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GROUP RADICALS AND STRONGLY COMPACT CARDINALS

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Abstract. We answer some natural questions about group radicals and torsion classes, which involve the existence of measurable cardinals, by constructing, relative to the existence of a supercompact cardinal, a model of ZFC in which the first ω_1-strongly compact cardinal is singular.

1. Introduction

Recall that an uncountable cardinal κ is *measurable* if it carries a non-trivial, two-valued κ-additive measure μ. Equivalently, if there is a non-principal ultrafilter $\mathcal{U}$ on κ, that is κ-complete.[1] Indeed, $\mathcal{U} = \{X \subseteq \kappa : \mu(X) = 1\}$. It is a well-known fact (see [14], 10.2) that the least cardinal κ that carries a non-principal ω_1-complete ultrafilter $\mathcal{U}$ is measurable and, in fact, $\mathcal{U}$ is κ-complete.

The existence of measurable cardinals cannot be proved in ZFC (provided ZFC is consistent), and therefore their existence has to be assumed as a large-cardinal axiom of set theory.

Measurable cardinals turned out to play an important role in infinite abelian group theory. For example, results of Eda ([8], [9]) show that the existence of a non-trivial homomorphism $h : \mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa} \to \mathbb{Z}$, for some uncountable cardinal κ, is equivalent to the existence of a measurable cardinal (see Theorem 3.1 below). Further results of Dugas-Göbel [7] and Dugas [6], on group radicals and torsion classes, respectively, use even stronger large cardinals, such as strongly compact and supercompact cardinals. See [11] for a survey on most of these results, as well as some other results on abelian groups and modules involving large cardinals.

This paper has two parts. In the first one (Sections 2-5) we give a small survey of some results in abelian group theory that involve the use of measurable, strongly compact, and supercompact cardinals. In some cases we give new proofs and we improve on some of the results by using weaker large-cardinal hypotheses. For instance, we give a proof of a theorem of Dugas ([6]) on torsion classes generated by a set by using δ-strongly compact cardinals, instead of the stronger assumption

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[1]For the definition of all standard set-theoretic notions see [14].

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of the existence of a supercompact cardinal. In the case of $\mathbb{Z}$ we show that some of the results actually follow from the existence of ω_1-strongly compact cardinals. Thus, some natural questions about the necessary use of large cardinal assumptions stronger than measurability, e.g., for the equivalence of radicals and κ-radicals in the case of $\mathbb{Z}$, or for the torsion class ${}^{\perp}\mathbb{Z}$ being generated by a set, reduce to the question of whether it is consistent that the first measurable cardinal, the first ω_1-strongly compact cardinal, and the first strongly compact cardinal are all different. In the second part of the paper (Section 6) we show that it is. The main result (Theorem 6.1) shows that, assuming the consistency of ZFC plus the existence of a supercompact cardinal, one can construct, using Radin forcing, a model of ZFC in which the first ω_1-strongly compact cardinal is singular, and therefore not measurable.

2. Preliminaries

For each n, let $e_n : \omega \to \{0,1\}$ be such that $e_n(m) = 1$ if and only if $m = n$. Many results in this paper rely on the following well-known fact, due to E. Specker.

Lemma 2.1. *If $h : \mathbb{Z}^\omega \to \mathbb{Z}$ is a homomorphism, then $h(e_n) = 0$ for all but finitely many n.*

Proof. Towards a contradiction, suppose $h(e_n) \neq 0$ for infinitely many n. Then we may assume $h(e_n) \neq 0$ for all n.

Let $k_1 = 1$, and choose $k_{n+1} > k_n!\,|h(e_n)|$.

Let $X = \{\sum x_n e_n \in \mathbb{Z}^\omega : \forall n(x_n = 0 \text{ or } x_n = k_n!)\}$.

We have $|X| = 2^{\aleph_0}$. Hence, there exist $\sum x_n e_n$ and $\sum y_n e_n$ distinct such that $h(\sum x_n e_n) = h(\sum y_n e_n)$.

Let m be least such that $x_m \neq y_m$. Then,

$$h((x_m - y_m)e_m) = -h(\sum_{i>m}(x_i - y_i)e_i).$$

But $x_m - y_m = \pm k_m!$, so

$$h((x_m - y_m)e_m) = \pm k_m!\, h(e_m).$$

Thus, k_{m+1} does not divide $h((x_m - y_m)e_m)$.

But k_{m+1} divides $h(\sum_{i>m}(x_i - y_i)e_i)$. A contradiction. □

Note, in contrast, that for every $N < \omega$, the map $h_N : \mathbb{Z}^\omega \to \mathbb{Z}$ given by

$$h_N(\sum_{n<\omega} r_n e_n) = \sum_{n \leq N} r_n$$

is a homomorphism that maps each e_n with $n \leq N$ to 1.

The lemma above holds for uncountable products of $\mathbb{Z}$ as well. Namely, given an uncountable cardinal κ, if $h : \mathbb{Z}^\kappa \to \mathbb{Z}$ is a homomorphism, then the set $\{\alpha < \kappa : h(e_\alpha) \neq 0\}$ is finite. Otherwise, choose a set $A = \{\alpha_n : n < \omega\}$ of ordinals less than κ such that $h(e_{\alpha_n}) \neq 0$, for all $n < \omega$. Then the restriction of h to A yields a homomorphism from $\mathbb{Z}^A$ into $\mathbb{Z}$. But since $\mathbb{Z}^A$ and $\mathbb{Z}^\omega$ are isomorphic, this contradicts the lemma.

Let us write $\prod_{n \geq m} \mathbb{Z}$ for the set of ω-sequences of integers with the first m elements equal to 0.

Lemma 2.2. *For every homomorphism $h : \mathbb{Z}^\omega \to \mathbb{Z}$, there exists $m < \omega$ such that $h[\prod_{n\geq m} \mathbb{Z}] = \{0\}$.*

Proof. Suppose otherwise. For each m choose $a^{(m)} \in \prod_{n\geq m} \mathbb{Z}$ such that $h(a^{(m)}) \neq 0$.

For each $(r_m)_{m<\omega}$ define $z((r_m)_{m<\omega}) \in \prod_{n<\omega} \mathbb{Z}$ by letting

$$z((r_m)_{m<\omega})(n) = \sum_{m\leq n} r_m a^{(m)}(n).$$

Define $\theta : \mathbb{Z}^\omega \to \mathbb{Z}$ as

$$\theta((r_m)_{m<\omega}) = h(z((r_m)_{m<\omega})).$$

θ is a homomorphism, and for all m, $\theta(e_m) = h(a^{(m)}) \neq 0$, contradicting the previous lemma. □

A direct consequence of Lemma 2.2 is that

$$Hom(\prod_{n=1}^{\infty} \mathbb{Z},\ \mathbb{Z}) \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}.$$

Indeed, one can easily check that the map that sends each homomorphism $h : \prod_{n=1}^{\infty} \mathbb{Z} \to \mathbb{Z}$ to $\sum_{n\leq m} r_n e_n$, where $r_n = h(e_n)$ and m is as in Lemma 2.2, is an isomorphism.

Another consequence is that $Hom(\mathbb{Z}^\omega/\mathbb{Z}^{<\omega},\ \mathbb{Z}) = \{0\}$. For given $h^* : \mathbb{Z}^\omega/\mathbb{Z}^{<\omega} \to \mathbb{Z}$, let $h : \mathbb{Z}^\omega \to \mathbb{Z}$ be the homomorphism given by

$$h(\sum r_n e_n) = h^*([\sum r_n e_n]).$$

Let m be such that $h[\prod_{n\geq m} \mathbb{Z}] = \{0\}$. Then,

$$h^*([\sum r_n e_n]) = h^*([\sum_{n\leq m} r_n e_n]) + h^*([\sum_{n>m} r_n e_n]) = 0.$$

3. Group homomorphisms and measurable cardinals

Let us now consider homomorphisms from the quotient group $\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}$ into $\mathbb{Z}$, for uncountable cardinals κ. The following theorem is due to Eda ([8], [9]), building on results of Łoś published in [12].

Theorem 3.1. *$Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z}) \neq \{0\}$ if and only if there exists an ω_1-complete non-principal ultrafilter on κ.*

Proof. First suppose κ is a cardinal and $\mathcal{U}$ is an ω_1-complete non-principal ultrafilter on κ.

Define $\varphi : \mathbb{Z}^\kappa/\mathbb{Z}^{<\omega} \to \mathbb{Z}$ by

$$\varphi([\sum_{\alpha<\kappa} r_\alpha e_\alpha]) = n \quad \text{if and only if} \quad \{\alpha : r_\alpha = n\} \in \mathcal{U}.$$

Since $\mathcal{U}$ is ω_1-complete and non-principal, φ is well-defined. So let us check that φ is a homomorphism.

Given $\sum r_\alpha e_\alpha,\ \sum s_\alpha e_\alpha \in \mathbb{Z}^\kappa$, let n_0 and n_1 be such that $\{\alpha : r_\alpha = n_0\}$, $\{\alpha : s_\alpha = n_1\} \in \mathcal{U}$. Since $\mathcal{U}$ is a filter, $\{\alpha : r_\alpha = n_0 \text{ and } s_\alpha = n_1\} \in \mathcal{U}$. Hence,

$$\varphi([\sum (r_\alpha + s_\alpha) e_\alpha]) = n_0 + n_1 = \varphi([\sum r_\alpha e_\alpha]) + \varphi([\sum s_\alpha e_\alpha]).$$

Notice that φ is a surjection.

For the converse, suppose $\varphi : \mathbb{Z}^\kappa/\mathbb{Z}^{<\omega} \to \mathbb{Z}$ is a non-zero homomorphism.

For $Y \subseteq \kappa$, let $\mathbb{Z}^\kappa \restriction Y$ be the set of all κ-sequences of integers whose set of non-zero coordinates is contained in Y. Let

$$S' = \{Y \subseteq \kappa : \varphi[\mathbb{Z}^\kappa \restriction Y/\mathbb{Z}^{<\omega}] \neq \{0\}\}.$$

Note that $\kappa \in S'$, and that for every $Y \in S'$ and every $Z \subseteq Y$, either $Z \in S'$ or $Y \setminus Z \in S'$ (or both).

Claim 3.2. Any set of pairwise-disjoint elements of S' is finite.

Proof of the claim. Suppose, to the contrary, that $\{Y_n : n \in \omega\}$ is a pairwise-disjoint set of elements of S'.

For each n, let $a^{(n)} \in \mathbb{Z}^\kappa \restriction Y_n$ be such that $\varphi([a^{(n)}]) \neq 0$.

Define $\theta : \mathbb{Z}^\omega \to \mathbb{Z}$ by

$$\theta((r_n)_{n\in\omega}) = \varphi([\sum r_n a^{(n)}]).$$

Then $\theta(e_n) = \varphi([a^{n)}]) \neq 0$, for all $n \in \omega$, contradicting Lemma 2.1. □

Now let

$$S = \{Y \in S' : \forall Z \subseteq Y \text{ (exactly one of } Z \text{ and } Y \setminus Z \text{ belongs to } S')\}.$$

Note that S is precisely the set of those $Y \in S'$ such that for all $Z \subseteq Y$, either $\varphi[\mathbb{Z}^\kappa \restriction Z/\mathbb{Z}^{<\omega}] = \{0\}$ or $\varphi[\mathbb{Z}^\kappa \restriction (Y \setminus Z)/\mathbb{Z}^{<\omega}] = \{0\}$.

By the claim above, $S \neq \emptyset$. Indeed, if $S = \emptyset$, then for every $Y \in S'$ there is $Z \subseteq Y$ such that both Z and its complement in Y belong to S'. Thus, starting with κ, which belongs to S', we can inductively build an infinite family of pairwise-disjoint elements of S'.

So let $Y \in S$, and define

$$D = \{X \subseteq \kappa : X \cap Y \in S'\}.$$

Claim 3.3. D is an ω_1-complete non-principal ultrafilter on κ.

Proof of the claim. Trivially $\emptyset \notin D$, and clearly D is upwards closed.

Suppose now, towards a contradiction, that $X_0, X_1 \in D$ and $X_0 \cap X_1 \notin D$. Without loss of generality, $X_0, X_1 \subseteq Y$.

For some $a \in \mathbb{Z}^\kappa$,

$$0 \neq \varphi([a \restriction X_0]) = \varphi([a \restriction (X_0 \cap X_1)]) + \varphi([a \restriction (X_0 \setminus X_1)]).$$

Hence, $\varphi([a \restriction (X_0 \setminus X_1)]) \neq 0$, and so $\varphi[\mathbb{Z}^\kappa \restriction (X_0 \setminus X_1)/\mathbb{Z}^{<\omega}] \neq 0$.

Similarly, $\varphi[\mathbb{Z}^\kappa \restriction (X_1 \setminus X_0)/\mathbb{Z}^{<\omega}] \neq \{0\}$.

But this contradicts $Y \in S$ (just take $Z = X_0 \setminus X_1$).

This shows that D is a filter. To see that D is an ultrafilter, suppose $X \notin D$, where $X \subseteq \kappa$.

Let $a \in \mathbb{Z}^\kappa \restriction Y$ be such that $\varphi([a]) \neq 0$. Then,

$$0 \neq \varphi([a]) = \varphi([a \restriction (X \cap Y)]) + \varphi([a \restriction (Y \setminus X)]) = \varphi([a \restriction (Y \setminus X)]).$$

This shows $Y \setminus X \in D$. Hence, $\kappa \setminus X \in D$.

Let us now see that D is ω_1-complete. Suppose, to the contrary, that $\{X_n : n \in \omega\} \subseteq D$ and $\bigcap_{n\in\omega} X_n \notin D$. Without loss of generality, $X_{n+1} \subseteq X_n \subseteq Y$, for all $n \in \omega$, and $\bigcap_{n\in\omega} X_n = \emptyset$.

Since $X_n \in D$, there exists $a^{(n)} \in \mathbb{Z}^\kappa \restriction X_n$ such that $\varphi([a^{(n)}]) \neq 0$.

Define $\theta : \mathbb{Z}^\omega \to \mathbb{Z}$ by letting

$$\theta((r_n)_{n\in\omega}) = \varphi([\sum r_n a^{(n)}]).$$

Since $\bigcap_{n\in\omega} X_n = \emptyset$, θ is a well-defined homomorphism.

But then $\theta(e_n) = \varphi([a^{(n)}]) \neq 0$, for all n, contradicting Lemma 2.1.

It remains to check that D is non-principal. But if $\{\alpha\} \in D$ for some $\alpha \in \kappa$, then for some $a \in \mathbb{Z}^\kappa \restriction \{\alpha\}$, $\varphi([a]) \neq 0$. This is impossible since $[a] = [0]$. □

This concludes the proof of Theorem 3.1. □

It follows that the least κ for which $Hom(Z^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z}) \neq \{0\}$ is a measurable cardinal.

Corollary 3.4. *There exists κ such that $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z}) \neq \{0\}$ if and only if there exists a measurable cardinal. The measurable cardinal is the least κ for which $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z}) \neq \{0\}$ holds.*

Recall that an ultrafilter $\mathcal{U}$ on an infinite cardinal κ is called *uniform* if every element of $\mathcal{U}$ has cardinality κ. The proof of the theorem above can be easily modified to show: $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \mathbb{Z}) \neq \{0\}$ if and only if there exists an ω_1-complete uniform ultrafilter on κ. Thus, if κ is measurable, then $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \mathbb{Z}) \neq \{0\}$. Also, the least κ for which $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \mathbb{Z}) \neq \{0\}$ is measurable.

Arguing similarly as in the proof of Theorem 3.1 we can now prove the following.

Theorem 3.5. *The following are equivalent:*

(1) *For every homomorphism $h : \mathbb{Z}^\kappa \to \mathbb{Z}$ there is a finite subset I of κ such that $h[\mathbb{Z}^\kappa \restriction (\kappa \setminus I)] = \{0\}$.*
(2) *There is no ω_1-complete non-principal ultrafilter on κ.*

Proof. If $\mathcal{U}$ is a ω_1-complete non-principal ultrafilter on κ, let $h : \mathbb{Z}^\kappa \to \mathbb{Z}$ be given by

$$h(\sum_{\alpha<\kappa} r_\alpha e_\alpha) = n \quad \text{if and only if} \quad \{\alpha : r_\alpha = n\} \in \mathcal{U}.$$

h is a homomorphism such that for every finite $I \subseteq \kappa$, $h[\mathbb{Z}^\kappa \restriction (\kappa \setminus I)] = \mathbb{Z}$.

For the converse, suppose h is a counterexample to (1) and let

$$S' = \{Y \subseteq \kappa : h[\mathbb{Z}^\kappa \restriction Y] \neq \{0\}\}.$$

Now define S as in Theorem 3.1, and for a fixed $Y \in S$ let

$$D = \{X \subseteq \kappa : X \neq \emptyset \text{ and } h[\mathbb{Z}^\kappa \restriction (X \cap Y)] \neq \{0\}\}.$$

Then D is an ω_1-complete ultrafilter on κ and is also non-principal because it contains all co-finite sets. □

Corollary 3.6. *Suppose there is no ω_1-complete non-principal ultrafilter on κ. Then*

$$Hom(\prod_{\alpha\in\kappa} \mathbb{Z},\ \mathbb{Z}) \cong \bigoplus_{\alpha\in\kappa} \mathbb{Z}.$$

Proof. The map that sends each $h : \prod_{\alpha\in\kappa} \mathbb{Z} \to \mathbb{Z}$ to $\sum_{\alpha\in I} r_\alpha e_\alpha$, where $r_\alpha = h(e_\alpha)$ and I is the finite subset of κ given by Theorem 3.5, is an isomorphism. □

4. On singly generated radicals and strongly compact cardinals

For X an abelian group, let $R_X : Ab \to Ab$ be the functor given by

$$R_X(A) = \bigcap\{Ker(f) : f \in Hom(A, X)\}.$$

R_X is called *the radical singly generated by* X.

Note that $R_X(X^\kappa) = \{0\}$. Also note that $Hom(A, X) = \{0\}$ if and only if $R_X(A) = A$.

Thus, from the results in the last section we have that $R_{\mathbb{Z}}(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}) = \mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$, for all κ less than the first measurable cardinal. Moreover, similarly as in the first part of the proof of Theorem 3.1, we can see that if κ is greater than or equal to the first measurable cardinal λ, then $R_{\mathbb{Z}}(\mathbb{Z}^\kappa/\mathbb{Z}^{<\lambda}) = 0$. Indeed, if $[a] \in \mathbb{Z}^\kappa/\mathbb{Z}^{<\lambda}$ is non-zero, then for some $X \subseteq \kappa$ of cardinality λ, $a(\alpha) \neq 0$ for all $\alpha \in X$. Let $\mathcal{U}$ be a λ-complete non-principal ultrafilter on X, and define $\varphi : \mathbb{Z}^\kappa/\mathbb{Z}^{<\lambda} \to \mathbb{Z}$ by

$$\varphi([\sum_{\alpha<\kappa} r_\alpha e_\alpha]) = n \quad \text{if and only if} \quad \{\alpha \in X : r_\alpha = n\} \in \mathcal{U}.$$

Then φ is a homomorphism and $[a] \notin Ker(\varphi)$.

For κ a cardinal, let

$$R_X^\kappa(A) = \sum\{R_X(B) : B \subseteq A,\ |B| < \kappa\}.$$

Notice that, since $B \subseteq A$ implies $R_X(B) \subseteq R_X(A)$, we have that $R_X^\kappa(A) \subseteq R_X(A)$, for all A. Also, trivially, $R_X^\kappa(A) = R_X(A)$ for all A of cardinality $< \kappa$. Note also that if $\kappa < \lambda$ and $R_X = R_X^\kappa$, then $R_X = R_X^\lambda$ as well.

Definition 4.1. For a group X, the *Dugas-Göbel cardinal for* X is the least cardinal κ such that $R_X = R_X^\kappa$.

Dugas and Göbel proved in [7] that the Dugas-Göbel cardinal for $\mathbb{Z}$, if it exists, is greater than or equal to the first measurable cardinal. We will give a proof of this result. But first let us prove the following general form of the Wald-Łoš Lemma (see [21]).

Lemma 4.2. *If I is an infinite set, G is a non-trivial group, F is a λ-complete filter over I, for some infinite cardinal λ, and B is a subgroup of $A = G^I/F$ of cardinality $< \lambda$, then B is embeddable into G^I.*

Proof of the lemma. Fix $B \subseteq A$ of cardinality $\kappa < \lambda$. Let $\pi : G^I \to A$ be the canonical projection homomorphism $\pi(a) = [a]$. For each $[a] \in A$, choose some $\overline{a} \in [a]$. Thus, $\pi(a) = [\overline{a}]$.

For $a \in G^I$ define $O(a) = \{\alpha : a(\alpha) = 0\}$.

If $[a], [b] \in A$, then $[a] + [b] - [a + b] = 0$, and therefore $O(\overline{a} + \overline{b} - (\overline{a + b})) \in F$. Let

$$D = \bigcap\{O(\overline{a} + \overline{b} - (\overline{a + b})) : [a], [b] \in B\}.$$

Since $\kappa < \lambda$, we have that $D \in F$. Now define $e : B \to G^I$ by $e([a]) = \overline{a} \restriction D$, where $\overline{a} \restriction D(\alpha) = \overline{a}(\alpha)$ for $\alpha \in D$, and $\overline{a} \restriction D(\alpha) = 0$ for $\alpha \notin D$.

Let us check that e is an embedding. First, e is clearly injective, for if $[a] \neq [b]$, then $\{\alpha : \overline{a}(\alpha) = \overline{b}(\alpha)\} \notin F$. Since $D \in F$, there exists $\alpha \in D$ such that $\overline{a}(\alpha) \neq \overline{b}(\alpha)$, which implies that $e([a]) \neq e([b])$. Second, $e([a] + [b]) = e([a + b]) = \overline{a + b} \restriction$

$D = (\overline{a} + \overline{b}) \restriction D$, because $D \subseteq O(\overline{a} + \overline{b} - (\overline{a+b}))$. Also, $(\overline{a} + \overline{b}) \restriction D = \overline{a} \restriction D + \overline{b} \restriction D = e([a]) + e([b])$. □

Corollary 4.3. *If λ is a regular cardinal and B is a subgroup of $A = \mathbb{Z}^\lambda/\mathbb{Z}^{<\lambda}$ of cardinality $< \lambda$, then B is embeddable into $\mathbb{Z}^\lambda$.*

Proof. Let F be the filter on λ consisting of the $X \subseteq \lambda$ whose complement has cardinality $< \lambda$. Note that F is non-principal and, since λ is regular, λ-complete. □

Theorem 4.4 ([7])**.** *The Dugas-Göbel cardinal for $\mathbb{Z}$, if it exists, is greater than or equal to the first measurable cardinal.*

Proof. Suppose κ is the Dugas-Göbel cardinal for $\mathbb{Z}$. Let $A = \mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$.

If κ is less than the first measurable cardinal, then by Corollary 3.4 and the remarks that follow it, $Hom(A, \mathbb{Z}) = \{0\}$. Hence, $R_\mathbb{Z}(A) = A$.

But since $R_\mathbb{Z}(\mathbb{Z}^\kappa) = \{0\}$, by the lemma above $R_\mathbb{Z}(B) = \{0\}$, for all $B \subseteq A$ of cardinality $< \kappa$. Therefore,

$$R^\kappa_\mathbb{Z}(A) = \{0\} \neq A = R_\mathbb{Z}(A),$$

which is impossible. □

The following is thus a natural question.

Question 1. Suppose κ is measurable. Does it follow that $R_\mathbb{Z} = R^\kappa_\mathbb{Z}$?

Note that Question 1 is equivalent to asking whether $R_\mathbb{Z} = R^\kappa_\mathbb{Z}$ holds for κ the first measurable cardinal. It is also equivalent to asking whether the Dugas-Göbel cardinal for $\mathbb{Z}$ is precisely the first measurable cardinal.

4.1. **Strongly compact cardinals.** Recall that an uncountable cardinal κ is *strongly compact* if for every set I, every κ-complete filter on I can be extended to a κ-complete ultrafilter on I.

Strong compactness implies regularity (see [15]). Thus, since for κ regular the filter consisting on all subsets of κ whose complement has cardinality less than κ is κ-complete and non-principal, every strongly compact cardinal is measurable.

The next theorem of Dugas shows that if X is an abelian group whose cardinality is smaller than some strongly compact cardinal κ, then $R_X = R^\kappa_X$, and so the Dugas-Göbel cardinal for X exists.

Theorem 4.5 ([6])**.** *If κ is a strongly compact cardinal and X is an abelian group of cardinality smaller than κ, then $R_X = R^\kappa_X$.*

We shall give a proof of a stronger result below. But first let us consider the following large cardinal notions.

Definition 4.6. If $\delta < \kappa$ are uncountable cardinals, which may be singular, we say that κ is *δ-strongly compact* if for every set I, every κ-complete filter on I can be extended to a δ-complete ultrafilter on I.

An uncountable limit cardinal κ is *almost strongly compact* if κ is δ-strongly compact for every uncountable cardinal $\delta < \kappa$.

To avoid confusion, let us note that in recent literature "κ is δ-strongly compact" is sometimes defined as: κ is regular, $\delta \geq \kappa$, and there is a κ-complete fine ultrafilter on $\mathcal{P}_\kappa(\delta)$ (see below).

Notice that if κ is δ-strongly compact and λ is a cardinal greater than κ, then λ is also δ-strongly compact. Also note that if κ is regular and ω_1-strongly compact, then there exists a measurable cardinal less than or equal to κ.

Suppose κ is δ-strongly compact. Let I be any non-empty set, and for every $a \in I$, let $X_a = \{x \in \mathcal{P}_\kappa(I) : a \in x\}$, where $\mathcal{P}_\kappa(I) = \{x \subseteq I : |x| < \kappa\}$. If κ is regular, then the set $\{X_a : a \in I\}$ generates a κ-complete filter on $\mathcal{P}_\kappa(I)$, which can be extended to a δ-complete ultrafilter on $\mathcal{P}_\kappa(I)$. A δ-complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(I)$ that contains the sets X_a, for $a \in I$, is called a *δ-complete fine measure* on $\mathcal{P}_\kappa(I)$. The *fineness* condition is that $X_a \in \mathcal{U}$ for all $a \in I$.

The following characterizations of δ-strong compactness will be of use later on. For the definition of the set-theoretic notions involved in the statement of the theorem and the proof, see [14] or [15].

Theorem 4.7. *The following are equivalent for any uncountable cardinals $\delta < \kappa$:*

(1) *κ is δ-strongly compact.*
(2) *For every α greater than or equal to κ there exists an elementary embedding $j : V \to M$, with M transitive, and a critical point greater than or equal to δ, such that j is definable in V, and there exists $D \in M$ such that $j''\alpha := \{j(\beta) : \beta < \alpha\} \subseteq D$ and $M \models |D| < j(\kappa)$.*
(3) *For every set I there exists a δ-complete fine measure on $\mathcal{P}_\kappa(I)$.*

Proof. (1)⇒(2): Assume κ is δ-strongly compact, and fix $\alpha \geq \kappa$. Suppose there exists a δ-complete fine measure $\mathcal{U}$ on $\mathcal{P}_\kappa(\alpha)$. If $j_\mathcal{U} : V \to Ult(V,\mathcal{U})$ is the corresponding ultrapower embedding, then since $\mathcal{U}$ is δ-complete, $Ult(V,\mathcal{U})$ is well-founded, and hence isomorphic to a transitive M. Moreover, by δ-completeness, the critical point of $j_\mathcal{U}$ is greater than or equal to δ. Let $\pi : Ult(V,\mathcal{U}) \to M$ be the transitive collapsing map, and let $j = \pi \circ j_\mathcal{U}$. We claim that j satisfies the conditions of (2). Let $D := \pi([Id]_\mathcal{U})$, where $Id : \mathcal{P}_\kappa(\alpha) \to V$ is the identity map. Thus $D \in M$ and, by fineness, $j''\alpha \subseteq D$. Clearly, $Ult(V,\mathcal{U}) \models |[Id]_\mathcal{U}| < j_\mathcal{U}(\kappa)$, and hence $M \models |D| < j(\kappa)$.

Thus, to prove (2) it will be enough to find, for every $\alpha \geq \kappa$, a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$. Notice that if $\kappa \leq \beta < \alpha$ and $\mathcal{U}$ is a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$, then the projection

$$\{X \subseteq \mathcal{P}_\kappa(\beta) : \{Y \in \mathcal{P}_\kappa(\alpha) : Y \cap \beta \in X\} \in \mathcal{U}\}$$

is a δ-complete fine measure on $\mathcal{P}_\kappa(\beta)$. So fix $\alpha \geq \kappa$ and assume, without loss of generality, that α is regular.

If κ is regular, then we have already observed above that a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$ does exist. So suppose κ is singular. Then κ^+ is regular and also δ-strongly compact. So let $\mathcal{U}^*$ be a δ-complete fine measure on $\mathcal{P}_{\kappa^+}(\alpha)$, and let $j_{\mathcal{U}^*} : V \to Ult(V,\mathcal{U}^*)$ be the ultrapower embedding, $\pi : Ult(V,\mathcal{U}^*) \cong M$ the transitive collapse, and $j := \pi \circ j_{\mathcal{U}^*}$. Note that the critical point of j is greater than or equal to δ. Letting $D := \pi([Id]_{\mathcal{U}^*})$, where $Id : \mathcal{P}_{\kappa^+}(\alpha) \to V$ is the identity map, we have that $D \in M$, $j"\alpha \subseteq D$, and $M \models$ "$|D| < j(\kappa^+) = j(\kappa)^+$".

Let $\beta = sup(j''\alpha)$. So, $\beta \cap D$ is cofinal in β. Hence, in M, the cofinality of β is at most $j(\kappa)$. In fact, since $M \models$ "$j(\kappa)$ is singular", $cof(\beta) < j(\kappa)$.

In M, let C be a closed unbounded subset of β of order-type $cof(\beta)$. Observe that $j''\alpha$ is an ω-closed subset of β. So, since $cof(\beta)$ is uncountable, $C \cap j''\alpha$ is unbounded in β. Hence, $I := \{\gamma < \alpha : j(\gamma) \in C\}$ is unbounded in α, and so $|I| = \alpha$.

Now define an ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(I)$ as follows:

$$X \in \mathcal{U} \text{ if and only if } X \subseteq \mathcal{P}_\kappa(I) \text{ and } j^*(I) \cap C \in j^*(X).$$

One can readily check that $\mathcal{U}$ is a δ-complete fine measure on $\mathcal{P}_\kappa(I)$ which, since $|I| = \alpha$, naturally induces a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$.

(2)⇒(3): Without loss of generality, we may assume I is some ordinal α greater than or equal to κ. Given $j : V \to M$ and D as in (2), for α, define $\mathcal{U}$ in V by

$$X \in \mathcal{U} \text{ if and only if } X \subseteq \mathcal{P}_\kappa(\alpha) \text{ and } D \in j(X).$$

Since $M \models |D| < j(\kappa)$, $\mathcal{U}$ is well-defined. It is easy to check that $\mathcal{U}$ is a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$.

(3)⇒(1): Suppose F is a κ-complete filter over some set I. We may assume that F is actually a filter over $\alpha = |I|$. Let $\mathcal{U}$ be a δ-complete fine measure on $\mathcal{P}_\kappa(F)$, and let $j : V \to M \cong Ult(V,\mathcal{U})$ be the corresponding ultrapower embedding, with M transitive. Let $\pi : Ult(V,\mathcal{U}) \to M$ be the transitive collapsing map, and set $D = \pi([Id]_\mathcal{U})$. By fineness, $j''F \subseteq D$, and clearly $M \models |D| < j(\kappa)$.

In M, $j(F)$ is $j(\kappa)$-complete. So there exists $a \in \bigcap(j(F) \cap D)$. Let $\mathcal{V}$ be given by

$$X \in \mathcal{V} \text{ if and only if } X \subseteq \alpha \text{ and } a \in j(X).$$

It is easy to see that $\mathcal{V}$ is a δ-complete ultrafilter on α. Also, it contains F, for if $X \in F$, then $j(X) \in D \cap j(F)$, and therefore $a \in j(X)$. □

If λ is the least measurable cardinal and κ is ω_1-strongly compact, κ not necessarily regular, then κ is λ-strongly compact. For if $\mathcal{U}$ is an ω_1-complete ultrafilter on a set I that is not λ-complete, then there is a partition $\{X_\alpha : \alpha < \beta\}$ of I, for some $\beta < \lambda$, such that none of the X_α belongs to $\mathcal{U}$. But then the set $\{X \subseteq \beta : \bigcup\{X_\alpha : \alpha \in X\} \in \mathcal{U}\}$ is a non-principal ω_1-complete ultrafilter on β, contradicting the minimality of λ.

Thus if κ is ω_1-strongly compact and is also the first measurable, a consistent situation as shown by Magidor [17], then κ is in fact strongly compact.

The following theorem from [10] improves Theorem 4.5 by using a weaker large-cardinal assumption. Here we give another proof.

Theorem 4.8 ([10]). *If κ is δ-strongly compact and X is an abelian group of cardinality smaller than δ, then $R_X = R_X^\kappa$. Hence, if κ is almost strongly compact, then $R_X = R_X^\kappa$ for every X of cardinality less than κ.*

Proof. Assume κ is δ-strongly compact and X is an abelian group of cardinality less than δ. We may assume, by taking an isomorphic copy of X if necessary, that $X \in H_\delta$, where H_δ is the set of all sets whose transitive closure has cardinality less than δ. Now fix an abelian group A and suppose $a \in A$ does not belong to $R_X^\kappa(A)$. We will show that $a \notin R_X(A)$.

For each $B \in \mathcal{P}_\kappa(A)$ with $a \in \langle B \rangle$, fix a homomorphism $f_B : \langle B \rangle \to X$ such that $f_B(a) \neq 0$. If $a \notin B$, set $f_B = 0$.

Let $\mathcal{U}$ be a δ-complete fine measure on $\mathcal{P}_\kappa(A)$, and let $j : V \to Ult(V,\mathcal{U})$ be the corresponding ultrapower embedding. Notice that by δ-completeness, $Ult(V,\mathcal{U})$ is well-founded and the composition of its transitive collapse with j is the identity on H_δ. Hence, $j(X) \cong X$.

In $Ult(V,\mathcal{U})$, the function $[f] := [\langle f_B : B \in \mathcal{P}_\kappa(A) \rangle]$ has as domain the subgroup $\bar{A} := [\prod_{B \in \mathcal{P}_\kappa(A)} \langle B \rangle]$ of $j(A)$ and values in $j(X)$. Moreover, since $\mathcal{U}$ is fine, the set

$\{B \in \mathcal{P}_\kappa(A) : b \in \langle B \rangle\}$ belongs to $\mathcal{U}$, for every $b \in A$. Thus $\bar{A}$ contains $j(b)$ for all $b \in A$, and in particular $j(a) \in \bar{A}$. Furthermore, $[f](j(a)) \neq 0$.

Now, letting $\pi : j(X) \to X$ be an isomorphism, we have that the map $\pi \circ [f] \circ j \restriction A : A \to X$ witnesses that $a \notin R_X(A)$. □

Corollary 4.9.

(1) *If κ is ω_1-strongly compact, then $R_{\mathbb{Z}} = R^\kappa_{\mathbb{Z}}$.*
(2) *If κ is ω_1-strongly compact and λ is the first measurable cardinal, then $R_X = R^\kappa_X$ for every X of cardinality smaller than λ.*

Recall that a group G is *slender* if for every homomorphism $h : \mathbb{Z}^\omega \to G$, $h(e_n) = 0$ for all but finitely many n. A weaker notion is the following.

Definition 4.10 ([11])**.** A group G is *residually slender* if for all non-zero $a \in G$ there is a subgroup H of G such that $a \notin H$ and G/H is slender.

For $X = \mathbb{Z}$ one can prove the following converse to the theorem above (cf. [11], IX 4.9).

Theorem 4.11.

(1) ([10]) *If $R_{\mathbb{Z}} = R^\kappa_{\mathbb{Z}}$, then κ is ω_1-strongly compact.*
(2) *Suppose λ is the first measurable cardinal and G is an infinite residually slender group. If $R_G = R^\kappa_G$, then κ is λ-strongly compact.*

Proof. We give the argument for (2). The proof of (1) is easier. As we observed above, it is enough to show that κ is ω_1-strongly compact. So fix a κ-complete filter F on some set I, and consider the group $A := G^I/F$.

By Lemma 4.2, every subgroup of A of cardinality less than κ can be embedded into G^I. But since $R_G(G^I) = \{0\}$, it follows that $R_G(B) = \{0\}$ for every subgroup B of A of cardinality less than κ.

Thus, since $R_G = R^\kappa_G$, we have $R_G(A) = \{0\}$. So fix a non-zero homomorphism $\varphi : A \to G$. Now an argument as in the proof of Theorem 3.1, replacing $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ by A, yields an ω_1-complete ultrafilter on I which extends F. The point is that, as one can easily check, F is contained in D, for D defined as in the proof of Theorem 3.1. □

Corollary 4.12 ([10])**.** *$R_{\mathbb{Z}} = R^\kappa_{\mathbb{Z}}$ if and only if κ is ω_1-strongly compact.*

Thus Question 1 above is equivalent to the following.

Question 2. Is the first ω_1-strongly compact cardinal the first measurable cardinal?

Let us next consider another equivalent question, involving non-principal ultrafilters on a cardinal κ.

Lemma 4.13. *Suppose κ is a cardinal and F is a λ-complete ultrafilter over κ. Let G be any group of cardinality less than λ. Define $\varphi : G^\kappa/F \to G$ by*

$$\varphi([\sum r_\alpha e_\alpha]) = x \quad \text{if and only if} \quad \{\alpha < \kappa : r_\alpha = x\} \in F.$$

Then φ is a homomorphism and $Ker(\varphi) = \{0\}$.

Proposition 4.14. *Suppose κ is a cardinal such that*

$$R_{\mathbb{Z}}(\mathbb{Z}^\kappa/F) = R^\kappa_{\mathbb{Z}}(\mathbb{Z}^\kappa/F)$$

for some non-principal filter F over κ. Then there is an ω_1-complete non-principal ultrafilter over κ.

Conversely, if F is an ω_1-complete non-principal ultrafilter on κ, then $\mathbb{Z}^\kappa/F$ is isomorphic to $\mathbb{Z}$, and so

$$R_{\mathbb{Z}}(\mathbb{Z}^\kappa/F) = R^\kappa_{\mathbb{Z}}(\mathbb{Z}^\kappa/F) = \{0\}.$$

Thus, $R_{\mathbb{Z}}(\mathbb{Z}^\kappa/F) = R^\kappa_{\mathbb{Z}}(\mathbb{Z}^\kappa/F)$, for some non-principal filter F over κ, if and only if there exists a ω_1-complete non-principal ultrafilter over κ

Proof. By the arguments given in the proofs of Theorems 4.11 and 3.1, and the lemma above. □

Thus, Questions 1 and 2 above are also equivalent to the following.

Question 3. Does $R_{\mathbb{Z}}(\mathbb{Z}^\kappa/F) = R^\kappa_{\mathbb{Z}}(\mathbb{Z}^\kappa/F)$, for some non-principal filter F over κ, imply $R_{\mathbb{Z}} = R^\kappa_{\mathbb{Z}}$?

The following is still open.

Question 4. Suppose κ is an uncountable cardinal such that $R_X = R^\kappa_X$, for all X of cardinality less than κ. Is κ almost strongly compact? (Recall that the converse was proved in Theorem 4.8.)

As we observed above, if λ is the least measurable cardinal and κ is ω_1-strongly compact, then κ is λ-strongly compact. So, the first measurable cardinal cannot be greater than some ω_1-strongly compact cardinal κ, for then κ would be strongly compact, and therefore measurable. But Magidor [17] built a model of ZFC, modulo the existence of a strongly compact cardinal, where the first strongly compact cardinal is also the first measurable cardinal, and therefore also the first ω_1-strongly compact cardinal.

We shall see that the answer to Questions 1, 2, and 3 is negative. However, a simple negative answer does not give much information about the size of the Dugas-Göbel cardinal for $\mathbb{Z}$, i.e., the first ω_1-strongly compact cardinal. Indeed, in Magidor's model no distinction is made between strongly compact and ω_1-strongly compact cardinals. So the relationship between the first ω_1-strongly compact cardinal, the first measurable cardinal, and the first strongly compact cardinal, in ZFC, is still unclear. In Section 6 we clarify the situation by building (modulo the existence of a supercompact cardinal and a strongly compact cardinal above it) a model of ZFC in which the first ω_1-strongly compact cardinal is singular, and therefore bigger than the first measurable cardinal, and smaller than the first strongly compact cardinal.

5. On torsion classes

For X an abelian group, the *torsion class cogenerated by* X is the class ${}^\perp X = \{A : Hom(A, X) = 0\}$. For example, ${}^\perp\mathbb{Z}$ is the class of all groups without free summands. Note that ${}^\perp X = \{A : R_X(A) = A\}$.

${}^\perp X$ is closed with respect to quotients, extensions, and direct sums. Typically, ${}^\perp X$ is a proper class, since it is closed under direct sums. So a natural question is whether ${}^\perp X$ can be generated by a set. That is, whether there exists a set $S \subseteq {}^\perp X$ such that every group in ${}^\perp X$ is isomorphic to a direct sum of groups from S. Equivalently, we have the question of whether there exists a cardinal κ such that every group in ${}^\perp X$ is isomorphic to a direct sum of groups in ${}^\perp X$, each of

them of cardinality less than κ. If such a cardinal κ exists we will say that ${}^{\perp}X$ *is* κ*-generated*.

A theorem of Dugas [6] implies that if there exists a proper class of supercompact cardinals (see the definition at the beginning of the next section), then ${}^{\perp}X$ can be generated by a set, for every abelian group X.

The next theorem yields a proof of this result under the weaker assumption of the existence of a proper class of strongly compact cardinals, which solves a Problem from [6].

Theorem 5.1. *If κ is a δ-strongly compact cardinal and X is an abelian group of cardinality smaller than δ, then ${}^{\perp}X$ is κ-generated.*

Proof. Fix X and $A \in {}^{\perp}X$. We may assume, by taking an isomorphic copy of X if necessary, that $X \in V_\delta$. We may also assume that the universe of A is some ordinal α.

Pick any $a \in A$. It will be enough to show that a belongs to some $G \subseteq A$ in ${}^{\perp}X$ of cardinality less than κ.

Let $j : V \to M$ be an elementary embedding as in Theorem 4.7 (2). So the critical point of j is greater than or equal to δ, hence $X \in M$, and there is $G \in M$ such that $j''A \subseteq G$ and $M \models |G| < j(\kappa)$. Since $j''A \subseteq j(A)$, we may assume G is actually a subgroup of $j(A)$.

In M define, by induction on ordinals, G_β as follows:

$G_0 = G$.
$G_{\beta+1} = R_X(G_\beta)$.
$G_\lambda = \bigcap_{\beta<\lambda} G_\beta$, if λ is a limit ordinal.

The sequence of G_β's stabilizes at some β', i.e., $G_{\beta'} = G_{\beta'+1}$. Therefore, $R_X(G_{\beta'}) = G_{\beta'}$.

By induction on $\beta \leq \beta'$ we will show that $j''A \subseteq G_\beta$. We only need to check for successor stages. So, assuming $j''A \subseteq G_\beta$, let us show that $j''A \subseteq G_{\beta+1}$.

Otherwise, there exists b in A such that $j(b) \notin G_{\beta+1}$. So there exists a homomorphism $h : G_\beta \to X$ in M such that $h(j(b)) \neq 0$. Now consider the map $h \circ j \restriction A$. This makes sense because $j''A \subseteq G_\beta$. Since j is elementary, $h \circ j \restriction : A \to X$ is a homomorphism. So, $h \circ j(b) \neq 0$, and therefore $A \notin {}^{\perp}X$. A contradiction.

The following hold in M:

(1) $G_{\beta'} \in {}^{\perp}X$. (Because $R_X(G_{\beta'}) = G_{\beta'}$.)
(2) $j(a) \in G_{\beta'}$.
(3) $|G_{\beta'}| < j(\kappa)$.

So, $M \models$ "$\exists G \in {}^{\perp}X(j(a) \in G \wedge |G| < j(\kappa))$". Hence, since j is elementary, $V \models$ "$\exists G \in {}^{\perp}X(a \in G \wedge |G| < \kappa)$". □

Thus, if there exists a proper class of almost strongly compact cardinals, then ${}^{\perp}X$ is generated by a set, for all abelian groups X.

Corollary 5.2. *If κ is an ω_1-strongly compact cardinal, then ${}^{\perp}\mathbb{Z}$ is κ-generated.*

We next prove the converse to Theorem 5.1, in the case of $\mathbb{Z}$.

Theorem 5.3. *If ${}^{\perp}\mathbb{Z}$ is κ-generated, then κ is ω_1-strongly compact.*

Proof. Let F be a κ-complete filter on a set I. Let $G := \mathbb{Z}^I/F$. By Lemma 4.2, every $H \subseteq G$ of cardinality less than κ can be embedded into $\mathbb{Z}^I$. But since $R_{\mathbb{Z}}(\mathbb{Z}^I) = \{0\}$, it follows that $R_{\mathbb{Z}}(H) = \{0\}$, and so $H \notin {}^{\perp}\mathbb{Z}$.

By our assumption, $G \notin {}^{\perp}\mathbb{Z}$. So fix a non-zero homomorphism $\varphi : G \to \mathbb{Z}$. Now an argument as in the proof of Theorem 3.1, replacing $\mathbb{Z}^{\kappa}/\mathbb{Z}^{<\kappa}$ by G, yields an ω_1-complete ultrafilter on I which extends F, as F is contained in D, for D defined as in Theorem 3.1. □

Corollary 5.4. *${}^{\perp}\mathbb{Z}$ is κ-generated if and only if κ is ω_1-strongly compact.*

An important precursor of Corollary 5.2 and Theorem 5.3 is some unpublished work of Bergman and Solovay (cf. [3]). In particular, they prove a theorem relating the notion of ω_1-strongly compact cardinals to a question of group theory, and their methods are similar to those used in the proofs of Corollary 5.2 and Theorem 5.3.

The following is still open.

Question 5. Does the converse to Theorem 5.1 hold? I.e., suppose that κ is an uncountable cardinal such that for every abelian group X of cardinality smaller than κ, ${}^{\perp}X$ is κ-generated. Is κ almost strongly compact?

6. A model of ZFC where the first ω_1-strongly compact cardinal is singular

Assuming the existence of a supercompact cardinal we shall construct a model of ZFC in which the first ω_1-strongly compact cardinal is singular, and therefore bigger than the first measurable cardinal.

Recall that a cardinal κ is *α-supercompact* if there exists an elementary embedding $j : V \to M$, with M transitive, such that κ is the critical point of j, $j(\kappa) > \alpha$, and M is closed under sequences of length α. A cardinal κ is *supercompact* if it is α-supercompact for all ordinals α. Equivalently, κ is *supercompact* if there is a κ-complete, *normal* fine measure on $\mathcal{P}_{\kappa}(A)$, for every set A of cardinality $\geq \kappa$. i.e., a κ-complete fine measure such that for every function $f : \mathcal{P}_{\kappa}(A) \to A$ such that $\{B \in \mathcal{P}_{\kappa}(A) : f(B) \in B\} \in \mathcal{U}$, there is a set in $\mathcal{U}$ on which f is constant. (See [15], 22 for a proof of the equivalence.)

Thus, every supercompact cardinal is strongly compact (see Theorem 4.7). But if there exists a measurable cardinal that is a limit of strongly compact cardinals, then there exists a strongly compact cardinal that is not supercompact, a result due to T. Menas ([18]; see also [15], 22.19). Thus, supercompactness is a stronger large cardinal property than strong compactness. However, Magidor [17] shows that it is consistent (assuming the consistency of the existence of a supercompact cardinal) that the least supercompact cardinal is also the first strongly compact cardinal. It is still an open question whether the consistency of ZFC plus the existence of a strongly compact cardinal implies the consistency of ZFC plus the existence of a supercompact cardinal.

We will use Radin forcing (see [5], [13], or [14]) to turn a supercompact cardinal κ into an ω_1-strongly compact cardinal, while preserving the measurability of some cardinal λ smaller than κ, so that κ becomes singular of cofinality λ. Let us recall some basic facts about Radin forcing that will be needed in the arguments below.

6.1. **Basic facts about Radin forcing.** Given an elementary embedding $j : V \to M$, with M transitive and critical point κ, define inductively a sequence $u^j = \langle u^j(\beta) : \beta < lh(u^j)\rangle$ of subsets of $\mathcal{P}(V_{\kappa})$ as follows:

$u^j(0) = \kappa$.

$u^j(\beta) = \{A \subseteq V_{\kappa} : u^j \restriction \beta \in j(A)\}$.

The sequence is defined as long as $u^j \restriction \beta$ is in M, so $lh(u^j)$ depends on the strength of j. One can easily check that each $u^j(\beta)$, for $\beta > 0$, is a well-defined κ-complete non-principal ultrafilter over V_κ. Moreover, if $\beta < \kappa$, then $u^j(\beta)$ concentrates on β-sequences.

One defines inductively the class U_∞ of *measure sequences* as follows:

$$U_0 = \{w : w = u^j \restriction \alpha, \text{ for some } j \text{ and some } \alpha < lh(u^j)\}.$$
$$U_{n+1} = \{w \in U_n : \forall 0 < \gamma < lh(w)(U_n \cap V_{w(0)} \in w(\gamma))\}.$$
$$U_\infty = \bigcap_{n<\omega} U_n.$$

The point is that if $w \in U_\infty$, then for $0 < \alpha < lh(w)$, $w(\alpha)$ concentrates on $U_\infty \cap V_{w(0)}$. Measure sequences exist, given the existence of sufficiently strong elementary embeddings j. For instance, if $j : V \to M$ is given by some (κ, δ)-extender so that $V_{\kappa+2} \subseteq M$ and M is closed under κ-sequences, then $lh(u^j) \geq (2^\kappa)^+$ and $u^j \restriction \alpha \in U_\infty$, for all $\alpha < (2^\kappa)^+$. (See [5] or [13] for details.)

For $u \in U_\infty$, let us write $\kappa(u)$ for $u(0)$, and let $\mathcal{F}(u) := \bigcap_{0<\alpha<lh(u)} u(\alpha)$ if $lh(u) > 1$, and $\mathcal{F}(u) := \{\emptyset\}$ otherwise. Notice that if $lh(u) > 1$, then $\mathcal{F}(u)$ is a κ-complete filter over V_κ.

Given a measure sequence u of length at least 2, one defines the *Radin forcing* R_u as follows. The elements of R_u (also called conditions) are finite sequences $p = \langle (u_0, A_0), \ldots, (u_n, A_n)\rangle$, where

1. For each $i \leq n$, $u_i \in U_\infty$, $A_i \subseteq U_\infty$, and $A_i \in \mathcal{F}(u_i)$.
2. For each $i < n$, $(u_i, A_i) \in V_{\kappa(u_{i+1})}$.
3. $u_n = u$.

(We say that $u_0, \ldots, u_n$ *occur in* p.)

The ordering on R_u is defined as follows. If $p = \langle (u_0, A_0), \ldots, (u_n, A_n)\rangle$ and $q = \langle (v_0, B_0), \ldots, (v_m, B_m)\rangle$ are in R_u, then $p \leq q$ if and only if

1. $\{v_0, \ldots, v_m\} \subseteq \{u_0, \ldots, u_n\}$.
2. For each $j \leq m$ and $i \leq n$, if $v_j = u_i$, then $A_i \subseteq B_j$.
3. If $i \leq n$ is such that $u_i \notin \{v_0, \ldots, v_m\}$ and if $j \leq m$ is the least such that $u_i(0) < v_j(0)$, then $u_i \in B_j$ and $A_i \subseteq B_j$.

One can easily check that $\leq$ is a partial ordering.

If $p = \langle (u_0, A_0), \ldots, (u_m, A_m), (u_{m+1}, A_{m+1}), \ldots, (u, A)\rangle$ is a condition in R_u, then it is clear from the definition of the ordering on R_u that $R_u \downarrow p := \{q \in R_u : q \leq p\}$ factors as

$$R_{u_m} \downarrow \langle (u_0, A_0), ..., (u_m, A_m)\rangle \times R_u \downarrow \langle (\langle \kappa(u_m)\rangle, \emptyset), (u_{m+1}, A_{m+1}), ..., (u, A)\rangle.$$

Given an R_u-generic ultrafilter G over V, let $g_G := \langle w_\alpha : \alpha < lh(g_G)\rangle$ be *the generic sequence given by* G. Namely, g_G is the unique sequence consisting of all $w \neq u$ such that for some B, $\langle w, B\rangle \in p$, for some p in G; and if $\alpha < \beta < lh(g_G)$, then $\kappa(w_\alpha) < \kappa(w_\beta)$. It is not hard to see that G can be recovered from g_G. Indeed, G consists of all $p \in R_u$ such that

(i) if w occurs in p and $w \neq u$, then $w = w_\alpha$, for some $\alpha < lh(g_G)$,
(ii) every w_α occurs in some $q \leq p$

The point is that the set of conditions satisfying (i) and (ii) is a filter that contains G, and therefore is equal to G.

The sequence g_G is *geometric*, which means it has the following three properties:

(1) The set $C_G := \{\kappa(w_\alpha) : \alpha < lh(g_G)\}$ is a club subset of $\kappa(u)$.

(2) $g_G \restriction \beta$ is a R_{w_β}-generic sequence over V, for every $\beta < lh(g_G)$ such that w_β has length at least 2. That is, the set $G \restriction \beta$ consisting of all $p \in R_{w_\beta}$ such that
   (i) If w occurs in p and $w \neq u$, then $w = w_\alpha$, for some $\alpha < \beta$,
   (ii) every w_α, for $\alpha < \beta$, occurs in some $q \leq p$
   is an R_{w_β}-generic filter over V.
(3) For every $A \subseteq V_{\kappa(u)}$,

$$A \in \mathcal{F}(u) \text{ if and only if } \exists \beta < lh(g_G)\, \forall \alpha > \beta\, (w_\alpha \in A).$$

An important fact, due to W. Mitchell [19], is that the converse is also true. Namely, if $g_G = \langle w_\alpha : \alpha < lh(g_G)\rangle$ is a sequence of measure sequences that satisfies (1)-(3), then the filter G defined as in (i) and (ii) above is R_u-generic over V. (See [5] or [13] for details.)

Radin forcing satisfies the *Prikry property*: if φ is a formula in the forcing language (i.e., it may contain names) and $\langle (u_0, A_0), \ldots, (u_n, A_n)\rangle$ is any condition, then for some $B_0 \subseteq A_0, \ldots, B_n \subseteq A_n$, either $\langle (u_0, B_0), \ldots, (u_n, B_n)\rangle$ forces φ or it forces $\neg\varphi$.

Forcing with R_u preserves cardinals. That is, if G is R_u-generic over V, then V and $V[G]$ have the same cardinals. However, the cofinality of κ may change. For example, if $\lambda = lh(u) < \kappa(u)$ is a regular uncountable cardinal, then for some $A \in \mathcal{F}(u)$, which consists of measure sequences of length $< \lambda$, the condition $\langle (u, A)\rangle$ forces that $lh(g_G) = \lambda$, and hence it forces $cof(C_G) = cof(\kappa(u)) = cof(\lambda)$. (See [13], 5.12.)

6.2. **The model.** We are now ready to prove the following.

Theorem 6.1. *Suppose the Generalized Continuum Hypothesis (GCH) holds, κ is a supercompact cardinal and $\lambda < \kappa$ is a measurable cardinal. Then in some generic extension of V that preserves cardinals and the measurability of λ, κ is the first ω_1-strongly compact cardinal and has cofinality λ.*

Proof. Since the GCH holds, we may assume by forcing over V if necessary, while preserving cardinals, the supercompactness of κ, and the measurability of λ, that for some unbounded subset $S \subseteq \kappa$ of cardinals, for every $\gamma \in S$ some stationary $S_\gamma \subseteq \{\eta \in [\gamma, \gamma^+) : cof(\eta) = \omega\}$ does not reflect, i.e., for every ordinal $\theta < \gamma^+$ of uncountable cofinality, $S_\gamma \cap \theta$ is not stationary in θ. This was first proved by Kimchi and Magidor ([16], unpublished). The first published proof appeared in [1]. (A stronger result, namely, that one can force $\square_\gamma$ on a stationary set of regular limit cardinals γ below κ, while preserving the supercompactness of κ and the measurability of λ, is due to A. Apter [2].) For the convenience of the reader, the following is a brief outline of the forcing construction.

For a successor cardinal γ^+, let Q_γ be the forcing notion for adding a non-reflecting stationary subset of γ^+ made up of points of cofinality ω. The elements of Q_γ are bounded subsets of the set $\{\eta \in [\gamma, \gamma^+) : cof(\eta) = \omega\}$ which are not stationary in any $\alpha < \gamma^+$, and the ordering is given by $p \leq q$ if and only if $p \supseteq q$. It is easily seen that the set introduced by this forcing is a stationary subset of γ^+ that does not reflect. Also, Q_γ is γ-strategically closed (see [4]), so it does not add any new subsets of γ. Let $f : \kappa \to V_\kappa$ be a Laver diamond function for κ. I.e., for every set y and every cardinal δ there is an elementary embedding $j : V \to M$, with M transitive and closed under δ-sequences, and with $crit(j) = \kappa$ and $j(f)(\kappa) = y$.

Let S be the set of $\gamma < \kappa$ that are a singular strong limit and such that V_γ is closed under f, and let $\mathbb{Q}$ be the Easton support iteration of Q_γ for $\gamma \in S$. In any forcing extension by $\mathbb{Q}$, for every $\gamma \in S$ there is a non-reflecting stationary subset of γ^+ made up of points of cofinality ω. Now, for every Mahlo cardinal $\mu < \kappa$ that is a limit point of S, $\mathbb{Q}$ splits as an iteration $\mathbb{Q}_\mu * \mathbb{Q}^\mu$, where $\mathbb{Q}_\mu$ is of size μ and satisfies the μ-cc and $\mathbb{Q}^\mu$ is $f(\mu)+1$-strategically closed. (Note that by the definition of S there are no members of S between μ and $f(\mu)$.) Using the definition of the function f one can show that forcing with $\mathbb{Q}$ preserves the supercompactness of κ. Let us note that since the GCH holds, the Easton support iteration of Q_γ for $\gamma \in S$ does not collapse $2^\gamma = \gamma^+$ because it adds no new subsets of γ (being strategically closed).

Continuing with the proof of the theorem, given an ordinal $\alpha \geq |V_{\kappa+2}|$, let $j_\alpha : V \to M_\alpha$ be the ultrapower elementary embedding coming from some κ-complete fine normal measure over $\mathcal{P}_\kappa(\alpha)$, so that j_α witnesses that κ is α-supercompact. Let $u_\alpha = \langle u_\alpha(\beta) : \beta < \lambda \rangle$ be the measure sequence of length λ obtained from j_α.

There is a proper class $\mathcal{C}$ of ordinals and a sequence $u = \langle u(\beta) : \beta < \lambda \rangle$ such that $u_\alpha = u$ for all α in $\mathcal{C}$.

Let R_u be the Radin forcing for u. We will show that some condition of R_u forces that κ is an ω_1-strongly compact cardinal.

Fix $\langle (u, A) \rangle$ in R_u which forces that $lh(g_{\dot{G}}) = \lambda$. So, A consists of measure sequences of length $< \lambda$. Moreover, we require that $A \in \mathcal{F}(u) \setminus V_{\lambda+1}$, which we can do because $\mathcal{F}(u)$ is κ-complete.

Now suppose α belongs to $\mathcal{C}$. Let $M := M_\alpha$, where, recall, $j_\alpha : V \to M_\alpha$ witnesses that κ is α-supercompact. Since λ is a measurable cardinal, in V we can pick a normal ultrafilter v on λ. Observe that $v \in M$. So we can let $j_v : M \to N \cong Ult(M, v)$, with N transitive, be the corresponding ultrapower embedding. Define $j^* : V \to N$ as $j_v \circ j_\alpha$.

Notice that $\langle (u, A) \rangle$ belongs to M.

Suppose now that G is an R_u-generic ultrafilter over V, with $\langle (u, A) \rangle \in G$. Let $g_G = \langle w_\alpha : \alpha < \lambda \rangle$ be the generic sequence given by G. Since $A \cap V_{\lambda+1} = \emptyset$, we must have $\kappa(w_\alpha) > \lambda$, for all $\alpha < \lambda$. Notice that each w_α, $\alpha < \lambda$, is a measure sequence of length $< \lambda$.

Claim 6.2. The sequence $\langle j_v(w_\alpha) : \alpha < \lambda \rangle$ is geometric, with respect to $j_v(u) \restriction \lambda$ and N. That is,

(1) The set $\{\kappa(j_v(w_\alpha)) : \alpha < \lambda\}$ is a club subset of $\kappa(j_v(u) \restriction \lambda)$.
(2) $\langle j_v(w_\alpha) : \alpha < \beta \rangle$ is an $R_{j_v(w_\beta)}$-generic sequence over N, for every $\beta < \lambda$ such that $j_v(w_\beta)$ has length at least 2.
(3) For every $A \in V_{\kappa+1} \cap N$, we have that $A \in \mathcal{F}(j_v(u) \restriction \lambda)$ if and only if $\exists \gamma < \lambda \, \forall \alpha > \gamma \, (j_v(w_\alpha) \in A)$.

Proof. Recall that $C_G = \{\kappa(w_\alpha) : \alpha < \lambda\}$ is a club subset of κ. So, (1) is clear, since $\kappa(w_\alpha)$ is an inaccessible cardinal greater than λ, and hence it is fixed by j_v. Thus, $\kappa(j_v(w_\alpha)) = \kappa(w_\alpha)$, for all $\alpha < \lambda$, and moreover $\kappa(j_v(u) \restriction \lambda) = \kappa$ and $j_v(\kappa) = \kappa$.

We will first prove (2) and (3) inductively on $\beta < \lambda$. So fix β. Since the proof is by induction, using the characterization of genericity given at the end of Subsection 6.1, we only need to prove (1) and (3) for w_β. Namely,

(1) The set $\{\kappa(j_v(w_\alpha)) : \alpha < \beta\}$ is a club subset of $\kappa(j_v(w_\beta))$.

(3) For every $A \in V_{\kappa(j_v(w_\beta))+1} \cap N$, we have that $A \in \mathcal{F}(j_v(w_\beta))$ if and only if $\exists \gamma < \beta \, \forall \alpha > \gamma \, (j_v(w_\alpha) \in A)$.

Again, (1) is clear, since $\{\kappa(w_\alpha) : \alpha < \beta\}$ is a club subset of $\kappa(w_\beta)$, and $\kappa(j_v(w_\alpha)) = \kappa(w_\alpha)$, for all $\alpha \leq \beta$.

For (3), fix $A \in V_{\kappa(j_v(w_\beta))+1} \cap N$ with $A \in \mathcal{F}(j_v(w_\beta))$. In $Ult(M, v)$, A is represented by some function $f : \lambda \to M$. Without loss of generality, $f(\eta) \in \mathcal{F}(w_\beta)$, for every $\eta < \lambda$.

Thus, since $\mathcal{F}(w_\beta)$ is a λ^+-complete filter, we have

$$\bar{A} := \bigcap_{\eta<\lambda} f(\eta) \in \mathcal{F}(w_\beta).$$

So for all $\eta < \lambda$, $\bar{A} \subseteq f(\eta)$, and therefore, $N \models j_v(\bar{A}) \subseteq A$. Also, since $\bar{A} \in \mathcal{F}(w_\beta)$, and the sequence $\langle w_\alpha : \alpha < \beta \rangle$ is R_{w_β}-generic over V, it is eventually contained in $\bar{A}$. Therefore, the sequence $\langle j_v(w_\alpha) : \alpha < \beta \rangle$ is eventually contained in $j_v(\bar{A})$, and hence in A.

Now suppose that $A \in V_{\kappa(j_v(w_\beta))+1} \cap N$, but $A \notin j_v(w_\beta)(\gamma)$, for some $0 < \gamma < lh(j_v(w_\beta))$. Then $A^c := V_{\kappa(j_v(w_\beta))} \setminus A$ belongs to $j_v(w_\beta)(\gamma)$. Note that since w_β has length $< \lambda$, we have that $lh(j_v(w_\beta)) = lh(w_\beta) < \lambda$, hence $\gamma < \lambda$.

In $Ult(M, v)$, A is represented by a function $f : \lambda \to M$, which we may assume takes only values in $V_{\kappa(w_\beta)+1}$. Also, since $j_v(\beta) = \beta$ and $j_v(\gamma) = \gamma$, we may also assume that the values of f do not belong to $w_\beta(\gamma)$. Let A^* be the union of all the $f(\eta)$, for $\eta < \lambda$. Since $w_\beta(\gamma)$ is λ^+-complete, A^* is also not in $w_\beta(\gamma)$. Hence A^* is not in $\mathcal{F}(w_\beta)$. Since the sequence $\langle w_\alpha : \alpha < \beta \rangle$ is geometric (after all, G is generic over V), for unboundedly many $\alpha < \beta$ we have that w_α is not in A^*. But if $w_\alpha \notin A^*$, then $j_v(w_\alpha) \notin j_v(A^*)$. By the definition of A^* this implies that $j_v(w_\alpha)$ is not in the union of sets of the form $j_v(f)(\eta)$, where $\eta < j_v(\lambda)$. In particular, for any such α, $j_v(w_\alpha)$ is not in $j_v(f)(\lambda)$, which is exactly A.

It only remains to prove (3) for λ. So suppose first that $A \in V_{\kappa+1} \cap N$ belongs to $\mathcal{F}(j_v(u) \restriction \lambda)$. We may assume $A = j_v(f)(\lambda)$, where $f : \lambda \to M$ and $f(\alpha) \in \mathcal{F}(u \restriction \alpha)$, for all $\alpha < \lambda$. For each $1 < \alpha < \lambda$, let $D_\alpha = \bigcap_{\alpha \leq \gamma} f(\gamma)$. Thus, $D_\alpha \in \mathcal{F}(u \restriction \alpha)$. Hence $D := \bigcup_{1<\alpha<\lambda} D_\alpha$ belongs to $\mathcal{F}(u \restriction \alpha)$, for all $1 < \alpha < \lambda$, and so it belongs to $\mathcal{F}(u)$. Now notice that if $w_\beta \in D$, then $j_v(w_\beta) \in j_v(f)(\lambda) = A$.

Suppose now that $A \notin \mathcal{F}(j_v(u) \restriction \lambda)$. That is, $A \notin j_v(u)(\gamma)$, for some $\gamma < \lambda$. We may argue as above. Namely, $A^c := V_{\kappa+1} \setminus A$ belongs to $j_v(u)(\gamma)$. Thus, we may assume $f(\eta) \notin u(\gamma)$, for all $\eta < \lambda$, and hence $A^* := \bigcup_{\eta<\lambda} f(\eta) \notin u(\gamma)$. So $A^* \notin \mathcal{F}(u)$, and hence for unboundedly many $\alpha < \lambda$, $w_\alpha \notin A^*$. But if $w_\alpha \notin A^*$, then $j_v(w_\alpha) \notin j_v(A^*)$, which implies that $j_v(w_\alpha) \notin \bigcup_{\eta<j_v(\lambda)} j_v(f)(\eta)$. In particular, $j_v(w_\alpha) \notin j_v(f)(\lambda) = A$. □

In $V[G]$, let H be the filter on $R_{j_v(u)\restriction\lambda}$ given by $j_v(g_G) \restriction \lambda$. Namely, H is the set of all $p \in R_{j_v(u)\restriction\lambda}$ such that

(i) If w occurs in p and $w \neq j_v(u) \restriction \lambda$, then $w = j_v(w_\alpha)$, for some $\alpha < \lambda$.
(ii) Every $j_v(w_\alpha)$, $\alpha < \lambda$, occurs in some $q \leq p$.

It follows from the claim above that H is $R_{j_v(u)\restriction\lambda}$-generic over N. Moreover, $\langle (j_v(u) \restriction \lambda, j_v(A)) \rangle$ belongs to H.

Claim 6.3. $p := \langle (j_v(u) \restriction \lambda, j_v(A)), (j^*(u), j^*(A)) \rangle$ belongs to $j^*(R_u)$ and extends $\langle (j^*(u), j^*(A)) \rangle$.

Proof. Since $A \in \mathcal{F}(u)$ and j_v is elementary, $j_v(A) \in \mathcal{F}(j_v(u) \restriction \lambda)$, and so $p \in j^*(R_u)$. To see that p extends $\langle (j^*(u), j^*(A)) \rangle$, notice first that $A \subseteq j_\alpha(A)$, and therefore, applying j_v, we have $j_v(A) \subseteq j^*(A)$. So it only remains to check that $j_v(u) \restriction \lambda \in j^*(A)$. Since $A \in \bigcap_{\beta < lh(u)} u(\beta)$, by applying j_v we have that

$$j_v(A) \in \bigcap_{\beta < lh(j_v(u))} j_v(u)(\beta).$$

Hence, since $\lambda < j_v(\lambda) = lh(j_v(u))$, we have $j_v(A) \in j_v(u)(\lambda)$. Recall that, by the definition of u, $A \in u(\beta)$ if and only if $u \restriction \beta \in j_\alpha(A)$, for all $\beta < lh(u)$. Thus, applying j_v and taking $\beta = \lambda < lh(j_v(u))$, we conclude that $j_v(u) \restriction \lambda \in j_v(j_\alpha(A)) = j^*(A)$, as required. $\square$

By the last claim, $j^*(R_u) \downarrow p$ factors as

$$R_{j_v(u)\restriction\lambda} \downarrow \langle (j_v(u) \restriction \lambda, j_v(A)) \rangle \times R_{j^*(u)} \downarrow \langle (\langle \kappa \rangle, \emptyset), (j^*(u), j^*(A)) \rangle.$$

We have already shown that $H_0 := H \cap R_{j_v(u)\restriction\lambda} \downarrow \langle (j_v(u) \restriction \lambda, j_v(A)) \rangle$ is $R_{j_v(u)\restriction\lambda} \downarrow \langle (j_v(u) \restriction \lambda, j_v(A)) \rangle$-generic over N.

Consider the set $j_\alpha[\mathcal{F}(u)] := \{j_\alpha(B) : B \in \mathcal{F}(u)\}$, which has cardinality 2^κ and is included in M. Since we picked α so that $\alpha \geq |V_{\kappa+2}| > 2^\kappa$, and M is closed under α-sequences, we have that $j_\alpha[\mathcal{F}(u)] \in M$. Moreover, since $j_\alpha[\mathcal{F}(u)] \subseteq j_\alpha(\mathcal{F}(u))$ and M thinks that $j_\alpha(\mathcal{F}(u))$ is a $j_\alpha(\kappa)$-complete filter and that $j_\alpha(\kappa) > \alpha$, we have that

$$C := \bigcap_{B \in \mathcal{F}(u)} j_\alpha(B) \in j_\alpha(\mathcal{F}(u)) = \mathcal{F}(j_\alpha(u)).$$

Let $A^* := j_v(C)$. Then

$$A^* \in j_v(j_\alpha(\mathcal{F}(u))) = \mathcal{F}(j^*(u)).$$

Also, if $B \in \mathcal{F}(u)$, then $A^* \subseteq j^*(B)$. In particular, $A^* \subseteq j^*(A)$.

So now let $\mathbb{P}$ be the forcing $R_{j^*(u)} \downarrow \langle (\langle \kappa \rangle, \emptyset), (j^*(u), j^*(A)) \rangle$, and suppose H_1 is $\mathbb{P}$-generic over $V[G]$ with $\langle (\langle \kappa \rangle, \emptyset), (j^*(u), A^*) \rangle \in H_1$. Note that H_1 is also $\mathbb{P}$-generic over $N[H_0]$.

Observe that if $q = \langle (u_0, A_0), \ldots, (u_n, A_n), (u, B) \rangle$ is a condition of R_u, then since j_α is the identity on V_κ, $j^*(q)$ can be extended (by Claim 6.3 and our choice of A^*) to

$$\langle (j_v(u_0), j_v(A_0)), \ldots, (j_v(u_n), j_v(A_n)), (j_v(u) \restriction \lambda, j_v(B)), (j^*(u), A^*) \rangle,$$

which factors as

$$\langle \langle (j_v(u_0), j_v(A_0)), \ldots, (j_v(u_n), j_v(A_n)), (j_v(u) \restriction \lambda, j_v(B)) \rangle,$$
$$\langle (\langle \kappa \rangle, \emptyset), (j^*(u), A^*) \rangle \rangle.$$

In particular, if $q \in G$, then q is of the form $\langle (w'_0, B_0), \ldots, (w'_n, B_n), (u, B) \rangle$, where the w'_i are among the w_α's that occur in the sequence g_G, and $B \subseteq A$. Thus, $j^*(q)$ can be extended to a condition that factors as

$$\langle \langle (j_v(w'_0), j_v(B_0)), \ldots, (j_v(w'_n), j_v(B_n)), (j_v(u) \restriction \lambda, j_v(B)) \rangle,$$
$$\langle (\langle \kappa \rangle, \emptyset), (j^*(u), A^*) \rangle \rangle,$$

which clearly belongs to $H_0 \times H_1$.

For $r \in R_{j^*(u)} \downarrow \langle (\langle \kappa \rangle, \emptyset), (j^*(u), j^*(A)) \rangle$, define $r^- := r - \{(\langle \kappa \rangle, \emptyset)\}$. Thus, letting $G^* := \{q \cup r^- : \langle q, r \rangle \in H_0 \times H_1\}$ we have that G^* is $j^*(R_u) \restriction p$-generic over

N. Moreover, $j^*[G] := \{j^*(q) : q \in G\}$ is contained in G^*. It follows that j^* lifts to an elementary embedding $j : V[G] \to N[G^*]$.

Now we would like to show, using the characterization of ω_1-strong compactness given in Theorem 4.7 (2), that j witnesses that κ is ω_1-strongly compact for α in $V[G]$. Unfortunately, however, j is not definable in $V[G]$.

But j is definable in $V[G][H_1]$ and therefore, in $V[G]$, there is a $\mathbb{P}$-name τ for j.

Let $D^* := \{j_\alpha(\beta) : \beta < \alpha\}$. Since M is closed under α-sequences, $D^* \in M$, and

$$M \models |D^*| < j_\alpha(\kappa).$$

Now let $D := j_v(D^*)$. So $D \in N$ and $\Vdash_{\mathbb{P}} \tau''\alpha = (j^*)''\alpha \subseteq D$.

In $V[G]$ we are going to define an ω_1-complete fine measure $\mathcal{U}$ on $\mathcal{P}_\kappa(\alpha)$. To wit,

$$X \in \mathcal{U} \text{ if and only if } \exists B(\langle(\langle\kappa\rangle, \emptyset), (j^*(u), B)\rangle \Vdash_{\mathbb{P}} D \in \tau(X)).$$

This definition makes sense because $\Vdash_{\mathbb{P}} D \in \mathcal{P}_{j^*(\kappa)}(j^*(\alpha)) = \mathcal{P}_{\tau(\kappa)}(\tau(\alpha))$.

Moreover, since $\mathcal{F}(j^*(u))$ is a filter, so is $\mathcal{U}$. Also, since $\mathbb{P}$ as a forcing notion in N satisfies the Prikry property in N, and $D, \tau(X) \in N$, $\mathcal{U}$ is an ultrafilter.

To show that $\mathcal{U}$ is ω_1-complete (actually, λ-complete), suppose that $\{X_n : n < \omega\} \subseteq \mathcal{P}(\mathcal{P}_\kappa(\alpha))$ and, for each $n < \omega$, $B_n \in \mathcal{F}(j^*(u))$ witnesses that $X_n \in \mathcal{U}$. The sequence $s := \langle B_n : n < \omega\rangle$ lives in $V[G]$, and so it has an R_u-name $\dot{s}$. For each n, there exists a maximal antichain I_n of R_u and a family $\mathcal{B}_n := \{B_n^p : p \in I_n\}$ such that $p \Vdash$ "$\dot{s}(n) = B_n^p$", for all $p \in I_n$. Since R_u is κ^+-cc (see [13], 5.5), $|I_n| \leq \kappa$, and so $\mathcal{B}_n \in M$. We need the following lemma.

Lemma 6.4. *Every subset of N of cardinality at most κ is included in a set in N whose N-cardinality is at most κ.*

Proof. Let $\{x_\gamma : \gamma < \kappa\}$ be a subset of N,. Since N is the transitive collapse of the ultrapower of M by the ultrafilter v, each x_γ is represented in the ultrapower by a function $f_\gamma : \lambda \to M$, which belongs to M. Since M is closed under κ-sequences, the function F on λ given by $F(\rho) = \{f_\gamma(\rho) : \gamma < \kappa\}$ belongs to M and represents in the ultrapower a set in N whose N-cardinality is at most $j_v(\kappa) = \kappa$ and which includes x_γ, for every $\gamma < \kappa$. □

By the lemma above we can find a set $\mathcal{B}_n^* \in N$ whose N-cardinality is at most κ, and which includes $\mathcal{B}_n$. Without loss of generality we may assume that $\mathcal{B}_n^* \subseteq \mathcal{F}(j^*(u))$. So, since $\mathcal{F}(j^*(u))$ is $j^*(\kappa)$-complete, $B'_n := \bigcap \mathcal{B}_n^* \in \mathcal{F}(j^*(u))$. Thus, every condition forces $B'_n \subseteq \dot{s}(n)$. Since N is closed under ω-sequences, $\{B'_n : n < \omega\} \in N$, and so $B := \bigcap_{n<\omega} B'_n \in \mathcal{F}(j^*(u))$. It follows that

$$\langle(\langle\kappa\rangle, \emptyset), (j^*(u), B)\rangle \Vdash_{\mathbb{P}} D \in \bigcap_{n<\omega} \tau(X_n) = \tau(\bigcap_{n<\omega} X_n),$$

which implies $\bigcap_{n<\omega} X_n \in \mathcal{U}$.

It remains to show that $\mathcal{U}$ is fine. So fix $\beta < \alpha$, and let $X_\beta = \{x \in \mathcal{P}_\kappa(\alpha) : \beta \in x\}$. Then, $\Vdash_{\mathbb{P}} \tau(X_\beta) = \{x \in \mathcal{P}_{j^*(\kappa)}(j^*(\alpha)) : j^*(\beta) \in x\}$. Hence, $\Vdash_{\mathbb{P}} D \in \tau(X_\beta)$, which implies $X_\beta \in \mathcal{U}$.

We have shown that for every $\alpha \in \mathcal{C}$, in $V[G]$ there is an ω_1-complete fine measure on $\mathcal{P}_\kappa(\alpha)$. Hence, by Theorem 4.7 (3), κ is an ω_1-strongly compact cardinal in $V[G]$.

Since we required that $A \in \mathcal{F}(u) \setminus V_{\lambda+1}$, all the measures in all of the measure sequences occurring in every $p \in R_u \downarrow \langle(u, A)\rangle$ are λ^+-complete. Therefore, by the Prikry property, neither new subsets of λ nor new subsets of $\mathcal{P}(\lambda)$ of cardinality $\leq \lambda$ appear in $V[G]$. Therefore, λ is still a measurable cardinal in $V[G]$.

An immediate consequence of [20], Theorem 4.8, is that if δ is an ω_1-strongly compact cardinal, then for every regular cardinal $\gamma > \delta$, every stationary subset of $\{\eta < \gamma : cof(\eta) = \omega\}$ reflects. Thus, to show that in $V[G]$ the cardinal κ is the least ω_1-strongly compact cardinal, it will be sufficient to see that, in $V[G]$, for every $\gamma \in S$ (see the beginning of the proof), S_γ is still a stationary subset of $\{\eta \in [\gamma, \gamma^+) : cof(\eta) = \omega\}$ and it does not reflect.

On the one hand, notice that for some $\alpha < \lambda$, $\kappa(w_\alpha) \leq \gamma < \gamma^+ < \kappa(w_{\alpha+1})$, where, we recall, $g_G = \langle w_\alpha : \alpha < \lambda \rangle$ is the generic sequence added by G. Using the Prikry property of R_u, it is not hard to show that every subset of γ^+ in $V[G]$ appears already in $V[\langle w_\beta : \beta < \alpha \rangle]$ (see [5], 6.5.4). Hence, since $V[\langle w_\beta : \beta < \alpha \rangle]$ is a $\kappa(w_\alpha)^+$-cc extension of V, S_γ is still stationary in $V[\langle w_\beta : \beta < \alpha \rangle]$, and therefore also in $V[G]$.

On the other hand, if C_θ is a closed unbounded subset of some ordinal $\theta < \gamma^+$ of uncountable cofinality such that $S_\gamma \cap C_\theta = \emptyset$, then this is still true in $V[G]$, which shows that S_γ does not reflect on θ in $V[G]$ either.

This concludes the proof of Theorem 6.1. □

Corollary 6.5. *If ZFC plus the existence of a supercompact cardinal is consistent, then so is ZFC plus the fact that the first ω_1-strongly compact cardinal is singular, and hence greater than the first measurable cardinal.*

Moreover, if ZFC plus the existence of two supercompact cardinals is consistent, then so is ZFC plus the fact that the first measurable cardinal is smaller than the first ω_1-strongly compact cardinal, which is smaller than the first strongly compact cardinal.

Proof. If κ is supercompact, then it is well known that one can force the GCH while preserving the supercompactness of κ. This is a folklore result that uses arguments due to Silver (see [14], 21.4). Then the conclusion of the first part of the corollary follows from Theorem 6.1.

As for the second part, if $\kappa < \mu$ are supercompact, then one can force GCH while preserving the supercompactness of κ and μ. Then the conclusion follows from the fact that the forcing of Theorem 6.1, being much smaller than μ, preserves the supercompactness, and hence the strong compactness of μ (see [14], 21.2). □

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