4.2.1 in [BM02] for the proof of the above statements. We also need the following result on Galois characters.

Lemma 2.1.5. *Let $\chi : G \to \mathcal{O}_E^\times$ be an ℓ-adic Galois character which is unramified outside a finite set S of finite primes excluding ℓ. Then χ is a finite character in the sense that the image of χ is a finite (cyclic) group.*

Proof. Let p be a prime in S. It suffices to show that if χ is ramified at p, then the image $\chi(I_p)$ is finite. Let I_p^w be the wild inertia subgroup of I_p, $I_p^t := I_p/I_p^w$ the tame inertia group, and $G^w := \chi(I_p^w)$. We claim that the map $\chi : G^w \hookrightarrow \mathcal{O}_E^\times \xrightarrow{q} (\mathcal{O}_E/\mathfrak{m}_E)^\times = k^\times$ is an injection. In fact, G^w is a pro-p-group and $\ker(q)$ is a pro-ℓ-group. So they can only have trivial intersection. Since G^w injects in $k^\times$, G^w is a finite group. Replacing $\mathbb{Q}$ by a suitable finite extension, we may assume that χ factors though I_p^t. Let τ be a lift of Fr_p and $\sigma \in I_p^t$. We have $\tau\sigma = \sigma^p\tau$. Applying χ to this equation, we see that $\chi(\sigma)^{p-1} = 1$. Thus $\chi(I_p^t)$ is contained in the group of $(p-1)$th-units in $\mathcal{O}_E^\times$, hence $\chi(I_p)$ is finite. □

Corollary 2.1.6. *Assume that $\chi : G \to \mathcal{O}_E^\times$ is a Galois character such that χ is unramified almost everywhere and $\chi|_{G_\ell}$ is crystalline. Then there exists a finite character ψ and an integer r such that $\chi = \psi\epsilon_\ell^r$.*

Proof. Using the p-adic Hodge theory on the classification of crystalline characters of G_ℓ, we see that $\chi|_{G_\ell} = \psi_\ell\epsilon_\ell^r$ with ψ_ℓ a character unramified at ℓ. Applying the above lemma to $\chi\epsilon_\ell^{-r}$, we prove the corollary. □

Remark 2.1.7. The above corollary may fail if G is replaced by G_K, with $K/\mathbb{Q}$ a finite extension. The problem is that the classification of crystalline characters of $G_{\mathfrak{l}}$ is much more complicated if ℓ is inert in K. Here is a more concrete example: Let K be an imaginary quadratic extension of $\mathbb{Q}$ and consider an elliptic curve E defined over K with complex multiplication by the ring $\mathcal{O}_K$. Choose a prime ideal $\mathfrak{l}$ of $\mathcal{O}_K$ generated by a prime ℓ inert in K. Then the Tate module $T_\ell(E)$ induces a Galois $\mathcal{O}_{K_{\mathfrak{l}}}$-character $\chi : G_K \to \mathcal{O}_{K_{\mathfrak{l}}}^\times$. Note that χ has Hodge-Tate weights 0 and 1. So it cannot be written as $\psi\epsilon_\ell^r$, which only has Hodge-Tate weight r.

Let $k = \mathcal{O}_E/\mathfrak{m}_E$ denote the residue field of $\mathcal{O}_E$, where $\mathfrak{m}_E$ is the maximal ideal of $\mathcal{O}_E$. Let $\bar\rho : G \to \mathrm{GL}_2(k)$ be the reduction of ρ. We distinguish three cases:

(1) $(\bar\rho \otimes_k \bar k)|_{G_\ell}$ is reducible but $\bar\rho \otimes_k \bar k$ is irreducible;
(2) $\bar\rho \otimes_k \bar k$ is reducible;
(3) $(\bar\rho \otimes_k \bar k)|_{G_\ell}$ is irreducible.

Now we proceed to prove Theorem 2.1.4 case by case. For case (1), $(\bar\rho \otimes_k \bar k)|_{G_\ell}$ is reducible, hence it is of type I. So $\rho|_{I_\ell} \sim \begin{pmatrix} \epsilon_\ell^r & * \\ 0 & 1 \end{pmatrix}$. Then the main theorem in the introduction of [SW99] and Theorem 2.1.2 imply that ρ is modular.

Next we consider case (2). Note that $\bar\rho \otimes_k \bar k$ is reducible, and so is $(\bar\rho \otimes_k \bar k)|_{G_\ell}$. As discussed above, ρ_ℓ is of type I and the semi-simplification $(\bar\rho \otimes_k \bar k)^{\mathrm{ss}}|_{I_\ell} = 1 \oplus \omega_1^r$. Hence $(\bar\rho \otimes_k \bar k)^{\mathrm{ss}} = \bar\psi \oplus \chi$ with $\bar\psi|_{I_\ell} = 1$. Twisting ρ by a finite character if necessary, we may assume that $(\bar\rho \otimes_k \bar k)^{\mathrm{ss}} = 1 \oplus \chi$ with $\chi|_{I_\ell} = \omega_1^r \neq 1$. So the Theorem in the introduction of [SW99] proves that ρ is modular.

Finally, case (3) follows from Theorem 0.3 in the introduction of [DFG04] and Theorem 2.1.2. Note that the condition $1 \le r \le \ell - 2$ and $\ell + 1 \nmid r$ implies that $\bar\rho$ restricted to $G_{\mathbb{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})}$ is absolutely irreducible, which is required by Theorem 0.3 in the introduction of [DFG04].

3. Galois representations endowed with Quaternion Multiplication

In this section we show that if a 4-dimensional ℓ-adic representation of G is endowed with Quaternion Multiplication over a quadratic or biquadratic field, then either it decomposes into the sum of two degree-2 representations, or it is induced from a degree-2 representation of an index 2 subgroup G_K of G. In the latter case, we show that this degree-2 representation of G_K, after twisting by a character, can be extended to a representation of G.

3.1. **Quaternion Multiplication.** If there is a quaternionic action on the space of a 4-dimensional ℓ-adic representation in the following sense, then a lot more can be said. In this section, F is always assumed to be a finite extension of $\mathbb{Q}_\ell$.

Definition 3.1.1. Let ρ_ℓ be an ℓ-adic representation of G acting on a 4-dimensional F-vector space W_ℓ. It is said to have Quaternion Multiplication (QM) if there are linear operators J_s and J_t on W_ℓ, parametrized by two distinct nonsquare integers s and t, satisfying

(a) $J_s^2 = J_t^2 = -id, J_sJ_t = -J_tJ_s$.
(b) For $u \in \{s,t\}$ and $g \in G$, we have $J_u\rho_\ell(g) = \pm\rho_\ell(g)J_u$ with a $+$ sign if and only if $g \in G_{\mathbb{Q}(\sqrt{u})}$.

In this case, we say that the representation has QM over $\mathbb{Q}(\sqrt{s},\sqrt{t})$.

Theorem 3.1.2. *Let ρ_ℓ be a 4-dimensional ℓ-adic representation over F with QM over $\mathbb{Q}(\sqrt{s},\sqrt{t})$. Then the following two statements hold:*

(1) *For a nonsquare $u \in \{s,t,st\}$, there are two 2-dimensional ℓ-adic representations σ_u and σ_u^- of $G_{\mathbb{Q}(\sqrt{u})}$ over $E = F(\sqrt{-1})$ such that*

$$\rho_\ell \otimes_F E \simeq \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^G \sigma_u, \quad (\rho_\ell \otimes_F E)|_{G_{\mathbb{Q}(\sqrt{u})}} = \sigma_u \oplus \sigma_u^-, \quad \text{and} \quad \sigma_u^- \cong \sigma_u \otimes \delta_u \cong \sigma_u^\tau,$$

where δ_u is the character of $G_{\mathbb{Q}(\sqrt{u})}$ with kernel $G_{\mathbb{Q}(\sqrt{s},\sqrt{t})}$ and $\tau \in G \setminus G_{\mathbb{Q}(\sqrt{u})}$. Consequently, if $\mathbb{Q}(\sqrt{s},\sqrt{t})$ is a biquadratic extension of $\mathbb{Q}$, then $\rho_\ell \otimes_F E$ is induced from three representations σ_u of $G_{\mathbb{Q}(\sqrt{u})}$ for $u \in \{s,t,st\}$. Further, σ_u is irreducible if $\rho_\ell \otimes_F E$ is.

(2) *If $\mathbb{Q}(\sqrt{s},\sqrt{t})$ is a quadratic extension of $\mathbb{Q}$, then there is a 2-dimensional ℓ-adic representation σ of G over $E = F(\sqrt{-1})$ such that*

$$\rho_\ell \otimes_F E = \sigma \oplus (\sigma \otimes \theta),$$

where θ is the quadratic character of G with kernel $G_{\mathbb{Q}(\sqrt{s},\sqrt{t})}$. In particular, ρ_ℓ is absolutely reducible.

There is a vast literature on abelian varieties attached to weight 2 congruence Hecke eigenform or more general modular motives possessing quaternion multiplication (QM) due to the existence of extra twists given by operators like Atkin-Lehner involutions (see [Rib80], [Mom81], [BG03], [GGJQ05], [Die03], etc.). Compared to these cases our situation distinguishes itself by the specific condition on the field of definition for each J operator.

Proof. We follow the proof of Theorem 6 in [ALL08]. Set $J_{st} = J_sJ_t$. It is easy to check that J_{st} satisfies property (b) in Definition 3.1.1 and $J_{st}^2 = -id$. Let $E = F(\sqrt{-1})$ and write $\rho_\mathfrak{l}$ for $\rho_\ell \otimes_F E$ for simplicity. Given a nonsquare $u \in \{s,t,st\}$, we first show that there exists an ℓ-adic 2-dimensional representation σ_u of $G_{\mathbb{Q}(\sqrt{u})}$

such that $\rho_{\mathfrak{l}} \simeq \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^{G} \sigma_u$. Since $J_u^2 = -id$, its eigenvalues are contained in E. Let $v \in W_\ell \otimes_F E$ be an eigenvector of J_u with eigenvalue i, a square root of -1. Then property (b) implies that for $g \in G \setminus G_{\mathbb{Q}(\sqrt{u})}$, $\rho_{\mathfrak{l}}(g)v$ is an eigenvector of J_u with the opposite eigenvalue $-i$. Thus J_u has eigenvalues $\pm i$ with 2-dimensional $\pm i$-eigenspace. Since J_u commutes with the action of $G_{\mathbb{Q}(\sqrt{u})}$, each $\pm i$-eigenspace affords a $G_{\mathbb{Q}(\sqrt{u})}$-action, denoted by σ_u, σ_u^-, respectively. When u is a nonsquare, by property (b), any $\tau \in G \setminus G_{\mathbb{Q}(\sqrt{u})}$ gives rise to an E-isomorphism from the $(-i)$-eigenspace to the i-eigenspace. On the $(-i)$-eigenspace, for all $g \in G_{\mathbb{Q}(\sqrt{u})}$, we have $\sigma_u^-(g) = \rho_{\mathfrak{l}}(g) = \rho_{\mathfrak{l}}(\tau^{-1}\tau g \tau^{-1}\tau) = \rho_{\mathfrak{l}}(\tau)^{-1}\sigma_u^\tau(g)\rho_{\mathfrak{l}}(\tau)$, which shows that σ_u^- and σ_u^τ are isomorphic $G_{\mathbb{Q}(\sqrt{u})}$-modules. Therefore $\rho_{\mathfrak{l}} \simeq \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^{G} \sigma_u$. Since $\rho_{\mathfrak{l}}$ is induced from σ_u, if $\rho_{\mathfrak{l}}$ is irreducible, then so is σ_u.

Let $\{v_1, v_2\}$ be a basis of the i-eigenspace of J_u and choose a nonsquare $v \in \{s, t, st\}$ not equal to u. Then $v_3 := J_v v_1$ and $v_4 := J_v v_2$ form a basis of the $-i$-eigenspace of J_u. With respect to the ordered basis $\{v_1, v_2, v_3, v_4\}$, we may express the operators by the matrices

$$J_u = \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix}, \ J_v = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}, \ J_v J_u = \begin{pmatrix} 0 & iI_2 \\ iI_2 & 0 \end{pmatrix}. \tag{3.1}$$

This gives a representation γ of the quaternion group generated by J_s and J_t.

First we prove (1). Let $N = G_{\mathbb{Q}(\sqrt{s},\sqrt{t})}$. It follows from (b) that with respect to the same basis, $\rho_{\mathfrak{l}}(g)$ is represented by $\begin{pmatrix} P(g) & 0 \\ 0 & P(g) \end{pmatrix}$ for $g \in N$, $\begin{pmatrix} P(g) & 0 \\ 0 & -P(g) \end{pmatrix}$ for $g \in G_{\mathbb{Q}(\sqrt{u})} \setminus N$, $\begin{pmatrix} 0 & P(g) \\ -P(g) & 0 \end{pmatrix}$ for $g \in G_{\mathbb{Q}(\sqrt{v})} \setminus N$, and $\begin{pmatrix} 0 & P(g) \\ P(g) & 0 \end{pmatrix}$ for $g \in G_{\mathbb{Q}(\sqrt{vu})} \setminus N$. This shows that the map sending $g \in G_{\mathbb{Q}(\sqrt{u})}$ to the matrix $P(g)$ is the 2-dimensional representation σ_u of $G_{\mathbb{Q}(\sqrt{u})}$ and the representation σ_u^- is given by $\sigma_u \otimes \delta_u$, where δ_u is the character of $G_{\mathbb{Q}(\sqrt{u})}$ with kernel $G_{\mathbb{Q}(\sqrt{s},\sqrt{t})}$. Next we prove (2). Since $\mathbb{Q}(\sqrt{s},\sqrt{t}) = \mathbb{Q}(\sqrt{t}) = \mathbb{Q}(\sqrt{s})$, we have $N = G_{\mathbb{Q}(\sqrt{s},\sqrt{t})} = G_{\mathbb{Q}(\sqrt{t})}$. In this case $\rho_{\mathfrak{l}}(g)$ is represented by $\begin{pmatrix} P(g) & 0 \\ 0 & P(g) \end{pmatrix}$ for $g \in N$ and $\begin{pmatrix} 0 & P(g) \\ P(g) & 0 \end{pmatrix}$ for $g \in G \setminus N$. Furthermore, $\{v_1 + v_3, v_2 + v_4\}$ is a basis of the i-eigenspace W of $J_s J_t$, and $\{v_1 - v_3, v_2 - v_4\}$ is a basis of the $(-i)$-eigenspace W^- of $J_s J_t$. Each space is G-invariant by property (b) since $\mathbb{Q}(\sqrt{s}) = \mathbb{Q}(\sqrt{t})$; denote the 2-dimensional representations on W and W^- by σ and σ^-, respectively. It is straightforward to check that with respect to the above bases of W and W^-, we have $\sigma(g) = P(g) = \sigma^-(g)$ for $g \in N$ and $\sigma(g) = P(g) = -\sigma^-(g)$ for $g \in G \setminus N$. This shows that $\sigma^- = \sigma \otimes \theta$, where θ is the quadratic character of G with kernel $G_{\mathbb{Q}(\sqrt{s},\sqrt{t})}$. □

If $\mathbb{Q}(\sqrt{s},\sqrt{t})$ is a quadratic extension of $\mathbb{Q}$, the characteristic polynomial $H_p(x)$ of $\rho_\ell(\mathrm{Frob}_p)$ is the product of the characteristic polynomials of $\sigma(\mathrm{Frob}_p)$ and $(\sigma \otimes \theta)(\mathrm{Frob}_p)$. We shall see that there is also a natural factorization for $H_p(x)$ when $\mathbb{Q}(\sqrt{s},\sqrt{t})$ is biquadratic.

Remark 3.1.3. Assume $\mathbb{Q}(\sqrt{s},\sqrt{t})$ is biquadratic over $\mathbb{Q}$. As stated in the theorem above, for any $\tau \in G \setminus G_{\mathbb{Q}(\sqrt{u})}$, we have $\sigma_u^\tau \simeq \sigma_u \otimes \delta_u$. Thus for any prime $p \neq \ell$ splitting into two places $\mathfrak{p}_\pm$ in $\mathbb{Q}(\sqrt{u})$, τ permutes the two places $\mathfrak{p}_\pm$; and if ρ_ℓ is unramified at p, the characteristic polynomial of $\sigma_u^\tau(\mathrm{Frob}_{\mathfrak{p}_+}) = \sigma_u(\mathrm{Frob}_{\mathfrak{p}_+})\delta_u(\mathrm{Frob}_{\mathfrak{p}_+})$

is equal to that of $\sigma_u(\mathrm{Frob}_{\mathfrak{p}_-})$. Therefore $\sigma_u(\mathrm{Frob}_{\mathfrak{p}_\pm})$ have the same characteristic polynomials if $\delta_u(\mathrm{Frob}_{\mathfrak{p}_\pm}) = 1$, which occurs if and only if p splits in $\mathbb{Q}(\sqrt{s}, \sqrt{t})$.

Statement (a) below on the factorization of the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_p)$ at an unramified place p follows immediately from Theorem 3.1.2 and the remark above.

Corollary 3.1.4. *Let ρ_ℓ be a 4-dimensional ℓ-adic representation of G with QM over $\mathbb{Q}(\sqrt{s}, \sqrt{t})$, a biquadratic extension of $\mathbb{Q}$. For $u \in \{s, t, st\}$, let σ_u be as in Theorem* 3.1.2.

(a) *Let p be a prime different from ℓ at which ρ_ℓ is unramified. Then the degree-4 characteristic polynomial $H_p(x)$ of $\rho_\ell(\mathrm{Frob}_p)$ and the degree-2 characteristic polynomial $H_{\mathfrak{p},u}(x)$ of $\sigma_u(\mathrm{Frob}_{\mathfrak{p}})$ at a place $\mathfrak{p}$ of $\mathbb{Q}(\sqrt{u})$ dividing p are related as follows:*
   (a1) *If p splits completely in $\mathbb{Q}(\sqrt{s}, \sqrt{t})$, then $H_p(x) = H_{\mathfrak{p},u}(x)^2$, where $H_{\mathfrak{p},u}(x) = x^2 - A(p)x + B(p)$ is independent of the choice of $\mathfrak{p}$ and u.*
   (a2) *If p splits in $\mathbb{Q}(\sqrt{u})$ and not in $\mathbb{Q}(\sqrt{s}, \sqrt{t})$, then there are two places $\mathfrak{p}_\pm$ of $\mathbb{Q}(\sqrt{u})$ dividing p, and $H_p(x) = H_{\mathfrak{p}_+,u}(x)H_{\mathfrak{p}_-,u}(x)$, where $H_{\mathfrak{p}_\pm,u}(x) = x^2 \mp A_u(p)x + B_u(p)$.*
   (a3) *If p is inert in $\mathbb{Q}(\sqrt{u})$, then there is one place $\mathfrak{p}$ of $\mathbb{Q}(\sqrt{u})$ dividing p, and $H_p(x) = H_{\mathfrak{p},u}(x^2)$.*

(b) *Suppose $u > 0$, $c \in G_{\mathbb{Q}(\sqrt{u})}$ is a complex conjugation, and $\tau \in G \setminus G_{\mathbb{Q}(\sqrt{u})}$. Then $\mathrm{Tr}\sigma_u(c) = \mathrm{Tr}\sigma_u^\tau(c)$ if $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is totally real, and $\mathrm{Tr}\sigma_u(c) = -\mathrm{Tr}\sigma_u^\tau(c)$ otherwise.*

Proof. It remains to prove part (b). This follows from the fact that $\sigma_u^\tau(c) = \sigma_u(c)\delta_u(c)$ and $\delta_u(c) = 1$ if and only if $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is totally real. □

Consequently, in case (a2) above, $H_p(x)$ is also the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_{\mathfrak{p}_\pm})$. Since a prime p splits in at least one of the quadratic fields contained in the biquadratic field $\mathbb{Q}(\sqrt{s}, \sqrt{t})$, when the representation ρ_ℓ admits QM over $\mathbb{Q}(\sqrt{s}, \sqrt{t})$, we obtain a natural factorization of the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_p)$ as a product of two quadratic polynomials.

Proposition 3.1.5. *Keep the same notation and assumptions as in Theorem* 3.1.2. *Assume that -1 is not a square in F. Let j denote a square root of -1 and set $E = F(j)$.*

(a) *For any $g \in G_{\mathbb{Q}(\sqrt{u})}$, let $H_{g,+}(x)$ and $H_{g,-}(x)$ be the characteristic polynomials of $\sigma_u(g)$ and $\sigma_u^-(g)$ respectively. Then $H_{g,+}(x)$ and $H_{g,-}(x)$ are conjugate under the map $j \mapsto -j$.*

(b) *Assume $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is biquadratic over $\mathbb{Q}$. For u, c and τ as in Corollary* 3.1.4 *(b), we have $\mathrm{Tr}\sigma_u(c) = \mathrm{Tr}\sigma_u^\tau(c)$. Therefore σ_u is odd if $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is not totally real.*

(c) *Assume $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is quadratic over $\mathbb{Q}$. Then $\mathrm{Tr}\sigma(c) = \mathrm{Tr}\sigma(c)\theta(c)$ for the complex conjugation $c \in G$.*

Consequently, if $\mathrm{Tr}\rho_\ell(c) = 0$ at the complex conjugation c, then σ is odd when $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is quadratic, and σ_u is odd for positive $u \in \{s, t, st\}$ when $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is biquadratic.

Proof. (a) Let w be a nonzero vector in the representation space W_ℓ of ρ_ℓ. Then w and $J_u w$ are linearly independent over F since the eigenvalues of J_u are outside

F by assumption. Let w' be a vector in W_ℓ and not in the F-span $\langle w, J_u w\rangle$. Claim that $w, w', J_u w, J_u w'$ are linearly independent over F. If not, then $J_u w' = \alpha w + \beta J_u w + \gamma w'$ for some $\alpha, \beta, \gamma \in F$ not all zero. Apply J_u to the above relation to get another relation $\gamma J_u w' = \beta w - \alpha J_u w - w'$. Comparing both relations yields $\gamma^2 = -1$, a contradiction. Therefore $\{w, w', J_u w, J_u w'\}$ forms an F-basis of W_ℓ. With respect to this ordered basis, J_u is represented by the block matrix

$$\begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

On the extension $W_\ell \otimes_F E$ the actions of $\rho_\ell(G)$ and J_u are E-linear. Recall that σ_u and σ_u^- are ρ_ℓ restricted to the $\pm j$-eigenspaces of J_u on $W_\ell \otimes_F E$. It is easy to check that $\{w - jJ_u w, w' - jJ_u w'\}$ forms an E-basis for the space of σ_u and $\{w + jJ_u w, w' + jJ_u w'\}$ forms a basis for the space of σ_u^-. In other words,

$$\begin{pmatrix} I_2 & -jI_2 \\ I_2 & jI_2 \end{pmatrix} \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} I_2 & -jI_2 \\ I_2 & jI_2 \end{pmatrix}^{-1} = \begin{pmatrix} jI_2 & 0 \\ 0 & -jI_2 \end{pmatrix}.$$

Since J_u commutes with $\rho_\ell(G_{\mathbb{Q}(\sqrt{u})})$ by condition (b) of QM, given any $g \in G_{\mathbb{Q}(\sqrt{u})}$, the matrix of $\rho_\ell(g)$ with respect to $\{w, w', J_u w, J_u w'\}$ is represented by the block matrix

$$\begin{pmatrix} P & R \\ -R & P \end{pmatrix}$$

for some 2×2 matrices P and R with entries in F. One checks that

$$\begin{pmatrix} I_2 & -jI_2 \\ I_2 & jI_2 \end{pmatrix} \begin{pmatrix} P & R \\ -R & P \end{pmatrix} \begin{pmatrix} I_2 & -jI_2 \\ I_2 & jI_2 \end{pmatrix}^{-1} = \begin{pmatrix} P + jR & 0 \\ 0 & P - jR \end{pmatrix}.$$

This shows that the action of $\rho_\ell(g)$ on the $\pm j$-eigenspaces of J_u are represented by matrices conjugate under $j \mapsto -j$, hence the characteristic polynomials $H_{g,\pm}(x)$ are as asserted.

(b) For $u > 0$, a complex conjugation c lies in $G_{\mathbb{Q}(\sqrt{u})}$. By applying (a) to $g = c$, we get that $\mathrm{Tr}\sigma_u(c)$ and $\mathrm{Tr}\sigma_u^\tau(c)$ are conjugate under $j \mapsto -j$. As $c^2 = id$, the possible values for $\mathrm{Tr}\sigma_u(c)$ are ± 2 and 0, which all lie in F. Hence $\mathrm{Tr}\sigma_u(c) = \mathrm{Tr}\sigma_u^\tau(c)$. When $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is not totally real, it follows from Corollary 3.1.4 (b) that the two traces are opposite, therefore they are equal to zero, and hence σ_u is odd.

(c) In this case, choose $u = st$, which is a square. The operator $J_u = J_{st} := J_s J_t$ commutes with $\rho_\ell(G)$. The representation σ_u is called σ and σ_u^- called $\sigma \otimes \theta$ in Theorem 3.1.2. Apply (a) to $g = c$, the complex conjugation in G. By the same argument as in (b) we conclude $\mathrm{Tr}\sigma(c) = \mathrm{Tr}\sigma(c)\theta(c)$.

Finally, using $0 = \mathrm{Tr}\rho_\ell(c) = \mathrm{Tr}\sigma(c) + \mathrm{Tr}\sigma(c)\theta(c) = 2\mathrm{Tr}\sigma(c)$ when $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is quadratic, and $0 = \mathrm{Tr}\rho_\ell(c) = \mathrm{Tr}\sigma_u(c) + \mathrm{Tr}\sigma_u^\tau(c) = 2\mathrm{Tr}\sigma_u(c)$ for positive $u \in \{s, t, st\}$ when $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ is biquadratic, we conclude the oddness of σ and σ_u respectively. □

3.2. **Quaternion Multiplication and extension of representations.** Let K be a finite extension of $\mathbb{Q}$ and V be an ℓ-adic representation of G_K. We say that V can be *extended* to an ℓ-adic representation V' of G if $V'|_{G_K} \sim V$. Note that V', if it exists, is not unique in general.

Assume that ρ_ℓ has QM over a biquadratic $\mathbb{Q}(\sqrt{s}, \sqrt{t})$. By Theorem 3.1.2, $\rho_\ell \sim \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^G \sigma_u$ for $u \in \{s, t, st\}$. We shall show that, for each $u \in \{s, t, st\}$, there

exists a finite character χ_u of $G_{\mathbb{Q}(\sqrt{u})}$ and a 2-dimensional representation η_u of G such that $\sigma_u \otimes \chi_u \sim \eta_u|_{G_{\mathbb{Q}(\sqrt{u})}}$. In other words, $\sigma_u \otimes \chi_u$ can be extended to η_u.

To see this, we return to the proof of Theorem 3.1.2 (1) with a chosen $u \in \{s, t, st\}$. The argument there shows that given $g, h \in G$, there exists a constant $\alpha_0(g, h) \in \{\pm 1\}$ such that $P(g)P(h) = \alpha_0(g, h)P(gh)$. Hence $g \mapsto P(g)$ defines a degree-2 irreducible projective representation $\tilde{\rho}_u$ of G whose restriction to $G_{\mathbb{Q}(\sqrt{u})}$ is the representation σ_u. Let $\tilde{\gamma}_u$ be the map sending gN for $g \in N$, $G_{\mathbb{Q}(\sqrt{u})} \smallsetminus N$, $G_{\mathbb{Q}(\sqrt{v})} \smallsetminus N$, and $G_{\mathbb{Q}(\sqrt{vu})} \smallsetminus N$ to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively. Note that $\tilde{\gamma}_u$ is a degree-2 irreducible projective representation of G trivial on N and

$$\rho_\ell = \tilde{\rho}_u \otimes \tilde{\gamma}_u. \tag{3.2}$$

By Tate's vanishing theorem $\mathrm{H}^2(G, \mathbb{C}^\times) = 0$ ([Ta62]), there is a representation $\gamma_u : G \to \mathrm{GL}_2(\mathbb{C})$ with finite image which lifts $\tilde{\gamma}_u$. Then the kernel H of γ_u is a normal subgroup of N with quotient N/H finite cyclic. Embed the finite image $\gamma_u(G)$ into $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ and regard γ_u as an ℓ-adic representation.

For any $g \in G$, there exists $s_u(g) \in \overline{\mathbb{Q}}_\ell^\times$ such that $\tilde{\gamma}_u(g) = s_u(g)\gamma_u(g)$. It follows from $\rho_\ell = \tilde{\rho}_u \otimes \tilde{\gamma}_u = \tilde{\rho}_u \cdot s_u \otimes s_u^{-1} \cdot \tilde{\gamma}_u = \tilde{\rho}_u \cdot s_u \otimes \gamma_u$ that $\eta_u := \tilde{\rho}_u \cdot s_u$ is also an ordinary representation of G. Since $\tilde{\rho}_u$ restricted to $G_{\mathbb{Q}(\sqrt{u})}$ is equal to the representation σ_u as noted above, s_u restricted to $G_{\mathbb{Q}(\sqrt{u})}$ is a character, denoted by χ_u, with kernel containing H. Hence η_u is an extension to G of the representation $\sigma_u \otimes \chi_u$ of $G_{\mathbb{Q}(\sqrt{u})}$. Note that γ_u and $\mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^G \chi_u^{-1}$ have the same restrictions to $G_{\mathbb{Q}(\sqrt{u})}$, so they differ at most by the quadratic character θ_u of G with kernel $G_{\mathbb{Q}(\sqrt{u})}$. Replacing γ_u by $\gamma_u \otimes \theta_u$ and η_u by $\eta_u \otimes \theta_u$ if necessary, we may assume that $\gamma_u = \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^G \chi_u^{-1}$ so that $\rho_\ell \sim \eta_u \otimes \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^G \chi_u^{-1}$. This proves

Theorem 3.2.1. *Let ρ_ℓ be a 4-dimensional ℓ-adic representation of G with QM over a biquadratic $\mathbb{Q}(\sqrt{s}, \sqrt{t})$. For $u \in \{s, t, st\}$, let σ_u be as in Theorem 3.1.2. Then there exists a finite character χ_u of $G_{\mathbb{Q}(\sqrt{u})}$ such that $\sigma_u \otimes \chi_u$ extends to a degree-2 representation η_u of G and*

$$\rho_\ell \sim \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^G \sigma_u \sim \eta_u \otimes \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^G \chi_u^{-1}.$$

Remark 3.2.2. As irreducible projective representations of $\mathrm{Gal}(\mathbb{Q}(\sqrt{s}, \sqrt{t})/\mathbb{Q})$, the $\tilde{\gamma}_u$'s are in fact equivalent. This can be shown directly (for instance, $\tilde{\gamma}_u$ conjugated by $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ is equivalent to $\tilde{\gamma}_v$) or seen from the fact that the Schur multiplier of $\mathrm{Gal}(\mathbb{Q}(\sqrt{s}, \sqrt{t})/\mathbb{Q})$ has only one nontrivial element. Hence the η_u's are also projectively equivalent. In particular, they have the same parity, equal to that of η_u with $u > 0$.

We close this section by discussing the oddness of the representations occurred.

Proposition 3.2.3. *With the same notation and assumption as in Theorem 3.2.1, we have*

(a) for positive $u \in \{s, t, st\}$, η_u is odd if and only if σ_u is odd at both real places of $\mathbb{Q}(\sqrt{u})$;

(b) the representation $\mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^{G}\chi_u^{-1}$ *of* G *corresponds to an automorphic representation of* GL_2 *over* $\mathbb{Q}$. *Moreover, it is odd if* $\mathbb{Q}(\sqrt{s},\sqrt{t})$ *is not totally real and even otherwise.*

Proof. Denote by c the complex conjugation in G.

(a) The relation $\eta_u|_{G_{\mathbb{Q}(\sqrt{u})}} = \sigma_u \otimes \chi_u$ implies $\det(\eta_u)|_{G_{\mathbb{Q}(\sqrt{u})}} = \det(\sigma_u)\cdot\chi_u^2$. As $\det(\eta_u)(g^{-1}cg)$ remains the same for all g in G, we see that σ_u and η_u have the same parity.

(b) The first statement follows from base change [AC89]. For $\mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{u})}}^{G}\chi_u^{-1}$ write γ_u as in the proof of Theorem 3.2.1. Assume that $\mathbb{Q}(\sqrt{s},\sqrt{t})$ is not totally real so that $c \notin G_{\mathbb{Q}(\sqrt{s},\sqrt{t})}$. By definition, there is a constant $b \in \overline{\mathbb{Q}}_\ell^\times$ such that $\gamma_u(c) = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$. In all cases, $\gamma_u(c)$ has trace zero, and hence is odd. When $\mathbb{Q}(\sqrt{s},\sqrt{t})$ is totally real, c fixes $\mathbb{Q}(\sqrt{s},\sqrt{t})$ and hence lies in $G_{\mathbb{Q}(\sqrt{s},\sqrt{t})}$. Thus $\gamma_u(c) = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$, which implies that γ_u is even. □

4. Galois representations attached to noncongruence cusp forms

4.1. **Modularity of Scholl representations when** $d = 1$. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a *noncongruence* subgroup, that is, Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ not containing any principal congruence subgroup $\Gamma(N)$. For any integer $\kappa \geq 2$, the space $S_\kappa(\Gamma)$ of weight κ cusp forms for Γ is finite-dimensional; denote by $d = d(\Gamma,\kappa)$ its dimension. Assume that the compactified modular curve $(\Gamma\backslash\mathfrak{H})^*$ is defined over $\mathbb{Q}$ and the cusp at infinity is $\mathbb{Q}$-rational. For even $\kappa \geq 4$ and any prime ℓ, in [Sch85] Scholl constructed an ℓ-adic Galois representation $\rho_\ell : G \to \mathrm{GL}_{2d}(\mathbb{Q}_\ell)$ attached to $S_\kappa(\Gamma)$. The representations ρ_ℓ form a *compatible system* in the sense that there exists a finite set S of finite primes of $\mathbb{Q}$ such that for any prime $p \notin S$ and primes ℓ and ℓ' different from p, the representations ρ_ℓ and $\rho_{\ell'}$ are unramified at p and the characteristic polynomials of Frob_p under ρ_ℓ and $\rho_{\ell'}$ have coefficients in $\mathbb{Z}$ and agree. Scholl also showed that all the roots of the characteristic polynomial of Frob_p have the same complex absolute value $p^{(\kappa-1)/2}$ (cf. §5.3 in [Sch85]). Scholl's results can be extended to odd weights under some extra hypotheses (e.g., $\pm(\Gamma\cap\Gamma(N)) = \pm(\Gamma)\cap\pm(\Gamma(N))$, where $\pm : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$ is the projection). The readers are referred to the end of [Sch85] for more details. In this subsection we always assume that ρ_ℓ does exist. Our main concern is

Is either ρ_ℓ *or its dual* $\rho_\ell^\vee$ *automorphic as predicted by the Langlands conjecture?*

If so, then the associated L-function coincides with the L-function of an automorphic representation of some adelic reductive group. When the reductive group is GL_2 over $\mathbb{Q}$, the representation is called "modular" in §2.

We provide an affirmative answer to the case $d = 1$.

Theorem 4.1.1. *When* $d = 1$, *the Scholl representation* ρ_ℓ *of* Γ *is modular.*

Proof. In fact, it has been known for a long time (see [Ser87] (4.8)) that the strong form of Serre's conjecture implies the modularity of 2-dimensional motives over $\mathbb{Q}$, which can be directly applied to our theorem. Here we use a slightly different method which will be useful later.

Since $\{\rho_\ell\}$ forms a compatible system, by the Chebotarev Density Theorem, it suffices to show that there exists an ℓ such that $\rho_\ell^\vee$ is modular. The plan is to

conclude this by applying Theorem 2.1.4 to $\rho_\ell^\vee$ for a large ℓ. Hence we have to show the existence of an ℓ such that $\rho_\ell^\vee$ satisfies the three hypotheses in Theorem 2.1.4. In [Sch96], Scholl proved that a general Scholl representation ρ_ℓ as above is the ℓ-adic realization of a certain motive over $\mathbb{Q}$ in the sense of Grothendieck.[3] In particular, there exists a smooth projective $\mathbb{Q}$-scheme X such that $\rho_\ell \simeq \mathrm{H}^{\kappa-1}_{\text{ét}}(X)$ as G-representations. Furthermore, he also proved that the Hodge type of $\mathrm{H}^{\kappa-1}_{\text{ét}}(X)$ is $(\kappa-1,0)^d$ and $(0,\kappa-1)^d$ (Theorem 2.12 in [Sch85]). Consequently, we know the Hodge-Tate weights of ρ_ℓ are 0 and $-(\kappa-1)$ and those of $\rho_\ell^\vee$ are 0 and $\kappa-1$. Pick $\ell > 2\kappa-2$ so that X has a smooth model over $\mathbb{Z}_{(\ell)}$, the localization of $\mathbb{Z}$ with respect to the prime ℓ. Then $\rho_\ell|_{G_\ell}$ is crystalline by Faltings' comparison theorem ([Fa89]). To complete the proof, it remains to show that ρ_ℓ is odd and absolutely irreducible.

Write $\mathrm{H}^{\kappa-1}(X)$ for $\mathrm{H}^{\kappa-1}(X,\mathbb{Q})$, the singular cohomology of X. It follows from the comparison theorem between singular cohomology and étale cohomology that $\mathrm{H}^{\kappa-1}(X)\otimes\mathbb{Q}_\ell \simeq \mathrm{H}^{\kappa-1}_{\text{ét}}(X)$ and the isomorphism is compatible with the action of complex conjugation c. Moreover, the comparison theorem between singular cohomology and de Rham cohomology implies $\mathrm{H}^{\kappa-1}(X)\otimes_{\mathbb{Q}}\mathbb{C} \simeq \mathrm{H}^{\kappa-1}_{\mathrm{dR}}(X)$. By Hodge decomposition, we write $\mathrm{H}^{\kappa-1}_{\mathrm{dR}}(X) \simeq \bigoplus_{p+q=\kappa-1} \mathrm{H}^{p,q}(X)$. Since the Hodge type of $\mathrm{H}^{\kappa-1}_{\mathrm{dR}}(X)$ is $(\kappa-1,0)^d$ and $(0,\kappa-1)^d$, we have $\mathrm{H}^{\kappa-1}_{\mathrm{dR}}(X) \simeq \mathrm{H}^{0,\kappa-1}(X)\bigoplus \mathrm{H}^{\kappa-1,0}(X)$. As $c(\mathrm{H}^{p,q}(X)) = \mathrm{H}^{q,p}(X)$ for any q,p, we conclude that half of the eigenvalues of c on $\mathrm{H}^{\kappa-1}_{\mathrm{dR}}(X)$ are 1 and the other half are -1. When $d=1$, this shows that the complex conjugation on the 2-dimensional space $\mathrm{H}^{\kappa-1}(X)$ has eigenvalues 1 and -1, and hence ρ_ℓ is odd.

Finally we prove that ρ_ℓ (hence $\rho_\ell^\vee$) is absolutely irreducible. Suppose otherwise, namely ρ_ℓ is absolutely reducible. Denote $\rho_\ell\otimes_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell$ by ρ_ℓ again and its semi-simplification by ρ_ℓ^{ss}. By assumption, $\rho_\ell^{\mathrm{ss}} \simeq \chi_1\oplus\chi_2$ for some Galois characters χ_i. Consider ρ_ℓ restricted to G_ℓ. Since $\rho_\ell|_{G_\ell}$ is crystalline with Hodge-Tate weights $\{0,-(\kappa-1)\}$, reducible and $\ell > 2\kappa-2 \geq \kappa$, from the discussion above Lemma 2.1.5, we see that $\rho_\ell|_{G_\ell}$ is of type I. Consequently, we have

$$\rho_\ell|_{I_\ell} \sim \begin{pmatrix} 1 & * \\ 0 & \epsilon_\ell^{-(\kappa-1)} \end{pmatrix}.$$

Hence without loss of generality, we may assume $\chi_1|_{I_\ell} \simeq \epsilon_\ell^{-(\kappa-1)}$ and $\chi_2|_{I_\ell}\simeq 1$. By Lemma 2.1.5, $\chi_1 \simeq \epsilon_\ell^{-(\kappa-1)}\psi_1$ and $\chi_2\simeq\psi_2$ where ψ_i are characters of finite order. Pick a prime p large enough such that ρ_ℓ is unramified at p. Then the eigenvalues of the characteristic polynomial of Frob_p under ρ_ℓ do not have the same complex absolute values, a contradiction. Thus ρ_ℓ is absolutely irreducible. This completes the proof of the theorem. □

Contained in the proof above are the following two useful results.

Corollary 4.1.2. *Let ρ_ℓ be a $2d$-dimensional ℓ-adic Scholl representation attached to a space $S_\kappa(\Gamma)$ of dimension d. Then the eigenvalues of the complex conjugation on the space of ρ_ℓ are 1 and -1, each of multiplicity d.*

[3] In fact, Scholl's construction is valid for the smallest field K_Γ over which the modular curve $\Gamma\backslash\mathfrak{H}^*$ is defined. Here we restrict ourselves to the case $K_\Gamma=\mathbb{Q}$.

Corollary 4.1.3. *Let ρ be a 2-dimensional ℓ-adic representation of G such that $\rho|_{G_\ell}$ is crystalline with Hodge-Tate weights $\{0, -(\kappa-1)\}$, where $\kappa \geq 2$. Suppose that $\ell > \kappa$ and for almost all primes p the eigenvalues of the characteristic polynomial of $\rho(\mathrm{Frob}_p)$ are algebraic numbers with the same complex absolute value. Then ρ is absolutely irreducible.*

4.2. **Automorphy of certain Scholl representations with $d \geq 2$.** In the remainder of this paper, we consider the automorphy of certain Scholl representations for the case $d \geq 2$.

In this subsection, we fix a degree d cyclic extension $K/\mathbb{Q}$. Let ρ_ℓ be a $2d$-dimensional subrepresentation of the Scholl representation of G attached to a space of weight κ cusp forms for a noncongruence subgroup Γ. We further assume that the Hodge-Tate weights of ρ_ℓ are $\{0, -(\kappa-1)\}$, each with multiplicity d. For an ℓ-adic Galois representation V of G_K and a prime $\mathfrak{l}$ of $\mathcal{O}_K$ above ℓ, denote by $\mathrm{HT}_{\mathfrak{l}}(V)$ the set of Hodge-Tate weights of the local Galois representation $V|_{G_{K_{\mathfrak{l}}}}$ and by $\mathrm{HT}(V)$ the union of $\mathrm{HT}_{\mathfrak{l}}(V)$ for all primes $\mathfrak{l}$ above ℓ.

Proposition 4.2.1. *(1) Assume that $\rho_\ell \sim \mathrm{Ind}_{G_K}^{G} \widetilde{\rho}$ for some 2-dimensional representation $\widetilde{\rho}$ of G_K. Then $\mathrm{HT}(\widetilde{\rho}) = \{0, -(\kappa-1)\}$.*

(2) Suppose there is a finite character χ of G_K such that $\widetilde{\rho} \otimes \chi$ can be extended to a degree-2 representation $\hat{\rho}$ of G, then $\mathrm{HT}(\hat{\rho}) = \{0, -(\kappa-1)\}$.

We begin by proving a lemma dealing with Hodge-Tate weights:

Lemma 4.2.2. *Let $E/\mathbb{Q}_\ell$ be a finite Galois extension and V be a finite-dimensional Hodge-Tate representation of $G_E := \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/E)$ over $\mathbb{Q}_\ell$. Then for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, the Hodge-Tate weights of V^σ are the same as those of V.*

Proof. Set $\mathrm{Fil}^i D = (V \otimes_{\mathbb{Q}_\ell} \mathrm{Fil}^i B_{\mathrm{HT}})^{G_E}$ and $\mathrm{Fil}^i D^\sigma = (V^\sigma \otimes_{\mathbb{Q}_\ell} \mathrm{Fil}^i B_{\mathrm{HT}})^{G_E}$, where $B_{\mathrm{HT}} = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}_p(m)$ is the *Hodge-Tate ring*. One can find a construction and discussion of B_{HT} in [Con09] (see Definition 2.3.6). Both of them are E-vector spaces. It suffices to show that they have the same dimension over E. For any $\sum_j v_j \otimes a_j \in \mathrm{Fil}^i D$, where $v_j \in V$ and $a_j \in \mathrm{Fil}^i B_{\mathrm{HT}}$, it can be easily checked that $\sum_j v_j \otimes \sigma(a_j) \in \mathrm{Fil}^i D^\sigma$. Hence σ induces a $\mathbb{Q}_\ell$-linear isomorphism between $\mathrm{Fil}^i D$ and $\mathrm{Fil}^i D^\sigma$. Thus $\mathrm{Fil}^i D$ and $\mathrm{Fil}^i D^\sigma$ have the same E-dimension. □

Proof of Proposition 4.2.1. Since Hodge-Tate weights are stable when restricted to a Galois subgroup of finite index, the representations $\hat{\rho}$, $\widetilde{\rho}$, and $\widetilde{\rho} \otimes \chi$ have the same Hodge-Tate weights. More precisely, suppose that V is an ℓ-adic Galois representation of G and F is a finite extension of $\mathbb{Q}$. If $V|_{G_\ell}$ is Hodge-Tate, then $\mathrm{HT}_\ell(V|_{G_\ell}) = \mathrm{HT}_{\mathfrak{l}}(V|_{G_{F_{\mathfrak{l}}}})$ for each prime $\mathfrak{l}$ of $\mathcal{O}_K$ above ℓ. Now it suffices to show that $\mathrm{HT}(\widetilde{\rho}) = \{0, -(\kappa-1)\}$. Select $\sigma \in G$ such that its image in $\mathrm{Gal}(K/\mathbb{Q})$ is a generator. Note that $\rho_\ell|_{G_K} \simeq \bigoplus_{i=0}^{d-1} \widetilde{\rho}^{\sigma^i}$. We claim that $\mathrm{HT}(\widetilde{\rho}^{\sigma^i})$ are the same for all i. If so, since ρ_ℓ has Hodge-Tate weights $\{0, -(\kappa-1)\}$ and each weight appears with multiplicity d, we get that $\mathrm{HT}(\widetilde{\rho}) = \{0, -(\kappa-1)\}$. To prove the claim, it suffices to show that for any prime $\mathfrak{l}$ of $\mathcal{O}_K$ we have $\mathrm{HT}_{\mathfrak{l}}(\widetilde{\rho}) = \mathrm{HT}_{\sigma^i(\mathfrak{l})}(\widetilde{\rho}^{\sigma^i})$. We should be careful that we cannot directly use Lemma 4.2.2 here because it is not clear that $\sigma^i(\mathfrak{l}) = \mathfrak{l}$.

Observe that $\widetilde{\rho}|_{G_{K_{\mathfrak{l}}}}$ is isomorphic to $\widetilde{\rho}^{\sigma^i}$ restricted to $\sigma^i G_{K_{\mathfrak{l}}} \sigma^{-i}$. Let ϖ be the maximal ideal of $\mathcal{O}_{\overline{\mathbb{Q}}}$ above $\mathfrak{l}$ which gives rise to the decomposition group $G_{K_{\mathfrak{l}}}$.

Then there exists $\tau \in G$ such that $G_{K_{\sigma^i(\mathfrak{l})}} = \tau G_{K_\mathfrak{l}} \tau^{-1}$. Note that $\sigma^i(\mathfrak{l}) = \tau(\mathfrak{l})$, so there exists $\lambda \in G_K$ such that $\lambda\sigma^i(\varpi) = \tau(\varpi)$. Since λ is in G_K, it is easy to see that $\widetilde{\rho}^{\sigma^i}$ restricted to $\sigma^i G_{K_\mathfrak{l}} \sigma^{-i}$ is isomorphic to $\widetilde{\rho}^{\lambda\sigma^i}$ restricted to $\lambda\sigma^i G_{K_\mathfrak{l}} (\lambda\sigma^i)^{-1}$, resulting from the isomorphism $\sigma^i G_{K_\mathfrak{l}} \sigma^{-i} \simeq \lambda\sigma^i G_{K_\mathfrak{l}} (\lambda\sigma^i)^{-1}$ given by conjugation. So without loss of generality, we may assume that $\sigma^i(\varpi) = \tau(\varpi)$. Thus $\sigma^i\tau^{-1}$ is in G_ℓ but may not be in $G_{K_\mathfrak{l}}$.

Identify $\sigma^i G_{K_\mathfrak{l}} \sigma^{-i}$ with $G_{K_{\sigma^i(\mathfrak{l})}} = \tau G_{K_\mathfrak{l}} \tau^{-1}$ via conjugation by $\sigma^i\tau^{-1}$. The identity map $\mathrm{Id} : \widetilde{\rho}^{\sigma^i} \to \widetilde{\rho}^{\tau}$ is an isomorphism of G_K-modules. Now we need to show that $\widetilde{\rho}^{\tau}$ and $\widetilde{\rho}^{\sigma^i}$ have the same Hodge-Tate weights on $G_{K_{\sigma^i(\mathfrak{l})}}$. This follows from Lemma 4.2.2 and the fact that $\widetilde{\rho}^{\tau} = (\widetilde{\rho}^{\sigma^i})^{\tau\sigma^{-i}}$. □

We are ready to prove the automorphy of certain Scholl representations. First suppose that the Scholl representation ρ_ℓ has degree-4 and admits QM over a quadratic field $\mathbb{Q}(\sqrt{s})$. By Theorem 3.1.2 (2), we have, over $\mathbb{Q}_\ell(\sqrt{-1})$,

$$\rho_\ell = \sigma \oplus (\sigma \otimes \theta) = \sigma \otimes \mathrm{Ind}_{\mathrm{G}_{\mathbb{Q}(\sqrt{s})}}^{\mathrm{G}} 1 = \mathrm{Ind}_{\mathrm{G}_{\mathbb{Q}(\sqrt{s})}}^{\mathrm{G}} \sigma|_{\mathrm{G}_{\mathbb{Q}(\sqrt{s})}}$$

for a degree-2 representation σ of G and a quadratic character θ of G with kernel $G_{\mathbb{Q}(\sqrt{s})}$. So σ and $\sigma \otimes \theta$ restricted to the decomposition group G_ℓ are crystalline, and they have the same Hodge-Tate weights $\{0, -(1-\kappa)\}$ by Proposition 4.2.1. We conclude their absolute irreducibility from Corollary 4.1.3. Corollary 4.1.2 says that ρ_ℓ has trace zero at the complex conjugation; hence by Proposition 3.1.5, both σ and $\sigma \otimes \theta$ are odd. It then follows from Theorem 2.1.4 that σ and $\sigma \otimes \theta$ are modular for large ℓ, and thus for all ℓ by compatibility. We record this in

Theorem 4.2.3. *Suppose that the Scholl representation ρ_ℓ has degree-4 and admits QM over a quadratic field $\mathbb{Q}(\sqrt{s})$. Then over $\mathbb{Q}_\ell(\sqrt{-1})$ it decomposes as a direct sum of two modular representations σ and $\sigma \otimes \theta$. Here θ is the quadratic character of* $\mathrm{Gal}(\mathbb{Q}(\sqrt{s})/\mathbb{Q})$.

Next we consider a more general situation.

Theorem 4.2.4. *Let K be a degree d cyclic extension of $\mathbb{Q}$ and ρ_ℓ be a $2d$-dimensional ℓ-adic subrepresentation of a Scholl representation of G as above. Assume that*

(a) *ρ_ℓ is induced from a 2-dimensional representation $\widetilde{\rho}$ of G_K;*
(b) *there exists a character χ of G_K of finite order such that $\widetilde{\rho} \otimes \chi$ can be extended to an ℓ-adic representation $\hat{\rho}$ of G;*
(c) *$\rho_\ell|_{G_\ell}$ is crystalline and χ is unramified at any prime above ℓ.*

Then $\hat{\rho}$ is absolutely irreducible. If we further assume that

(d) *K is unramified over ℓ and $\ell > 2\kappa - 2$;*
(e) *$\hat{\rho}$ is odd,*

then the dual of $\hat{\rho}$ is isomorphic to a modular representation $\rho_{g,\lambda}$ of G attached to a weight κ cuspidal newform g, and the following relations on L-series hold:

$$L(s, \rho_\ell^\vee) = L(s, \widetilde{\rho}^\vee) = L(s, (\rho_{g,\lambda}|_{G_K}) \otimes \chi).$$

Remark 4.2.5. The character χ of G_K corresponds to an automorphic form π_χ of $\mathrm{GL}_d(\mathbb{A}_\mathbb{Q})$ by base change [AC89]. The above expression shows that the semi-simplification of $\rho_\ell^\vee$ is automorphic, arising from the form $g \times \pi_\chi$ of the group

$\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_d(\mathbb{A}_\mathbb{Q})$. This is because

$$(\rho_\ell^\vee)^{\mathrm{ss}} = (\mathrm{Ind}_{G_K}^G(\widetilde{\rho}^\vee)^{\mathrm{ss}})^{\mathrm{ss}} = (\mathrm{Ind}_{G_K}^G(\rho_{g,\lambda}|_{G_K} \otimes \chi))^{\mathrm{ss}},$$

and as representations of G, $\mathrm{Ind}_{G_K}^G(\rho_{g,\lambda}|_{G_K} \otimes \chi)$ is isomorphic to $\rho_{g,\lambda} \otimes \mathrm{Ind}_{G_K}^G \chi$ (cf. Theorem 38.5 of [CR62, pp. 268]).

When $d = 2$, it follows from the work of Ramakrishnan [Ram00] that an automorphic form for $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ corresponds to an automorphic form for $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$. In this case the L-function attached to the degree-4 representation $\rho_\ell^\vee$ is an automorphic L-function for $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$. This fact also holds for general d by base change [AC89, §3.6]. More precisely, that $\hat{\rho}^\vee$ comes from an automorphic form of GL_2 over $\mathbb{Q}$ implies that $\tilde{\rho}^\vee$ comes from an automorphic form of GL_2 over K, which in turn implies that $\rho_\ell^\vee$ corresponds to an automorphic form of GL_{2d} over $\mathbb{Q}$.

Proof of Theorem 4.2.4. We first show that $\hat{\rho}$ is absolutely irreducible. Suppose otherwise. Enlarging the field of coefficients if necessary, we may assume that $(\hat{\rho})^{\mathrm{ss}} = \eta_1 \oplus \eta_2$ for some characters η_1 and η_2 of G. Note that $(\hat{\rho}|_{G_K})^{\mathrm{ss}} = (\widetilde{\rho} \otimes \chi)^{\mathrm{ss}}$. Since ρ_ℓ is crystalline at ℓ, $\widetilde{\rho}$ is crystalline at any prime above ℓ, which in turn implies that $\widetilde{\rho} \otimes \chi$ is crystalline at any prime above ℓ because χ is unramified at any prime dividing ℓ. Consequently, each η_i is crystalline at ℓ. Hence $\eta_i = \mu_i \epsilon_\ell^{\kappa_i}$, with μ_i a finite character and κ_i the Hodge-Tate weight of η_i by Corollary 2.1.6. Since $\hat{\rho}$ has Hodge-Tate weights 0 and $-(\kappa-1)$ by Proposition 4.2.1, we see that each η_i has Hodge-Tate weight either 0 or $-(\kappa-1)$. Therefore $\widetilde{\rho}^{\mathrm{ss}} = (\mu_1\epsilon_1 \oplus \mu_2 \epsilon_\ell^{-(\kappa-1)})|_{G_K}$. This contradicts the fact that the roots of the characteristic polynomial of Frob_p under the representation $\rho_\ell = \mathrm{Ind}_{G_K}^G \widetilde{\rho}$ have the same complex absolute value. So $\hat{\rho}$ cannot be absolutely reducible.

Now we prove the remaining assertions under the additional assumptions. As shown above, $\hat{\rho}$ is absolutely irreducible and has Hodge-Tate weights $\{0, -(\kappa-1)\}$. Further, since K is unramified above ℓ, $\hat{\rho}|_{G_\ell}$ is crystalline. Finally, since $\hat{\rho}$ is odd by assumption, we conclude from Theorem 2.1.4 that $\hat{\rho}^\vee$ comes from a weight κ cuspidal newform g as described in Theorem 2.1.1. Hence $\widetilde{\rho} = \rho_{g,\lambda}^\vee|_{G_K} \otimes \chi^{-1}$ and the relations on L-functions hold. □

Remark 4.2.6. Since we have a compatible family of Scholl representations ρ_ℓ constructed from geometry, there always exists a prime ℓ large enough such that the conditions (c) and (d) are satisfied. However, the oddness of $\hat{\rho}$ is not automatic.

4.3. **Degree-4 Scholl representations with QM and Atkin-Swinnerton-Dyer conjecture.** Hecke operators played a fundamental role in the arithmetic of congruence modular forms. It was shown in [Th89, Be94, Sch97] that similarly defined Hecke operators yielded little information on genuine noncongruence forms. When the modular curve of a noncongruence subgroup Γ has a model over $\mathbb{Q}$, Atkin and Swinnerton-Dyer in [ASD71] predicted a "p-adic" Hecke theory in terms of 3-term congruence relations on Fourier coefficients of weight κ cusp forms in $S_\kappa(\Gamma)$ as follows. Suppose the cusp at ∞ is a $\mathbb{Q}$-rational point with cusp width μ. The d-dimensional space $S_\kappa(\Gamma)$ admits a basis whose Fourier coefficients are in a finite extension of $\mathbb{Q}$. Moreover, there is an integer M such that for each prime $p \nmid M$, these Fourier coefficients are p-adically integral, that is, integral over $\mathbb{Z}_p$. Atkin and Swinnerton-Dyer conjectured in [ASD71] that, given $\kappa \geq 2$ even, for almost all primes p, $S_\kappa(\Gamma)$ has a p-adically integral basis $f_j(z) = \sum_{n\geq 1} a_j(n) q^{n/\mu}$, where

$q = e^{2\pi iz}$ and $1 \le j \le d$, depending on p, whose Fourier coefficients satisfy the 3-term congruences

$$a_j(pn) - A_j(p)a_j(n) + p^{(\kappa-1)/2}a_j(n/p) \equiv 0 \mod p^{(\kappa-1)(1+ord_p n)} \quad \forall n \ge 1 \tag{4.1}$$

for some algebraic integers $A_j(p)$ satisfying the Ramanujan bound. The congruence (4.1) is carefully explained in [ASD71] and [Sch85]. It can be summed up as the quotient $(a_j(pn) - A_j(p)a_j(n) + p^{(\kappa-1)/2}a_j(n/p))/p^{(\kappa-1)(1+ord_p n)}$ is integral over $\mathbb{Z}_p$, as in [ALL08]. In what follows, this is the interpretation of ASD congruences we shall adopt. The $\kappa = 2$ case of this conjecture was verified in [ASD71] for $d = 1$ and discussed in [Car71] using formal group language.

Scholl showed in [Sch85] that, for $\kappa \ge 4$ even and p large enough, all p-adically integral cusp forms $f = \sum_{n\ge1} a(n)q^{n/\mu}$ in $S_\kappa(\Gamma)$ satisfy $(2d+1)$-term congruences

$$a(p^d n) + C_1(p)a(p^{d-1}n) + \cdots + C_{2d}(p)a(n/p^d) \equiv 0 \mod p^{(\kappa-1)(1+ord_p n)} \quad \forall n \ge 1,$$

in which $H_p(x) := x^{2d} + C_1(p)x^{2d-1} + \cdots + C_{2d}(p) \in \mathbb{Z}[x]$ is the characteristic polynomial of the associated Scholl representation ρ_ℓ at Frob_p. A similar statement holds for κ odd. Hence the ASD conjecture holds for $d = 1$. For arbitrary d, proving the ASD conjecture amounts to factoring $H_p(x)$ as a product of d quadratic polynomials $x^2 - A_j(p)x + B_j(p)$, $1 \le j \le d$, and finding a basis $f_j = \sum_{n\ge1} a_j(n)q^{n/\mu}$ of $S_\kappa(\Gamma)$ with p-adically integral coefficients such that the 3-term congruences

$$a_j(pn) - A_j(p)a_j(n) + B_j(p)a_j(n/p) \equiv 0 \mod p^{(\kappa-1)(1+ord_p n)} \quad \forall n \ge 1$$

hold for $1 \le j \le d$.

Assume that $S_\kappa(\Gamma)$ has a 2-dimensional subspace S to which one can associate a family of compatible degree-4 Scholl representations $\{\rho_\ell\}$ of G over $\mathbb{Q}_\ell$. Suppose that there is a finite extension F of $\mathbb{Q}$ and there are operators J_s and J_t acting on extensions over fields in $\mathbb{Q}_\ell \otimes F$ of the representation spaces of all ρ_ℓ that satisfy the QM conditions in Definition 3.1.1 over a biquadratic $\mathbb{Q}(\sqrt{s}, \sqrt{t})$. Then, as stated in Corollary 3.1.4, for almost all primes p they give rise to a factorization of the characteristic polynomial $H_p(x)$ of $\rho_\ell(\mathrm{Frob}_p)$ into a product of two quadratic polynomials $H_{\mathfrak{p}_\pm,u}(x) = x^2 - A_{\pm,u}(p)x + B_u(p)$ by choosing a $u \in \{s, t, st\}$ such that p splits into two places $\mathfrak{p}_\pm$ in $\mathbb{Q}(\sqrt{u})$. Recall that $H_{\mathfrak{p}_\pm,u}(x)$ are the respective characteristic polynomials of $\sigma_u(\mathrm{Frob}_{\mathfrak{p}_+})$ and $\sigma_u(\mathrm{Frob}_{\mathfrak{p}_-}) = (\sigma_u \otimes \delta_u)(\mathrm{Frob}_{\mathfrak{p}_+})$, where σ_u and δ_u are as in Theorem 3.1.2.

For $p \nmid M$ and unramified in $\mathbb{Q}(\sqrt{s}, \sqrt{t})$, let V_p be the 4-dimensional p-adic Scholl space containing the space S_p of p-adically integral forms in S as a subspace and its dual $(S_p)^\vee$ as a quotient (cf. [Sch85]). On V_p there is the action of F_p, the Frobenius at p. Scholl proved in [Sch85] that, for $\ell \ne p$ and ρ_ℓ unramified at p, $\rho_\ell(\mathrm{Frob}_p)$ and F_p have the same characteristic polynomial.

Assume that the operators J_s and J_t also act on S, preserving forms whose Fourier coefficients are in a suitable number field K which are p-adically integral for almost all p. For such a p, their actions on S_p extend to V_p, again denoted by J_s and J_t. Suppose that they satisfy condition (a) in Definition 3.1.1 and

(b)$'$ For $u \in \{s, t, st\}$, the operator J_u commutes with F_p for p split in $\mathbb{Q}(\sqrt{u})$, and $J_u F_p = -F_p J_u$ for p inert in $\mathbb{Q}(\sqrt{u})$. Here $J_{st} = J_s J_t$.

We say that V_p admits QM over $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ for simplicity. The same argument as in the proof of Theorem 3.1.2 shows that over any field in $\mathbb{Q}_p \otimes K(\sqrt{-1})$, for every $u \in \{s, t, st\}$, each $\pm i$-eigenspace $V_{p,\pm,u}$ of J_u is 2-dimensional, containing a form in S_p, and J_v, where $v \in \{s, t, st\}$ and $v \ne u$, permutes the two eigenspaces.

If p splits completely in $\mathbb{Q}(\sqrt{s},\sqrt{t})$, then $V_{p,\pm,u}$ are invariant under F_p and J_v commutes with F_p. Therefore the actions of F_p on $V_{p,\pm,u}$ are isomorphic and hence have the same characteristic polynomial equal to $H_{\mathfrak{p}_\pm,u}(x)$. If p does not split in $\mathbb{Q}(\sqrt{s},\sqrt{t})$, choose an element $u \in \{s,t,st\}$ such that p splits in $\mathbb{Q}(\sqrt{u})$. Then $V_{p,\pm,u}$ are invariant under F_p. On the ℓ-adic side, the representation space W_ℓ of ρ_ℓ decomposes similarly into two $\pm i$-eigenspaces $W_{\ell,p,\pm,u}$ of J_u, each is invariant under $\rho_\ell(\mathrm{Frob}_p)$. Moreover, viewed as $J_u\rho_\ell(\mathrm{Frob}_p)$ invariant spaces, they are intertwined by J_v because of condition (b) in Definition 3.1.1. Therefore $J_u\rho_\ell(\mathrm{Frob}_p)$ has the same characteristic polynomial on $W_{\ell,p,\pm,u}$.

A normalizer of Γ acts on the space S and hence V_p. Further, its action on the modular curve X_Γ induces an action on the ℓ-adic cohomology group, which is the representation space of ρ_ℓ. When J_s and J_t arise from normalizers of Γ, or algebraic real linear combinations of such, we conclude from the argument in [Sch85, Prop. 4.4 and proof] that J_uF_p on V_p and $J_u\rho_\ell(\mathrm{Frob}_p)$ on W_ℓ have the same characteristic polynomials. Combined with the fact that $\rho_\ell(\mathrm{Frob}_p)$ and F_p have the same characteristic polynomials, we get, by using the argument of Lemma 8 of [ALL08], that when p splits in $\mathbb{Q}(\sqrt{u})$, F_p on $V_{p,\pm,u}$ and $\rho_\ell(\mathrm{Frob}_p)$ on $W_{\ell,p,\pm,u}$ have the same characteristic polynomials $H_{\mathfrak{p}_\pm,u}(x) = x^2 - A_{\pm,u}(p)x + B_u(p)$.

As observed above, for almost all primes p, $V_{p,\pm,u}$ contain a nonzero $f_{\pm,u} = \sum_{n\geq 1} a_{\pm,u}(n)q^{n/\mu} \in S$ with p-adically integral Fourier coefficients. They form a basis of S. It follows from [Sch85, FHL+] that the ASD congruences hold, namely,

(4.2)
$$a_{\pm,u}(pn) - A_{\pm,u}(p)a_{\pm,u}(n) + B_u(p)a_{\pm,u}(n/p) \equiv 0 \mod p^{(\kappa-1)(1+ord_p n)} \quad \forall n \geq 1.$$

Now we turn to the modularity of the degree-4 representations ρ_ℓ with QM over $\mathbb{Q}(\sqrt{s},\sqrt{t})$ as above. Theorem 3.2.1 says that for each $u \in \{s,t,st\}$ there is a finite ℓ-adic character χ_u of $G_{\mathbb{Q}(\sqrt{u})}$ such that $\sigma_u \otimes \chi_u$ extends to a representation η_u of G. We know from Corollary 4.1.2 that all ρ_ℓ have trace 0 at the complex conjugation. If there is a large ℓ such that the representation ρ_ℓ is over a field not containing a square root of -1, then by combining Propositions 3.1.5 and 3.2.3, we get that η_u is odd for this and hence all ℓ because of compatibility. We then conclude from Theorem 4.2.4 that the dual of ρ_ℓ is automorphic. In this case the dual of σ_u corresponds to an automorphic form h_u of GL_2 over $\mathbb{Q}(\sqrt{u})$. As σ_s, σ_t, σ_{st} have the same restrictions to $G_{\mathbb{Q}(\sqrt{s},\sqrt{t})}$, the three automorphic forms h_u of GL_2 over $\mathbb{Q}(\sqrt{u})$ for $u \in \{s,t,st\}$ base change to the same automorphic form h of GL_2 over $\mathbb{Q}(\sqrt{s},\sqrt{t})$.

We summarize the above discussions in the theorems below.

Theorem 4.3.1. *Let $\{\rho_\ell\}$ be a family of compatible degree-4 Scholl representations of G associated to a 2-dimensional subspace S of $S_\kappa(\Gamma)$ for a noncongruence subgroup Γ. Suppose that there exists a finite real extension F of $\mathbb{Q}$ and operators J_s and J_t acting on the spaces of ρ_ℓ extended over fields in $\mathbb{Q}_\ell \otimes F$ satisfying the QM conditions in Definition 3.1.1 over a biquadratic field $\mathbb{Q}(\sqrt{s},\sqrt{t})$. Then for each $u \in \{s,t,st\}$, there is an automorphic form h_u of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}(\sqrt{u})})$ such that the following L-functions have the same local factors over all primes of $\mathbb{Z}$:*

$$L(s,\rho_\ell^\vee) = L(s,h_s) = L(s,h_t) = L(s,h_{st}).$$

In addition to these three automorphic interpretations, $L(s,\rho_\ell^\vee)$ also agrees with the L-functions of an automorphic form of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})\times\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ and an automorphic form of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$, as remarked after Theorem 4.2.4.

If ρ_ℓ is unramified at a prime p which splits in $\mathbb{Q}(\sqrt{u})$, then h_u is an eigenfunction of the Hecke operators at the two places $\mathfrak{p}_\pm$ of $\mathbb{Q}(\sqrt{u})$ above p, and the product of the local L-factors attached to h_u at these two places is

$$\frac{1}{(1-A_{+,u}(p)p^{-s}+B_u(p)p^{-2s})(1-A_{-,u}(p)p^{-s}+B_u(p)p^{-2s})},$$

where $H_{\mathfrak{p}_\pm,u}(x) = x^2 - A_{\pm,u}(p)x + B_u(p)$ are the characteristic polynomials of $\sigma_u(\mathrm{Frob}_{\mathfrak{p}_\pm})$.

Theorem 4.3.2. *Keep the same notation and hypotheses as in Theorem* 4.3.1. *Assume the modular curve X_Γ has a model defined over $\mathbb{Q}$ such that the cusp at ∞ is a $\mathbb{Q}$-rational point with cusp width μ. Further, assume that J_s and J_t arise from algebraic real linear combinations of normalizers of Γ whose actions on V_p admit QM over $\mathbb{Q}(\sqrt{s},\sqrt{t})$ for almost all p. Then there exists a finite set of primes T, including ramified primes, primes $< 2\kappa - 2$, and primes where X_Γ has bad reductions, such that for each $p \notin T$, S has a p-adically integral basis $f_{\pm,p} = \sum_{n\geq 1} a_{\pm,u}(n)q^{n/\mu}$ which are $\pm i$-eigenfunctions of J_u for some $u \in \{s,t,st\}$ such that p splits in $\mathbb{Q}(\sqrt{u})$. The Atkin and Swinnerton-Dyer congruences* (4.2) *hold with $A_{\pm,u}(p)$ and $B_u(p)$ coming from the characteristic polynomials $H_{\mathfrak{p}_\pm,u}(x) = x^2 - A_{\pm,u}(p)x + B_u(p)$ of $\rho_\ell(\mathrm{Frob}_p)$ on the $\pm i$-eigenspaces of J_u on the space of ρ_ℓ. Moreover, $(1-A_{+,u}(p)p^{-s}+B_u(p)p^{-2s})^{-1}$ and $(1-A_{-,u}(p)p^{-s}+B_u(p)p^{-2s})^{-1}$ are the local factors at the two places $\mathfrak{p}_\pm$ of $\mathbb{Q}(\sqrt{u})$ above p of an automorphic form h_u for GL_2 over $\mathbb{Q}(\sqrt{u})$.*

Observe that the basis for which the ASD conjecture holds depends on p modulo the discriminant of $\mathbb{Q}(\sqrt{s},\sqrt{t})$. Further, the ASD congruences (4.2) describe congruence relations between Fourier coefficients of noncongruence forms $f_{\pm,p}$ and those of congruence forms h_u.

Remark 4.3.3. Under the same assumptions as above except that the field $\mathbb{Q}(\sqrt{s},\sqrt{t})$ is a quadratic extension of $\mathbb{Q}$, the same argument shows that, for almost all primes p, if each eigenspace of J_s contains a nonzero form in S, then the ASD conjecture on S holds at such p.

5. Applications

As applications, we exhibit some examples of Scholl representations which admit QM. We use Theorem 4.3.1 to conclude the automorphy of the representation and Theorem 4.3.2 to conclude the ASD congruences. We expect that there is a wide variety of cases to which Theorem 4.2.4 could be applied.

5.1. **Old cases.** We first re-establish several known automorphy results, previously proved using Faltings-Serre modularity criterion. While the new argument is more conceptual, no information on the congruence modular form giving rise to the Galois representation is revealed.

In a series of papers [LLY05, ALL08, Lon08], a sequence of genus 0 normal subgroups, denoted by Γ_n, of $\Gamma^1(5)$ are considered. Their modular curves are n-fold covers of the modular curve for $\Gamma^1(5)$, ramified only at two cusps ∞ and -2 of $\Gamma^1(5)$ with ramification degree n. The group Γ_n is noncongruence when $n \neq 1, 5$, its modular curve is defined over $\mathbb{Q}$ with the cusp at ∞ a $\mathbb{Q}$-rational point, and an explicit basis for $S_3(\Gamma_n)$ with rational coefficients was constructed (cf. [ALL08, Prop. 1]).

To the $(n-1)$-dimensional space $S_3(\Gamma_n)$ Scholl has attached a compatible family of $2(n-1)$-dimensional ℓ-adic Galois representations of G. The functions in $S_3(\Gamma_n)$ belonging to a nontrivial supergroup Γ_m of Γ_n are called "old" forms; denote by $S_3(\Gamma_n)^{\text{new}}$ the subspace of forms in $S_3(\Gamma_n)$ orthogonal to the "old" forms. Accordingly, there is a compatible family of degree $2\varphi(n)$-dimensional ℓ-adic representations $\rho^{\text{new}}_{3,\mathfrak{l},n}$ of G attached to $S_3(\Gamma_n)^{\text{new}}$. Here $\varphi(n)$ stands for the Euler phi-function (cf. [Lon08, §3]). For almost all ℓ, $\rho^{\text{new}}_{3,\mathfrak{l},n}|_{G_\ell}$ is crystalline with Hodge Tate weights $\{0,-2\}$, each with multiplicity $\varphi(n)$, and the action of complex conjugation has trace zero. Let $A = \begin{pmatrix} -2 & -5 \\ 1 & -2 \end{pmatrix}$ and $\zeta = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$. They induce operators A^* and ζ^* on the cohomology level whose actions on the space of $\rho^{\text{new}}_{3,\mathfrak{l},n}$ satisfy the following relations:

$$(A^*)^2 = -I, \quad (\zeta^*)^n = I, \quad \text{and} \quad \zeta^* A^* \zeta^* = A^*.$$

Theorem 5.1.1 ([LLY05, ALL08, Lon08])**.** *When $n = 3, 4, 6$, the degree-4 representations $\rho^{new}_{3,\mathfrak{l},n}$ are automorphic and the ASD conjecture on $S_3(\Gamma_n)^{new}$ holds.*

Proof. When $n = 3$, A^* and ζ^* are both defined over $\mathbb{Q}(\sqrt{-3})$. The operators $J_{-3} := A^*$ and $J_{-27} := \frac{1}{\sqrt{3}}(\zeta^* - (\zeta^*)^{-1})$ define QM over $\mathbb{Q}(\sqrt{-3})$ on the representation space of $\rho^{\text{new}}_{3,\mathfrak{l},3}$. The automorphy of this representation follows from Theorem 4.2.3.

When $n = 4$, the 4-dimensional space of $\rho^{\text{new}}_{3,\mathfrak{l},4}$ admits QM over $\mathbb{Q}(\sqrt{-1}, \sqrt{-2})$ by $J_{-1} = A^*$ and $J_{-2} = \frac{1}{\sqrt{2}} A^*(1 + \zeta^*)$. We can use $F = \mathbb{Q}(\sqrt{2})$ in Theorem 4.3.1. Similarly, when $n = 6$, the space of $\rho^{\text{new}}_{3,\mathfrak{l},6}$ admits QM over $\mathbb{Q}(\sqrt{-1}, \sqrt{-3})$ by $J_{-1} = A^*$ and $J_{-3} = \frac{1}{\sqrt{3}}(\zeta^* - (\zeta^*)^{-1})$. In this case, $F = \mathbb{Q}(\sqrt{3})$. By Theorem 4.3.1 $\rho^{\text{new}}_{3,\mathfrak{l},n}$ is automorphic for $n = 4$ and 6.

Moreover, the ASD conjecture on $S_3(\Gamma_n)^{\text{new}}$ holds for $n = 3, 4, 6$, as it is easy to verify that the assumptions in Theorem 4.3.2 and the remark following it are satisfied. □

Remark 5.1.2. The above automorphy argument does not work for $n > 6$. However, we conjecture that when $n = p$ is an odd prime, the representation $\rho_{3,\mathfrak{l},p}$ is related to a Hilbert modular form.

Another known example of 4-dimensional Scholl representation with QM by a biquadratic field has been investigated in [HLV10].

5.2. **A new example with QM over a biquadratic field.** The group $\Gamma^1(6)$ is normalized by W_6, $W_2 = \begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$ and $W_3 = \begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix}$. It is a torsion-free index-12 genus 0 subgroup of $PSL_2(\mathbb{Z})$ with four cusps: ∞, $0 = W_6^{-1} \cdot \infty$, $-3 = W_3^{-1} \cdot \infty$, and $-2 = W_2{}^{-1} \cdot \infty$. Its space of weight 3 cusp forms is 0-dimensional. The function $F := \dfrac{\eta(z)^4 \eta(2z) \eta(6z)^5}{\eta(3z)^4}$ is a weight 3 Eisenstein series for $\Gamma^1(6)$ which vanishes at every cusp except -2.

Let $B = \frac{\eta(2z)^3 \eta(3z)^9}{\eta(z)^3 \eta(6z)^9}$ and Γ be an index-6 subgroup of $\Gamma^1(6)$ whose modular curve is a 6-fold cover of the modular curve for $\Gamma^1(6)$, ramified totally at the two cusps ∞ and $-2 = W_2^{-1}\infty$ and unramified elsewhere. I.e., Γ admits $(B)^{1/6}$ as a Hauptmodul. The space $S_3(\Gamma)$ has a basis $\langle F_j = B^{(6-j)/6} \cdot F \rangle_{j=1}^5$. The functions

F_j with $2 \le j \le 4$ are for certain supergroups of Γ, and thus are regarded as "old" forms. The "new" space $S_3(\Gamma)^{\text{new}}$ for Γ is spanned by

$$F_1 = (B)^{5/6} \cdot F \quad \text{and} \quad F_5 = (B)^{1/6} \cdot F.$$

On the p-adic side, the action of W_2 on $S_3(\Gamma)^{\text{new}}$ is as follows. By noting $B|W_2 = \frac{-8}{B}$, we claim

$$F_1|W_2 = 2e^{2\pi i 5/12} F_5 \quad \text{and} \quad F_5|W_2 = \frac{1}{2} e^{2\pi i/12} F_1. \tag{5.1}$$

Indeed, by definition, $B = c\frac{(F|W_2)}{F}$ for some nonzero constant c, $F_1 = B^{5/6}F = c^{5/6}(F|W_2)^{5/6}F^{1/6}$, and $F_5 = B^{1/6}F = c^{1/6}F^{5/6}(F|W_2)^{1/6}$. Since

$$(W_2)^2 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

and F has weight 3, we have $F|W_2W_2 = -F$. Therefore

$$F_1|W_2 = c^{5/6}(-1)^{5/6}F^{5/6}(F|W_2)^{1/6} = c^{2/3}(-1)^{5/6}F_5$$

and

$$F_5|W_2 = c^{1/6}(F|W_2)^{5/6}(-1)^{1/6}F^{1/6} = c^{-2/3}(-1)^{1/6}F_1.$$

To compute the constant c, we use the fact that $B|W_2 = \frac{-8}{B}$. This gives $c^2 = 8$ and hence $c^{2/3} = 2$.

Let $\zeta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is easy to verify that

$$F_1|\zeta = e^{\pi i/3}F_1 \quad \text{and} \quad F_5|\zeta = e^{-\pi i/3}F_5. \tag{5.2}$$

Let V be the 4-dimensional p-adic Scholl space which contains $\langle F_1, F_5\rangle$ as a subspace and their dual $\langle F_1^\vee, F_5^\vee\rangle$ as a quotient (cf. [Sch85]). Let $J_{-2} = \zeta W_2, J_{-3} = \frac{1}{\sqrt{3}}(2\zeta - 1), J_6 = J_{-2}J_{-3}$. With respect to a suitable basis J_{-2} and J_{-3} can be represented by the following matrices respectively:

$$\begin{pmatrix} 0 & \frac{1}{2}iI_2 \\ 2iI_2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix}.$$

Thus J_{-2} and J_{-3} generate a quaternion algebra over $\mathbb{Q}$ with $J_{-2}^2 = -I, J_{-3}^2 = -I, J_{-3}J_{-2} = -J_{-2}J_{-3}$. It is easy to check that

- F_1 and F_5 are eigenvectors of J_{-3} with eigenvalues $\pm i$;
- $F_1 \pm 2F_5$ are eigenvectors of J_{-2} with eigenvalues $\pm i$;
- $F_1 \pm 2iF_5$ are eigenvectors of J_6 with eigenvalues $\pm i$.

We now turn to the ℓ-adic side. Under the action of W_2, $B \mapsto -8/B$. This gives rise to an involution on the modular curves for $\Gamma^1(6)$ since W_2 normalizes $\Gamma^1(6)$. Consider two elliptic surfaces $\mathcal{E}_{\Gamma^1(6)}$ and $\mathcal{E}_{W_2^{-1}\Gamma^1(6)W_2}$ fibred over the modular curve $X_{\Gamma^1(6)}$ parametrized by B defined by the following equations:

$$\begin{aligned} \mathcal{E}_{\Gamma^1(6)} \quad &: \quad Y^2 = X^3 + a(B)X^2 + b(B)X, \\ \mathcal{E}_{W_2^{-1}\Gamma^1(6)W_2} \quad &: \quad Y^2 = X^3 - 2a(\frac{-8}{B})X^2 + \left(a(\frac{-8}{B})^2 - 4b(\frac{-8}{B})\right)X, \end{aligned}$$

where

$$a(B) := \frac{2}{27} - \frac{5}{27}B - \frac{1}{108}B^2 \quad \text{and} \quad b(B) := \frac{1}{729} + \frac{1}{243}B + \frac{1}{729}B^3 + \frac{1}{243}B^2.$$

Then

$$(X,Y,B)\mapsto\left(\frac{Y^2}{X^2},\frac{Y(b(B)-X^2)}{X^2},\frac{-8}{B}\right)$$

gives a 2-isogeny from $\mathcal{E}_{\Gamma^1(6)}$ to $\mathcal{E}_{W_2^{-1}\Gamma^1(6)W_2}$.

By letting $t^6=B$ we obtain an explicit defining equation for the elliptic surface $\mathcal{E}_\Gamma$ fibred over the modular curve of Γ. A similar argument yields the following degree-2 isogeny between $\mathcal{E}_\Gamma$ and $\mathcal{E}_{W_2^{-1}\Gamma W_2}$: Again, W_2 normalizes Γ and gives rise to the following map:

$$W_2:\quad (X,Y,t)\mapsto\left(\frac{Y^2}{X^2},\frac{Y(b(t^6)-X^2)}{X^2},\sqrt[6]{-8}/t\right). \tag{5.3}$$

Also on $\mathcal{E}_\Gamma$ and $\mathcal{E}_{W_2^{-1}\Gamma W_2}$, we have

$$\zeta:\quad (X,Y,t)\mapsto(X,Y,e^{-\pi i/3}t).$$

It is straightforward to check that

$$\zeta W_2\zeta=W_2, \tag{5.4}$$

where the ζ on the left acts on $\mathcal{E}_{W_2^{-1}\Gamma W_2}$ and the one on the right acts on $\mathcal{E}_\Gamma$.

The composition $T_{-2}=\zeta\circ W_2$ is an isogeny defined over $\mathbb{Q}_\ell(\sqrt{-2})$. Let $\widehat{T}_{-2}$ denote its dual isogeny.

We first note that the above map T_{-2} induces a natural homomorphism g_1 from the parabolic cohomology $H^1(X_\Gamma\otimes\overline{\mathbb{Q}},i_*R^1h^0_*\mathbb{Q}_\ell)$ to that of $X_{W_2^{-1}\Gamma W_2}$. Here $i:\mathfrak{H}/\Gamma\to X_\Gamma=(\mathfrak{H}/\Gamma)^*$ is the inclusion map, $h:\mathcal{E}_\Gamma\to(\mathfrak{H}/\Gamma)^*$ is the natural projection and h^0 is the restriction of h to $\mathfrak{H}/\Gamma$. We further let $w_2:X_\Gamma\to X_{W_2^{-1}\Gamma W_2}$ denote the isomorphism which sends t to $\sqrt[6]{-8}/t$. Compared with T_{-2}, this map is on the base curve level. Using w_2, we have a natural isomorphism

$$g_2:H^1(X_\Gamma\otimes\overline{\mathbb{Q}},i_*R^1h^0_*\mathbb{Q}_\ell)\cong H^1(X_{W_2^{-1}\Gamma W_2}\otimes\overline{\mathbb{Q}},(w_2)_*i_*R^1h^0_*\mathbb{Q}_\ell),$$

and the cohomology on the right is isomorphic to the parabolic cohomology attached to weight 3 cusp forms of $W_2^{-1}\Gamma W_2$. The composition $g_2^{-1}\circ g_1$ is an endomorphism on $H^1(X_\Gamma\otimes\overline{\mathbb{Q}},i_*R^1h^0_*\mathbb{Q}_\ell)$ and is again denoted by T_{-2}. On $H^1(X_\Gamma\otimes\overline{\mathbb{Q}},i_*R^1h^0_*\mathbb{Q}_\ell)$ the map $(T_{-2})^2=-2\ id$ is induced by the fiber-wise multiplication by 2 map $\widehat{T}_{-2}\circ T_{-2}$.

Denote by ρ_ℓ^{new} the 4-dimensional ℓ-adic Scholl representation of G corresponding to the "new" forms of $S_3(\Gamma)$ (cf. [Lon08]). Let $J_{-2}=\frac{1}{\sqrt{2}}T_{-2}$ and let J_{-3} denote $\frac{1}{\sqrt{3}}(2\zeta-1)^*$ on ρ_ℓ^{new}. As linear operators on the representation space of ρ_ℓ^{new}, J_{-2} and J_{-3} generate a quaternion group. We know from the above discussions that the actions of J_δ are defined over $\mathbb{Q}(\sqrt{\delta})$ for $\delta=-2,-3$. Choose the field F in Theorem 4.3.1 to be $\mathbb{Q}(\sqrt{2},\sqrt{3})$. The space of ρ_ℓ^{new} over fields in $\mathbb{Q}_\ell\otimes F$ is endowed with QM over $\mathbb{Q}(\sqrt{-2},\sqrt{-3})$. By Theorem 4.3.1, we conclude the automorphy of ρ_ℓ^{new}.

We have also seen that J_{-2} and J_{-3} act on the space $\langle F_1,F_5\rangle$ and that the space V_p admits QM over $\mathbb{Q}(\sqrt{-2},\sqrt{-3})$. Since J_{-2} and J_{-3} arise from normalizers of $\Gamma^1(6)$, by Theorems 4.3.1 and 4.3.2, we have

Theorem 5.2.1. *The representation ρ_ℓ^{new} is automorphic, and the ASD congruences hold on the space $\langle F_1,F_5\rangle$.*

In particular, ρ_ℓ^{new} is isomorphic to $\eta_{-2} \otimes \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{-2})}}^{G} \chi_{-2}^{-1}$, which corresponds to an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, where χ_{-2} is a character of $G_{\mathbb{Q}(\sqrt{-2})}$. Similar to [ALL08], we conclude this section by exhibiting, from numerical data, a congruence modular form f which gives rise to η with $\chi_{-2} = \xi_1\xi_2$ being the product of the following two characters of $G_{\mathbb{Q}(\sqrt{-2})}$: ξ_1 is a quartic character of $\mathrm{Gal}(\mathbb{Q}(\sqrt{-2}, \sqrt{1+\sqrt{2}})/\mathbb{Q}(\sqrt{-2}))$ and ξ_2 is the quadratic character of $\mathrm{Gal}(\mathbb{Q}(\sqrt{1+\sqrt{-2}})/\mathbb{Q}(\sqrt{-2}))$.

To motivate the expression for f, we list explicit factorizations of the characteristic polynomials of Frob_p at small primes.

p	Char. poly.	Factorization
5	$x^4 + 4x^2 + 5^4$	$(x^2 - 3x\sqrt{-6} - 25)(x^2 + 3x\sqrt{-6} - 25)$
7	$x^4 + 10x^2 + 7^4$	$(x^2 - 6x\sqrt{-3} - 49)(x^2 + 6x\sqrt{-3} - 49)$
11	$x^4 - 170x^2 + 11^4$	$(x^2 - 6x\sqrt{-2} - 121)(x^2 + 6x\sqrt{-2} - 121)$
13	$x^4 - 230x^2 + 13^4$	$(x^2 - 6x\sqrt{-3} - 13^2)(x^2 + 6x\sqrt{-3} - 13^2)$
17	$x^4 - 128x^2 + 17^4$	$(x^2 - 15x\sqrt{-2} - 17^2)(x^2 + 15x\sqrt{-2} - 17^2)$
19	$(x^2 + 20x + 19^2)^2$	$(x^2 + 20x + 19^2)^2$
23	$x^4 - 842x^2 + 23^4$	$(x^2 + 6x\sqrt{-6} - 23^2)(x^2 - 6x\sqrt{-6} - 23^2)$
29	$x^4 - 332x^2 + 29^4$	$(x^2 + 15x\sqrt{-6} - 841)(x^2 - 15x\sqrt{-6} - 841)$

The congruence form $f(z)$ is a Hecke eigenform in $S_3(\Gamma_0(576), (-6/\cdot))$ whose Fourier coefficients occur at positive integers coprime to 24. Express it as a linear combination of forms f_j whose Fourier coefficients occur at integers congruent to j mod 24 as follows:

$$f(z/24) := f_1 + 3jk \cdot f_5 + 6k \cdot f_7 + 6j \cdot f_{11} + 6ik \cdot f_{13} + 3ij \cdot f_{17} - 4i \cdot f_{19} - 6ijk \cdot f_{23},$$

where $i = \sqrt{-1}, j = \sqrt{2}$ and $k = \sqrt{3}$. Here

$$f_5 = \frac{\eta(z)^5\eta(6z)^5}{\eta(3z)^2\eta(12z)^2}, \qquad f_7 = \frac{\eta(z)^5\eta(4z)\eta(6z)^2}{\eta(2z)\eta(12z)},$$
$$f_{13} = \frac{\eta(z)^4\eta(2z)^2\eta(3z)\eta(12z)}{\eta(4z)\eta(6z)}, \qquad f_{23} = \frac{\eta(z)^5\eta(12z)^2}{\eta(6z)},$$

which span a space invariant under W_9 and W_{64}. The remaining forms f_1, f_{11}, f_{17}, f_{19} are uniquely determined by applying the Hecke operators to the eigenform f. Listed below are their initial Fourier coefficients:

$$\begin{aligned}
f_1 &= q^{1/24}(q + 29q^2 + 59q^3 + 20q^4 + 40q^5 - 49q^6 - 270q^7 + 61q^8 + \cdots),\\
f_{11} &= q^{11/24}(q + 9q^2 + 4q^3 + 15q^4 - 20q^5 + 6q^6 - 45q^7 + 6q^8 + \cdots),\\
f_{17} &= q^{17/24}(5q + 17q^2 + 18q^3 + 3q^4 - 15q^5 - 25q^6 - 36q^7 - 72q^8 + \cdots),\\
f_{19} &= q^{19/24}(5q + 10q^2 + 25q^3 - 27q^4 + 27q^5 - 43q^6 - 40q^7 - 45q^8 + \cdots).
\end{aligned}$$

Choosing different signs of i, j, k gives rise to 8 eigenforms that are conjugate to each other under G.

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