TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 3, March 2014, Pages 1195–1211
S 0002-9947(2013)05750-9
Article electronically published on August 19, 2013

SL(n)-CONTRAVARIANT L_p-MINKOWSKI VALUATIONS

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ABSTRACT. All SL(n)-contravariant L_p-Minkowski valuations on polytopes are completely classified. The prototypes of such valuations turn out to be the asymmetric L_p-projection body operators.

1. INTRODUCTION

A map Φ defined on the set of convex bodies, i.e. nonempty compact convex subsets of $\mathbb{R}^n$, is called a valuation if it satisfies

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L)$$

for all convex bodies K, L such that $K \cup L$ is also convex. Ever since Hadwiger's famous characterization theorem (see e.g. [39]), which describes all rigid motion invariant scalar valuations, classifications of valuations have become the focus of intense research (see e.g. [2–7, 19–21, 29, 30, 37]). In the last few years a theory of convex body valued valuations emerged (see e.g. [1, 12–15, 23–28, 38, 40, 42, 43, 48]). One important type of such valuations are Minkowski valuations, i.e. valuations with respect to Minkowski addition, which is given by $K + L = \{x + y : x \in K, y \in L\}$. A prominent example of such a valuation is the projection body operator, which was introduced by Minkowski at the turn of the previous century (see e.g. [39]). The projection body of a convex body K encodes the information obtained from the $(n-1)$-dimensional volumes of projections of K onto $(n-1)$-dimensional subspaces into a single convex body. Projection bodies and their generalizations found applications in many areas such as convexity, stochastic geometry, functional analysis and geometric tomography (see e.g. [1, 9, 10, 17, 18, 31, 33–36, 41, 43, 46, 50]).

In two far-reaching articles Ludwig [23, 26] characterized the projection body operator up to a constant as the only continuous and homogeneous Minkowski valuation which is SL(n)-contravariant. A Minkowski valuation Φ is called SL(n)-contravariant if it satisfies

$$\Phi(\phi K) = \phi^{-t}\, \Phi\, K$$

for all convex bodies K and all ϕ in the special linear group SL(n). Moreover, Ludwig proved similar characterizations for Minkowski valuations defined on polytopes without continuity assumptions. Very recently, Haberl [15] showed that the homogeneity assumptions in Ludwig's characterization results are not necessary.

Another important class of convex body valued valuations are L_p-Minkowski valuations for $p > 1$. They are valuations with respect to L_p-Minkowski addition,

Received by the editors August 29, 2011 and, in revised form, November 7, 2011.
2010 *Mathematics Subject Classification.* Primary 52A20, 52B45.
The author was supported by Austrian Science Fund (FWF): P22388 and Austrian Science Fund (FWF): P23639.

which is given by

$$h\left(K +_p L, .\right)^p = h\left(K, .\right)^p + h\left(L, .\right)^p,$$

where $h\left(K, u\right) := \max_{x \in K} \langle x, u \rangle$ denotes the support function of K. This addition is the basis of the L_p-Brunn-Minkowski theory (see e.g. [8,16,18,26,31–35,44,45,49]). Important operators in this theory are the symmetric and asymmetric L_p analogs of the projection body. The symmetric L_p-projection body was first introduced by Lutwak, Yang and Zhang in [31]. Volume inequalities for symmetric and asymmetric L_p-projection bodies are the geometric core of affine Sobolev inequalities, which turn out to be stronger than their classical counterparts (see e.g. [9,17,34,50]). In [26] Ludwig also extended her characterization of the projection body operator to the L_p case.

In this article we remove the assumption of homogeneity in this characterization of the L_p-projection body operators. The set of convex polytopes (respectively convex bodies) containing the origin is denoted by $\mathcal{P}_o^n$ (respectively $\mathcal{K}_o^n$). The asymmetric L_p-projection body operator $\Pi_p^+ \colon \mathcal{P}_o^n \to \mathcal{K}_o^n$ is defined by

$$h\left(\Pi_p^+ P, .\right)^p = \sum_{\substack{v \in \mathcal{N}(P) \\ o \notin F(P,v)}} \mathrm{vol}_{n-1}(F(P,v)) h\left(P, v\right)^{1-p} \langle v, . \rangle_+^p$$

for all $P \in \mathcal{P}_o^n$. Here $\mathcal{N}(P)$ denotes the set of all outer unit normals of facets of P and $F(P,v)$ denotes the facet corresponding to $v \in \mathcal{N}(P)$. Moreover, $\langle v, . \rangle_+$ denotes the positive part of $\langle v, . \rangle$. Similarly we define Π_p^- by replacing $\langle v, . \rangle_+$ with $\langle v, . \rangle_-$, which denotes the negative part of $\langle v, . \rangle$.

Theorem. *An operator $\Phi \colon \mathcal{P}_o^n \to \mathcal{K}_o^n$, where $n \geq 3$, is an $\mathrm{SL}(n)$-contravariant L_p-Minkowski valuation if and only if there exist constants $c_1, c_2 \geq 0$ such that*

$$\Phi\, P = c_1\, \Pi_p^+\, P +_p c_2\, \Pi_p^-\, P$$

for all $P \in \mathcal{P}_o^n$.

All the above characterizations deal with valuations defined on convex polytopes (or convex bodies) containing the origin. Building on Ludwig's characterization of the projection body operator, Schuster and Wannerer [43] obtained a classification of Minkowski valuations on the set of *all* convex bodies and showed that an additional operator arises. Our second main theorem establishes a corresponding classification for L_p-Minkowski valuations defined on the set of *all* convex polytopes, denoted by $\mathcal{P}^n$. As it turns out, also in this case, new operators arise. For the definitions of the operators $\Pi_p^{+>}$, $\Pi_p^{->}$, $\Pi_p^{+<}$ and $\Pi_p^{-<}$ see Section 4.

Theorem. *An operator $\Phi \colon \mathcal{P}^n \to \mathcal{K}_o^n$, where $n \geq 3$, is an $\mathrm{SL}(n)$-contravariant L_p-Minkowski valuation if and only if there exist constants $c_1, c_2, c_3, c_4 \geq 0$ such that*

$$\Phi\, P = c_1\, \Pi_p^{+>}\, P +_p c_2\, \Pi_p^{->}\, P +_p c_3\, \Pi_p^{+<}\, P +_p c_4\, \Pi_p^{-<}\, P$$

for all $P \in \mathcal{P}^n$.

2. Preliminaries

As a general reference for the concepts introduced in the following sections see [11,22,39]. Throughout this article $n \geq 1$ will denote the dimension of the Euclidean space $\mathbb{R}^n$. The vectors $e_1, \ldots, e_n$ are the standard basis vectors. We denote by $\langle x, y \rangle$ the inner product of two vectors $x, y \in \mathbb{R}^n$. The orthogonal complement of x is

denoted by $x^\perp$ and the norm induced by the inner product is denoted by $\|x\|$. The unit ball in $\mathbb{R}^n$ is written as B^n and its boundary as S^{n-1}. The m-dimensional volume in an m-dimensional subspace will be written as vol_m for $1 \leq m \leq n$. The general linear group and the special linear group are denoted by $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, respectively. We denote by lin the linear hull and by conv the convex hull of a subset of $\mathbb{R}^n$.

A nonempty compact convex subset of $\mathbb{R}^n$ is called a convex body. The set of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. We denote by dim the dimension of a convex body. The convex hull of a finite set of points in $\mathbb{R}^n$ is called a convex polytope. The set of all convex polytopes in $\mathbb{R}^n$ is denoted by $\mathcal{P}^n$. Clearly $\mathcal{P}^n \subseteq \mathcal{K}^n$. The subset of all convex bodies and convex polytopes containing the origin o are denoted by $\mathcal{K}^n_o$ and $\mathcal{P}^n_o$, respectively.

The Minkowski sum of two convex bodies K and L, denoted $K + L$, is defined by

$$K + L = \{x + y : x \in K, y \in L\}.$$

The scalar multiple of a convex body K and $s \geq 0$, denoted sK, is defined by

$$sK = \{sx : x \in K\}.$$

Note that $\mathcal{K}^n$, $\mathcal{P}^n$, $\mathcal{K}^n_o$ and $\mathcal{P}^n_o$ are closed under these operations. We equip $\mathcal{K}^n$ with the Hausdorff metric δ, defined by

$$\delta(K, L) = \min\{\epsilon > 0 : K + \epsilon B^n \subseteq L, L + \epsilon B^n \subseteq K\}$$

for all $K, L \in \mathcal{K}^n$. Note that the above operations are continuous and that $\mathcal{K}^n$ is a complete metric space.

Every $K \in \mathcal{K}^n$ is characterized by its support function

$$h\,(K, u) := \max_{x \in K} \langle x, u \rangle, \quad u \in \mathbb{R}^n.$$

A function $h\colon \mathbb{R}^n \to \mathbb{R}$ is a support function if and only if it is sublinear, i.e.

$$h(u + v) \leq h(u) + h(v) \quad \text{and} \quad h(su) = sh(u)$$

for all $u, v \in \mathbb{R}^n$ and $s > 0$. Note that sublinearity implies convexity and therefore continuity. Because of the homogeneity, a support function is determined by its values on S^{n-1}. Minkowski addition and scalar multiplication are compatible with the map $K \mapsto h\,(K, .)$, i.e.

$$h\,(K + L, .) = h\,(K, .) + h\,(L, .) \quad \text{and} \quad h\,(sK, .) = sh\,(K, .)$$

for all $K, L \in \mathcal{K}^n$ and $s \geq 0$. The Hausdorff distance of two convex bodies $K, L \in \mathcal{K}^n$ can be calculated by

$$\delta(K, L) = \|h\,(K, .) - h\,(L, .)\,\|_\infty,$$

where $\|.\|_\infty$ denotes the infinity norm on $C(S^{n-1})$. In particular, we can think of $\mathcal{K}^n$ with Minkowski addition as a subsemigroup of the abelian group $C(\mathbb{R}^n)$.

Throughout this article $p > 1$ will denote a real number. Note that a convex body K contains the origin if and only if $h\,(K, .) \geq 0$. It is easy to see that $\sqrt[p]{h\,(K, .)^p + h\,(L, .)^p}$ defines a nonnegative sublinear function for all $K, L \in \mathcal{K}^n_o$. It is therefore the support function of a unique convex body in $\mathcal{K}^n_o$. The L_p-Minkowski sum of $K, L \in \mathcal{K}^n_o$, denoted $K +_p L$, is defined by

$$h\,(K +_p L, .)^p = h\,(K, .)^p + h\,(L, .)^p\,.$$

By identifying $K \in \mathcal{K}_o^n$ with $h(K,.)^p$ we can think of $\mathcal{K}_o^n$ with L_p-Minkowski addition as a subsemigroup of $C(\mathbb{R}^n)$. Clearly $h(K,.)^p$ is a p-homogeneous function for all $K \in \mathcal{K}_o^n$. We denote by $C_p(\mathbb{R}^n)$ the set of all p-homogeneous functions in $C(\mathbb{R}^n)$.

Cauchy's functional equation

$$f(a+b) = f(a) + f(b) \quad \forall a, b \in \mathbb{R} \tag{1}$$

will be important for us. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nonlinear function which satisfies (1). It is a well-known fact that the graph of such a function f is dense in $\mathbb{R}^2$. An equivalent statement is that every bounded open interval has a dense image under f.

Let $f\colon (0,+\infty) \to \mathbb{R}$ be a nonlinear function which satisfies (1) for all $a, b \in (0,+\infty)$. It is easy to see that we can extend f to an odd function $f\colon \mathbb{R} \to \mathbb{R}$ which satisfies (1) for all $a, b \in \mathbb{R}$. Therefore every bounded open interval which is a subset of $(0,+\infty)$ has a dense image under f.

3. Valuations

Let $\mathcal{Q}^n$ be a subset of $\mathcal{K}^n$ and let A be an abelian semigroup. A map $\Phi\colon \mathcal{Q}^n \to A$ is called a valuation if it satisfies

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L)$$

for all $K, L \in \mathcal{Q}^n$ with $K \cup L, K \cap L \in \mathcal{Q}^n$. Furthermore, if A has an identity 0, we assume $\Phi(\emptyset) = 0$, even if $\emptyset \notin \mathcal{Q}^n$. If A is $\mathcal{K}^n$ with Minkowski addition, then Φ is called a Minkowski valuation. Note that Φ is a Minkowski valuation if and only if $K \mapsto h(\Phi K,.) \in C(\mathbb{R}^n)$ is a valuation. If A is $\mathcal{K}_o^n$ with L_p-Minkowski addition, then Φ is called an L_p-Minkowski valuation. Note that Φ is an L_p-Minkowski valuation if and only if $K \mapsto h(\Phi K,.)^p \in C_p(\mathbb{R}^n)$ is a valuation. It is easy to see that $K \cup L \in \mathcal{K}^n$ implies $K \cap L \in \mathcal{K}^n$ for all $K, L \in \mathcal{K}^n$. The same holds for $\mathcal{K}_o^n$.

Let A be an abelian monoid with identity 0. A valuation $\Phi\colon \mathcal{Q}^n \to A$ is called simple if $\Phi K = 0$ for all $K \in \mathcal{Q}^n$ with $\dim K < n$.

A k-dimensional simplex is the convex hull of $k+1$ affinely independent points for $k \in \{0, \dots, n\}$. The n-dimensional standard simplex, denoted T^n, is defined by

$$T^n = \operatorname{conv}\{o, e_1, \dots, e_n\}.$$

We need some general theorems on valuations. With the exception of the next theorem due to Volland [47] (see also [22]) we give proofs for the sake of completeness.

3.1. **Theorem.** *Let A be an abelian group and $\Phi\colon \mathcal{P}^n \to A$ a valuation. Then Φ satisfies the inclusion exclusion principle, i.e.*

$$\Phi(P_1 \cup \ldots \cup P_m) = \sum_{\emptyset \neq S \subseteq \{1,\dots,m\}} (-1)^{|S|-1}\, \Phi\left(\bigcap_{i \in S} P_i\right)$$

for all $m \in \mathbb{N}$ and $P_1, \dots, P_m \in \mathcal{P}^n$ with $P_1 \cup \ldots \cup P_m \in \mathcal{P}^n$.

3.2. **Lemma.** *Let A be an abelian group and $\Phi\colon \mathcal{P}^n \to A$ a valuation. Then Φ is determined by its values on n-simplices.*

Proof. Assume that Φ vanishes on n-simplices. Let $k \in \{0, \dots, n-1\}$. Every k-dimensional simplex T can be written as the intersection of two $(k+1)$-dimensional simplices T_1, T_2. We can do this in such a way that $T_1 \cup T_2$ is also a $(k+1)$-dimensional simplex. Using induction from n to 0 shows that Φ vanishes on all simplices. Now, using Theorem 3.1 and induction from 0 to n finishes the proof. □

3.3. **Lemma.** *Let A be an abelian group and $\Phi\colon \mathcal{P}_o^n \to A$ a valuation. Then Φ satisfies the inclusion exclusion principle, i.e.*

$$\Phi(P_1 \cup \ldots \cup P_m) = \sum_{\emptyset \neq S \subseteq \{1,\ldots,m\}} (-1)^{|S|-1}\, \Phi\left(\bigcap_{i \in S} P_i\right)$$

for all $m \in \mathbb{N}$ and $P_1, \dots, P_m \in \mathcal{P}_o^n$ with $P_1 \cup \ldots \cup P_m \in \mathcal{P}_o^n$.

Proof. Extend Φ to $\mathcal{P}^n$ by

$$\Phi\, P = \Phi(P_o)$$

for all $P \in \mathcal{P}^n$, where $P_o := \operatorname{conv}(\{o\} \cup P)$. It is easy to see that this defines a valuation on $\mathcal{P}^n$. The assertion now follows from Theorem 3.1. □

3.4. **Lemma.** *Let A be an abelian group and $\Phi\colon \mathcal{P}_o^n \to A$ a valuation. Then Φ is determined by its values on n-simplices with one vertex at the origin and its value on $\{o\}$.*

Proof. Assume that Φ vanishes on n-simplices with one vertex at the origin and on $\{o\}$. Similar to the proof of Lemma 3.2, it follows that Φ vanishes on all simplices with one vertex at the origin. Note that a k-dimensional polytope which contains the origin can be dissected into k-simplices with one vertex at the origin for all $k \in \{0, \dots, n\}$. Now, using Lemma 3.3 and induction from 0 to n finishes the proof. □

3.5. **Lemma.** *Let A be an abelian group and $\Phi\colon \mathcal{P}^n \to A$ a simple valuation. Then Φ is determined by its values on $\mathcal{P}_o^n$.*

Proof. Let $P \in \mathcal{P}^n$. Denote by $F_1, \dots, F_m$ the facets of P which face towards the origin. Here we say that a facet F is facing towards the origin if $h\,(P, v) < 0$, where v is the corresponding outer unit normal. Clearly

$$P_o = P \cup (F_1)_o \cup \ldots \cup (F_m)_o,$$

where $P_o := \operatorname{conv}(\{o\} \cup P)$. Note that $P_o, (F_1)_o, \dots, (F_m)_o \in \mathcal{P}_o^n$. Furthermore the intersection of two convex bodies of the right hand side is lower dimensional. Since Φ is simple, Theorem 3.1 implies

$$\Phi\, P_o = \Phi\, P + \Phi(F_1)_o + \ldots + \Phi(F_m)_o$$

or equivalently

$$\Phi\, P = \Phi\, P_o - \Phi(F_1)_o - \ldots - \Phi(F_m)_o.$$

□

4. SL(n)-contravariance

Let $\mathcal{Q}^n$ denote $\mathcal{K}^n$,$\mathcal{P}^n$,$\mathcal{K}_o^n$ or $\mathcal{P}_o^n$. A map $\Phi\colon \mathcal{Q}^n \to \mathcal{K}^n$ is called SL(n)-contravariant if it satisfies

$$\Phi(\phi K) = \phi^{-t}\,\Phi\,K$$

for all $K \in \mathcal{Q}^n$ and $\phi \in \mathrm{SL}(n)$. A map $\Phi\colon \mathcal{Q}^n \to C(\mathbb{R}^n)$ is called SL(n)-contravariant if it satisfies

$$\Phi(\phi K) = \Phi(K) \circ \phi^{-1}$$

for all $K \in \mathcal{Q}^n$ and $\phi \in \mathrm{SL}(n)$. Since

$$h\left(\phi^{-t}\,\Phi\,K, u\right) = h\left(\Phi\,K, \phi^{-1}u\right)$$

holds for all $K \in \mathcal{Q}^n$, $u \in \mathbb{R}^n$ and $\phi \in \mathrm{SL}(n)$, we see that a map $\Phi\colon \mathcal{Q}^n \to \mathcal{K}^n$ is SL(n)-contravariant if and only if $K \mapsto h\left(\Phi\,K, .\right)$ is SL(n)-contravariant. Similarly a map $\Phi\colon \mathcal{Q}^n \to \mathcal{K}_o^n$ is SL(n)-contravariant if and only if $K \mapsto h\left(\Phi\,K, .\right)^p$ is SL(n)-contravariant.

We will now define some SL(n)-contravariant L_p-Minkowski valuations. The SL(n)-contravariance and the valuation property will be proven below. The asymmetric L_p-projection body operator $\Pi_p^+\colon \mathcal{P}_o^n \to \mathcal{K}_o^n$ is defined by

$$h\left(\Pi_p^+\,P, .\right)^p = \sum_{\substack{v\in\mathcal{N}(P)\\ o\notin F(P,v)}} \mathrm{vol}_{n-1}(F(P,v))h\left(P,v\right)^{1-p}\,\langle v, .\rangle_+^p$$

for all $P \in \mathcal{P}_o^n$. Here $\mathcal{N}(P)$ denotes the set of all outer unit normals of facets of P and $F(P,v)$ denotes the facet corresponding to $v \in \mathcal{N}(P)$. More generally we define $F(P,v) = P \cap \{x \in \mathbb{R}^n : \langle x, v\rangle = h\left(P,v\right)\}$. Furthermore $\langle v, .\rangle_+$ denotes the positive part of $\langle v, .\rangle$, i.e. $\max\{\langle v, .\rangle, 0\}$. Note that $\langle v, .\rangle_+$ is the support function of the straight line segment $[o, v]$. The map $\Pi_p^{+>}\colon \mathcal{P}^n \to \mathcal{K}_o^n$ is defined by

$$h\left(\Pi_p^{+>}\,P, .\right)^p = \sum_{\substack{v\in\mathcal{N}(P)\\ h(P,v)>0}} \mathrm{vol}_{n-1}(F(P,v))h\left(P,v\right)^{1-p}\,\langle v, .\rangle_+^p$$

for all $P \in \mathcal{P}^n$. Note that $\Pi_p^{+>}$ is an extension of Π_p^+ to $\mathcal{P}^n$. The map $\Pi_p^{+<}\colon \mathcal{P}^n \to \mathcal{K}_o^n$ is defined by

$$h\left(\Pi_p^{+<}\,P, .\right)^p = \sum_{\substack{v\in\mathcal{N}(P)\\ h(P,v)<0}} \mathrm{vol}_{n-1}(F(P,v))|h\left(P,v\right)|^{1-p}\langle v, .\rangle_+^p$$

for all $P \in \mathcal{P}^n$. Note that $\Pi_p^{+<}$ vanishes on $\mathcal{P}_o^n$. Similarly we define Π_p^-, $\Pi_p^{->}$ and $\Pi_p^{-<}$ with $\langle v, .\rangle_+^p$ replaced by $\langle v, .\rangle_-^p$, where $\langle v, .\rangle_- = \max\{-\langle v, .\rangle, 0\}$.

The map $\Delta_p^+\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ is defined by

$$\begin{aligned}\Delta_p^+\,P &= h\left(\Pi_p^{+>}\,P, .\right)^p - h\left(\Pi_p^{-<}\,P, .\right)^p\\ &= \sum_{\substack{v\in\mathcal{N}(P)\\ h(P,v)\neq 0}} \mathrm{vol}_{n-1}(F(P,v))\,\mathrm{sgn}\,h\left(P,v\right)|h\left(P,v\right)|^{1-p}\langle v, .\rangle_{\mathrm{sgn}\,h(P,v)}^p\end{aligned}$$

for all $P \in \mathcal{P}^n$. Note that Δ_p^+ is a simple extension of $P \mapsto h\left(\Pi_p^+\,P, .\right)^p$ to $\mathcal{P}^n$.

Similarly we define Δ_p^- by

$$\Delta_p^- P = h\left(\Pi_p^{->} P, .\right)^p - h\left(\Pi_p^{+<} P, .\right)^p$$

for all $P \in \mathcal{P}^n$.

4.1. **Lemma.** *Let V be a real vector space and $f\colon \mathbb{R}\times S^{n-1} \to V$ a function. Define $\Phi\colon \mathcal{P}^n \to V$ by*

$$\Phi P = \sum_{v\in\mathcal{N}(P)} \mathrm{vol}_{n-1}(F(P,v)) f(h(P,v), v)$$

for all $P \in \mathcal{P}^n$. Then Φ is a valuation.

Proof. We need to show that

$$\Phi(P\cup Q) + \Phi(P\cap Q) = \Phi P + \Phi Q$$

for all $P, Q \in \mathcal{P}^n$ with $P\cup Q \in \mathcal{P}^n$. We distinguish three sets of normal vectors:

$$\begin{aligned} I_1 &:= \{v \in S^{n-1} : h(P,v) < h(Q,v)\},\\ I_2 &:= \{v \in S^{n-1} : h(P,v) = h(Q,v)\},\\ I_3 &:= \{v \in S^{n-1} : h(P,v) > h(Q,v)\}. \end{aligned}$$

For $v \in I_1$ we have $F(P\cup Q, v) = F(Q,v)$, $h(P\cup Q, v) = h(Q,v)$, $F(P\cap Q, v) = F(P,v)$ and $h(P\cap Q, v) = h(P,v)$. Analogous for I_3. It follows that the above equation is equivalent to

$$\begin{aligned} &\sum_{\substack{v\in\mathcal{N}(P\cup Q)\\ v\in I_2}} \mathrm{vol}_{n-1}(F(P\cup Q,v)) f(h(P\cup Q,v), v)\\ &+ \sum_{\substack{v\in\mathcal{N}(P\cap Q)\\ v\in I_2}} \mathrm{vol}_{n-1}(F(P\cap Q,v)) f(h(P\cap Q,v), v)\\ &= \sum_{\substack{v\in\mathcal{N}(P)\\ v\in I_2}} \mathrm{vol}_{n-1}(F(P,v)) f(h(P,v), v)\\ &+ \sum_{\substack{v\in\mathcal{N}(Q)\\ v\in I_2}} \mathrm{vol}_{n-1}(F(Q,v)) f(h(Q,v), v). \end{aligned}$$

Note that $f(h(P\cup Q,v), v) = f(h(P\cap Q,v), v) = f(h(P,v), v) = f(h(Q,v), v)$ for $v \in I_2$. Furthermore $\mathcal{N}(P\cup Q)\cup\mathcal{N}(P\cap Q) = \mathcal{N}(P)\cup\mathcal{N}(Q)$. Since

$$P \mapsto \mathrm{vol}_{n-1}(F(P,v)),\ P \in \mathcal{P}^n,$$

is a valuation for fixed $v \in S^{n-1}$ (as is easy to verify), this implies the desired result. □

4.2. **Lemma.** *The map $\Pi_p^{+>}\colon \mathcal{P}^n \to \mathcal{K}_o^n$ is an* SL(n)*-contravariant L_p-Minkowski valuation.*

Proof. The valuation property follows directly from the definition and Lemma 4.1 with $f\colon \mathbb{R}\times S^{n-1} \to C_p(\mathbb{R}^n)$ defined by

$$f(t,v) := \begin{cases} 0 & \text{if } t \le 0,\\ t^{1-p}\langle v, .\rangle_+^p & \text{if } t > 0 \end{cases}$$

for all $t \in \mathbb{R}$ and $v \in S^{n-1}$. To show the $\mathrm{SL}(n)$-contravariance let $\phi \in \mathrm{SL}(n)$. Note that

$$v \in \mathcal{N}(P) \Longleftrightarrow \tilde{v} \in \mathcal{N}(\phi P)$$

with $\tilde{v} := \|\phi^{-t}v\|^{-1}\phi^{-t}v$ and that

$$\mathrm{vol}_{n-1}(F(\phi P, \tilde{v})) = \|\phi^{-t}v\| \, \mathrm{vol}_{n-1}(F(P, v)).$$

Furthermore

$$\begin{aligned} f(h\left(\phi P, \tilde{v}\right), \tilde{v}) &= f(h\left(P, \phi^t \tilde{v}\right), \tilde{v}) \\ &= f(\|\phi^{-t}v\|^{-1} h\left(P, v\right), \|\phi^{-t}v\|^{-1}\phi^{-t}v) \\ &= \|\phi^{-t}v\|^{-1} f(h\left(P, v\right), \phi^{-t}v). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \mathrm{vol}_{n-1}(F(\phi P, \tilde{v})) f(h\left(\phi P, \tilde{v}\right), \tilde{v}) &= \mathrm{vol}_{n-1}(F(P, v)) f(h\left(P, v\right), \phi^{-t}v) \\ &= \mathrm{vol}_{n-1}(F(P, v)) f(h\left(P, v\right), v) \circ \phi^{-1}. \end{aligned}$$

This implies the desired result. □

4.3. *Remark.* Analogous to Lemma 4.2 we see that $\Pi_p^{->}$, $\Pi_p^{+<}$ and $\Pi_p^{-<}$ are $\mathrm{SL}(n)$-contravariant L_p-Minkowski valuations. Furthermore this implies that Π_p^+ and Π_p^- are $\mathrm{SL}(n)$-contravariant L_p-Minkowski valuations. A similar result holds for Δ_p^+ and Δ_p^-.

4.4. **Example.** For further reference, we evaluate our valuations at some special polytopes. We start with T^n. The only facet in T^n which does not contain the origin is $F := \mathrm{conv}\{e_1, \ldots, e_n\}$. The outer unit normal at F is $v := \frac{1}{\sqrt{n}}(e_1 + \ldots + e_n)$. Note that $\mathrm{vol}_{n-1} F = \frac{\sqrt{n}}{(n-1)!}$ and that $h\left(T^n, v\right) = \frac{1}{\sqrt{n}}$. Thus, we have

$$\begin{aligned} h\left(\Pi_p^+ T^n, u\right)^p &= \mathrm{vol}_{n-1}(F) h\left(T^n, v\right)^{1-p} \langle v, u\rangle_+^p \\ &= \frac{\sqrt{n}}{(n-1)!}\left(\frac{1}{\sqrt{n}}\right)^{1-p} \left\langle \frac{1}{\sqrt{n}}(e_1 + \ldots + e_n), u\right\rangle_+^p \\ &= \frac{1}{(n-1)!}\langle e_1 + \ldots + e_n, u\rangle_+^p \end{aligned}$$

for all $u \in \mathbb{R}^n$. Let $i \in \{1, \ldots, n\}$. It follows that

$$\begin{aligned} h\left(\Pi_p^+ T^n, e_i\right)^p &= \frac{1}{(n-1)!}, \\ h\left(\Pi_p^+ T^n, -e_i\right)^p &= 0. \end{aligned}$$

Similarly we get

$$\begin{aligned} h\left(\Pi_p^- T^n, e_i\right)^p &= 0, \\ h\left(\Pi_p^- T^n, -e_i\right)^p &= \frac{1}{(n-1)!}. \end{aligned}$$

Next we consider $e_1 + T^n$. There are two facets such that the origin is not contained in the affine hull of the facet. The first one is $F_1 := e_1 + \mathrm{conv}\{e_1, \ldots, e_n\}$ with outer unit normal $v_1 := \frac{1}{\sqrt{n}}(e_1 + \ldots + e_n)$. Note that $\mathrm{vol}_{n-1} F_1 = \frac{\sqrt{n}}{(n-1)!}$ and that

$h\left(e_1+T^n, v_1\right)=\frac{2}{\sqrt{n}}$. The second one is $F_2:=e_1+\operatorname{conv}\{o, e_2, \ldots, e_n\}$ with outer unit normal $v_2:=-e_1$. Note that $\operatorname{vol}_{n-1} F_2=\frac{1}{(n-1)!}$ and that $h\left(e_1+T^n, v_2\right)=-1$. Thus, we have

$$\begin{aligned} h\left(\Pi_p^{+>}(e_1+T^n), u\right)^p &= \operatorname{vol}_{n-1}(F_1) h\left(e_1+T^n, v_1\right)^{1-p}\langle v_1, u\rangle_+^p \\ &= \frac{\sqrt{n}}{(n-1)!}\left(\frac{2}{\sqrt{n}}\right)^{1-p}\left\langle \frac{1}{\sqrt{n}}(e_1+\ldots+e_n), u\right\rangle_+^p \\ &= \frac{2^{1-p}}{(n-1)!}\langle e_1+\ldots+e_n, u\rangle_+^p \end{aligned}$$

and

$$\begin{aligned} h\left(\Pi_p^{+<}(e_1+T^n), u\right)^p &= \operatorname{vol}_{n-1}(F_2)\left|h\left(e_1+T^n, v_2\right)\right|^{1-p}\langle v_2, u\rangle_+^p \\ &= \frac{1}{(n-1)!}\langle -e_1, u\rangle_+^p \end{aligned}$$

for all $u \in \mathbb{R}^n$. It follows that

$$\begin{aligned} h\left(\Pi_p^{+>}(e_1+T^n), e_2-e_1\right)^p &= 0, \\ h\left(\Pi_p^{+>}(e_1+T^n), -e_2+e_1\right)^p &= 0, \\ h\left(\Pi_p^{+<}(e_1+T^n), e_2-e_1\right)^p &= \frac{1}{(n-1)!}, \\ h\left(\Pi_p^{+<}(e_1+T^n), -e_2+e_1\right)^p &= 0. \end{aligned}$$

Similarly we get

$$\begin{aligned} h\left(\Pi_p^{->}(e_1+T^n), e_2-e_1\right)^p &= 0, \\ h\left(\Pi_p^{->}(e_1+T^n), -e_2+e_1\right)^p &= 0, \\ h\left(\Pi_p^{-<}(e_1+T^n), e_2-e_1\right)^p &= 0, \\ h\left(\Pi_p^{-<}(e_1+T^n), -e_2+e_1\right)^p &= \frac{1}{(n-1)!}. \end{aligned}$$

4.5. **Lemma.** *Let $\Phi\colon \mathcal{Q}^n \to C_p(\mathbb{R}^n)$ be $\mathrm{SL}(n)$-contravariant, where $\mathcal{Q}^n$ is either $\mathcal{P}_o^n$ or $\mathcal{P}^n$. Furthermore let $\phi \in \mathrm{GL}(n)$ with $\det \phi > 0$. Then*

$$\Phi(\phi P)=\det(\phi)^{\frac{p}{n}}\, \Phi\left(\det(\phi)^{\frac{1}{n}} P\right) \circ \phi^{-1}$$

for all $P \in \mathcal{Q}^n$.

Proof. Since $\det(\phi)^{-\frac{1}{n}}\phi \in \mathrm{SL}(n)$, this follows directly from the $\mathrm{SL}(n)$-contravariance of Φ and the p-homogeneity of the functions in $C_p(\mathbb{R}^n)$. □

5. Main results on $\mathcal{P}_o^n$

5.1. **Lemma.** *Let $\Phi\colon \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ be an $\mathrm{SL}(n)$-contravariant valuation. If $n \geq 3$, then Φ is simple.*

Proof. Let $P \in \mathcal{P}_o^n$ with $d:=\dim P \leq n-1$. Because of the $\mathrm{SL}(n)$-contravariance, we can assume without loss of generality that

$$\operatorname{aff} P=\{e_1, \ldots, e_{n-d}\}^{\perp}=\operatorname{lin}\{e_{n-d+1}, \ldots, e_n\}.$$

Define $\phi \in \mathrm{SL}(n)$ by

$$\phi = \begin{pmatrix} A & 0 \\ B & I \end{pmatrix},$$

where $A \in \mathbb{R}^{(n-d)\times(n-d)}$ is a matrix with $\det A = 1$, $B \in \mathbb{R}^{d\times(n-d)}$ is an arbitrary matrix, $0 \in \mathbb{R}^{(n-d)\times d}$ is the zero matrix and $I \in \mathbb{R}^{d\times d}$ is the identity matrix. Let $x \in \mathbb{R}^n$. Write $x = (x', x'')^t \in \mathbb{R}^{n-d} \times \mathbb{R}^d$ and assume $x' \neq 0$. Since $\phi P = P$ and since Φ is $\mathrm{SL}(n)$-contravariant, we have

$$\Phi(P)(x) = \Phi(\phi P)(x) = \Phi(P)(\phi^{-1}x). \tag{2}$$

A simple calculation yields

$$\phi^{-1}x = \begin{pmatrix} A^{-1} & 0 \\ -BA^{-1} & I \end{pmatrix} \cdot \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} A^{-1}x' \\ -BA^{-1}x' + x'' \end{pmatrix}. \tag{3}$$

In the case $d \leq n-2$, we can choose A with $\det A = 1$ such that $A^{-1}x'$ is any nonzero vector. After choosing A we can choose B such that $-BA^{-1}x' + x''$ is any vector. Combining (2) and (3) it follows that $\Phi(P)$ is constant on a dense subset of $\mathbb{R}^n$. Because $\Phi(P)$ is continuous, it must be constant everywhere. Since $\Phi(P)(0) = 0$, we get $\Phi(P) = 0$.

Now let $d = n-1$. Using Lemmas 4.5 and 3.4 in dimension $n-1$ it is enough to show that $\Phi(sT^{n-1}) = 0$ for $s > 0$, where T^{n-1} denotes the standard simplex in $e_1^\perp$. Analogous to the case $d \leq n-2$ we see that $\Phi(sT^{n-1})(x) = \Phi(sT^{n-1})\left((x', 0)^t\right)$. Note that $x' \in \mathbb{R}$. Because $\Phi(sT^{n-1})$ is p-homogeneous, we only need to show that $\Phi(sT^{n-1})(\pm e_1) = 0$.

Let $\lambda \in (0,1)$ and denote by H_λ the hyperplane through o with normal vector $\lambda e_2 - (1-\lambda)e_3$. Since Φ is a valuation, we have

$$\Phi(sT^{n-1}) + \Phi(sT^{n-1} \cap H_\lambda) = \Phi(sT^{n-1} \cap H_\lambda^+) + \Phi(sT^{n-1} \cap H_\lambda^-),$$

where H_λ^+ and H_λ^- denote the two halfspaces bounded by H_λ. Because $sT^{n-1} \cap H_\lambda$ has dimension $n-2$, the case $d \leq n-2$ shows that $\Phi(sT^{n-1} \cap H_\lambda) = 0$ and we obtain

$$\Phi(sT^{n-1}) = \Phi(sT^{n-1} \cap H_\lambda^+) + \Phi(sT^{n-1} \cap H_\lambda^-). \tag{4}$$

Define $\phi_\lambda \in \mathrm{SL}(n)$ by

$$\phi_\lambda e_1 = \frac{1}{\lambda} e_1, \quad \phi_\lambda e_2 = e_2, \quad \phi_\lambda e_3 = (1-\lambda)e_2 + \lambda e_3, \quad \phi_\lambda e_k = e_k \quad \text{for } 4 \leq k \leq n$$

and define $\psi_\lambda \in \mathrm{SL}(n)$ by

$$\psi_\lambda e_1 = \frac{1}{1-\lambda} e_1, \quad \psi_\lambda e_2 = (1-\lambda)e_2 + \lambda e_3, \quad \psi_\lambda e_3 = e_3, \quad \psi_\lambda e_k = e_k \quad \text{for } 4 \leq k \leq n.$$

Since

$$sT^{n-1} \cap H_\lambda^+ = \phi_\lambda(sT^{n-1}) \quad \text{and} \quad sT^{n-1} \cap H_\lambda^- = \psi_\lambda(sT^{n-1}),$$

relation (4) becomes

$$\Phi(sT^{n-1}) = \Phi(\phi_\lambda(sT^{n-1})) + \Phi(\psi_\lambda(sT^{n-1})).$$

We rewrite the last equation at e_1 using the SL(n)-contravariance of Φ and the p-homogeneity of the functions in $C_p(\mathbb{R}^n)$ as

$$\begin{aligned}\Phi(sT^{n-1})(e_1) &= \Phi(\phi_\lambda(sT^{n-1}))(e_1) + \Phi(\psi_\lambda(sT^{n-1}))(e_1)\\ &= \Phi(sT^{n-1})(\phi_\lambda^{-1}e_1) + \Phi(sT^{n-1})(\psi_\lambda^{-1}e_1)\\ &= \Phi(sT^{n-1})(\lambda e_1) + \Phi(sT^{n-1})((1-\lambda)e_1)\\ &= \lambda^p\,\Phi(sT^{n-1})(e_1) + (1-\lambda)^p\,\Phi(sT^{n-1})(e_1)\\ &= (\lambda^p + (1-\lambda)^p)\,\Phi(sT^{n-1})(e_1).\end{aligned}$$

Since $p > 1$, the resulting equation can only hold if $\Phi(sT^{n-1})(e_1) = 0$. Similarly we see that $\Phi(sT^{n-1})(-e_1) = 0$. □

5.2. **Theorem.** *Let $\Phi\colon \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ be an* SL(n)*-contravariant valuation. Assume further that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $I_y \subseteq (0,+\infty)$ such that $\{\Phi(sT^n)(y) : s \in I_y\}$ is not dense in $\mathbb{R}$. If $n \geq 3$, then there exist constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\Phi\,P = c_1 h\left(\Pi_p^+\,P, .\right)^p + c_2 h\left(\Pi_p^-\,P, .\right)^p$$

for all $P \in \mathcal{P}_o^n$.

Proof. By Lemmas 4.5 and 3.4, it is sufficient to prove that

$$\Phi(sT^n) = c_1 h\left(\Pi_p^+(sT^n), .\right)^p + c_2 h\left(\Pi_p^-(sT^n), .\right)^p \tag{5}$$

for $s > 0$.

1. Functional equation: Let $\lambda \in (0,1)$ and denote by H_λ the hyperplane through o with normal vector $\lambda e_1 - (1-\lambda)e_2$. Since Φ is a valuation, we have

$$\Phi(sT^n) + \Phi(sT^n \cap H_\lambda) = \Phi(sT^n \cap H_\lambda^+) + \Phi(sT^n \cap H_\lambda^-).$$

Because $sT^n \cap H_\lambda$ has dimension $n-1$, Lemma 5.1 shows that $\Phi(sT^n \cap H_\lambda) = 0$ and we obtain

$$\Phi(sT^n) = \Phi(sT^n \cap H_\lambda^+) + \Phi(sT^n \cap H_\lambda^-). \tag{6}$$

Define $\phi_\lambda \in \mathrm{GL}(n)$ by

$$\phi_\lambda e_1 = e_1, \quad \phi_\lambda e_2 = (1-\lambda)e_1 + \lambda e_2, \quad \phi_\lambda e_k = e_k \quad \text{for } 3 \leq k \leq n$$

and define $\psi_\lambda \in \mathrm{GL}(n)$ by

$$\psi_\lambda e_1 = (1-\lambda)e_1 + \lambda e_2, \quad \psi_\lambda e_2 = e_2, \quad \psi_\lambda e_k = e_k \quad \text{for } 3 \leq k \leq n.$$

Note that

$$\det(\phi_\lambda) = \lambda \quad \text{and} \quad \det(\psi_\lambda) = 1-\lambda. \tag{7}$$

Since

$$T^n \cap H_\lambda^+ = \phi_\lambda T^n \quad \text{and} \quad T^n \cap H_\lambda^- = \psi_\lambda T^n,$$

relation (6) becomes

$$\Phi(sT^n) = \Phi(\phi_\lambda sT^n) + \Phi(\psi_\lambda sT^n).$$

Using Lemma 4.5 and (7), we can rewrite the last equation as

$$\Phi(sT^n)(x) = \lambda^{\frac{p}{n}}\,\Phi\left(\lambda^{\frac{1}{n}} sT^n\right)(\phi_\lambda^{-1}x) + (1-\lambda)^{\frac{p}{n}}\,\Phi\left((1-\lambda)^{\frac{1}{n}} sT^n\right)(\psi_\lambda^{-1}x) \tag{8}$$

for all $x \in \mathbb{R}^n$.

2. Homogeneity: For $y \in \{e_1, e_2\}^\perp$ this becomes

$$\Phi(sT^n)(y) = \lambda^{\frac{p}{n}}\, \Phi\left(\lambda^{\frac{1}{n}} sT^n\right)(y) + (1-\lambda)^{\frac{p}{n}}\, \Phi\left((1-\lambda)^{\frac{1}{n}} sT^n\right)(y).$$

Define $g(s) = \Phi\left(s^{\frac{1}{n}} T^n\right)(y)$; then we have

$$g(s) = \lambda^{\frac{p}{n}} g(\lambda s) + (1-\lambda)^{\frac{p}{n}} g((1-\lambda)s).$$

Let $a, b > 0$. Setting $s = a+b$ and $\lambda = \frac{a}{a+b}$ we obtain

$$g(a+b) = \left(\frac{a}{a+b}\right)^{\frac{p}{n}} g(a) + \left(\frac{b}{a+b}\right)^{\frac{p}{n}} g(b),$$
$$(a+b)^{\frac{p}{n}} g(a+b) = a^{\frac{p}{n}} g(a) + b^{\frac{p}{n}} g(b).$$

Thus, $s \mapsto s^{\frac{p}{n}} g(s)$ solves Cauchy's functional equation for $s > 0$. By the assumption that there is a bounded open interval I_y such that $g(I_y)$ is not dense in $\mathbb{R}$, it follows that $s \mapsto s^{\frac{p}{n}} g(s)$ is linear. This implies $s^{\frac{p}{n}} g(s) = sg(1)$ and hence $g(s) = s^{1-\frac{p}{n}} g(1)$. The definition of g therefore yields

$$\Phi(sT^n)(y) = g(s^n) = s^{n-p} g(1) = s^{n-p}\, \Phi(T^n)(y).$$

Since $n \geq 3$ and since we can repeat the above argument for any two standard basis vectors, we obtain

$$\Phi(sT^n)(\pm e_i) = s^{n-p}\, \Phi(T^n)(\pm e_i) \quad \text{for } i = 1, \ldots, n. \tag{9}$$

3. Constants: Let $i \in \{1, \ldots, n\}$. Since $n \geq 3$, we can find a permutation of the coordinates $\phi \in \mathrm{SL}(n)$ such that $\phi e_1 = e_i$. It follows that

$$\Phi(T^n)(e_i) = \Phi(T^n)(\phi^{-1} e_1) = \Phi(\phi T^n)(e_1) = \Phi(T^n)(e_1). \tag{10}$$

Similarly we get $\Phi(T^n)(-e_i) = \Phi(T^n)(-e_1)$. Set

$$c_1 = (n-1)!\, \Phi(T^n)(e_1) \quad \text{and} \quad c_2 = (n-1)!\, \Phi(T^n)(-e_1). \tag{11}$$

4. Induction: We are now going to show by induction on the number m of coordinates of x not equal to zero that

$$\Phi(sT^n)(x) = c_1 h\left(\Pi_p^+(sT^n), x\right)^p + c_2 h\left(\Pi_p^-(sT^n), x\right)^p \tag{12}$$

for $s > 0$ and for all $x \in \mathbb{R}^n$. Note that since $P \mapsto c_1 h\left(\Pi_p^+(P), .\right)^p + c_2 h\left(\Pi_p^-(P), .\right)^p$ satisfies the assumptions of the theorem it also satisfies (8) and (9).

The case $m = 0$ is trivial. The case $m = 1$ is also easy to verify with (9), (10), (11) and Example 4.4. Let $m \geq 2$ and write $x = (x_1, \ldots, x_n)^t$. Assume without loss of generality that $x_1, x_2 \neq 0$ and that $|x_1| \leq |x_2|$. Since the functions in $C_p(\mathbb{R}^n)$ are continuous, we can further assume that $|x_1| < |x_2|$.

First consider the case where x_1, x_2 have the same sign. Set $\lambda = \frac{x_2}{x_1+x_2} \in (0,1)$ and calculate

$$\begin{aligned}
&\phi_\lambda\left((x_1+x_2)e_2 + x_3 e_3 + \ldots + x_n e_n\right) \\
&= (x_1+x_2)(1-\lambda)e_1 + (x_1+x_2)\lambda e_2 + x_3 e_3 + \ldots + x_n e_n \\
&= x_1 e_1 + x_2 e_2 + x_3 e_3 + \ldots + x_n e_n \\
&= x
\end{aligned}$$

or equivalently

$$\phi_\lambda^{-1} x = (x_1+x_2)e_2 + x_3 e_3 + \ldots + x_n e_n.$$

Similarly we calculate

$$\psi_\lambda^{-1} x = (x_1 + x_2)e_1 + x_3 e_3 + \ldots + x_n e_n.$$

Using (8) and the induction hypotheses yields the desired result.

Now consider the case where x_1, x_2 have different signs. Set $\lambda = 1 + \frac{x_1}{x_2} \in (0,1)$ and calculate

$$\begin{aligned}\phi_\lambda x &= x_1 e_1 + x_2(1-\lambda)e_1 + x_2 \lambda e_2 + x_3 e_3 + \ldots + x_n e_n \\ &= (x_1 + x_2)e_2 + x_3 e_3 + \ldots + x_n e_n.\end{aligned}$$

Similarly we calculate

$$\psi_\lambda^{-1} \phi_\lambda x = (x_1 + x_2)e_2 + x_3 e_3 + \ldots + x_n e_n.$$

Using (8) with x replaced by $\phi_\lambda x$ and using the induction hypotheses yields the desired result.

This completes the induction and proves (12) and thus (5). □

5.3. *Remark.* The additional regularity assumption in the last theorem is only used for the vectors of the standard basis and their inverses in the proof. By reasoning similar to step three in the proof of Theorem 5.2 it is enough to have this assumption for only one standard basis vector and its reflection at the origin.

5.4. **Corollary.** *Let $\Phi\colon \mathcal{P}_o^n \to \mathcal{K}_o^n$ be an* SL(n)*-contravariant L_p-Minkowski valuation. If $n \geq 3$, then there exist constants $c_1, c_2 \geq 0$ such that*

$$\Phi\, P = c_1\, \Pi_p^+\, P +_p c_2\, \Pi_p^-\, P$$

for all $P \in \mathcal{P}_o^n$.

Proof. Since $\Phi(sT^n) \in \mathcal{K}_o^n$, we have $h\left(\Phi(sT^n), y\right) \geq 0$ for $s > 0$ and for all $y \in \mathbb{R}^n$. Therefore the map

$$P \mapsto h\left(\Phi\, P, .\right)^p, \quad P \in \mathcal{P}_o^n,$$

satisfies the assumptions of Theorem 5.2. Thus there exist constants $d_1, d_2 \in \mathbb{R}$ such that

$$h\left(\Phi\, P, .\right)^p = d_1 h\left(\Pi_p^+\, P, .\right)^p + d_2 h\left(\Pi_p^-\, P, .\right)^p.$$

According to (11) these constants are given by

$$\begin{aligned} d_1 &= (n-1)! h\left(\Phi\, T^n, e_1\right)^p, \\ d_2 &= (n-1)! h\left(\Phi\, T^n, -e_1\right)^p. \end{aligned}$$

It follows that $d_1, d_2 \geq 0$. Defining $c_1 = \sqrt[p]{d_1}$ and $c_2 = \sqrt[p]{d_2}$ completes the proof. □

The first main theorem from the introduction now follows from Corollary 5.4 and the fact that Π_p^+ and Π_p^- have the desired properties.

6. Main results on $\mathcal{P}^n$

6.1. **Lemma.** *Let $\Phi\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ be an* SL(n)*-contravariant valuation. If $n \geq 3$, then $\Phi\, P = 0$ for all $P \in \mathcal{P}^n$ with $\dim P \leq n-2$ and for all $P \in \mathcal{P}^n$ with $\dim P = n-1$ and $o \in \operatorname{aff} P$.*

Proof. Since Φ is SL(n)-contravariant, it is sufficient to prove $\Phi\, P = 0$ for all $P \in \mathcal{P}^n$ with $P \subseteq e_1^\perp$. Let $d := \dim P$. There are several different cases:

(i) $d \leq n-3$,
(ii) $d = n-2$ and $o \in \operatorname{aff} P$,

(iii) $d = n-2$ and $o \notin \operatorname{aff} P$,
(iv) $d = n-1$.

In cases (i) and (ii) we can use the $\mathrm{SL}(n)$-contravariance of Φ to assume that $P \subseteq \{e_1, e_2\}^\perp$. Here we can use the same reasoning as for the case $d = n-2$ in the proof of Lemma 5.1.

Now consider case (iii). Using Lemmas 4.5 and 3.2 in $\operatorname{aff} P$ it is enough to show that $\Phi(sT^{n-2}) = 0$ for $s > 0$, where $T^{n-2} := \operatorname{conv}\{e_2, \dots, e_n\}$. Here we can use the same reasoning as for the case $d = n-1$ in the proof of Lemma 5.1.

Finally consider case (iv). Lemma 5.1 shows that $\Phi\, P = 0$ if $o \in P$. Using cases (i)-(iii) and Lemma 3.5 in dimension $n-1$ gives the desired result. □

6.2. **Theorem.** *Let $\Phi\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ be a simple $\mathrm{SL}(n)$-contravariant valuation. Assume further that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $I_y \subseteq (0, +\infty)$ such that $\{\Phi(sT^n)(y) : s \in I_y\}$ is not dense in $\mathbb{R}$. If $n \geq 3$, then there exist constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\Phi\, P = c_1\, \Delta_p^+\, P + c_2\, \Delta_p^-\, P$$

for all $P \in \mathcal{P}^n$.

Proof. Theorem 5.2 implies that there are constants $c_1, c_2 \in \mathbb{R}$ such that

$$\Phi\, P = c_1 h\left(\Pi_p^+\, P, .\right)^p + c_2 h\left(\Pi_p^-\, P, .\right)^p$$

for all $P \in \mathcal{P}_o^n$. Since $c_1\, \Delta_p^+ + c_2\, \Delta_p^-$ is a simple $\mathrm{SL}(n)$-contravariant valuation which coincides with Φ on $\mathcal{P}_o^n$, the assertion follows from Lemma 3.5. □

6.3. **Theorem.** *Let $\Phi\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ be an $\mathrm{SL}(n)$-contravariant valuation. Assume further that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $I_y \subseteq (0, +\infty)$ such that $\{\Phi(sT^n)(y) : s \in I_y\}$ is not dense in $\mathbb{R}$ and that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $J_y \subseteq (0, +\infty)$ such that $\{\Phi(sT^{n-1})(y) : s \in J_y\}$ is not dense in $\mathbb{R}$, where $T^{n-1} := \operatorname{conv}\{e_1, \dots, e_n\}$. If $n \geq 3$, then there exist constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that*

$$\Phi\, P = c_1 h\left(\Pi_p^{+>}\, P, .\right)^p + c_2 h\left(\Pi_p^{->}\, P, .\right)^p + c_3 h\left(\Pi_p^{+<}\, P, .\right)^p + c_4 h\left(\Pi_p^{-<}\, P, .\right)^p$$

for all $P \in \mathcal{P}^n$.

Proof. By replacing T^n with T^{n-1} and Lemmas 5.1 with 6.1 in the proof of Theorem 5.2 we see that there exist constants d_3, d_4 such that

$$\Phi(sT^{n-1}) = d_3 h\left(\Pi_p^{+>}(sT^{n-1}), .\right)^p + d_4 h\left(\Pi_p^{->}(sT^{n-1}), .\right)^p$$

for $s > 0$. The constants are given by

$$d_3 = (n-1)!\, \Phi(T^{n-1})(e_1) \quad \text{and} \quad d_4 = (n-1)!\, \Phi(T^{n-1})(-e_1).$$

Define $\Psi\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ by

$$\Psi\, P = \Phi\, P - d_3 h\left(\Pi_p^{+>}\, P, .\right)^p - d_4 h\left(\Pi_p^{->}\, P, .\right)^p$$

for all $P \in \mathcal{P}^n$. Note that Ψ is an $\mathrm{SL}(n)$-contravariant valuation. Using Lemma 6.1, $\Psi(sT^{n-1}) = 0$, the $\mathrm{SL}(n)$-contravariance of Ψ and Theorem 3.1 we see that Ψ is simple. Now Theorem 6.2 implies that there exist constants $d_1, d_2 \in \mathbb{R}$ such that

$$\Psi\, P = d_1\, \Delta_p^+\, P + d_2\, \Delta_p^-\, P$$

for all $P \in \mathcal{P}^n$. The definition of Ψ yields

$$\begin{aligned}\Phi\, P &= \Psi\, P + d_3 h\left(\Pi_p^{+>} P, .\right)^p + d_4 h\left(\Pi_p^{->} P, .\right)^p \\ &= d_1\, \Delta_p^+\, P + d_2\, \Delta_p^-\, P + d_3 h\left(\Pi_p^{+>} P, .\right)^p + d_4 h\left(\Pi_p^{->} P, .\right)^p \\ &= d_1\left(h\left(\Pi_p^{+>} P, .\right)^p - h\left(\Pi_p^{-<} P, .\right)^p\right) + d_2\left(h\left(\Pi_p^{->} P, .\right)^p - h\left(\Pi_p^{+<} P, .\right)^p\right) \\ &\quad + d_3 h\left(\Pi_p^{+>} P, .\right)^p + d_4 h\left(\Pi_p^{->} P, .\right)^p \\ &= (d_1 + d_3) h\left(\Pi_p^{+>} P, .\right)^p + (d_2 + d_4) h\left(\Pi_p^{->} P, .\right)^p \\ &\quad - d_2 h\left(\Pi_p^{+<} P, .\right)^p - d_1 h\left(\Pi_p^{-<} P, .\right)^p\end{aligned}$$

for all $P \in \mathcal{P}^n$. Define $c_1 = d_1 + d_3$, $c_2 = d_2 + d_4$, $c_3 = -d_2$ and $c_4 = -d_1$ to complete the proof. □

6.4. *Remark.* Similar to the $\mathcal{P}_o^n$ case, the additional regularity assumptions in Theorems 6.2 and 6.3 each just have to hold for only one standard basis vector and its reflection at the origin.

6.5. **Corollary.** *Let $\Phi\colon \mathcal{P}^n \to \mathcal{K}_o^n$ be an* SL(n)*-contravariant L_p-Minkowski valuation. If $n \geq 3$, then there exist constants $c_1, c_2, c_3, c_4 \geq 0$ such that*

$$\Phi\, P = c_1\, \Pi_p^{+>} P +_p c_2\, \Pi_p^{->} P +_p c_3\, \Pi_p^{+<} P +_p c_4\, \Pi_p^{-<} P$$

for all $P \in \mathcal{P}^n$.

Proof. Since $\Phi\, P \in \mathcal{K}_o^n$, we have $h\left(\Phi\, P, y\right) \geq 0$ for all $P \in \mathcal{P}^n$ and $y \in \mathbb{R}^n$. Therefore the map

$$P \mapsto h\left(\Phi\, P, .\right)^p, \quad P \in \mathcal{P}^n,$$

satisfies the assumptions of Theorem 6.3. Thus there exist constants $d_1, d_2, d_3, d_4 \in \mathbb{R}$ such that

$$h\left(\Phi\, P, .\right)^p = d_1 h\left(\Pi_p^{+>} P, .\right)^p + d_2 h\left(\Pi_p^{->} P, .\right)^p + d_3 h\left(\Pi_p^{+<} P, .\right)^p + d_4 h\left(\Pi_p^{-<} P, .\right)^p$$

for all $P \in \mathcal{P}^n$. Using Example 4.4 we obtain

$$\begin{aligned} d_1 &= (n-1)! h\left(\Phi\, T^n, e_1\right)^p, \\ d_2 &= (n-1)! h\left(\Phi\, T^n, -e_1\right)^p, \\ d_3 &= (n-1)! h\left(\Phi(e_1 + T^n), e_2 - e_1\right)^p, \\ d_4 &= (n-1)! h\left(\Phi(e_1 + T^n), e_1 - e_2\right)^p. \end{aligned}$$

It follows that $d_1, d_2, d_3, d_4 \geq 0$. Defining $c_1 = \sqrt[p]{d_1}, c_2 = \sqrt[p]{d_2}, c_3 = \sqrt[p]{d_3}$ and $c_4 = \sqrt[p]{d_4}$ completes the proof. □

The second main theorem from the introduction now follows from Corollary 6.5 and the fact that $\Pi_p^{+>}$, $\Pi_p^{->}$, $\Pi_p^{+<}$ and $\Pi_p^{-<}$ have the desired properties.

6.6. *Remark.* Corollary 6.5 implies Corollary 5.4. To see this, note that we can extend an SL(n)-contravariant L_p-Minkowski valuation Φ on $\mathcal{P}_o^n$ to $\mathcal{P}^n$ by

$$\Phi\, P = \Phi(P_o)$$

for all $P \in \mathcal{P}^n$, where $P_o := \operatorname{conv}(\{o\} \cup P)$.

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