

DUALITY OF HOLOMORPHIC FUNCTION SPACES AND SMOOTHING PROPERTIES OF THE BERGMAN PROJECTION

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ABSTRACT. Let $\Omega \Subset \mathbb{C}^n$ be a domain with smooth boundary, whose Bergman projection B maps the Sobolev space $H^{k_1}(\Omega)$ (continuously) into $H^{k_2}(\Omega)$. We establish two smoothing results: (i) the full Sobolev norm $\|Bf\|_{k_2}$ is controlled by L^2 derivatives of f taken along a *single, distinguished direction* (of order $\leq k_1$), and (ii) the projection of a *conjugate holomorphic* function in $L^2(\Omega)$ is automatically in $H^{k_2}(\Omega)$. There are obvious corollaries for when B is globally regular.

1. INTRODUCTION

In this paper, we consider the range of the Bergman projection, B , associated to a smoothly bounded domain $\Omega \Subset \mathbb{C}^n$ through the following question: what functions defined on Ω does B map to elements of $C^\infty(\overline{\Omega})$? Our particular interest is in finding families of functions \mathcal{F} , $\mathcal{F} \not\subset C^\infty(\overline{\Omega})$, such that $B(\mathcal{F}) \subset C^\infty(\overline{\Omega})$.

We present two partial answers to this general question, both under the hypothesis that B satisfies Condition R of Bell–Ligocka [3], which says that $B(C^\infty(\overline{\Omega})) \subseteq C^\infty(\overline{\Omega})$. In the first, we show that an L^2 function f which has square-integrable derivatives of all orders only in a *single, distinguished direction* is necessarily mapped by B to a function in $C^\infty(\overline{\Omega})$. The direction is distinguished by being both tangential to $b\Omega$, the boundary of Ω , and not contained in the complex tangent space to $b\Omega$ (the last property is called *complex transversal*, for short). Note that no smoothness about f is assumed except in this one direction, yet Bf is smooth up to the boundary in all directions. This ‘partial smoothing’ property of B has a different character than traditionally studied mapping properties of B , which concentrate on whether B preserves various Banach space structures, and was discovered recently by the first two authors [13]. Our first result here generalizes the main theorem in [13], which established partial smoothing of B under the hypothesis that the $\bar{\partial}$ -Neumann operator on Ω is compact. The second result in this paper says that all *conjugate holomorphic* functions in $L^2(\Omega)$ are mapped to $C^\infty(\overline{\Omega})$ by B . This result differs from the first as functions in $A^0(\Omega)$, where $A^0(\Omega)$ denotes the space of square-integrable holomorphic functions on Ω , need have no derivatives in $L^2(\Omega)$.

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We now state our results more precisely. $H^k(\Omega)$ denotes the standard L^2 Sobolev space of order k , and $H_T^k(\Omega)$ denotes the Sobolev space of order k involving only derivatives in direction T ; see Definition 2.11.

Theorem 1.1. *Let $\Omega \Subset \mathbb{C}^n$ be a smoothly bounded domain and let T be a smooth vector field on $\overline{\Omega}$ that is tangential and complex transversal at the boundary. Suppose that there exist a pair $(k_1, k_2) \in \mathbb{N} \times \mathbb{N}$ and a constant $C_1 > 0$ such that*

$$(1.2) \quad \|Bf\|_{k_2} \leq C_1 \|f\|_{k_1} \quad \forall f \in H^{k_1}(\Omega).$$

Then there exists a constant $C_2 > 0$, such that

$$(1.3) \quad \|Bf\|_{k_2} \leq C_2 \|f\|_{k_1, T} \quad \forall f \in H_T^{k_1}(\Omega).$$

Note that in (1.2), $k_1 \geq k_2$ automatically. We emphasize that (1.3) is a genuine estimate: if the right hand side is finite, i.e., f has T derivatives up to order k_1 in $L^2(\Omega)$, then $Bf \in H^{k_2}(\Omega)$, and the estimate holds.

Condition R is equivalent to the statement that for each $k_2 \in \mathbb{N}$, there exists $k_1 \in \mathbb{N}$ such that $B : H^{k_1}(\Omega) \rightarrow H^{k_2}(\Omega)$. Combined with the Sobolev Lemma to the effect that $\bigcap_{k=0}^\infty H^k(\Omega) = C^\infty(\overline{\Omega})$, Theorem 1.1 implies in particular the $C^\infty(\overline{\Omega})$ result described in the second paragraph above:

Corollary 1.4. *Assume Ω and T are as in Theorem 1.1, and that Condition R holds for Ω . Then B maps $H_T^\infty := \bigcap_{k=0}^\infty H_T^k(\Omega)$ (continuously) to $C^\infty(\overline{\Omega})$.*

The method we use to prove Theorem 1.1 is rather general and applicable in other situations, i.e., to other operators connected to elliptic PDEs and to other spaces of holomorphic functions.

The proof consists of two distinct steps. The first step is to show that when B satisfies the regularity condition (1.2), the $H^{k_2}(\Omega)$ norm of a holomorphic function g can be estimated by pairing g , in the ordinary L^2 inner product, against holomorphic functions contained in the unit ball of $H^{-k_1}(\Omega)$ (see Proposition 2.3 below). This is a special fact about holomorphic functions (but it can also be established for solutions to other homogeneous elliptic systems; see [5, 20, 16] and Appendix B in [8]). This special type of duality for holomorphic functions originated in the work of Bell [4, 6], and was subsequently extended and refined in [5, 7, 15, 20, 2]. For our purposes, duality is used to reduce the problem of estimating $\|Bf\|_{k_2}$ significantly. In order to illustrate how this works, assume for simplicity that $k_1 = k_2 = 1$. Then, for some constant $c > 0$,

$$(1.5) \quad \begin{aligned} \|Bf\|_1 &\leq c \sup \{ |(Bf, h)| : h \in A^1(\Omega), \|h\|_{-1} \leq 1 \} \\ &= c \sup \{ |(f, h)| : h \in A^1(\Omega), \|h\|_{-1} \leq 1 \}. \end{aligned}$$

The key point is that the self-adjointness of B in the ordinary L^2 inner product, which yields the last equality, ‘eliminates’ B from the right hand side of (1.5). This obviates the need to study the commutator of B with differential operators in order to bound the left hand side of (1.5); estimating such commutators is a difficult problem in general since B is an abstractly given integral operator. (This effect also occurs when one can use vector fields with holomorphic coefficients to control Sobolev norms of Bf ; see [1].) A similar use of duality seems applicable to other self-adjoint operators, and also to B itself in other scales of Banach spaces besides $H^k(\Omega)$.

Let T be the vector field from Theorem 1.1. Once (1.5) is in hand, the second step (still assuming $k_1 = k_2 = 1$) consists of replacing h , on the right hand side of (1.5), by the sum of the \bar{T} derivative of a function, \mathcal{H}_1 , and of another function, \mathcal{H}_2 , both of whose L^2 norms are uniformly bounded (when $\|h\|_{-1} \leq 1$). Since T is tangential, it follows that

$$\begin{aligned} (f, h) &= (f, \bar{T}\mathcal{H}_1 + \mathcal{H}_2) = (\bar{T}^* f, \mathcal{H}_1) + (f, \mathcal{H}_2) \\ &\leq C (\|Tf\| + \|f\|), \end{aligned}$$

where the last inequality follows after noticing that the L^2 adjoint \bar{T}^* differs from $-T$ by a term bounded in L^2 . In this very simple way, the full Sobolev norm of Bf of order 1 is controlled by the L^2 norm of the derivative of f in the special direction T , provided $\mathcal{H}_1, \mathcal{H}_2$ exist.

Proving the existence of \mathcal{H}_1 and \mathcal{H}_2 , and similar functions for higher powers of \bar{T} , is conceptually simple, but, for higher powers, is somewhat technical. This fact accounts for much of the length of this paper. To simplify matters temporarily suppose that T is of the form $T = T_1 + L$, where both T_1 and L are tangential at the boundary, T_1 is real, and L is of type $(1, 0)$. Note that T_1 is complex transversal because T is. Consequently, if J denotes the complex structure map on \mathbb{C}^n , JT_1 is transversal to the boundary of Ω . Now let $h \in A^1(\Omega)$ with $\|h\|_{-1} \leq 1$ be given. The goal is to write $h = \bar{T}\mathcal{H}_1 + \mathcal{H}_2$, where $\|\mathcal{H}_1\|_{L^2}, \|\mathcal{H}_2\|_{L^2}$ are bounded by constants depending only on Ω and T . To this end, first consider $\mathfrak{A} \circ (JT_1 h)$, where \mathfrak{A} is the operator of ‘anti-differentiation along the direction JT_1 ’. Then

$$(1.6) \quad h(z) = \mathfrak{A} \circ JT_1 h(z) + h(q),$$

where $q = q(z)$ varies in a fixed compact subset of Ω . Because h is holomorphic, the contributions given by $h(q)$ are easily shown to be bounded in L^2 and are folded into the function \mathcal{H}_2 . Furthermore, the Cauchy–Riemann equations show that $JT_1(h) = -iT_1(h)$. Thus

$$\begin{aligned} \mathfrak{A} \circ JT_1(h) &= \mathfrak{A} \circ (-i)T_1 h \\ (1.7) \quad &= -i (\mathfrak{A} \circ \bar{T}(h) - \mathfrak{A}(\bar{L}h)) . \end{aligned}$$

But $\bar{L}h$ vanishes since h is holomorphic. As a last step, we commute \bar{T} with \mathfrak{A} in (1.7):

$$(1.8) \quad \mathfrak{A} \circ \bar{T}(h) = \bar{T} \circ \mathfrak{A}(h) + [\mathfrak{A}, \bar{T}](h).$$

The commutator in (1.8) is straightforward to analyze since both \mathfrak{A} and \bar{T} are explicit operators; this term forms the final component of the function \mathcal{H}_2 . The term $-i\mathfrak{A}(h)$ is the function \mathcal{H}_1 . The needed inequality $\|\mathfrak{A}(h)\| \leq C\|h\|_{-1}$ is a consequence of Hardy’s inequality, which says that \mathfrak{A} gains a factor of the boundary distance $d(z)$, together with the fact that such a factor gains a derivative in the case of holomorphic functions:

$$(1.9) \quad \|\mathfrak{A}(h)\| \leq c_1 \|d(z)h\| \leq c_2 \|h\|_{-1} .$$

Coming to our second result, we start with an observation about the Bergman projection on the unit disk. Modulo constants, conjugate holomorphic functions on the unit disc are orthogonal to the Bergman space. Equivalently, their Bergman projections are constant functions. Of course, this fails on general (even planar) domains. Our second result says that nevertheless, if the Bergman projection satisfies

a regularity estimate such as (1.2), projections of conjugate holomorphic functions are still as good as projections of functions smooth up to the boundary: they belong to $H^{k_2}(\Omega)$. In particular, if Condition R holds, these projections are themselves smooth up to the boundary. In fact, more is true; the case of a conjugate holomorphic function corresponds to $h \equiv 1$ in the following result.¹

Theorem 1.10. *Let $\Omega \Subset \mathbb{C}^n$ be a smoothly bounded domain. Suppose that the Bergman projection B on Ω satisfies the regularity condition (1.2). Then for every $h \in C^\infty(\overline{\Omega})$, there is a constant $C_h > 0$ such that*

$$(1.11) \quad \|B(fh)\|_{k_2} \leq C_h \|f\|$$

for all conjugate holomorphic functions f in $L^2(\Omega)$.

In contrast to Theorem 1.1, smoothing here takes place in all directions. But just as with Theorem 1.1, there is an immediate corollary for when Condition R holds. We only formulate the special case of projections of conjugate holomorphic functions.

Corollary 1.12. *Let Ω be as in Theorem 1.10 and assume that Condition R holds. Then for every $k \in \mathbb{N}$ there is a constant $C_k > 0$ such that*

$$(1.13) \quad \|Bf\|_k \leq C_k \|f\|$$

for f conjugate holomorphic in $L^2(\Omega)$. In particular, Bf is smooth up to the boundary.

The proof of Theorem 1.10, say again for $k_1 = k_2 = 1$, also starts with (1.5). But now, estimating $|(fh, g)|$, $g \in A^1(\Omega)$, means we are estimating (the absolute value of) the integral over Ω of a holomorphic function (namely $\overline{f}g$) times a function in $C^\infty(\overline{\Omega})$ (namely \overline{h}). Such an integral can be dominated by $\|\overline{f}g\|_{-m} \|h\|_m$, for any $m \in \mathbb{N}$ ([5, 7, 15, 20]). Finally, the equivalence, for holomorphic functions, of membership in a negative Sobolev space and polynomial boundedness in the reciprocal of the boundary distance ([5, 20]) gives the estimate $\|\overline{f}g\|_{-m} \leq C \|f\| \|g\|_{-1}$ for m large enough. We remark that the last two steps are valid for functions in the kernel of more general elliptic operators (systems); see [17] and [18], respectively, for the relevant results. However, the first step after invoking (1.5) may fail: for example, the product of two harmonic functions need not be harmonic.

Regularity properties like (1.2) are known to hold on a large class of domains. When the domain Ω is pseudoconvex, these properties are essentially equivalent to corresponding regularity properties of the $\overline{\partial}$ -Neumann operator. For example, when the domain is of finite type, or when it admits a defining function that is plurisubharmonic at boundary points (in particular, when the domain is convex), (1.2) holds for the pair (k, k) for all $k \in \mathbb{N}$. Non-pseudoconvex domains on which regularity estimates for the Bergman projection hold include Reinhardt domains, complete Hartogs domains in \mathbb{C}^2 , and domains with ‘approximate symmetries’. For these results and for further information on the L^2 Sobolev regularity theory of the Bergman projection and of the $\overline{\partial}$ -Neumann operator, we refer the reader to [10, 9, 21] and their references. [9] also contains a discussion of the connection between the regularity theory of the Bergman projection and duality of holomorphic function spaces.

¹We are indebted to the referee for a remark that led to this stronger version.

The paper is laid out as follows. In Section 2 we define the notions needed for Theorem 1.1, collect some standard definitions, and give a new proof of (a portion of) Bell’s duality theorem on holomorphic functions in Sobolev spaces. In Section 3, we state the anti-differentiation result, Proposition 3.1, then give a proof of Theorem 1.1 modulo a proof of Proposition 3.1. Section 4 is devoted to a proof of the anti-differentiation result, broken down into several subsections. We develop the algebra of operators associated to anti-differentiation along integral curves to transverse vector fields in these subsections in some detail, as we need these results for our proof of Proposition 3.1; we also mention that the results in Section 4 can be applied elsewhere. In Section 5 we prove Theorem 1.10.

2. FUNCTION SPACES, DUALITY, AND COMPLEX TRANSVERSALITY

Throughout the paper, Ω will denote a domain with smooth boundary $b\Omega$, contained in \mathbb{C}^n . The standard L^2 inner product and norm on functions defined on Ω will be denoted

$$(f, g) = \int_{\Omega} f \bar{g} \quad \text{and} \quad \|f\| = \left(\int_{\Omega} |f|^2 \right)^{1/2},$$

where the integrals are taken with respect to the Euclidean volume element. If k is a positive integer, we let $H^k(\Omega)$ denote the usual Sobolev space of complex-valued functions whose norm $\|\cdot\|_k$ is induced by the inner product

$$(f, g)_k = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g),$$

for α a multi-index and D^α denoting differentiation of order α . Let $C_0^\infty(\Omega)$ denote the set of smooth functions with compact support in Ω and $C^\infty(\bar{\Omega})$ the set of functions smooth up to $b\Omega$. As is well-known, $C^\infty(\bar{\Omega})$ is dense in $H^k(\Omega)$. We let $H_0^k(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$.

The dual space of $H_0^k(\Omega)$ will be denoted $H^{-k}(\Omega)$. Because $C_0^\infty(\Omega)$ is dense in $H_0^k(\Omega)$, $H^{-k}(\Omega)$ imbeds naturally into the space of distributions $\mathcal{D}'(\Omega) = (C_0^\infty(\Omega))^*$ on Ω , and $L \in \mathcal{D}'(\Omega)$ belongs to $H^{-k}(\Omega)$ precisely when

$$(2.1) \quad \|L\|_{-k} = \sup \{ |\langle L, \phi \rangle| : \phi \in C_0^\infty(\Omega), \|\phi\|_k \leq 1 \}$$

is finite. Here $\langle L, \phi \rangle$ denotes the action of the distribution L on the test function ϕ . We shall only need to compute (2.1) on certain $L \in L^2(\Omega)$, in which case $\langle L, \phi \rangle = (L, \bar{\phi})$.

The subspace of $H^k(\Omega)$ consisting of holomorphic functions will be denoted $A^k(\Omega)$. For consistency of notation, we let $A^0(\Omega)$ denote the Bergman space: $A^0(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$, where $\mathcal{O}(\Omega)$ denotes the space of holomorphic functions on Ω . For functions f in $A^{-k}(\Omega)$, the norm $\|f\|_{-k}$ is comparable to a weighted L^2 norm of f , with weight equal to the corresponding positive power of the distance to $b\Omega$ ([16, 8]). If r is a smooth defining function for Ω , i.e., $\Omega = \{r < 0\}$ and $dr \neq 0$ when $r = 0$, we shall use the following version of (half of) this fact: for $k \in \mathbb{N}$ there exists a constant $\beta_k > 0$ such that

$$(2.2) \quad \|(-r)^k h\| \leq \beta_k \|h\|_{-k} \quad \forall h \in A^0(\Omega).$$

The Bergman projection is the orthogonal projection of $L^2(\Omega)$ onto $A^0(\Omega)$. We say that B satisfies *Condition R* (see [3]) if it maps $C^\infty(\overline{\Omega})$ to itself (by continuity in $L^2(\Omega)$ and the closed graph theorem for Fréchet spaces, this map is then automatically continuous). This property is often also referred to as *global regularity*. An equivalent formulation is: for each $k_2 \in \mathbb{N}$ there exists $k_1 \in \mathbb{N}$ such that B maps $H^{k_1}(\Omega)$ (continuously) into $H^{k_2}(\Omega)$. The special case where (1.2) holds with $k_1 = k_2$ is usually referred to as *exact regularity* at level $k = k_1 = k_2$. We remark that, rather intriguingly, no instance is known where B satisfies Condition R, but is not exactly regular at all levels.

As mentioned in the introduction, a crucial fact for our proof of Theorem 1.1 is that regularity of the Bergman projection as in (1.2) implies that the Sobolev norm of a function $f \in A^{k_2}(\Omega)$ is controlled by pairing f with a family of holomorphic functions in the *ordinary* L^2 inner product. Duality results of this kind are known, [4, 6, 5, 20, 7, 15, 2], but they are formulated for exact regularity, rather than (1.2), or for Condition R (i.e., assuming (1.2) holds for all $k_2 \in \mathbb{N}$). In these situations, a clean cut formulation is possible: for example, $A^k(\Omega)$ and $A_{cl}^{-k}(\Omega)$, the closure of $A^0(\Omega)$ in $A^{-k}(\Omega)$, are mutually dual, under a natural extension of the L^2 pairing, when B is exactly regular at level k . When there is a loss of derivatives, that is, when $k_2 < k_1$ in (1.2), this duality has a somewhat less striking formulation. For this reason, we only state below what we need, and give, for the reader's convenience, a straightforward proof (which seems to be new even for the case $k_1 = k_2$).

Proposition 2.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain with smooth boundary. Suppose that (1.2) holds. Then there exists a constant $c > 0$ such that*

$$(2.4) \quad \|f\|_{k_2} \leq c \sup \{ |(f, h)| : h \in A^{k_2}(\Omega), \|h\|_{-k_1} \leq 1 \}$$

holds for all $f \in A^0(\Omega)$. The constant c depends on C_1 from (1.2), n and k_2 .

Remark 2.5. It is part of (2.4) that if the right hand side is finite, then so is the left hand side. That is, (2.4) is a genuine estimate, as opposed to an *a priori* estimate.

Proof. Let $f \in A^0(\Omega) \subset C^\infty(\Omega)$ and let α be a fixed multi-index with $|\alpha| \leq k_2$. Then $D^\alpha f \in L^2(\Omega)$ if and only if

$$\sup \{ |(D^\alpha f, g)| : g \in C_0^\infty(\Omega), \|g\| \leq 1 \} < \infty .$$

Moreover, in this case, $\|D^\alpha f\|$ is given by this supremum. Because $g \in C_0^\infty(\Omega)$ and $f \in C^\infty(\Omega)$, integration by parts yields

$$|(D^\alpha f, g)| = \left| (-1)^{|\alpha|} (f, D^\alpha g) \right| = |(Bf, D^\alpha g)| = |(f, BD^\alpha g)| .$$

Thus,

$$(2.6) \quad \|D^\alpha f\| = \sup \{ |(f, BD^\alpha g)| : g \in C_0^\infty(\Omega), \|g\| \leq 1 \} .$$

Note that (1.2) implies that $BD^\alpha g \in A^{k_2}(\Omega)$. The aim now is to see that (1.2) forces $BD^\alpha g$ to be uniformly bounded in $H^{-k_1}(\Omega)$. For that, let $\varphi \in C_0^\infty(\Omega)$ with $\|\varphi\|_{k_1} \leq 1$. Integration by parts and Cauchy–Schwarz give

$$(2.7) \quad |(\varphi, BD^\alpha g)| = |(D^\alpha B\varphi, g)| \leq \|B\varphi\|_{k_2} \|g\| \leq C_1 \|\varphi\|_{k_1} \|g\| \leq C_1 ,$$

where (1.2) was used again. Thus, (2.1) shows $\|BD^\alpha g\|_{-k_1} \leq C_1$. Returning to (2.6), we now obtain that

$$\begin{aligned} \|D^\alpha f\| &= C_1 \sup \left\{ \left| \left(f, BD^\alpha \frac{g}{C_1} \right) \right| : g \in C_0^\infty(\Omega), \|g\| \leq 1 \right\} \\ &\leq C_1 \sup \{ |(f, h)| : h \in A^{k_2}(\Omega), \|h\|_{-k_1} \leq 1 \}. \end{aligned}$$

Summing over $|\alpha| \leq k_2$ gives (2.4) with $c = \sum_{j=0}^{k_2} \binom{2n}{j} C_1$. □

Remark 2.8. Proposition 2.3 implies that when (1.2) holds, then

$$\begin{aligned} \|Bf\|_{k_2} &\leq c \sup \{ |(Bf, h)| : h \in A^{k_2}(\Omega), \|h\|_{-k_1} \leq 1 \} \\ (2.9) \qquad &= c \sup \{ |(f, h)| : h \in A^{k_2}(\Omega), \|h\|_{-k_1} \leq 1 \} \end{aligned}$$

for all $f \in H^{k_1}(\Omega)$. It is (2.9) that will be used in the proof of Theorem 1.1 (compare (1.5) above).

Let $\Omega \Subset \mathbb{C}^n = \{\rho < 0\}$ be a smoothly bounded domain and denote by $L_n := \sum_{j=1}^n (\partial\rho/\partial\bar{z}_j)(\partial/\partial z_j)$ the complex normal field of type (1,0). Set $T_0 = i(L_n - \bar{L}_n)$. Then T_0 is real and tangential to $b\Omega$, and it spans (over \mathbb{C}) the orthogonal complement of $T^{1,0}(b\Omega) \oplus T^{0,1}(b\Omega)$ in $T(b\Omega) \otimes \mathbb{C}$. In particular, a vector field T , with coefficients in $C^\infty(\bar{\Omega})$, that is tangential to $b\Omega$, can be written as

$$T = aT_0 + Y_1 + \bar{Y}_2,$$

where a is a smooth complex-valued function, and Y_1, Y_2 are of type (1,0) and tangential to $b\Omega$. The vector field T is called *complex transversal* if it is transversal to $T^{1,0}(b\Omega) \oplus T^{0,1}(b\Omega)$, i.e., if a is nowhere vanishing on $b\Omega$. Writing \bar{Y}_2 as $a((1/a)\bar{Y}_2 + (1/\bar{a})Y_2) - (a/\bar{a})Y_2$ shows that near $b\Omega$, T can be written in the form

$$(2.10) \qquad T = a(T_1 + L),$$

where T_1 is real and complex transversal, L is of type (1,0), and both T_1 and L are tangential to $b\Omega$.

The Hilbert spaces $H_T^k(\Omega)$ that occur in Theorem 1.1 are the usual Sobolev spaces with respect to differentiation in the direction T^k , where the latter is the k -fold differentiation with respect to T : $T^k(f) = T(T(\dots(Tf)\dots))$.

Definition 2.11. For $k \in \mathbb{N}$, set

$$H_T^k(\Omega) = \{ f \in L^2(\Omega) : T^j f \in L^2(\Omega), j \in \{1, \dots, k\} \},$$

where $T^j f$ is taken in the sense of distributions.

For fixed k , $H_T^k(\Omega)$ is a Hilbert space with respect to the inner product

$$(f, g)_{k,T} := \sum_{j=0}^k (T^j f, T^j g) \qquad \forall f, g \in H_T^k(\Omega),$$

and $C^\infty(\bar{\Omega})$ is dense in $H_T^k(\Omega)$ with respect to the norm

$$\|f\|_{k,T}^2 = \sum_{j=0}^k \|T^j f\|^2$$

induced by this inner product. The completeness of $H_T^k(\Omega)$ is proved in the same way as for ordinary Sobolev spaces; density of $C^\infty(\bar{\Omega})$ is a standard application

of Friedrichs' Lemma (see for example [10], Lemma D.1 and Corollary D.2). The spaces $H_T^k(\Omega)$, $k \geq 1$, depend on the choice of the tangential, complex transversal vector field T ; cf. Section 5 in [13]. However, if b is a smooth, non-vanishing, complex-valued function, then $H_T^k(\Omega)$ and $H_{bT}^k(\Omega)$ are equal, and an inductive argument gives, e.g., the somewhat rough estimate

$$(2.12) \quad \|(bT)^k f\| \leq b_k \|f\|_{k,T} \quad \text{for } b_k = \max_{\Omega, 0 \leq \ell \leq k} \left\{ |T^\ell b|^k, 1 \right\}.$$

This implies in particular that Theorem 1.1 holds for T if and only if it holds for bT . Moreover, (2.12) indicates how the constant in (1.3) changes (although there are other quantities that determine the constant in (1.3)).

The Fréchet space $H_T^\infty(\Omega) = \bigcap_{k=0}^\infty H_T^k(\Omega)$ is equipped with the (metrizable) topology induced by the family of norms $\{\|\cdot\|_{k,T} : k \in \mathbb{N}\}$ (see e.g. [19]). It inherits completeness from the $H_T^k(\Omega)$. For a class of examples of the fact that $C^\infty(\overline{\Omega}) \subsetneq H_T^\infty(\Omega)$, see Section 5 in [13].

3. PROOF OF THEOREM 1.1

As indicated in the introduction, the second step in proving Theorem 1.1 relies on a representation of a holomorphic function h in terms of \overline{T}^j -derivatives, $j \leq k$, of functions whose L^2 norms are controlled by $\|h\|_{-k}$ for $k \in \mathbb{N}$.

Proposition 3.1. *Let $\Omega \Subset \mathbb{C}^n$ be a smoothly bounded domain and let T be a vector field which is tangential to $b\Omega$ and complex transversal. Let $k \in \mathbb{N}$.*

There exists an open neighborhood U of $b\Omega$, a function $\zeta \in C_0^\infty(\overline{\Omega} \cap U)$ which equals 1 near $b\Omega$, and constants $C_{j,k} > 0$ for $j \in \{0, \dots, k\}$, such that for all $h \in A^0(\Omega)$ there exist functions $\mathcal{H}_j^k \in H^k(\Omega) \cap C^\infty(\Omega)$, $j \in \{0, \dots, k\}$, satisfying

- (i) $\zeta h = \sum_{j=0}^k \overline{T}^j \mathcal{H}_j^k$ on Ω ,
- (ii) $\|\mathcal{H}_j^k\| \leq C_{j,k} \|h\|_{-k}$ for all $j \in \{0, \dots, k\}$.

Remark 3.2. For simplicity, we have only stated what we need in Proposition 3.1. As usual in such situations, membership in $H^k(\Omega)$ actually comes with a norm estimate that can be worked out with a little additional care. In fact, the arguments in §4.4 can be refined to show that for some constant $C_{s,k} > 0$, $\|\mathcal{H}_j^k(h)\|_s \leq C_{s,k} \|h\|_{s-k}$, for $s \in \mathbb{R}$.

The proof of Proposition 3.1 will be given in Section 4. Assuming it is true, we now give the proof of Theorem 1.1. We will use the notation $A \lesssim B$ to denote the existence of a constant C independent of f , but allowed to depend on Ω and T , such that $A \leq C \cdot B$.

Proof of Theorem 1.1. First assume $f \in C^\infty(\overline{\Omega})$. Then $Bf \in A^{k_2}(\Omega)$, by (1.2). Inequality (2.9) states that

$$(3.3) \quad \|Bf\|_{k_2} \leq c \sup \{ |(f, h)| : h \in A^{k_2}(\Omega), \|h\|_{-k_1} \leq 1 \}$$

holds for some constant $c > 0$ independent of f .

We use Proposition 3.1 for $k = k_1$ to estimate the right hand side of (3.3) as follows. For T as in Theorem 1.1, choose the neighborhood U of $b\Omega$ and the cut-off function ζ described in Proposition 3.1. Using the partition of unity $\{\zeta, 1 - \zeta\}$

yields

$$(3.4) \quad \begin{aligned} |(f, h)| &\leq |(f, \zeta h)| + |(f, (1 - \zeta)h)| \\ &= \left| \left(f, \sum_{j=0}^{k_1} \bar{T}^j \mathcal{H}_j^{k_1} \right) \right| + |(f, (1 - \zeta)h)|. \end{aligned}$$

As $(1 - \zeta)$ is identically zero near $b\Omega$, it follows from the Cauchy–Schwarz inequality, (2.2), and $\|h\|_{-k_1} \leq 1$ that

$$(3.5) \quad |(f, (1 - \zeta)h)| \leq \tilde{\beta}_k \|f\| \cdot \|h\|_{-k_1} \leq \tilde{\beta}_k \|f\|,$$

where the constant $\tilde{\beta}_k > 0$ depends on β_k (see (2.2)) and ζ . Since T is tangential, integration by parts (justified since $\mathcal{H}_j^{k_1} \in H^{k_1}(\Omega)$) gives

$$\left| \left(f, \bar{T}^j \mathcal{H}_j^{k_1} \right) \right| = \left| \left(Tf, \bar{T}^{j-1} \mathcal{H}_j^{k_1} \right) + \left(g_0 f, \bar{T}^{j-1} \mathcal{H}_j^{k_1} \right) \right|,$$

where g_0 , depending on the first order derivatives of the coefficients of T , is a smooth function on $\bar{\Omega}$. Continuing this procedure yields

$$\left| \left(f, \bar{T}^j \mathcal{H}_j^{k_1} \right) \right| \leq \sum_{\ell=0}^j \|g_\ell T^\ell f\| \cdot \|\mathcal{H}_j^{k_1}\|$$

for functions $g_\ell \in C^\infty(\bar{\Omega})$, $\ell \in \{0, \dots, j-1\}$, and $g_j = 1$. By part (ii) of Proposition 3.1 it now follows that

$$\left| \left(f, \bar{T}^j \mathcal{H}_j^{k_1} \right) \right| \leq C_{j,k} \sum_{\ell=0}^j \|g_\ell T^\ell f\|.$$

Summing over $j \leq k_1$ gives, in view of (3.3), (3.4), and (3.5), that

$$(3.6) \quad \|Bf\|_{k_2} \leq C \|f\|_{k_1, T} \quad \forall f \in C^\infty(\bar{\Omega}).$$

To remove the smoothness assumption on f , approximate f with respect to $\|\cdot\|_{k_1, T}$ by functions in $C^\infty(\bar{\Omega})$. Then invoke (3.6) and the continuity of B in $L^2(\Omega)$ to obtain (1.3); for more details see Lemma 4.2 in [13]. □

4. ANTI-DIFFERENTIATION ALONG A TRANSVERSE DIRECTION

4.1. General set-up. In this and the next two sections we work in the real setting of \mathbb{R}^n . Let Ω be a bounded domain with smooth boundary. Denote by $\mathcal{N} = \sum_{j=1}^n \mathcal{N}_j \frac{\partial}{\partial x_j}$ a vector field with smooth coefficients in a neighborhood of Ω . Suppose that \mathcal{N} is transversal to $b\Omega$. Then there exists an open, bounded neighborhood $V \subset \mathbb{R}^n$ of $b\Omega$ on which \mathcal{N} is non-vanishing. Moreover, for each $x \in V$ there exist a scalar $\tau_x > 0$ and a smooth integral curve $\varphi_x : (-\tau_x, \tau_x) \rightarrow \mathbb{R}^n$ satisfying $\varphi_x(0) = x$ and

$$\frac{\partial \varphi_x}{\partial t}(t) = \left\langle \mathcal{N}_1(\varphi_x(t)), \dots, \mathcal{N}_n(\varphi_x(t)) \right\rangle$$

for all $t \in (-\tau_x, \tau_x)$. Because the curve φ_x intersects $b\Omega$ transversally, there exists a scalar $\tau_0 > 0$ such that φ_p is defined on $(-\tau_0, \tau_0)$ for all $p \in b\Omega$. After possibly rescaling \mathcal{N} , it may be assumed that $\tau = 2$.

For each $x \in V$ define t_x to be the (unique) scalar for which $\varphi(t_x, x) \in b\Omega$, and note that $|t_x| = \mathcal{O}(d_{b\Omega}(x))$. It may be assumed that $t_x > 0$ for all $x \in V \cap \Omega$. Set $U := \{x : |t_x| < 1\}$. Then U is an open neighborhood of $b\Omega$. Moreover, the flow

map, $\varphi(t, x) := \varphi_x(t)$, is a smooth map on $(-1, 1) \times U$ satisfying $\varphi(0, x) = x$ as well as

$$(4.1) \quad \frac{\partial \varphi_\ell}{\partial t}(t, x) = \mathcal{N}_\ell(\varphi(t, x)) \quad \forall \ell \in \{1, \dots, n\}.$$

We can now make precise the anti-differentiation operator \mathfrak{A} from Section 1. Denote $C_{\overline{U}}^\infty(\Omega) := \{f \in C^\infty(\Omega) \mid f \equiv 0 \text{ on } \Omega \setminus \overline{U}\}$. Then we define \mathfrak{A} as an operator from $C_{\overline{U}}^\infty(\Omega)$ to itself:

$$\mathfrak{A}[g](x) = \begin{cases} \int_{-1}^0 (g \circ \varphi)(s, x) \, ds & \text{if } x \in \Omega \cap U, \\ 0 & \text{if } x \in \Omega \setminus U. \end{cases}$$

\mathfrak{A} belongs to a class of operators denoted by $\mathcal{A}_{0,0}^1$ below; see Definition 4.4, where the mapping properties are also discussed. \mathfrak{A} inverts \mathcal{N} in the following sense.

Lemma 4.2. *Let $g \in C_{\overline{U}}^\infty(\Omega)$. Then*

$$(4.3) \quad g(x) = \mathfrak{A}[\mathcal{N}g](x), \quad x \in \Omega.$$

Proof. When $x \in \Omega \setminus U$, both sides of (4.3) equal zero. When $x \in \Omega \cap U$, then $g(\varphi(-1, x)) = 0$ because $\varphi(-1, x) \in \Omega \setminus U$. Note that (4.1) implies

$$((\mathcal{N}g) \circ \varphi)(s, x) = \frac{\partial}{\partial s} ((g \circ \varphi)(s, x)),$$

hence an application of the Fundamental Theorem of Calculus completes the proof of (4.3). □

4.2. The spaces $\mathcal{A}_{*,*}^*$ and their L^2 mapping properties. In order to organize the proof of Proposition 3.1, we now define several spaces of operators related to, but more general than, \mathfrak{A} . We group the operators both according to their form as well as according to their mapping properties.

Definition 4.4. (1) Denote by $\mathcal{A}_{\mu,0}^1$, $\mu \in \mathbb{N}_0$ the space of operators acting on $C_{\overline{U}}^\infty$ which are of the form

$$(4.5) \quad A[g](x) = \begin{cases} \int_{-1}^0 s^\mu \gamma_A(s, x) \cdot (g \circ \varphi)(s, x) \, ds & \text{if } x \in \Omega \cap U, \\ 0 & \text{if } x \in \Omega \setminus U \end{cases}$$

for some $\gamma_A \in C^\infty([-1, 0] \times \overline{U})$. Then $A[g] \in C_{\overline{U}}^\infty(\Omega)$.

(2) Denote by $\mathcal{A}_{\mu,\nu}^1$, $\mu \in \mathbb{N}_0$, $\nu \in \mathbb{N}$, the space of operators on $C_{\overline{U}}^\infty(\Omega)$ spanned by operators of the form $A_\mu \circ D^\beta$ for $A_\mu \in \mathcal{A}_{\mu,0}^1$ and $|\beta| \leq \nu$.

These operators have mapping properties in weighted L^2 spaces that will be very useful for our purposes. We introduce the following definition:

Definition 4.6. An operator A on $C_{\overline{U}}^\infty(\Omega)$ is said to belong to \mathcal{S}_ν^k if there exists a constant $C > 0$ such that

$$(4.7) \quad \|t_x^\ell \cdot A[g]\| \leq C \sum_{|\beta| \leq \nu} \|t_x^{\ell+k} \cdot D^\beta g\| \quad \forall \ell \in \mathbb{N}_0, \quad g \in C_{\overline{U}}^\infty(\Omega).$$

Here, C does not depend on g or ℓ . Note that because t_x is defined on $\Omega \cap \overline{U}$ and both g and $A[g]$ vanish on $\Omega \setminus \overline{U}$, both sides of (4.7) are well-defined.

The following lemma is the key to the mapping properties of the anti-derivative operator \mathfrak{A} and its generalizations; in particular, it makes precise the notion of ‘gaining’ a factor of the boundary distance. As mentioned in Section 1, it is a consequence of one of a group of inequalities due to Hardy ([12], sections 9.8, 9.9).

Lemma 4.8. *For given $\mu \in \mathbb{N}_0$, define the operator B_μ on $C^\infty_{\overline{U}}(\Omega)$ by*

$$(4.9) \quad B_\mu[g](x) = \begin{cases} \int_{-1}^0 (t\varphi(s,x))^\mu \cdot |g(\varphi(s,x))| \, ds & \text{if } x \in \Omega \cap U, \\ 0 & \text{if } x \in \Omega \setminus U. \end{cases}$$

Then $B_\mu \in \mathcal{S}_0^{\mu+1}$.

Proof. It is convenient in this proof (but not later on) to rewrite B_μ using coordinates $(\tau, p) \in [0, 2] \times b\Omega$ on $\overline{\Omega} \cap \overline{V}$ given by $(\tau, p) \rightarrow \varphi(-\tau, p)$. Expressing B_μ in these coordinates and changing variables gives

$$(4.10) \quad B_\mu[g](\tau, p) = \int_\tau^{\tau+1} \sigma^\mu |g(\sigma, p)| \, d\sigma, \quad (\tau, p) \sim x = \varphi(-\tau, p) \in \Omega \cap U.$$

Estimate (4.7) now follows from the inequality of Hardy (see [12], Theorem 330, section 9.9) for $f \geq 0$:

$$(4.11) \quad \int_0^\infty \left(x^r \int_x^\infty f(t) \, dt \right)^2 dx \leq \frac{4}{(2r+1)^2} \int_0^\infty (t^{r+1} f(t))^2 \, dt, \quad r > -\frac{1}{2}.$$

Indeed, to verify (4.7) for B_μ , it suffices to observe that for $x \sim (\tau, p) \in \Omega \cap U$,

$$(4.12) \quad |\tau^\ell B[g](\tau, p)| \leq \tau^\ell \int_\tau^2 \sigma^\mu |g(\sigma, p)| \, d\sigma,$$

then to apply (4.11) with $f(t) = t^\mu |g(t, p)|$, $0 \leq t \leq 2$, and $f(t) = 0$, $t > 2$, $r = \ell$, for $p \in b\Omega$ fixed, and lastly, to integrate over $b\Omega$. We also use the equivalence, with a constant that depends only on Ω and \mathcal{N} , of the volume elements dV and $d\tau \times d_{b\Omega}$ on $\Omega \cap \overline{V}$, where dV and $d_{b\Omega}$ denote the Euclidean volume elements on \mathbb{R}^n and $b\Omega$, respectively. Also note that for $r = \ell \geq 0$, the constant on the right hand side of (4.11) is less than or equal to 4. Replacing it by 4 yields a constant in (4.7) that does not depend on ℓ . \square

If A is of the form $A = A_\mu \circ D^\beta$, then the inequality $|s| \leq |s| + t_x = t_{\varphi(s,x)}$ and the boundedness of $|\gamma_A|$ on $[-1, 0] \times \overline{U}$ imply

$$(4.13) \quad \|t_x^\ell A[g]\| = \|t_x^\ell A_\mu[D^\beta g]\| \lesssim \|t_x^\ell B_\mu[D^\beta g]\| \lesssim \|t_x^{\ell+\mu+1} D^\beta g\|,$$

where the last inequality in (4.13) follows from Lemma 4.8. This proves:

Lemma 4.14. $\mathcal{A}_{\mu,\nu}^1 \subset \mathcal{S}_\nu^{\mu+1}$, $\mu, \nu \in \mathbb{N}_0$.

We will also need notation for compositions of operators:

Definition 4.15. (1) For a multi-index $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{N}_0^\ell$, $\ell \in \mathbb{N}$, define

$$\mathcal{A}_{\alpha,0}^\ell = \left\langle A_1 \circ \dots \circ A_\ell : A_j \in \mathcal{A}_{\alpha_j,0}^1 \right\rangle.$$

(2) Denote by $\mathcal{A}_{\alpha,\nu}^\ell$, $\alpha \in \mathbb{N}_0^\ell$, $\nu \in \mathbb{N}$ and $\ell \in \mathbb{N}$, the space of operators on $C^\infty_{\overline{U}}(\Omega)$ spanned by operators of the form $A_\alpha^\ell \circ D^\beta$ for $A_\alpha^\ell \in \mathcal{A}_{\alpha,0}^\ell$ and $|\beta| \leq \nu$.

With Lemma 4.14 above, we have the following weighted mapping properties of these operators:

Lemma 4.16. $\mathcal{A}_{\alpha,\nu}^\ell \subset \mathcal{S}_\nu^{\ell+|\alpha|}$, where $|\alpha| = \sum_{j=1}^\ell \alpha_j$.

4.3. On the algebra of commutators of $\mathcal{A}_{*,*}^*$ and differential operators.

Throughout this subsection, $X = \sum_{j=1}^n X_j \frac{\partial}{\partial x_j}$ denotes a vector field with smooth coefficients on (a neighborhood of) $\bar{\Omega}$. We collect a series of elementary lemmas that give control over various commutators involving the operators introduced in the previous subsection and X or D^β . The notation $+$, or \sum , will be used to indicate sums of operators in the indicated spaces.

Lemma 4.17. Let $A \in \mathcal{A}_{\mu,0}^1$ for some $\mu \in \mathbb{N}_0$. Then

$$(4.18) \quad [A, X] \in \mathcal{A}_{\mu,0}^1 + \mathcal{A}_{\mu+1,1}^1.$$

Proof. Let $g \in C_{\bar{\Omega}}^\infty(\Omega)$. Then

$$(4.19) \quad (X \circ A)[g](x) = \int_{-1}^0 s^\mu \left((X_x(\gamma_A)) \cdot (g \circ \varphi) \right)(s, x) ds \\ + \int_{-1}^0 s^\mu \gamma_A(s, x) \cdot X_x(g \circ \varphi)(s, x) ds.$$

Let A_0 be the operator in $\mathcal{A}_{\mu,0}^1$ such that the first term on the right hand side of (4.19) equals $-A_0[g](x)$. Then it follows that

$$(4.20) \quad [A, X][g](x) = A_0[g](x) + \int_{-1}^0 s^\mu \gamma_A(s, x) \left((Xg) \circ \varphi - X_x(g \circ \varphi) \right)(s, x) ds.$$

The fact that the term on the right hand side of (4.20) given by the integral belongs to $\mathcal{A}_{\mu+1,1}^1$ can be seen as follows: because $\varphi(0, x) \equiv x$, $\partial\varphi_j/\partial x_k = \delta_{j,k} + O(s)$. Therefore, the chain rule shows that $\left((Xg) \circ \varphi - X_x(g \circ \varphi) \right)(s, x)$ is a sum of terms each of which is of the form $s\gamma_\beta(s, x)D^\beta g(\varphi(s, x))$, where γ_β is smooth (depending on first derivatives of $X \circ \varphi$ and second derivatives of φ) and $|\beta| = 1$. \square

Lemma 4.21. Let $A_{\mu,\nu} \in \mathcal{A}_{\mu,\nu}^1$ for some $\mu \in \mathbb{N}_0$, $\nu \in \mathbb{N}$. Then

$$(4.22) \quad [A_{\mu,\nu}, X] \in \mathcal{A}_{\mu,\nu}^1 + \mathcal{A}_{\mu+1,\nu+1}^1.$$

Proof. By definition of $\mathcal{A}_{\mu,\nu}^1$, $A_{\mu,\nu}$ can be written as a linear combination of operators of the form $A_\mu \circ D^\beta$, where $A_\mu \in \mathcal{A}_{\mu,0}^1$ and $|\beta| \leq \nu$. It suffices to show (4.22) for the latter operators. For that, note that

$$\begin{aligned} [A_\mu \circ D^\beta, X] &= A_\mu \circ D^\beta \circ X - X \circ A_\mu \circ D^\beta \\ &= A_\mu \circ [D^\beta, X] + A_\mu \circ X \circ D^\beta - X \circ A_\mu \circ D^\beta \\ &= A_\mu \circ [D^\beta, X] + [A_\mu, X] \circ D^\beta. \end{aligned}$$

Since $[D^\beta, X]$ is a differential operator of order $|\beta| - 1$, $A_\mu \circ [D^\beta, X]$ belongs to $\mathcal{A}_{\mu,|\beta|-1}^1$. Furthermore, (4.18) yields that $[A_\mu, X] \circ D^\beta \in \mathcal{A}_{\mu,|\beta|}^1 + \mathcal{A}_{\mu+1,|\beta|+1}^1$. \square

Lemma 4.17 extends to iterated commutators as follows. Set

$$C_X^\nu(A) = [C_X^{\nu-1}(A), X]$$

with $C_X^1(A) = [A, X]$. That is, $C_X^\nu(A)$ equals the ν -fold iterated commutator $[\dots[A, X], X], \dots, X]$.

Lemma 4.23. *Let A be given as in Lemma 4.17. Then $C_X^\nu(A) \in \sum_{j=0}^\nu \mathcal{A}_{\mu+j,j}^1$.*

Proof. The proof is done via induction on ν . Note first that the case $\nu = 1$ is Lemma 4.17. Next suppose that $C_X^{\nu-1}(A) \in \sum_{j=0}^{\nu-1} \mathcal{A}_{\mu+j,j}^1$ holds for some $\nu \in \mathbb{N}$. It follows from Lemma 4.21 that $[A_{\mu+j,j}, X] \in \mathcal{A}_{\mu+j,j}^1 + \mathcal{A}_{\mu+j+1,j+1}^1$ for $A_{\mu+j,j} \in \mathcal{A}_{\mu+j,j}^1$, which completes the proof. \square

We next consider commutators with higher order derivatives.

Lemma 4.24. *If $A \in \mathcal{A}_{\mu,0}^1$ for some $\mu \in \mathbb{N}_0$, then $[A, D^\beta] \in \mathcal{A}_{\mu,|\beta|-1}^1 + \mathcal{A}_{\mu+1,|\beta|}^1$.*

Proof. The proof is done via induction on $|\beta|$. The case $|\beta| = 1$ follows from Lemma 4.17. Suppose now that $[A, D^\sigma] \in \mathcal{A}_{\kappa,\nu-2}^1 \cup \mathcal{A}_{\kappa+1,\nu-1}^1$ holds for any multi-index σ of length $\nu - 1$ for some $\nu \in \mathbb{N}$ and for all $\kappa \in \mathbb{N}_0$. Let β be a multi-index of length ν . Then D^β may be written as $D^{\beta-\sigma} \circ D^\sigma$ for some multi-index σ of length $\nu - 1$. It then follows that

$$\begin{aligned} [A, D^\beta] &= [A, D^{\beta-\sigma}] \circ D^\sigma + D^{\beta-\sigma} \circ A \circ D^\sigma - D^{\beta-\sigma} \circ D^\sigma \circ A \\ &= [A, D^{\beta-\sigma}] \circ D^\sigma + D^{\beta-\sigma} \circ [A, D^\sigma]. \end{aligned}$$

It follows from Lemma 4.17 and the fact that $|\beta - \sigma| = 1$ that $[A, D^{\beta-\sigma}] \circ D^\sigma \in \mathcal{A}_{\mu,|\beta|-1}^1 + \mathcal{A}_{\mu+1,|\beta|}^1$. The induction hypothesis furnishes two operators $A_1 \in \mathcal{A}_{\mu,\nu-2}^1$ and $A_2 \in \mathcal{A}_{\mu+1,\nu-1}^1$ so that $[A, D^\sigma] = A_1 + A_2$. Then

$$\begin{aligned} D^{\beta-\sigma} \circ [A, D^\sigma] &= D^{\beta-\sigma} \circ (A_1 + A_2) \\ &= A_1 \circ D^{\beta-\sigma} + A_2 \circ D^{\beta-\sigma} + [D^{\beta-\sigma}, A_1] + [D^{\beta-\sigma}, A_2]. \end{aligned}$$

The four terms on the right hand side are in, respectively, $\mathcal{A}_{\mu,\nu-1}^1$, $\mathcal{A}_{\mu+1,\nu}^1$, $\mathcal{A}_{\mu,\nu-2}^1 + \mathcal{A}_{\mu+1,\nu-1}^1$, and $\mathcal{A}_{\mu+1,\nu-1}^1 + \mathcal{A}_{\mu+2,\nu}^1$. We have used Lemma 4.21 for the last two terms ($|\beta - \sigma| = 1!$). Taking into account the (trivial) inclusions $\mathcal{A}_{\mu,\nu-2}^1 \subset \mathcal{A}_{\mu,\nu-1}^1$, $\mathcal{A}_{\mu+1,\nu-1}^1 \subset \mathcal{A}_{\mu,\nu-1}^1$, and $\mathcal{A}_{\mu+2,\nu}^1 \subset \mathcal{A}_{\mu+1,\nu}^1$, shows that $D^{\beta-\sigma} \circ [A, D^\sigma] \in \mathcal{A}_{\mu,\nu-1}^1 + \mathcal{A}_{\mu+1,\nu}^1$. This completes the induction. \square

For compositions of operators, Lemma 4.24 takes on the following form.

Lemma 4.25. *If $A_{\alpha,\nu} \in \mathcal{A}_{\alpha,\nu}^\ell$ for some multi-index $\alpha \in \mathbb{N}_0^\ell$, $\nu, \ell \in \mathbb{N}$, then*

$$(4.26) \quad [A_{\alpha,\nu}, D^\beta] \in \mathcal{A}_{\alpha,\nu+|\beta|-1}^\ell + \sum_{j=1}^\ell \mathcal{A}_{\alpha+e_j,\nu+|\beta|}^\ell,$$

where e_j is the standard j -th unit vector.

Proof. We first consider the case $\nu = 0$; in this case, the proof is by induction on ℓ . The case $\ell = 1$ is Lemma 4.24. The induction step is analogous to that in the proof of Lemma 4.24, with the multi-index α now playing the role of β there. We leave the details to the reader.

When $\nu > 0$, $A_{\alpha,\nu}$ is a linear combination of operators of the form $A_{\alpha,0} \circ D^\gamma$ with $|\gamma| \leq \nu$. Thus $[A_{\alpha,\nu}, D^\beta]$ is a linear combination of terms of the form

$$[A_{\alpha,0} \circ D^\gamma, D^\beta] = A_{\alpha,0} \circ D^\gamma \circ D^\beta - D^\beta \circ A_{\alpha,0} \circ D^\gamma = [A_{\alpha,0}, D^\beta] \circ D^\gamma$$

because $D^\gamma \circ D^\beta = D^\beta \circ D^\gamma$. (4.26) now follows from the case $\nu = 0$ (already shown) applied to $[A_{\alpha,0}, D^\beta]$. \square

4.4. Proof of Proposition 3.1. Let Ω and T be as in Proposition 3.1. Near $b\Omega$, $T = a(T_1 + L)$, with both T_1 and L tangential, T_1 real, L of type $(1, 0)$, and a a smooth function that does not vanish near $b\Omega$ (see (2.10)). It is easy to see that the conclusion of Proposition 3.1 holds for T if and only if it holds for $T_1 + L$ (however, the \mathcal{H}_j^k 's in (i) change, and so do the constants $C_{j,k}$ in (ii); see also the short discussion surrounding (2.12)). Therefore, it may be assumed that $T = T_1 + L$, with T_1 and L as above. Set $\mathcal{N} := JT_1$. Because T_1 is complex transversal, \mathcal{N} is transversal to $b\Omega$, and so the general set-up of §4.1 applies.

The proof of Proposition 3.1 will be achieved in two steps. The first step consists in replacing \mathcal{N} in Lemma 4.2 by $i\bar{T}$, adding the necessary correction, and then iterating the result. The key for the proof of Proposition 3.1 is that when applied to a holomorphic function, the correction term is benign, as a result of the Cauchy–Riemann equations; see (4.39) below.

Lemma 4.27. *Let $h \in C^\infty(\Omega)$, $k \in \mathbb{N}$. Let $\zeta \in C_0^\infty(\bar{\Omega} \cap U)$ be a non-negative function which is identically 1 in a neighborhood of $b\Omega$ contained in U . Then*

$$(4.28) \quad (\zeta h)(x) = i^k (\mathfrak{A} \circ \bar{T})^k [\zeta h] + \sum_{j=0}^{k-1} i^j (\mathfrak{A} \circ \bar{T})^j \circ \mathfrak{A} \circ (\mathcal{N} - i\bar{T})[\zeta h].$$

Proof. The proof is done by induction on k . Suppose first that $k = 1$. Then Lemma 4.2 yields

$$(4.29) \quad \begin{aligned} \zeta h &= \mathfrak{A}[\mathcal{N}(\zeta h)] = \mathfrak{A}[i\bar{T}(\zeta h)] + \mathfrak{A}[(\mathcal{N} - i\bar{T})(\zeta h)] \\ &= i (\mathfrak{A} \circ \bar{T}) [\zeta h] + \mathfrak{A} \circ (\mathcal{N} - i\bar{T})[\zeta h]. \end{aligned}$$

For the induction step suppose that

$$(4.30) \quad \zeta h = i^{k-1} (\mathfrak{A} \circ \bar{T})^{k-1} [\zeta h] + \sum_{j=0}^{k-2} i^j (\mathfrak{A} \circ \bar{T})^j \circ \mathfrak{A} \circ (\mathcal{N} - i\bar{T})[\zeta h]$$

holds. Using identity (4.29) to replace ζh in the first term of the right hand side of (4.30) gives the result. \square

In the second step, we will write the powers $(\mathfrak{A} \circ \bar{T})^\ell$ in terms of compositions of the form $(\bar{T})^m \circ \tilde{\mathfrak{A}}$ with good control of $\tilde{\mathfrak{A}}$. This is accomplished in the following lemma; its proof relies heavily on the machinery of §4.2 and §4.3.

Lemma 4.31. *Let $A \in \mathcal{A}_{0,0}^1$ and X be a vector field with smooth coefficients on $\bar{\Omega}$. Then, for any $\ell \in \mathbb{N}$, there exist operators G_m^ℓ , $m \in \{0, \dots, \ell\}$, which belong to $\sum_{\{\nu \leq \ell - m, |\alpha| - \nu \geq 0\}} \mathcal{A}_{\alpha,\nu}^\ell$ such that*

$$(A \circ X)^\ell = \sum_{m=0}^{\ell} X^m \circ G_m^\ell.$$

Proof. The proof is again by induction on ℓ . For $\ell = 1$, commuting A by X yields

$$A \circ X = X \circ A + [A, X] =: X \circ G_1^1 + G_0^1.$$

It follows from the definition of A that $G_1^1 = A \in \mathcal{A}_{0,0}^1$. Furthermore, Lemma 4.17 gives that G_0^1 belongs to $\mathcal{A}_{0,0}^1 + \mathcal{A}_{1,1}^1$.

For the induction step suppose that

$$(A \circ X)^{\ell-1} = \sum_{m=0}^{\ell-1} X^m \circ G_m^{\ell-1}$$

holds for some $G_m^{\ell-1} \in \sum_{\{\nu \leq \ell-1-m, |\alpha|-\nu \geq 0\}} \mathcal{A}_{\alpha, \nu}^{\ell-1}$. Then

$$\begin{aligned} (A \circ X)^\ell &= \sum_{m=0}^{\ell-1} A \circ X^{m+1} \circ G_m^{\ell-1} = \sum_{m=1}^{\ell} A \circ X^m \circ G_{m-1}^{\ell-1} \\ (4.32) \quad &= \sum_{m=1}^{\ell} (X^m \circ A \circ G_{m-1}^{\ell-1} + [A, X^m] \circ G_{m-1}^{\ell-1}). \end{aligned}$$

Since $A \in \mathcal{A}_{0,0}^1$, it follows that

$$A \circ G_{m-1}^{\ell-1} \in \sum_{\{\nu \leq \ell-m, |\alpha|-\nu \geq 0\}} \mathcal{A}_{\alpha, \nu}^\ell.$$

To deal with the second term on the right hand side of (4.32), we use the formula

$$(4.33) \quad [A, X^m] = \sum_{j=0}^{m-1} \binom{m}{j} X^j \circ C_X^{m-j}(A),$$

where the iterated commutators C_*^* are defined just before Lemma 4.23. ((4.33) is purely algebraic and is easily proved by induction on m ; alternatively, see [11], Lemma 2 or [21], formula (3.54).) What must be shown then is that

$$C_X^{m-j}(A) \circ G_{m-1}^{\ell-1} \in \sum_{\{\nu \leq \ell-j, |\alpha|-\nu \geq 0\}} \mathcal{A}_{\alpha, \nu}^\ell.$$

For that, recall first that Lemma 4.23 says that $C_X^{m-j}(A) \in \sum_{k=0}^{m-j} \mathcal{A}_{k,k}^1$. Furthermore, it follows from Lemma 4.25 that for $|\beta| \leq k$,

$$D^\beta \circ G_{m-1}^{\ell-1} = G_{m-1}^{\ell-1} \circ D^\beta + [D^\beta, G_{m-1}^{\ell-1}] \in \sum_{\{\nu \leq \ell-m, |\alpha|-\nu \geq 0\}} \mathcal{A}_{\alpha, \nu+k}^{\ell-1}.$$

It then follows that $C_X^{m-j}(A) \circ G_{m-1}^{\ell-1}$ is contained in

$$\sum_{\{\nu \leq \ell-m, |\alpha|-\nu \geq 0, 0 \leq k \leq m-j\}} \mathcal{A}_{(k, \alpha), \nu+k}^\ell.$$

Set $\tilde{\nu} = \nu + k$; then $\nu \leq \ell - m$ implies that $\tilde{\nu} \leq \ell - j$, since $k \leq m - j$. Moreover, setting $\tilde{\alpha} = (k, \alpha)$ yields $|\tilde{\alpha}| - \tilde{\nu} = |\alpha| - \nu \geq 0$. Hence

$$C_X^{m-j}(A) \circ G_{m-1}^{\ell-1} \in \sum_{\{\tilde{\nu} \leq \ell-j, |\tilde{\alpha}|-\tilde{\nu} \geq 0\}} \mathcal{A}_{\tilde{\alpha}, \tilde{\nu}}^\ell,$$

which completes the proof. □

We are now ready to prove Proposition 3.1. This amounts to combining Lemmas 4.27 and 4.31 to obtain representation (i) with $\mathcal{H}_j^k \in C^\infty(\Omega)$; this works for $h \in C^\infty(\Omega)$. The final step then consists in obtaining the required estimates of (ii) and membership in $H^k(\Omega)$ when h is holomorphic.

Proof of Proposition 3.1. For h and ζ given as in Lemma 4.27, use Lemmas 4.27 and 4.31 (with \mathfrak{A} and \bar{T} in place of A and X , respectively) in (4.28) to obtain that

$$\zeta h = \sum_{m=0}^k \bar{T}^m \mathcal{H}_m^k,$$

where

$$(4.34) \quad \mathcal{H}_k^k := G_k^k [i^k \zeta h]$$

and

$$(4.35) \quad \mathcal{H}_m^k := \left(i^k G_m^k + \sum_{j=m}^{k-1} i^j G_m^j \circ \mathfrak{A} \circ (\mathcal{N} - i\bar{T}) \right) [\zeta h], \quad 0 \leq m \leq k-1.$$

Because $h \in C^\infty(\Omega)$, so are the \mathcal{H}_m^k , $0 \leq m \leq k$. This establishes representation (i) in Proposition 3.1, except for membership in $H^k(\Omega)$.

We now prove the estimates of (ii). By Lemmas 4.31 and 4.16, it follows that

$$\|\mathcal{H}_k^k\| \lesssim \|t_x^k \cdot \zeta h\| \lesssim \|h\|_{-k},$$

where the second step follows from (2.2); here we use that h in Proposition 3.1 is holomorphic. By analogous reasoning, we have

$$(4.36) \quad \|\mathcal{H}_m^k[\zeta h]\| \lesssim \sum_{|\beta| \leq |\alpha|} \|t_x^{k+|\alpha|} D^\beta h\| \lesssim \|D^\beta h\|_{-k-|\alpha|} \lesssim \|h\|_{-k},$$

where α is such that $G_m^k \in \mathcal{A}_{\alpha,\nu}^k$ for some $\nu \leq |\alpha|$. Note that $D^\beta h$ is also holomorphic so that (2.2) applies. The last inequality holds because $|\beta| \leq |\alpha|$.

To obtain the claimed estimates for the remaining terms in \mathcal{H}_m^k , first note that because $G_m^j \in \sum_{\nu \leq j-m, |\alpha|-\nu \geq 0} \mathcal{A}_{\alpha,\nu}^j$, $G_m^j \circ \mathfrak{A}$ is a sum of terms of the form

$$A_{\alpha_1}^1 \circ A_{\alpha_2}^2 \circ \dots \circ A_{\alpha_j}^1 \circ D^\gamma \circ \mathfrak{A} = A_{\alpha_1}^1 \circ \dots \circ A_{\alpha_j}^1 \circ \mathfrak{A} \circ D^\gamma + A_{\alpha_1}^1 \circ \dots \circ A_{\alpha_j}^1 \circ [D^\gamma, \mathfrak{A}],$$

where $|\gamma| \leq |\alpha| \leq \nu \leq j-m \leq j \leq k-1$. Because $[D^\gamma, \mathfrak{A}] \in \mathcal{A}_{0,|\gamma|-1}^1 + \mathcal{A}_{1,|\gamma|}^1$ (Lemma 4.24), it follows, in view of Lemma 4.14, that

$$(4.37) \quad \|G_m^j \circ \mathfrak{A} (\mathcal{N} - i\bar{T}) [\zeta h]\| \lesssim \|(\mathcal{N} - i\bar{T}) [\zeta h]\|_{k-1}.$$

The Cauchy–Riemann equations for h yield

$$(4.38) \quad \mathcal{N}h = iT_1 h = i\bar{T}_1 h = i\bar{T}h$$

since $T_1 = \bar{T}_1$ and $\bar{L}h = 0$. Consequently,

$$(4.39) \quad (\mathcal{N} - i\bar{T})[\zeta h] = ((\mathcal{N} - i\bar{T})[\zeta]) h.$$

Since $((\mathcal{N} - i\bar{T})[\zeta])$ has compact support in $\Omega \cap U$ that does not depend on h , it follows (for example from (2.2) by using that factors of r are bounded away from zero on this support, so that introducing them will at most ‘increase’ the norm) that there exists a constant C_k such that

$$(4.40) \quad \|(\mathcal{N} - i\bar{T})[\zeta h]\|_{k-1} = \|((\mathcal{N} - i\bar{T})[\zeta]) h\|_{k-1} \leq C_k \|h\|_{-k}.$$

Combining (4.36), (4.37), and (4.40) gives that

$$\|\mathcal{H}_m^k\| \lesssim \|h\|_{-k}$$

for all $m \in \{0, \dots, k-1\}$. This concludes the proof of (ii).

It remains to see that $\mathcal{H}_m^k \in H^k(\Omega)$. First note that because $(\mathcal{N} - i\bar{T})[\zeta h] \in C_0^\infty(\Omega)$, it follows easily from the form of the operators G_m^j that $G_m^j \circ \mathfrak{A} \circ (\mathcal{N} - i\bar{T})[\zeta h] \in C^\infty(\bar{\Omega}) \subset H^k(\Omega)$, $j \leq m \leq (k-1)$. The remaining contributions in (4.34) and (4.35) that need to be checked are of the form $G_m^k[\zeta h]$, $0 \leq m \leq k$. That is, we need to show that $D^\beta G_m^k[\zeta h] \in L^2(\Omega)$ for $|\beta| \leq k$. But $D^\beta G_m^k = G_m^k D^\beta - [G_m^k, D^\beta]$. The argument is now analogous to the discussion above. For example, in view of Lemmas 4.31, 4.25, and 4.16, the commutator $[G_m^k, D^\beta]$ is a sum of terms in $\mathcal{S}_{\nu+|\beta|-1}^{k+|\alpha|} + \mathcal{S}_{\nu+|\beta|}^{k+|\alpha|+1} \subseteq \mathcal{S}_{\nu+|\beta|}^{k+|\alpha|}$, with $\nu \leq k - m$ and $|\alpha| \geq \nu$. Therefore, arguing as in (4.36), we see that the contribution of each of these terms to $\|[G_m^k, D^\beta](\zeta h)\|$ is dominated by $\sum_{|\gamma| \leq \nu+|\beta|} \|D^\gamma h\|_{-k-|\alpha|}$. Because $|\gamma| \leq \nu + |\beta| \leq |\alpha| + |\beta| \leq |\alpha| + k$, all these terms are indeed dominated by $\|h\|$.

The argument for $G_m^k D^\beta$ is similar. This concludes the proof of Proposition 3.1. □

5. PROOF OF THEOREM 1.10

Proof of Theorem 1.10. As in the proof of Theorem 1.1, we invoke duality via Proposition 2.3 and Remark 2.8: it suffices to show that

$$(5.1) \quad \left| \int_{\Omega} fh\bar{g} \right| \leq C_h \|f\| \cdot \|g\|_{-k_1}$$

for $f \in \overline{A^0}$, $h \in C^\infty(\bar{\Omega})$, and $g \in A^0(\Omega)$. We now use that for holomorphic functions, membership in $A^{-m}(\Omega)$ for some m is equivalent to having a blow up rate near the boundary of at most a power of $1/d_{b\Omega}(z)$, where $d_{b\Omega}(z)$ is the boundary distance function. For a holomorphic function q , we have the estimates

$$(5.2) \quad C_m^1 \|q\|_{-m-2n-2} \leq \sup_{z \in \Omega} |q(z)| \cdot d_{b\Omega}(z)^{m+2n} \leq C_m^2 \|q\|_{-m}.$$

In both estimates, if the right hand side is finite, then so is the left hand side (and the estimate holds; that is, the estimates are genuine estimates as opposed to *a priori* estimates). The inequalities (5.2) are essentially Lemma 2 in [5], except that the norm in the leftmost term there is the $(-m - 4n)$ -norm. The stronger version given here is in [20], Theorems 1.1 and 1.3; see in particular the proof of the implication (iii) \Rightarrow (iv) in Theorem 1.1.

Applying the second inequality in (5.2) to \bar{f} (for $m = 0$) and to g (for $m = k_1$) yields

$$\sup_{z \in \Omega} |f(z)| \cdot d_{b\Omega}(z)^{2n} \leq C_0^2 \|f\| \quad \text{and} \quad \sup_{z \in \Omega} |g(z)| \cdot d_{b\Omega}(z)^{k_1+2n} \leq C_{k_1}^2 \|g\|_{-k_1}.$$

Multiply these inequalities, then apply the first inequality in (5.2) to $\bar{f}g$ (which is holomorphic). The conclusion is that $\bar{f}g$, hence $f\bar{g}$, belongs to $H^{-k_1-4n-2}(\Omega)$, and

$$(5.3) \quad \|\bar{f}g\|_{-k_1-4n-2} \leq C_{k_1} \|f\| \cdot \|g\|_{-k_1}.$$

On the other hand, the following estimate also holds:

$$(5.4) \quad \left| \int_{\Omega} fh\bar{g} \right| = \left| \int_{\Omega} \bar{f}g\bar{h} \right| \leq \tilde{C}_{k_1} \|\bar{f}g\|_{-k_1-4n-2} \|\bar{h}\|_{k_1+4n+2}.$$

The inequality in (5.4) is Proposition 1.9 in [20], which gives this estimate for the pairing of a harmonic function in some $H^{-m}(\Omega)$ with an arbitrary function in $C^\infty(\bar{\Omega})$. Note that since both f and g are in $L^2(\Omega)$, $\bar{f}g\bar{h}$ is integrable, and the

integral denoted by \tilde{f} in [20] coincides with the ordinary integral over Ω . Combining (5.3) and (5.4) completes the proof of Theorem 1.10. \square

Remark 5.5. When Condition R holds, one can extend B by duality to a projection \tilde{B} from the dual $(C^\infty(\bar{\Omega}))^*$ into $\bigcup_{k=1}^\infty A^{-k}(\Omega)$. This was observed in [14], in a note added in proof (where the idea is attributed to Nirenberg and Trèves). The conclusion of Theorem 1.10 remains true for this extended projection: when f is in $\overline{\bigcup_{k=1}^\infty A^{-k}(\Omega)}$ and h is in $C^\infty(\bar{\Omega})$, then $\tilde{B}(fh)$ is smooth up to the boundary (see [20], Theorem 3.4, for the ‘canonical’ inclusion $\overline{\bigcup_{k=1}^\infty A^{-k}(\Omega)} \hookrightarrow (C^\infty(\bar{\Omega}))^*$). A similar discussion applies when (1.2) holds.

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