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FINITELY PRESENTED EXPANSIONS OF GROUPS, SEMIGROUPS, AND ALGEBRAS

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Abstract. Finitely presented algebraic systems, such as groups and semigroups, are of foundational interest in algebra and computation. Finitely presented algebraic systems necessarily have a computably enumerable (c.e. for short) word equality problem and these systems are finitely generated. Call finitely generated algebraic systems with a c.e. word equality problem computably enumerable. Computably enumerable finitely generated algebraic systems are not necessarily finitely presented. This paper is concerned with finding finitely presented expansions of finitely generated c.e. algebraic systems. The method of expansions of algebraic systems, such as turning groups into rings or distinguishing elements in the underlying algebraic systems, is an important method used in algebra, model theory, and in various areas of theoretical computer science. Bergstra and Tucker proved that all c.e. algebraic systems with decidable word problem possess finitely presented expansions. Then they, and, independently, Goncharov asked if every finitely generated c.e. algebraic system has a finitely presented expansion. In this paper we build examples of finitely generated c.e. semigroups, groups, and algebras that fail to possess finitely presented expansions, thus answering the question of Bergstra-Tucker and Goncharov for the classes of semigroups, groups and algebras. We also construct an example of a residually finite, infinite, and algorithmically finite group, thus answering the question of Miasnikov and Osin. Our constructions are based on the interplay between key concepts and known results from computability theory (such as simple and immune sets) and algebra (such as residual finiteness and the theorem of Golod-Shafaverevich).

1. Introduction

1.1. **Computably enumerable universal algebras.** Finitely presented algebraic systems, such as groups and semigroups, are of foundational interest in algebra and computation. Finitely presented algebraic systems necessarily have a computably enumerable (c.e.) word equality problem and these systems are finitely generated. We call finitely generated algebraic systems with a c.e. word equality problem computably enumerable (see a formal definition later). Computably enumerable finitely generated algebraic systems are not necessarily finitely presented. For instance, the wreath product of the group of integers with itself is a finitely generated group that is not finitely presented. This paper is concerned with finding finitely presented expansions of finitely generated c.e. algebraic systems. The method of expansions of algebraic systems, such as turning groups into rings or distinguishing elements in the underlying algebraic systems, is an important method

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used in algebra, model theory, and in various areas of theoretical computer science, e.g. in the development of the algebraic theory of abstract data types. Bergstra and Tucker [1], [2], motivated by research in abstract data types and their algebraic specifications, proved that all c.e. algebraic systems with decidable word problem possess finitely presented expansions. In fact, later Bergstra and Tucker proved that as few as six new operations and four equations would suffice to build finitely presented expansions of such algebraic systems. Then they [1], [2] and, independently, Goncharov [7] asked if every finitely generated c.e. algebraic system has a finitely presented expansion. In this paper we build examples of finitely generated c.e. semigroups, groups, and algebras that fail to possess finitely presented expansions, thus answering the question of Bergstra-Tucker and Goncharov for the classes of semigroups, groups and algebras. We also construct examples of residually finite, infinite, and algorithmically finite groups. This answers the question posed by Miasnikov and Osin in [17]. Our constructions are based on the interplay between key concepts and known results from computability theory (such as simple and immune sets) and algebra (such as residual finiteness and the theorem of Golod-Shafaverevich).

A *universal algebra* $\mathcal{A}$ is a tuple $(A; f_1, \ldots, f_n)$, where A is a non-empty set called the *domain* of $\mathcal{A}$ and each f_i is a total function $A^{k_i} \to A$ called a *basic operation* of arity k_i. The *signature* σ of $\mathcal{A}$ is the sequence $f_1, \ldots, f_n$ of symbols representing the operations. Operations of arity 0 are called *constants*, and we denote them by symbols c or d (possibly with indices). We always assume that the signature σ contains a function symbol of arity greater than 0, and at least one constant. An *expansion* of the universal algebra $\mathcal{A}$ is any universal algebra of the form $\mathcal{A}' = (A; f_1, \ldots, f_n, h_1, \ldots, h_k)$. The signature of $\mathcal{A}'$ is then the expansion $f_1, \ldots, f_n, h_1, \ldots, h_k$ of the signature of $\mathcal{A}$. We sometimes refer to each of the operations h_1, …, h_k as a new operation. For a background on universal algebras see [6]. Background from a computer science perspective is given in [15]. Often universal algebras are called algebras for short. However, by an *algebra* we always mean a ring, not necessarily commutative, that forms a vector space over a field k. Note that every algebra $(A; +, \times, 0, 1)$ over a finite field k can be identified with the universal algebra $(A; +, \times, f_\alpha, 0, 1)_{\alpha \in k}$, where each f_α is a unary operation that represents the scalar multiplication by $\alpha \in k$, that is, $f_\alpha(a) = \alpha \cdot a$ for all $a \in A$. We will be using this identification.

A universal algebra $\mathcal{A}$ is *computable* if the domain A is a computable set and each basic operation of $\mathcal{A}$ is a computable function. For simplicity, when we are given such a computable universal algebra, we can assume that A is a subset of ω; indeed, if A is infinite, we can always assume that $A = \omega$. When convenient we can also take the domain A to be a computable subset of any space that can be naturally identified with ω via a standard coding, such as the set $\{0, 1\}^*$ of binary strings or the set of polynomials with coefficients over integers or over finite rings. Thus, for instance, a computable group is a group $(G; \cdot)$ where G is a computable subset of ω (or any space that can be naturally identified with ω) and $\cdot$ is a computable group operation. A. Mal′cev [13] and M. Rabin [19] were the first who initiated the development of the theory of computable algebra. This is now one of the core subjects in computability and model theory [8], [9]. V. Stoltenberg-Hansen and J. V. Tucker [21] studied computable algebras from a computer science perspective.

There are many examples of computable universal algebra; for instance the groups $(Z^n; +)$ or the arithmetic $(\omega; +\times, 0, 1)$. There are many other important examples, however, in which the domain and basic operations can be made computable only if we are willing to identify elements of the domain via a computably enumerable equivalence relation. Examples include finitely presented universal algebras (such as finitely presented groups) and the Lindenbaum Boolean algebras of computably enumerable first order theories (such as the Peano arithmetic). To capture this class of universal algebras, we have the notion of a computably enumerable (c.e.) universal algebra, which will be our main object of study in this paper. To define c.e. algebras, we begin with a few auxiliary definitions.

Let E be an equivalence relation on a set B. Denote the equivalence class of x by $[x]_E$, and the set of all such equivalence classes by B/E. We say that a function $f : B^n \to B$ *respects* E if whenever $[x_i]_E = [y_i]_E$ for all $i < n$, we have $[f(x_0, \ldots, x_{n-1})]_E = [f(y_0, \ldots, y_{n-1})]_E$. If f respects E, then f naturally defines the function, that we also denote by f, from $(B/E)^n$ to B/E as follows. For all $[x_0], \ldots, [x_{n-1}] \in B/E$, set $f([x_0]_E, \ldots, [x_{n-1}]_E) = [f(x_0, \ldots, x_{n-1})]_E$.

Let $\mathcal{B}$ be a universal algebra. An equivalence relation E on the domain B is called a *congruence relation* of $\mathcal{B}$ if every basic operation of $\mathcal{B}$ respects E. For $\mathcal{B}$ and its congruence relation E, denote $\mathcal{B}/E$ to be the quotient of the algebra $\mathcal{B}$ by E. Note that the universal algebras $\mathcal{B}$ and $\mathcal{B}/E$ are of the same signature.

Definition 1.1 (Computably enumerable universal algebra)**.** A universal algebra is *computably enumerable* (*c.e.*) if it is of the form $\mathcal{B}/E$ for a computable universal algebra $\mathcal{B}$ and a c.e. congruence relation E of $\mathcal{B}$.

For a fixed universal c.e. algebra $\mathcal{A} = \mathcal{B}/E$, we often drop the subscript and write $[x]$ in place of $[x]_E$. We also identify elements x of $\mathcal{B}$ with the elements $[x]$ of $\mathcal{A}$. We say that two elements x and y are *equivalent* if $(x, y) \in E$, and we speak of E as the *equality relation* of $\mathcal{A}$. It is easy to see that we can always assume that the domain of $\mathcal{B}$ is infinite, and hence that it is in fact ω. We note that in literature computably enumerable universal algebras are sometimes called *semicomputable*; see for instance [1], [2] and [21].

The dual of c.e. universal algebras are co-c.e. universal algebras defined below. Recall that a set is co-c.e. if its complement is c.e.

Definition 1.2 (Co-computably enumerable universal algebra)**.** A *co-computably enumerable (co-c.e.) universal algebra* is of the form $\mathcal{B}/E$ for a computable universal algebra $\mathcal{B}$ and a co-c.e. congruence relation E on the domain of $\mathcal{B}$.

An example of a co-c.e. universal algebra is the group G generated by a finite number of computable permutations $g_1, \ldots, g_k$ of the set ω of natural numbers. If g and g' are elements of this group, then their non-equality is confirmed by the existence of an n such that $g(n) \neq g'(n)$.

Let E be an equivalence relation on set B that has an order $\leqslant$ of type ω. For instance, B can be the set of all binary strings that are ordered length-lexicographically. The natural representatives of the E-equivalence classes are then the minimal elements (with respect to $\leqslant$) of the equivalence classes.

Definition 1.3 (Transversal)**.** The *transversal* of the equivalence relation E on B is the following set:

$$tr(E) = \{x \in B \mid \forall y(y < x \to (x, y) \notin E)\}.$$

The following proposition easily follows from the definitions.

Proposition 1.4. *Consider the quotient universal algebra $\mathcal{A} = \mathcal{B}/E$, where $\mathcal{B}$ is computable.*

(1) *If E is c.e., then $tr(E)$ is computable iff E is computable.*
(2) *If E is co-c.e., then the transversal $tr(E)$ is c.e.*
(3) *If E is c.e. or co-c.e. and $\mathcal{A}$ is finite, then E is computable.*
(4) *$\mathcal{A}$ is computable iff E is c.e. and co-c.e.* □

For modern treatment of computable algebras and computable model theory, see for instance [8].

1.2. **Homomorphisms and congruence relations.** Let $\mathcal{B}$ and $\mathcal{C}$ be universal algebras of the same signature σ. We recall the following definition.

Definition 1.5. A *homomorphism* from universal algebra $\mathcal{B}$ into $\mathcal{C}$ is a map $h : B \to C$ such that for all f functions of arity k from the signature and $b_1, \ldots, b_k \in B$, we have $h(f(b_1, \ldots, b_k)) = f(h(b_0), \ldots, h(b_k))$.

In particular, if h is a homomorphism from universal algebra $\mathcal{B}$ into $\mathcal{C}$, then the values of constants in $\mathcal{B}$ are mapped to the values of the corresponding constants in $\mathcal{C}$.

There is a one-to-one correspondence between homomorphisms of a universal algebra $\mathcal{B}$ and its congruence relations. Namely, for any universal algebra $\mathcal{B}$ and its congruence relation E, we have a homomorphism from $\mathcal{B}$ onto $\mathcal{B}/E$ defined by $b \to [b]_E$. Conversely, any homomorphism h from $\mathcal{B}$ determines the congruence relation E on $\mathcal{B}$ as follows: $E = \{(x, y) \mid h(x) = h(y)\}$. Thus, from the definitions above it is clear that every c.e. universal algebra is a homomorphic image of a computable universal algebra.

Definition 1.6 (Terms)**.** Let σ be a signature. *Terms* of the signature are defined inductively as follows. Each constant symbol and a variable x is a term. If $t_1, \ldots, t_k$ are terms and f is a function symbol in σ of arity k, then the expression $f_i(t_1, \ldots, t_{k_i})$ is a term.

Let $\mathcal{A}$ be a universal algebra. For each term t, its *interpretation* $i(t)$ in $\mathcal{A}$ is an operation in $\mathcal{A}$ defined in the obvious way; that is, $i(c) = c_a$ and $i(x) = id_A(x)$ in the base case (where id_A is the identity function), and $i(f(t_1, \ldots, t_k)) = f(i(t_1), \ldots, i(t_k))$ in the inductive step. We write every term t whose variables are among $x_1, \ldots, x_n$ as $t(x_1, \ldots, x_n)$. This term, as we have just explained, defines an n-ary operation on A. We use this in the following definition:

Definition 1.7 (Algebraic terms)**.** An *algebraic term* of the universal algebra $\mathcal{A}$ is a mapping of the form $t(a_1, ..., a_{i-1}, x, a_i, \ldots, a_{n-1})$, where $a_1 \ldots, a_{n-1}$ are elements of $\mathcal{A}$ and $t(x_1, \ldots, x_n)$ is a term.

Here is a simple lemma connecting algebraic terms with congruence relations.

Lemma 1.8. *Let $\mathcal{A}$ be an algebra and E be an equivalence relation on $\mathcal{A}$. The relation E is a congruence relation of $\mathcal{A}$ if and only if any algebraic term of $\mathcal{A}$ respects E.*

Proof. It is clear that if E is a congruence relation, then any algebraic term of $\mathcal{A}$ respects E. Suppose that every algebraic term respects E. Consider any n-tuple of pairs $(a_1, b_1), \ldots, (a_n, b_n) \in E$ and an n-ary basic operation f. Then

$(f(a_1, a_2, \ldots, a_n), f(b_1, a_2, \ldots, a_n)) \in E$ since $f(x, a_2, \ldots, a_n)$ is an algebraic term. Similarly, $f(b_1, x, a_3, \ldots, a_n)$ is an algebraic term, and thus $(f(b_1, a_2, a_3, ..., a_n), f(b_1, b_2, a_3, \ldots, a_n)) \in E$. Continuing this, by transitivity of E, we obtain that the pair $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n))$ belongs to E. □

1.3. **Term algebra.** We fix a signature σ. We call a term of the signature a *ground term* if it contains no variables. The set T_G of all ground terms can be turned into a universal algebra of the signature σ as follows. For each function symbol f (of arity k) and a tuple $(t_1, \ldots, t_k)$ of ground terms, set the value of f on the tuple to be $f_i(t_1, \ldots, t_k)$. We call the resulting algebra the *term algebra* and denote it by $\mathcal{T}_G$.

Recall that $i(t)$ denotes the interpretation of the term t in a given universal algebra $\mathcal{A}$. In particular, in the term algebra $\mathcal{T}_G$, we have $i(t) = t$ for all $t \in T$. We say that $\mathcal{A}$ is *generated by the constants* if every element of $\mathcal{A}$ is the interpretation of some ground term. In addition, we say that $\mathcal{A}$ is *freely generated* if whenever t, t' are different ground terms, we have $i(t) \neq i(t')$. It is not hard to see that the term algebra $\mathcal{T}_G$ has the following properties (see for instance [6]):

(1) The term algebra $\mathcal{T}_G$ is finitely generated and computable; the generators are the constants of the signature.
(2) Any universal algebra $\mathcal{A}$ of the given signature generated by the (values of the) constants is a homomorphic image of $\mathcal{T}_G$.
(3) A universal algebra $\mathcal{A}$ is freely generated if and only if it is isomorphic to $\mathcal{T}_G$.

We elaborate on property (2). Take a universal algebra $\mathcal{A}$ generated by the values of constants. Then there exists a congruence relation E on $\mathcal{T}_G$ such that $\mathcal{A}$ is isomorphic to $\mathcal{T}_G/E$. The congruence relation E is often called *the word problem for* $\mathcal{A}$. If the word problem for $\mathcal{A}$ is decidable, then the algebra is clearly computable (see Proposition 1.4). In this sense, the word problem for $\mathcal{A}$ represents computability-theoretic complexity of the universal algebra $\mathcal{A}$.

1.4. **Finitely presented universal algebras.** The concept of finitely presented groups can easily be translated into the context of universal algebras. We will explain this here.

Definition 1.9 (Equational specifications)**.** An *equation* is an expression of the form $t = t'$, where t and t' are terms possibly with variables. An *equational specification* is a finite set of equations. We denote equational specifications by S.

A good example of equational specifications are group axioms; in this case the signature of groups consists of the group operation, inverse, and the constant symbol for the identity. The set of axioms for rings also forms an equational specification. We note that equational specifications are finite objects.

Let $\mathcal{A}$ be a universal algebra and S an equational specification. We say that the algebra *satisfies* S if for all equations $t = p$ from S, we have $i(t) = i(p)$ in $\mathcal{A}$ for all interpretations i. The class of all universal algebras that satisfy a given equational specification is closed under subalgebras, homomorphisms, and products (see [6], [13]).

Let S be an equational specification. In the term algebra $\mathcal{T}_G$ consider the congruence relation $E(S)$ *generated* by S:

$$E(S) = \{(t,p) \mid t,p \in T_G \text{ and the equality } t = p \text{ can be deduced from the equations in } S\}.$$

Thus, one can naturally consider the following quotient algebra:

$$\mathcal{T}(S) = \mathcal{T}_G/E(S).$$

The following is a central definition of this paper:

Definition 1.10 (Finitely presented universal algebra)**.** We say that a universal algebra $\mathcal{A}$ is *finitely presented* if there exists an equational specification S such that $\mathcal{A}$ is isomorphic to $\mathcal{T}(S)$.

Examples of finitely presented universal algebras are finitely presented groups and semigroups. It is not hard to see that the finitely presented universal algebra $\mathcal{T}(S)$ satisfies the properties below (see, for instance [13]):

(1) $\mathcal{T}(S)$ is generated by the constants of the signature.
(2) $\mathcal{T}(S)$ is computable enumerable.
(3) $\mathcal{T}(S)$ satisfies S.
(4) Any universal algebra $\mathcal{A}$ generated by the (values of the) constants and that satisfies S is a homomorphic image of $\mathcal{T}_G$.
(5) If $\mathcal{B}$ satisfies S, generated by the constants, and satisfies property (4) above, then $\mathcal{T}(S)$ is isomorphic to $\mathcal{B}$.

Thus, finitely presented universal algebras are all computably enumerable universal algebras. Property (4) expresses the fact that the universal algebra $\mathcal{T}(S)$ is an initial object in the class of all systems that are generated by the constants and that satisfy S. The last property expresses the fact that the universal algebra $\mathcal{T}(S)$ is unique.

1.5. **The equational specification problem.** To motivate the problem we first present a simple example of Bergstra and Tucker [1]. Consider the universal algebra $(\omega; 0, S, 2^x)$, where S is the successor function. It is not too hard to see that this universal algebra is not finitely presented. For the proof the reader can consult [1]. However, the expansion

$$(\omega; 0, S, 2^x, +, \times)$$

is finitely presented. The equational presentation of this expanded universal algebra is given by the following set of equations:

$$x + y = y + x,\ (x + y) + z = x + (y + z),\ x + 0 = x,\ x + S(y) = S(x + y),$$
$$x \times 0 = 0,\ x \times S(0) = x,\ x \times S(y) = x \times y + x,\ 2^0 = S(0),\ 2^{x+S(0)} = 2^x \times S(S(0)).$$

These are just inductive definitions of the addition, multiplication, and the exponentiation operations on the set of natural numbers. It turns out that this method of expanding signatures allows us to build finitely presented expansions in many cases. Namely, Bergstra and Tucker proved that any computable universal algebra has a finitely presented expansion [1]. This fact led Bergstra and Tucker [1], [2], and, independently, Goncharov [7] to ask the following question known as the *equational specification problem.*

Does every finitely generated computably enumerable universal algebra possess a finitely presented expansion?

Contrary to the case of computable universal algebras, Kassymov constructs an example of a finitely generated c.e. universal algebra that has no finitely presented expansion [10]. A similar example, using Kolmogorov complexity, is constructed by Khoussainov in [11]. These examples are built specifically in order to answer the question, and the signatures of these examples consist of unary operation symbols only. The methods that construct such examples do not directly carry over to build, for instance, semigroups, groups or algebras with no finitely presented expansions. Hence, a natural problem arises to find such examples among semigroups, groups and algebras. Khoussainov and Hirschfeldt have recently constructed a finitely generated c.e. semigroup that has no finitely presented expansion [12]. Their method, however, cannot be applied to build examples of groups that do not have finitely presented expansions. In the rest of the paper, we construct such examples of groups and algebras. Our constructions are based on the interplay between some of the key concepts and results of computability theory (e.g. simple and immune sets) and algebra (e.g. residual finiteness and the theorem of Golod-Shafarevich).

2. Algorithmically finite universal algebras

2.1. **Definitions and some lemmas.** Let $\mathcal{A}$ be a universal algebra of the form $\mathcal{B}/E$, where E is a congruence relation of a computable universal algebra $\mathcal{B}$. Recall that elements of $\mathcal{A}$ are E-equivalence classes denoted by $[x]$, where x is an element of $\mathcal{B}$. Generally, many infinite c.e. algebras are effectively infinite, in the sense that one can effectively list an infinite sequence of pairwise distinct elements of the algebra. For example, any infinite computable algebra is clearly effectively infinite. We formally define this property as follows.

Definition 2.1 (Algorithmically finite universal algebra)**.** The universal algebra $\mathcal{A} = \mathcal{B}/E$ is *effectively infinite* if there exists an algorithm that enumerates a sequence $b_0, b_1, \ldots$ such that $[b_i] \neq [b_j]$ for all $i \neq j$. If $\mathcal{A}$ is not effectively infinite, then we call $\mathcal{A}$ *algorithmically finite*.

Thus, all finite universal algebras are examples of algorithmically finite algebras. But these are trivial examples. We will be constructing computably enumerable, infinite, finitely generated and algorithmically finite algebras. Note that our definition above does not require the algebra to be finitely generated. The term *algorithmically finite* was coined in [17] and we follow it in this paper.

Generally, constructing algorithmically finite but infinite universal algebras is not too hard. Here is one simple example. Let $a_0 < a_1 < \ldots$ be an infinite list of natural numbers such that no computable function f exists for which $a_i < f(i)$ for all sufficiently large i. Such sets exist [20]. The sequence $a_0 < a_1 < \ldots$ defines the following equivalence relation E:

$$E = \{(n, m) \mid \exists i(a_i \leqslant n < a_{i+1} \ \&\ a_i \leqslant m < a_{i+1})\}.$$

Note that the transversal $tr(E)$ of E is the set $\{a_0, a_1, \ldots\}$. It is not hard to see that all universal algebras whose domains coincide with ω/E are algorithmically finite.

Now we would like to prove several interesting properties of algorithmically finite universal algebras. We start with the following easy proposition whose proof is left to the reader.

Proposition 2.2. *Let $\mathcal{A}$ be an algorithmically finite universal algebra. Then $\mathcal{A}$ satisfies the following properties:*

(1) *Every expansion of $\mathcal{A}$ is algorithmically finite.*
(2) *All subalgebras of $\mathcal{A}$ are algorithmically finite.*
(3) *For every term $t(x)$ and element a of $\mathcal{A}$, the sequence*

$$a, t(a), t(t(a)), \ldots$$

is eventually periodic. In particular, if a group is algorithmically finite, then it is a periodic group.
(4) *All homomorphic images of $\mathcal{A}$ are algorithmically finite.* □

We need the following definition borrowed from group theory.

Definition 2.3 (Residually finite universal algebra)**.** A universal algebra $\mathcal{A}$ is *residually finite* if for any two distinct elements a and b of $\mathcal{A}$, there is a homomorphism of $\mathcal{A}$ onto a finite universal algebra such that the images of a and b under the homomorphism are distinct.

The lemma below does not assume algorithmic finiteness of underlying universal algebras.

Lemma 2.4 (Separation Lemma I)**.** *Let $\mathcal{A} = \mathcal{T}_G/E$ be a residually finite universal algebra. For all distinct elements x and y of $\mathcal{A}$ there exists a set $S(x, y) \subseteq T_G$ such that each of the following is satisfied:*

(1) *$S(x, y)$ is a computable set.*
(2) *$x \in S(x, y)$ and $y \notin S(x, y)$.*
(3) *The set $S(x, y)$ is E-closed, that is, for each element $u \in S(x, y)$ the E-equivalence class $[u]_E$ of u is a subset of $S(x, y)$.*

Proof. Note that there is no any requirement on E being computably enumerable. Let $\mathcal{B}$ be a finite universal algebra such that there exists a homomorphism h from $\mathcal{A}$ onto $\mathcal{B}$ for which we have $h(x) \neq h(y)$. Set

$$S(x, y) = \{u \in T_G \mid h([u]_E) = h(x)\}.$$

Since $\mathcal{B}$ is generated by the values of constants and $\mathcal{B}$ is finite, it is clear that the set $S(x, y)$ is computable. By definition, $x \in S(x, y)$ and $y \notin S(x, y)$. The set $S(x, y)$ is E-closed because h is a homomorphism, and the universal algebras $\mathcal{T}_G$, $\mathcal{A}$ and $\mathcal{B}$ are all generated by the (values of) constants of the signature. □

2.2. **Algorithmic and residual finiteness.** In this subsection we prove one of the main results of the paper. We prove that for computably enumerable algorithmically finite and finitely generated universal algebras, the residual finiteness property is invariant under expansions. Namely, we prove the following theorem.

Theorem 2.5. *If $\mathcal{A} = \mathcal{T}_G/E$ is a computably enumerable, algorithmically finite, and residually finite universal algebra, then all its c.e. expansions are also residually finite.*

Proof. If $\mathcal{A}$ is finite, then the theorem is obviously true. So, we assume that $\mathcal{A}$ is infinite. Let $\mathcal{A}' = (\mathcal{A}, h_1, \ldots, h_k)$ be a c.e. expansion of $\mathcal{A}$. Select two distinct elements x and y of $\mathcal{A}$ (that will clearly be also distinct in $\mathcal{A}'$). We want to show

that there exists a finite homomorphic image of the universal algebra $\mathcal{A}'$ in which the images of x and y are distinct. To prove this we need several definitions and lemmas.

Let a, b be two elements of $\mathcal{A}'$. There exists the minimal congruence relation, denoted by $C_{(a,b)}$, that contains the pair (a, b). In particular, if $a = b$ in $\mathcal{A}$, then $C_{(a,b)}$ coincides with the equality relation E on $\mathcal{A}'$. The congruence relation $C_{(a,b)}$ is the intersection of all congruence relations that contain the pair (a, b). The following lemma due to Mal′cev gives an algorithmic description of the congruence relation $C_{(a,b)}$. The proof can easily be checked by the reader.

Lemma 2.6 ([14]). *For all c, d from $\mathcal{A}$, the pair (c, d) belongs to $C_{(a,b)}$ if and only if there exists a finite sequence $e_0, \ldots, e_n$ of elements in $\mathcal{A}$ and algebraic terms $t_0, \ldots, t_{n-1}$ such that $c = e_0$, $d = e_n$, and for all $i = 0, \ldots, n-1$ we have $\{e_i, e_{i+1}\} = \{t_i(a), t_i(b)\}$.* □

Thus, from the lemma it is clear that there exists an effective process that, given a pair of elements (a, b), extracts an algorithm that generates the congruence relation $C_{(a,b)}$ defined by the pair (a, b). Hence, $C_{(a,b)}$ is a c.e. congruence relation.

For the selected elements x and y, consider the E-closed set $S(x, y)$ constructed in the Separation Lemma above (Lemma 2.4). Now we define the following binary relation $E_{(x,y)}$ on the expanded universal algebra $\mathcal{A}'$. For elements a and b, we have $(a, b) \in E_{(x,y)}$ if and only if the congruence relation $C_{(a,b)}$ identifies no elements in $S(x, y)$ with elements in the complement of $S(x, y)$. In other words, $(a, b) \in E_{(x,y)}$ if and only if for all u in $S(x, y)$ and for all v not in $S(x, y)$ we have $(u, v) \notin C_{(a,b)}$.

Lemma 2.7. *The binary relation $E_{(x,y)}$ is a congruence relation of the expanded universal algebra $\mathcal{A}'$.*

Indeed, first we show that $E_{(x,y)}$ is an equivalence relation. Reflexivity and symmetry properties of $E_{(x,y)}$ are clear. Thus, it suffices to prove that $E_{(x,y)}$ is a transitive relation. Assume that $(a, b), (c, d) \in E_{(x,y)}$ but $(a, c) \notin E_{(x,y)}$. This implies that the congruence relation $C_{(a,c)}$ contains (u, v) such that $u \in S(x, y)$ and $v \notin S(x, y)$. From Lemma 2.6, we can assume that (u, v) are selected such that $\{u, v\} = \{p(a), p(c)\}$ for some algebraic term p. Say $p(a) = u$, and hence $p(a) \in S(x, y)$ and $p(c) \notin S(x, y)$. Then, we have $p(b) \in S(x, y)$ since $(a, b) \in E_{(x,y)}$, and similarly, $p(b) \notin S(x, y)$ since $(b, c) \in E_{(x,y)}$. This is a contradiction. Hence $E_{(x,y)}$ is an equivalence relation.

To show that $E_{(x,y)}$ is a congruence relation, we apply Lemma 1.8. Assume that $(a, b) \in E_{(x,y)}$. Then for any algebraic term t we have $(t(a), t(b)) \in E_{(x,y)}$. Otherwise, by Lemma 2.6, there is an algebraic term p such that one of the elements among $p(t(a))$ and $p(t(b))$ belongs to $S(x, y)$ and the other belongs to the complement of $S(x, y)$. Note that the composition of algebraic terms is again an algebraic term. This implies that $(a, b) \notin E_{(x,y)}$. This is a contradiction that proves the lemma.

Lemma 2.8. *The relation $E_{(x,y)}$ is a co-c.e. congruence relation.*

The lemma follows from the definition of $E_{(x,y)}$. Indeed, the pair (a, b) does not belong to $E_{(x,y)}$ if and only if the congruence relation $C_{(a,b)}$ contains a pair (u, v) such that $u \in S(x, y)$ and $v \notin S(x, y)$. Since the set $S(x, y)$ is computable and $C_{(a,b)}$ is a c.e. relation, we see that the complement of $E_{(x,y)}$ is computably enumerable. We have proved the lemma.

Now we finish the proof of the theorem. Consider the quotient universal algebra $\mathcal{A}/E_{(x,y)}$. This universal algebra, as we have just proved in the lemma above, is a co-c.e.universal algebra. If $\mathcal{A}/E_{(x,y)}$ is infinite, then by part (2) of Lemma 1.4, it is an effectively infinite algebra. This contradicts part (4) of Proposition 2.2 since $\mathcal{A}$ is algorithmically finite. Hence, $\mathcal{A}/E_{(x,y)}$ is finite. In this quotient universal algebra, by the choice of $S(x,y)$, the images of x and y are distinct, which was required to be proved. □

2.3. **The non-finite presentability (NFP) theorem.** Here we show that all finitely generated, infinite, computably enumerable, algorithmically finite and residually finite universal algebras fail to possess finitely presented expansions. We start with one simple lemma that also goes back to Mal′cev.

Lemma 2.9 (Mal′cev's lemma [13])**.** *If a universal algebra $\mathcal{A}$ is finitely presented and residually finite, then the word problem in $\mathcal{A}$ is decidable.*

Proof. Since $\mathcal{A}$ is finitely presented, it is of the form $\mathcal{T}(S) = T_G/E(S)$ as in Definition 1.10. So, $\mathcal{A}$ is a c.e. algebra with equality relation $E(S)$ being a c.e. set. To prove the lemma, it suffices to show that $E(S)$ is co-c.e. To enumerate the complement of E, we enumerate all finite algebras that satisfy S and are generated by the values of the constants of the signature. For each such algebra and each pair of distinct elements b and b' of that algebra, there are ground terms t and t' such that b and b' are the interpretations of t and t', respectively. Recall that all of these finite algebras are homomorphic images of $\mathcal{A}$ since $\mathcal{A}$ is specified by S. Let a and a' be the interpretations of the ground terms t and t' in $\mathcal{A}$. Then we know that $(a, a') \notin E(S)$. The fact that $\mathcal{A}$ is residually finite ensures that every pair of elements of the domain of $\mathcal{A}$ that is not in E is eventually discovered in this fashion. □

Theorem 2.10 (The NFP theorem)**.** *If $\mathcal{A} = \mathcal{T}_G/E$ is an infinite, finitely generated, computably enumerable, algorithmically finite and residually finite universal algebra, then no expansion of $\mathcal{A}$ is finitely presented.*

Proof. Assume that $\mathcal{A}'$ is a finitely presented expansion of $\mathcal{A}$. Then $\mathcal{A}'$ is residually finite by Theorem 2.5. Hence by Lemma 2.9, the word problem in $\mathcal{A}'$, that is, the equality relation E, is decidable. This is impossible since $\mathcal{A}'$ is algorithmically finite. □

In the next sections we apply the NFP theorem to provide examples of semigroups, groups, and algebras that fail to possess finitely presented expansions.

3. A semigroup example

We construct an infinite algorithmically finite, computably enumerable, finitely generated and residually finite semigroup. By the NFP theorem, this semigroup has no finitely presented expansions. The example is from [12]. In [12] this example of a semigroup is shown to not have finitely presented expansions. The proof in [12] does not use the NFP theorem but rather uses the more involved Kassymov lemma from [10] specifically suited for a constructed semigroup. We will use this example in the last section of this paper as well, where we build an effectively infinite algebra without finitely presented expansions.

Recall that a semigroup is a universal algebra with exactly one associative binary operation. Let $\{x, y\}^\star$ be the set of all finite binary strings over the alphabet $\{x, y\}$. We will denote the empty string by λ, and the length of a string u by $|u|$. Let $\cdot$ be the concatenation operation on strings. Then the universal algebra

$$\mathcal{A} = (\{x, y\}^\star; \cdot)$$

is a semigroup. It is a free two generated semigroup with generators x and y. Every semigroup with two generators is a homomorphic image of $\mathcal{A}$. The semigroup $\mathcal{A}$ is obviously computable.

We will construct our semigroup as a quotient of $\mathcal{A}$ by an appropriate congruence relation. We need a few definitions. A string $v \neq \lambda$ over the alphabet $\{x, y\}$ is a *substring* of u if u is of the form $u = u_1 v u_2$.

Definition 3.1. Let Z be a subset of $\{x, y\}^\star$. We say that a string u *realizes* Z if u contains a substring from Z. Otherwise, we say that u *avoids* Z. We denote the set of all words that realize Z by $R(Z)$.

It is clear that $Z \subseteq R(Z)$ and $R(Z) = R(R(Z))$ for all Z. Note that if a string u avoids Z, then all of its prefixes avoid Z. Hence the complement of $R(Z)$, that is, the set $\{x, y\}^\star \setminus R(Z)$ that we denote by $Av(Z)$, is a tree under the prefix relation. With each $Z \subseteq \{x, y\}^\star$ we associate the following equivalence relation:

$$\eta_Z = \{(p, q) \mid p = q \text{ or } p, q \in R(Z)\}.$$

Each equivalence class of η_Z is either a singleton or is the set $R(Z)$.

Lemma 3.2. *The equivalence relation η_Z is a congruence relation of the semigroup $\mathcal{A}$.*

Proof. Indeed, assume that (u_1, u_2) and (v_1, v_2) are in η_Z. If none of the elements among u_1, u_2, v_1, v_2 realizes Z, then $u_1 = u_2$ and $v_1 = v_2$. Hence, $u_1 \cdot v_1 = u_2 \cdot v_2$. Assume one of the strings, say u_1, realizes Z. Then u_2 must also realize Z. Hence both $u_1 \cdot v_1$ and $u_2 \cdot v_2$ must realize Z. Therefore, $(u_1 \cdot v_1, u_2 \cdot v_2) \in \eta_Z$. □

We denote the quotient semigroup $\mathcal{A}/\eta_Z$ by $\mathcal{A}(Z)$. It is clear that this semigroup is finitely generated by the elements $[x]$ and $[y]$. The following lemma can easily be derived from the definitions. We stress the last property of the lemma.

Lemma 3.3. *The semigroup $\mathcal{A}(Z)$ satisfies the following properties:*

(1) *$\mathcal{A}(Z)$ is c.e. if and only if the set $R(Z)$ is c.e.*
(2) *$\mathcal{A}(Z)$ is computable if and only if the set $R(Z)$ is computable.*
(3) *$\mathcal{A}(Z)$ is finite if and only if $Av(Z)$ is finite.*
(4) *$\mathcal{A}(Z)$ is residually finite.* □

We seek the desired semigroup among the semigroups of the form $\mathcal{A}(Z)$. Therefore our goal is to ensure that the semigroup $\mathcal{A}(Z)$ we build is infinite. To construct an infinite semigroup of the form $\mathcal{A}(Z)$, we need to make Z sparse. Cenzer, Dashti, and King [3] showed that there is a computable sequence $l_0, l_1, \ldots$ such that if Z contains at most one string of each length l_i, and no strings of any other length, then there is an infinite binary string that avoids all strings in Z, and hence $R(Z)$ is co-infinite. The sequence of lengths they give is $6 \cdot 2^{2i(i+3)}$. They did not need to be efficient, and Millar [18] proved the following sharp result that we use in our construction of the semigroup $\mathcal{A}(Z)$.

Lemma 3.4 (Miller's lemma [18])**.** *Consider the sequence* $5, 6, 7, 8, \ldots$. *If* $Z \subseteq \{x, y\}^\star$ *contains at most one string of each length* i *and no strings of lengths* 0, 1, 2, 3 *and* 4, *then there is an infinite binary string that avoids all strings in* Z, *and hence* $R(Z)$ *is co-infinite.* □

We use Miller's lemma above to prove the following result:

Lemma 3.5. *Let* $Z \subset \{0, 1\}^\star$. *If for each* k *there are at most* k *many strings of length* $\leqslant k + 4$ *in* Z, *then* $R(Z)$ *is co-infinite.*

Proof. List the elements $z_0, z_1, \ldots$ of Z so that $|z_i| \geqslant i + 5$ for all i. Note that avoiding a string implies avoiding all its extensions. Now, for each z_i, let x_i be the string of length $i+5$ such that z_i is an extension of x_i. In particular, if $|x_i| = i+5$, then we can select $x_i = z_i$. Consider the set $X = \{x_0, x_1, x_2, \ldots\}$. The set X contains exactly one string of each length $i+5$. Moreover, any string that avoids X also avoids Z. By Miller's lemma above, there are infinitely many strings avoiding X, whence $R(Z)$ is co-infinite. □

Now we need one computability-theoretic concept, the notion of a simple set. This notion will be used to build infinite and algorithmically finite semigroups, groups, and algebras.

Definition 3.6. A co-infinite c.e. subset of $\{x, y\}^*$ (or of ω) is *simple* if its complement does not contain infinite c.e. subsets.

Simple sets are not computable, and they exist [20]. In fact, every non-zero c.e. Turing degree contains a simple set [20]. It is easy to see that any co-infinite c.e. set that contains a simple set is again simple. We recall that infinite sets that contain no infinite c.e. subsets are called *immune* [20]. Thus, simple sets are c.e. sets whose complements are immune. Our goal is to build a simple set Z such that the semigroup $\mathcal{A}(Z)$ is infinite. Note that in this case $Z \subseteq R(Z)$, and whence $R(Z)$ is also a simple set.

Lemma 3.7. *There exists a simple set* Z *such that the semigroup* $\mathcal{A}(Z)$ *is infinite. In particular,* $R(Z)$ *is also a simple set.*

Proof. Let $W_0, W_1, \ldots$ be a standard enumeration of all c.e. subsets of $\{x, y\}^\star$. Let Z be the set of all u such that, for some i, the string u is the first string of length $\geqslant i + 5$ to be enumerated into W_i. Clearly, Z is a c.e. set. We show that both Z and $R(Z)$ are simple. This will imply that $\mathcal{A}(Z)$ is infinite.

It follows easily from the definition of Z that for each k, there are at most k many strings of length $\leqslant k + 4$ in Z. By Lemma 3.5, the set $R(Z)$ is co-infinite. If the set W_i is infinite, then it contains a u such that $|u| \geqslant i + 5$. Therefore, the set Z contains an element of W_i. Thus the complement of Z does not contain infinite c.e. sets, whence Z is simple. Therefore $R(Z)$ is also simple. □

Note that our construction of the set Z is a version of the original Post construction of a simple set (see for instance, [20]). We now fix the semigroup $\mathcal{A}(Z)$ constructed in Lemma 3.7 above.

Lemma 3.8. *There exists a finitely generated, infinite, computably enumerable, algorithmically finite and residually finite semigroup.*

Proof. Consider the semigroup $\mathcal{A}(Z)$. We have already proved that the semigroup $\mathcal{A}(Z)$ is c.e., infinite, and residually finite. Clearly, the semigroup is finitely generated. If $v_0, v_1, \ldots$ are such that $(v_i, v_j) \notin \eta_Z$ for all $i \neq j$, then at most one v_i is in $R(Z)$, so $v_0, v_1, \ldots$ cannot be a computably enumerable sequence, as the complement of $R(Z)$ contains no infinite c.e. sequence. Hence, $\mathcal{A}(Z)$ is algorithmically finite. □

Thus, by the NFP theorem, our semigroup $\mathcal{A}(Z)$ has no finitely presented expansions:

Theorem 3.9. *There exists a finitely generated, algorithmically finite, residually finite and computably enumerable semigroup that fails to possess finitely presented expansions.* □

4. Infinite algorithmically finite algebras and groups

4.1. **A setup and Golod-Shafarevich Theorem.** Recall that by an *algebra* we mean an associative, non-necessarily commutative, ring that forms a vector space over a field k. We will always assume that the underlying field k is finite. Consider the ring

$$\mathcal{F} = k\{x_1, \ldots, x_d\}$$

of polynomials $p(x_1, \ldots, x_d)$ over the field k, where $d > 1$. For this ring $\mathcal{F}$, we always assume that the variables x_1, …, x_d are non-commuting. This ring $\mathcal{F}$ is clearly an algebra, and in fact, it is a free algebra over the field k. A *monomial* of degree n is simply a word of length n over the alphabet $\{x_1, \ldots, x_d\}$. A typical representation of the polynomials p from $\mathcal{F}$ is of the form

$$p(x_1, \ldots, x_d) = k_1 w_1 + \ldots + k_n w_n,$$

where each k_i is from the field k and all w_1, …, w_n are pairwise distinct monomials (words) over the variables $\{x_1, \ldots, x_d\}$ whose degrees are not decreasing. In this representation of p, we can assume that the monomials w_1, …, w_n are ordered length-lexicographically. We call each w_i a *monomial component* of the polynomial p. There are clearly d^n monomials of degree n. We can now represent $\mathcal{F}$ as the following infinite direct sum:

$$\mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \ldots,$$

where $\mathcal{F}_n$ is the subspace of $\mathcal{F}$ spanned over d^n monomials of degree n. In particular, $\mathcal{F}_0 = k$. *A homogeneous polynomial of degree* n is a non-zero polynomial that belongs to $\mathcal{F}_n$. Thus, a homogeneous polynomial of degree n is simply a linear combination of monomials of degree n. Representation of $\mathcal{F}$ as the direct sum $\mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \ldots$ tells us that every polynomial p in $\mathcal{F}$ can be written as the sum of homogeneous polynomials:

$$p = p_1 + p_2 + \ldots + p_k,$$

where $p_1 \in \mathcal{F}_{n_1}$, …, $p_k \in F_{n_k}$ and $n_1 < n_1 < \ldots < n_k$. We refer to the polynomials p_1,…, p_k as *homogeneous components* of p.

A subset H of $\mathcal{F}$ is called *homogeneous* if each polynomial in H is homogeneous. Note that homogeneous subsets do not require all polynomials to be of the same degree. A (two-sided) ideal I of $\mathcal{F}$ is called a *homogeneous ideal* if it is generated by a homogeneous set H. The following lemma is easy and we leave its proof to the reader.

Lemma 4.1. *An ideal I of $\mathcal{F}$ is homogenous if and only if it contains homogeneous components for all of its members.* □

Let I be the ideal generated by a homogeneous set H. Assume that elements of H are polynomials h_1, h_2, h_3, ... ordered so that the degrees of the polynomials are non-decreasing. We also assume that all the degrees are $\geqslant 2$. The following now follows from Lemma 4.1.

Lemma 4.2. *The quotient algebra $\mathcal{A} = \mathcal{F}/I$ can be represented as*

$$\mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots,$$

where $\mathcal{A}_n = (\mathcal{F}_n + I)/I$ for all n. Note that $\mathcal{A}_0 = k$ and $\mathcal{A}_1 = \mathcal{F}_1$. □

As above, let I be the ideal generated by a homogeneous set H. Let r_n be the number of polynomials of degree n in H. Since the field k is finite, r_n is a finite number. Golod and Shafarevich in their famous paper [5] studied sufficient conditions (in terms of r_n) under which the algebra $\mathcal{A} = \mathcal{F}/I$ is infinite dimensional. We need one of those conditions formulated in the following theorem.

Theorem 4.3 (Golod-Shafarevich Theorem, [5])**.** *Assume that for some ϵ such that $0 < \epsilon \leqslant d/2$ we have the following inequality:*

$$r_n \leqslant \epsilon^2 (d - 2\epsilon)^{n-2} \quad \textit{for all } n \geqslant 2.$$

Then the algebra $\mathcal{A} = \mathcal{F}/I$ has an infinite dimension. □

We will use this theorem of Golod-Shafarevich in constructing the desired algebras and groups with various properties.

4.2. **Algebra example.** Our goal is to construct an infinite quotient of $\mathcal{F}$ which is residually finite, algorithmically finite and computably enumerable. As the semigroup example, our construction can be viewed as Post's construction of a simple set where the underlying domain is the algebra $\mathcal{F}$.

Consider the algebra $\mathcal{F}$. The algebra $\mathcal{F}$ is obviously a computable algebra. Let $W_0, W_1, \ldots$ be a standard enumeration of all c.e. subsets of the algebra $\mathcal{F}$. Uniformly on i, for each W_i proceed as follows. The process constructs the set H of homogeneous polynomials uniformly on i.

(1) Enumerate W_i.
(2) Wait for the first two polynomials f and g that appear in W_i such that the following conditions are satisfied:
    (a) Both f and g are of the form $f = f_1 + h_1$, $g = g_1 + h_2$, where $f_1 = g_1$ and f_1 and g_1 are allowed to be equal to 0.
    (b) The degrees of all homogeneous components occurring in h_1 and h_2 are greater than $i + 10$.
(3) Put the homogeneous components present in h_1 and h_2 into H.

If in the above process h_1 and h_2 are put into H, then we say that W_i *is forced* to put both h_1 and h_2 into H. Now we claim the following properties of H:

(1) H is a homogeneous set.
(2) For each $n \geqslant 0$, there are at most $2n$ polynomials of degree at most $n + 10$ in H.

The first property of H is obvious since only homogeneous polynomials are put into H. For each i, where $0 \leqslant i \leqslant n$, W_i can force at most two polynomials of degree $> i + 10$ to be put into H. Thus, in H there are at most $2n$ homogeneous polynomials of degree not exceeding $n+10$. In the notation of the Golod-Shafarevich Theorem, for the homogeneous set H constructed we have $r_n \leqslant 2n$ for all $n \geqslant 2$ with $r_0 = r_1 = \ldots = r_{10} = 0$. Let I be the ideal generated by H.

Now assume that $d = 2$, that is, our free algebra $\mathcal{F}$ is $k\{x_1, x_2\}$. Set $\epsilon = 1/4$. Then, it is not too hard to check that $0 < \epsilon \leqslant 1$ and $r_n \leqslant \epsilon^2(d - 2\epsilon)^{n-2}$ for all $n \geqslant 2$. The ideal H and the sequence r_n satisfies the conditions of the Golod-Shafarevich Theorem. Therefore, by the theorem, the algebra $\mathcal{A} = \mathcal{F}/I$ is infinite. Also, note that the set H is computably enumerable. Hence, the algebra $\mathcal{A}$ is an infinite computably enumerable and finitely generated algebra.

Theorem 4.4. *The constructed algebra $\mathcal{A}$ is finitely generated, infinite, computably enumerable, algorithmically finite, and residually finite. Moreover, the algebra $\mathcal{A}$ is a nil-algebra, that is, for every element $a \in A$ that does not have a non-trivial homogeneous component of degree* 0 *there is an n such that $a^n = 0$.*

Proof. We already noted that $\mathcal{A}$ is a finitely generated, infinite, computably enumerable algebra. Assume that there exists an infinite computable sequence $p_0, p_1, \ldots$ of polynomials in $\mathcal{F}$ representing pairwise distinct elements of the algebra $\mathcal{A}$. Suppose that this sequence is witnessed by the c.e. set W_i. Since the field k is finite, there are two polynomials f and g that appear in W_i such that the following conditions are satisfied:

(1) Both f and g can be written as $f = f_1 + h_1$, $g = g_1 + h_2$, where $f_1 = g_1$,
(2) degrees of homogeneous components of f_1 and g_1 are strictly smaller than the degrees of homogeneous components of h_1 and h_2, and
(3) degrees of homogeneous components occurring in h_1 and h_2 are all greater than $i + 10$.

Hence, $f = g$ in the algebra $\mathcal{A}$. This implies that the algebra is algorithmically finite. Using Lemma 4.2, one easily sees that the algebra $\mathcal{A}$ is residually finite.

To show that $\mathcal{A}$ is a nil-algebra, take any element a of the algebra that does not have a non-trivial homogeneous component of degree 0. We can represent the element a as the sum $p_1 + p_2 + \ldots + p_k$ of homogeneous polynomials all not in I such that each $p_i \in \mathcal{F}_{n_i}$ and $0 < n_1 < n_2 < \ldots < n_k$. Consider the sequence a, a^2, a^3, ... This is a computable sequence witnessed by some c.e. set W_i. There is a stage at which a^{i+11} and a^{i+12} appear in W_i. By that stage (or at that stage), by construction of H, all homogeneous components of a^{i+11} and a^{i+12} are put into H. Hence $a^{i+11} \in I$. □

Applying the NFP theorem and using the theorem above, we derive the next theorem as a corollary:

Theorem 4.5. *There exists a finitely generated, algorithmically finite, residually finite and computably enumerable algebra that fails to possess finitely presented expansions.* □

4.3. **Group example.** Theorem 4.5 allows us to build examples of finitely generated computably enumerable groups that fail to possess finitely presented expansions. The main theorem of this subsection is thus the following result.

Theorem 4.6. *For every prime number p there exists a finitely generated, algorithmically finite, residually finite and computably enumerable p-group that fails to possess finitely presented expansions.*

Proof. We use notation from Theorem 4.4 and follow the argument of Golod from [4]. Take the algebra $\mathcal{A} = \mathcal{F}/I$ constructed in Theorem 4.4, where k is the field $\mathbb{Z}_p$ with p elements. As was mentioned in Lemma 4.2 $\mathcal{A}$ is a graded algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots$. Put $\mathcal{A}^+ = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots$. It was shown in Theorem 4.4 that every element u in $\mathcal{A}^+$ is nilpotent, i.e. $u^{n(u)} = 0$ for some $n(u) \in \mathbb{N}$. Hence $1 + u$ is invertible in $\mathcal{A}$. Let x_1 and x_2 be the standard generators of the algebra $\mathcal{F}$ and let $\bar{x}_1$ and $\bar{x}_2$ be their images in $\mathcal{F}/I$. Then the elements $1 + \bar{x}_1$ and $1 + \bar{x}_2$ are invertible in $\mathcal{A}$ so they generate a subgroup, say $G(\mathcal{A})$, in the group of units of $\mathcal{A}$. The group $G(\mathcal{A})$ is infinite. Indeed, the subalgebra generated by $G(\mathcal{A})$ and $\mathcal{A}_0 = \mathbb{Z}_p$ contains $\bar{x}_1$ and $\bar{x}_2$, so it is equal to $\mathcal{A}$. Since $\mathcal{A}$ is infinite, so is the group $G(\mathcal{A})$. Clearly, $G(\mathcal{A})$ is finitely generated. Moreover, $G(\mathcal{A})$ is c.e. To see this observe that $(1+\bar{x}_i)^{-1}$ is a finite sum of the type $1-\bar{x}_i+\bar{x}_i^2-\ldots$. Therefore every finite product γ of the generators of $G(\mathcal{A})$ and their inverses can be effectively presented in $\mathcal{A}$ by a polynomial $q(\gamma)$ from $\mathcal{F}$. Since the word problem for $\mathcal{A}$ is c.e., one can enumerate all the products γ such that $q(\gamma) - 1 = 0$ in $\mathcal{A}$. Furthermore, for $u \in \mathcal{A}^+$ one has $p^{n(u)} > n(u)$ so $u^{p^{n(u)}} = 0$ in $\mathcal{A}$. It follows that

$$(1 + u)^{p^{n(u)}} = 1 + u^{p^{n(u)}} = 1,$$

so every element from $G(\mathcal{A})$ has a finite p-order, so $G(\mathcal{A})$ is a p-group. Since $\mathcal{A}$ is algorithmically finite, the group $G(\mathcal{A})$ is also algorithmically finite, because every c.e. subset in $G(\mathcal{A})$ is also a c.e. subset $W \subseteq \mathcal{A}$ (enumerate one-by-one elements $\gamma_0, \gamma_1, \gamma_2, \ldots$ of W as products in the generators of $G(\mathcal{A})$ and for each γ_i compute a polynomial of the type $q(\gamma_i)$ representing γ_i in $\mathcal{A}$).

It remains to be seen that $G(\mathcal{A})$ is residually finite. Let $\mathcal{M}$ be the ideal in $\mathcal{A}$ generated by $\bar{x}_1, \bar{x}_2$. By Lemma 4.2 the algebra $\mathcal{A}$ is a graded algebra, so we have $\bigcap_n \mathcal{M}^n = 0$. Let G_n be the n-th term of the lower central series of $G(\mathcal{A})$. Here G_n is defined by induction: $G_1 = G(\mathcal{A}), G_{n+1} = [G_n, G_1]$. It easily follows from the construction of $G(\mathcal{A})$ by induction on n that $G_n \subseteq 1 + \mathcal{M}^n$ for every $n \in \mathbb{N}$. Hence if $g = 1 + u \in \bigcap_n G_n$, where $u \in \mathcal{A}^+$, then $g - 1 \in \bigcap_n \mathcal{M}^n = 0$, so $g = 1$ in $G(\mathcal{A})$. Therefore, $\bigcap_n G_n = 1$. Hence, the group $G(\mathcal{A})$ is residually nilpotent. Since $G(\mathcal{A})$ is a p-group the quotients $G(\mathcal{A})/G_n$ are finitely generated nilpotent p-groups. Therefore, they are finite. We conclude that $G(\mathcal{A})$ is residually finite, as claimed.

Finally, we apply the NFP theorem to the group constructed, which explains why the group has no finitely presented expansions. □

We mention that Miasnikov and Osin in [17] construct an example of a finitely generated, infinite, algorithmically finite, and computably enumerable group. They pose a question as to whether such groups exist in the class of all residually finite groups. The example constructed above gives a positive answer to the question.

5. Effectively infinite algebra without finitely presented expansions

Based on the results of the previous sections, it is natural to ask if algorithmic finiteness is a necessary condition for an algebra or a group not to have finitely presented expansions. In other words, are there examples of finitely generated and

c.e. algebras that are effectively infinite (that is, not algorithmically finite) yet fail to possess finitely presented expansions? It turns out that one can construct such examples of algebras using the Golod Shafarevich Theorem, the simple set Z constructed in Lemma 3.7, and the proof of Theorem 2.5. Here is the main theorem of this section:

Theorem 5.1. *There exists a finitely generated, residually finite, computably enumerable and effectively infinite algebra that fails to possess finitely presented expansions.*

Proof. Consider the computable free algebra $\mathcal{F} = k\{x, y\}$. We say that an ideal I is *strongly homogeneous* if it is generated by monomials. We have the following result that recasts Lemma 4.1 but for strongly homogeneous ideals:

Lemma 5.2. *An ideal I of $\mathcal{F}$ is strongly homogeneous if and only if all monomial components of each member of I belongs to I.* □

Let Z be the set constructed in Lemma 3.7. Recall that Z is a co-infinite c.e. set whose complement has no infinite computably enumerable subset. Let I be the strongly homogeneous ideal generated by Z. Consider the algebra $\mathcal{A} = \mathcal{F}/I$. Our goal is to show that the algebra $\mathcal{A}$ satisfies the theorem.

As our algebra example in Section 4.2, it is easy to see that the ideal I satisfies the conditions of the Golod-Shafarevich Theorem. Note that for the ideal I we have $r_n \leqslant n$ for all $n = 5, 6, 7, \ldots$. Hence the algebra $\mathcal{A}$ is infinite. Also, the algebra is computably enumerable because Z is a c.e. set.

Every element a of the algebra $\mathcal{A}$ can be written in the form

$$k_1 w_1 + \ldots + k_n w_n + k'_1 w'_1 + \ldots + k'_m w'_m,$$

where $w_1, \ldots, w_n, w_1, \ldots, w'_m$ are monomials such that $w_1, \ldots, w_n \notin Z$ and $w'_1, \ldots, w'_m \in Z$. We call these sums *representatives* of the element a, and refer to the sum

$$k_1 w_1 + \ldots + k_n w_n$$

as the *true representative* of a. By Lemma 5.2, any non-empty proper subsum, that is, a sum of the form $k_{i_1} w_{i_1} + \ldots + k_{i_t} w_{i_t}$, of monomial components of any true representation of a is a true representation of some other element.

Lemma 5.3. *No algorithm exists that enumerates an infinite sequence of pairwise distinct true representatives of elements of $\mathcal{A}$.*

We prove the lemma by assuming that there exists an algorithm that enumerates an infinite sequence a_0, a_1, ... of pairwise distinct true representatives of elements of $\mathcal{A}$. Since the field k is finite, we can effectively select a subsequence a_{i_1}, a_{i_2}, ... such that any two elements in this subsequence have no monomial components in common. By selecting a monomial component from each member of the subsequence, we thus produce an effectively infinite list of elements from the complement of Z. This contradicts the choice of Z, thus proving the lemma.

Lemma 5.4. *The algebra $\mathcal{A}$ is effectively infinite.*

For each $n \geqslant 1$, consider the following element:

$$a_n = \sum_{w \in \{x,y\}^n} w.$$

Note that for each $n \geqslant 1$ we have $a_n \neq 0$. Otherwise $\mathcal{A}$ would be a finite algebra. Moreover, for each $n \geqslant 1$ there exists a monomial w_n of degree n such that $w_n \notin Z$. Hence, the sequence $a_1, a_2, \ldots$ is computably enumerable such that $a_i \neq a_j$ for all $i \neq j$. This proves that the algebra $\mathcal{A}$ is effectively infinite.

Now we prove a result akin to Lemma 2.4.

Lemma 5.5 (Separation Lemma II)**.** *For all distinct elements f and g of the algebra $\mathcal{A}$ there exists a set $S(f,g) \subseteq \mathcal{F}$ such that each of the following is satisfied:*

(1) *$S(f,g)$ is a computable set.*
(2) *$f \in S(f,g)$ and $g \notin S(f,g)$.*
(3) *The set $S(f,g)$ is I-closed, that is, for each element $u \in S(f,g)$ the I-equivalence class of u is a subset of $S(f,g)$.*

We now prove the lemma. Let t_f and t_g be true representatives of f and g, respectively. Without loss of generality we can assume that t_g contains monomial components that are not present in t_f. Otherwise, we can just replace the roles of f and g. So, let $k_1w_1, \ldots, k_nw_n$ be all monomials present in t_g but not in t_f. Consider the following set:

$$S(f,g) = \{p \mid p \text{ contains all monomials of } t_f \text{ and contains none of the monomials } k_1w_1, \ldots, k_nw_n\}.$$

Computability of the set $S(f,g)$ is obvious. By construction, it is clear that $f \in S(f,g)$ and $g \notin S(f,g)$. Take any $p \in S(f,g)$. Then any polynomial p' equal to p is of the form

$$t_p + k_1'w_1 + \ldots + k_m'w_m',$$

where t_p is a true representative of p and all $w_1', \ldots, w_m'$ are in Z. Clearly, p' contains all monomials of t_f and contains none of the monomials $k_1w_1, \ldots, k_nw_n$. This proves the lemma.

The next lemma essentially repeats the proof of Theorem 2.5. We use notation from that proof. In particular, $C_{(a,b)}$ denotes the minimal congruence relation that contains the pair (a,b) in a given universal algebra. Recall that by Lemma 2.6 $C_{(a,b)}$ is a c.e. congruence relation.

Lemma 5.6. *All c.e. expansions of the algebra $\mathcal{A}$ are residually finite.*

To prove the lemma, let $\mathcal{A}' = (\mathcal{A}, h_1, \ldots, h_k)$ be a c.e. expansion of the algebra $\mathcal{A}$. Select two distinct elements f and g of $\mathcal{A}'$. We want to show that there exists a finite homomorphic image of $\mathcal{A}'$ in which the images of f and g are distinct.

For the selected elements f and g, consider the I-closed set $S(f,g)$ constructed in Lemma 5.5. Define the following binary relation $E_{(f,g)}$ on the expansion $\mathcal{A}'$. For elements a and b, we have $(a,b) \in E_{(f,g)}$ if and only if the congruence relation $C_{(a,b)}$ identifies no element in $S(f,g)$ with an element in the complement of $S(f,g)$. The following are proved exactly as in the proof of Theorem 2.5:

(1) The binary relation $E_{(f,g)}$ is a congruence relation of the universal algebra $\mathcal{A}'$.
(2) The congruence relation $E_{(f,g)}$ is a co-c.e. equivalence relation.

Consider the quotient universal algebra $\mathcal{A}/E_{(f,g)}$. As we have just noted, this algebra is co-c.e. To arrive at a contradiction, assume that $\mathcal{A}/E_{(f,g)}$ is infinite.

We call an element $a = k_1w_1 + \ldots + k_nw_n$ of the algebra $\mathcal{A}$ *special* if $n > 0$ and no non-trivial sum of its monomials equals a in the universal algebra $\mathcal{A}/E_{(f,g)}$. In

particular, if a is special, then it must be a true representative of some element of $\mathcal{A}$. The set S of all special elements must be computably enumerable because $E_{(f,g)}$ is a co-c.e. congruence relation. Moreover, the set S is infinite. We conclude that there exists an infinite computably enumerable sequence of pairwise non-equal special elements in the universal algebra $\mathcal{A}/E_{(f,g)}$. Since special elements are true representatives of elements of $\mathcal{A}$, we have a contradiction with Lemma 5.3. Thus, the algebra $\mathcal{A}/E_{(f,g)}$ is finite, in which the images of f and g are distinct. This proves the lemma.

Now the proof of the theorem follows easily. Assume that the algebra $\mathcal{A}$ has a finitely presented expansion $\mathcal{A}'$. Since $\mathcal{A}'$ is residually finite, by Mal′cev's lemma (Lemma 2.9), the word problem in $\mathcal{A}'$ is decidable. Hence, we can decide the word problem in the original algebra $\mathcal{A}$. But the ideal I is not computable by construction. □

We note that we do not know whether there exist finitely generated, computably enumerable, effectively infinite groups that fail to have finitely presented expansions.

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