TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 4, April 2014, Pages 2019–2040
S 0002-9947(2013)05916-8
Article electronically published on December 6, 2013

GRADED CARTAN DETERMINANTS OF THE SYMMETRIC GROUPS

SHUNSUKE TSUCHIOKA

Abstract. We give the graded Cartan determinants of the symmetric groups. Based on that, we propose a gradation of Hill's conjecture which is equivalent to Külshammer-Olsson-Robinson's conjecture on the generalized Cartan invariants of the symmetric groups.

1. Introduction

Let $\mathfrak{g}_0$ be a finite-dimensional simple simply-laced complex Lie algebra. Our aim is to give the Shapovalov determinants of the basic representation $V(\Lambda_0)$ of the untwisted quantum affinization $U_v(\widehat{\mathfrak{g}}_0)$ (see Theorem 4.4) and apply them to the graded (resp. generalized) modular representation theory of the symmetric groups (see §5 (resp. §6)).

1.1. **Kashiwara's problem on the specialization for quantum groups.** The motivation of this paper comes from a problem of Kashiwara. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and let $\lambda \in \mathcal{P}^+$ be a dominant integral weight. In [Kas, Problem 2], Kashiwara asks at which specialization $v = \xi \in \mathbb{C}^\times$ (see Remark 3.14) the integrable highest weight $U_v(\mathfrak{g})$-module $V(\lambda)$ remains irreducible. When $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra, it is known that the specialized module remains irreducible if ξ is not a root of unity [APW, Theorem 6.4]. More precisely, for an ℓ-th root of unity $\xi = \exp(\frac{2\pi\sqrt{-1}}{\ell})$ where $\ell \geq 1$, the specialized $U_v(\mathfrak{g})|_{v=\xi}$-module $V(\lambda)|_{v=\xi}$ is irreducible if $\ell < (\alpha, \lambda)$ for any positive root $\alpha \in \Delta^+$ of $\mathfrak{g}$ [APW, Theorem 6.7].

When $\mathfrak{g}$ is an affine Kac-Moody Lie algebra, almost nothing is known about Kashiwara's problem. Chari-Jing proved that the specialized $U_v(\widehat{\mathfrak{g}}_0)|_{v=\xi}$-module $V(\Lambda_0)|_{v=\xi}$ is irreducible if ξ is not a root of unity. More precisely, for an ℓ-th root of unity $\xi = \exp(\frac{2\pi\sqrt{-1}}{\ell})$ where $\ell \geq 1$, if ℓ is coprime to the Coxeter number N of $\mathfrak{g}_0$, then the specialized module is irreducible [CJ, Theorem 4]. Recall that $N = n$ (resp. $2(m-1), 12, 18, 30$) when $\mathfrak{g}_0$ is of type A_{n-1} for $n \geq 2$ (resp. D_m for $m \geq 4$, E_6, E_7, E_8). The main result of this paper (Theorem 4.4) gives an answer to Kashiwara's problem when $\mathfrak{g} = \widehat{\mathfrak{g}}_0$ and λ is level 1.

Received by the editors June 5, 2012 and, in revised form, June 13, 2012 and July 3, 2012.
2010 *Mathematics Subject Classification.* Primary 81R50; Secondary 20C08.
Key words and phrases. Symmetric groups, Iwahori-Hecke algebras, graded representation theory, quantum groups, Shapovalov forms, quiver Hecke algebras, generalized blocks.
This research was supported by Grant-in-Aid for Research Activity Startup 22840026 and Research Fellowships for Young Scientists 23·8363, Japan Society for the Promotion of Science.

©2013 American Mathematical Society
Reverts to public domain 28 years from publication

Theorem 1.1. *Let $\mathfrak{g}_0$ be a finite-dimensional simple simply-laced complex Lie algebra of type X. Then, for $\ell \geq 1$ the specialized $U_v(\widehat{\mathfrak{g}}_0)|_{v=\exp(\frac{2\pi\sqrt{-1}}{\ell})}$-module $V(\Lambda_0)|_{v=\exp(\frac{2\pi\sqrt{-1}}{\ell})}$ is irreducible if and only if*

(a) $\mathsf{gcd}(\ell, 2n) = 1, 2$ *when* $X = A_{n-1}$ *for* $n \geq 2$,
(b) $\ell \notin 4\mathbb{Z}$ *(resp.* $\notin 3\mathbb{Z}, \notin 4\mathbb{Z}, \notin 60\mathbb{Z}$*) when* $X = D_n$ *for* $n \geq 4$ *(resp.* E_6, E_7, E_8*).*

1.2. **Graded Cartan determinants.** Recently, Khovanov-Lauda and Rouquier independently introduced a remarkable family of algebras (the KLR algebras) that categorifies the negative half of the quantized enveloping algebras associated with symmetrizable Kac-Moody Lie algebras [KL1, KL2, Rou]. An application of the KLR algebras is the homogeneous presentation of the symmetric group algebras [BK2, Rou] which quantize [BK1] Ariki's categorification of the Kostant $\mathbb{Z}$-form of the basic $\widehat{\mathfrak{sl}}_p$-module $V(\Lambda_0)^{\mathbb{Z}} \cong \bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{Proj}(\mathbb{F}_p\mathfrak{S}_n))$. Since the Shapovalov-like form (see Proposition 3.8) on the left hand side and the graded Cartan pairing on the right hand side are compatible [BK3], the main result of this paper (Theorem 4.4) can be interpreted as the graded Cartan determinants of the symmetric groups. The story is also valid for its q-analog, the Iwahori-Hecke algebra of type A (see §5).

Theorem 1.2. *Let $p \geq 2$ be a prime number (resp. $p \geq 2$ an arbitrary integer). Then, the graded Cartan determinant of a block of the symmetric group algebra over characteristic p (resp. the Iwahori-Hecke algebra of type A over quantum characteristic p) whose p-weight is d is given by $\prod_{s=1}^{d}[p]_s^{N_{p,d,s}}$, where*

$$N_{p,d,s} = \sum_{\lambda\in\mathsf{Par}(d)} \frac{m_s(\lambda)}{p-1} \prod_{u\geq 1} \binom{m_u(\lambda)+p-2}{m_u(\lambda)} = \sum_{(\lambda_i)_{i=1}^{p-1}\in\mathsf{Par}_{p-1}(d)} m_s(\lambda_1).$$

For the notation on partitions, see Notation at the end of §1.

By virtue of the classical results of the modular representation theory of finite groups, the Cartan determinant (see Definition 5.1) of (a block of) a group algebra of a finite group G over characteristic $p \geq 2$ is a power of p [Bra, Theorem 1] and its degree reflects p-defects of p-regular conjugacy classes of G [BrNe, Part III,§16] (see Theorem 6.2). Based on Olsson's calculation of p-defects of the symmetric groups [Ols], Bessenrodt-Olsson gave an explicit formula for the Cartan determinants of (a block of) a symmetric group algebra over characteristic p in terms of generating functions [BO1, Theorem 3.3, Theorem 3.4].

On the other hand, Brundan-Kleshchev arrived at the same formula by a different method [BK4, Corollary 1]. They utilized Ariki's categorification and turned the calculation of the p-Cartan determinants of the symmetric groups into the calculation of the Shapovalov determinants of the basic $\widehat{\mathfrak{sl}}_p$-module $V(\Lambda_0)$. Their block-theory-free method has an advantage in that it is applicable to the Iwahori-Hecke algebra of type A. In fact, Brundan-Kleshchev settled affirmatively the conjecture of Mathas that the Cartan determinants of (a block of) the Iwahori-Hecke algebra of type A over quantum characteristic ℓ is a power of ℓ for arbitrary $\ell \geq 2$ (see [Don]).

Theorem 1.2 (and Theorem 4.4) can be seen as a gradation of Brundan-Kleshchev's result and our method is similar to theirs. A difference is that we use

a bi-additive form that obeys a rule slightly different from that of the Shapovalov form (see Proposition 3.8).

We remark that Brundan-Kleshchev also calculated the Shapovalov determinants of the basic module over a twisted affinization of $\mathfrak{g}_0$. Although our method should be applicable in the quantum setting, there are some obstructions to overcome (see Remark 4.9). We discuss them and give conjectural Shapovalov determinants (see Conjecture 4.11).

Recently, the authors [KKT] introduced a superversion of the KLR algebras which they call quiver Hecke superalgebras and showed "Morita superequivalence" between the spin symmetric group algebras and the cyclotomic quiver Hecke superalgebras [KKT, Theorem 5.4]. From the results of [BK5, Tsu], it is expected that the quiver Hecke superalgebras categorify the quantum groups (see also [HW]), and through this the Shapovalov determinants of the basic module of $U_v(A^{(2)}_{p-1})$ for odd prime $p \geq 3$ coincides with the graded Cartan determinants of the spin symmetric group algebras.

1.3. **Külshammer-Olsson-Robinson's conjecture and graded Cartan invariants.** Based on an observation that combinatorial notions for partitions (such as p-cores) that appear in the description of the p-modular representation-theoretic invariants of the symmetric groups make sense when p is not a prime, Külshammer-Olsson-Robinson initiated the study of "ℓ-modular representations of the symmetric groups" for any $\ell \geq 2$ [KOR].

Their main result is a generalization of the Nakayama-Brauer-Robinson classification of the p-blocks of $\mathfrak{S}_n$ for any $\ell \geq 2$ [KOR, Theorem 5.13]. Based on it, they gave a conjectural generalization of the Cartan invariants of the symmetric groups for an arbitrary $\ell \geq 2$ [KOR, Conjecture 6.4] (see Conjecture 6.5 which we call the KOR conjecture for short). As evidence, they checked that the KOR conjecture implies the Cartan determinants of the Iwahori-Hecke algebras of type A over quantum characteristic ℓ due to Brundan-Kleshchev.

Hill affirmatively settled the KOR conjecture for any $\ell \geq 2$ such that each prime factor p of ℓ divides ℓ at most p times [Hil, Theorem 1.3]. In his course of proof, Hill gave a conjecture [Hil, Conjecture 10.5] (see Conjecture 6.7). Hill proved that his conjecture implies the KOR conjecture. In fact, they are equivalent.

As the graded Cartan determinants calculated in this paper, it is reasonable to expect that a proof of Hill's conjecture works similarly in a graded setting. Motivated by this, we propose a gradation of Hill's conjecture (see Conjecture 6.8). For support, we check that Conjecture 6.8 is compatible with the graded Cartan determinants (see Theorem 6.11). The proof itself may be of interest. We expect that our gradation is correct and gives an insight to future trials of the proof of Hill's conjecture (and thus the KOR conjecture).

Notation. $\mathbb{N} = \mathbb{Z}_{\geq 0}$ (resp. $\mathbb{N}_+ = \mathbb{Z}_{\geq 1}$) means the set of non-negative (resp. positive) integers. We denote the empty partition by ϕ and reserve the symbol $\emptyset$ for the empty set. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, we define $m_k(\lambda) = |\{i \geq 1 \mid \lambda_i = k\}|$ for $k \geq 1$. We also define $|\lambda| = \sum_{i \geq 1} \lambda_i$ and $\ell(\lambda) = \sum_{i \geq 1} m_i(\lambda)$. We denote by $\mathsf{Par}(n)$ the set of partitions of $n \geq 0$ and define $\mathsf{Par} = \bigsqcup_{n \geq 0} \mathsf{Par}(n)$. For $m, n \geq 0$, we denote by $\mathsf{Par}_m(n)$ the multipartitions of n, i.e., $\mathsf{Par}_m(n) = \{(\lambda_i)_{i=1}^m \in \mathsf{Par}^m \mid \sum_{i=1}^m |\lambda_i| = n\}$, and put $u(m, n) = |\mathsf{Par}_m(n)|$. Note that $u(0, 0) = 1$ and $u(0, n) = 0$ for $n \geq 1$.

Organization of the paper

In §2, we develop a linear algebra for Gram determinants following [BK4]. In §3, we give a brief review on quantum groups. In §4, we calculate the Shapovalov determinants of the basic representations of the untwisted affine A,D,E quantum groups and in §5 using the KLR algebras we interpret it as graded Cartan determinants of the symmetric groups and its Iwahori-Hecke algebras. Finally, in §6 we propose a gradation on Hill's conjecture and make some justifications.

2. Linear algebra on Gram determinants

Let $\mathbf{k}$ be a field and let V be a $\mathbf{k}$-vector space. We say that a map $B : V \times V \to \mathbf{k}$ is bi-additive if we have $B(w_1+w_2, w) = B(w_1, w)+B(w_2, w)$ and $B(w, w_1+w_2) = B(w, w_1)+B(w, w_2)$ for all $w, w_1, w_2 \in V$. The radical $\mathsf{Rad}(B)$ of a bi-additive map B is defined to be one of the following two additive subgroups of V:

$$\{w_1 \in V \mid \forall w_2 \in V, B(w_1, w_2) = 0\}, \quad \{w_2 \in V \mid \forall w_1 \in V, B(w_1, w_2) = 0\}$$

if they are equal (otherwise, we do not define $\mathsf{Rad}(B)$). We say that a bi-additive map B is non-degenerate if $\mathsf{Rad}(B)$ is defined and $\mathsf{Rad}(B) = 0$.

Definition 2.1. Let $(\mathbf{k}, \tau, \mathscr{A}, V, S, V^{\mathscr{A}})$ be a 6-tuple such that

(a) $\mathbf{k}$ is a field with a ring involution $\tau : \mathbf{k} \xrightarrow{\sim} \mathbf{k}$,
(b) $\mathscr{A}$ is a subring of $\mathbf{k}$ such that $\tau(\mathscr{A}) \subseteq \mathscr{A}$,
(c) V is a finite-dimensional $\mathbf{k}$-vector space,
(d) $V^{\mathscr{A}}$ is an $\mathscr{A}$-lattice of V, i.e., $V^{\mathscr{A}}$ is a free $\mathscr{A}$-module with an isomorphism $\mathbf{k} \otimes_{\mathscr{A}} V^{\mathscr{A}} \xrightarrow{\sim} V$,
(e) $S : V \times V \to \mathbf{k}$ is a bi-additive non-degenerate map such that $S(aw_1, w_2) = \tau(a)S(w_1, w_2)$ and $S(w_1, aw_2) = aS(w_1, w_2)$ for all $w_1, w_2 \in V$ and $a \in \mathbf{k}$.

To such a 6-tuple, we define the Gram matrix

$$\mathsf{GM}_{\mathbf{k},\tau,\mathscr{A}}(V, S, V^{\mathscr{A}}) = (S(w_i, w_j))_{1 \le i,j \le \dim V}$$

and define the Gram determinant $\mathsf{Gd}_{\mathbf{k},\tau,\mathscr{A}}(V, S, V^{\mathscr{A}}) = \det \mathsf{GM}_{\mathbf{k},\tau,\mathscr{A}}(V, S, V^{\mathscr{A}})$, where $(w_i)_{1 \le i \le \dim V}$ is an $\mathscr{A}$-basis of $V^{\mathscr{A}}$.

Remark 2.2. Let X and Y be two Gram matrices associated with two choices of $\mathscr{A}$-bases of $V^{\mathscr{A}}$. Clearly, there exists $P \in \mathrm{GL}_{\dim V}(\mathscr{A})$ such that $X = \tau({}^{\mathrm{tr}}P)YP$. Thus, $\mathsf{Gd}_{\mathbf{k},\tau,\mathscr{A}}(V, S, V^{\mathscr{A}})$ is uniquely determined as an element of $\mathbf{k}^\times/\{a\tau(a) \mid a \in \mathscr{A}^\times\}$.

Applying the argument in [BK4, §5], we have the following comparison result.

Proposition 2.3. *Let I be a finite set and let $\mathbf{k}$ be a field. We regard a polynomial ring $V = \mathbf{k}[y_n^{(i)} \mid i \in I, n \ge 1]$ as a graded $\mathbf{k}$-algebra via $\deg y_n^{(i)} = n$ and denote by V_d the $\mathbf{k}$-vector subspace of V consisting of homogeneous elements of degree d for $d \ge 0$. Assume $\operatorname{char} \mathbf{k} = 0$, and we are given the following data:*

(a) *a subring $\mathscr{A}$ of $\mathbf{k}$ and a ring involution $\tau : \mathbf{k} \xrightarrow{\sim} \mathbf{k}$ such that $\tau(\mathscr{A}) \subseteq \mathscr{A}$,*
(b) *a family of invertible matrices $A = (A^{(m)})_{m \ge 1}$, where $A^{(m)} = (a_{ij}^{(m)})_{i,j \in I} \in \mathrm{GL}_I(\mathscr{A}^\tau)$,*
(c) *two bi-additive non-degenerate maps $\langle , \rangle_S, \langle , \rangle_K : V \times V \to \mathbf{k}$ such that*
   - *$\langle af, g\rangle_X = \tau(a)\langle f, g\rangle_X$, $\langle f, ag\rangle_X = a\langle f, g\rangle_X$ and $\langle f, g\rangle_X = \tau(\langle g, f\rangle_X)$,*

- $\langle 1,1\rangle_S = \langle 1,1\rangle_K$ *and* $\langle my_m^{(i)} f, g\rangle_S = \langle f, \sum_{j\in I} a_{ij}^{(m)} \frac{\partial g}{\partial y_m^{(j)}}\rangle_S$, $\langle my_m^{(i)} f, g\rangle_K = \langle f, \frac{\partial g}{\partial y_m^{(i)}}\rangle_K$,

for $X \in \{S, K\}$ *and* $f, g \in V, a \in \mathbf{k}, m \geq 1$,

(d) *a family of new variables* $(x_n^{(i)})_{\substack{i\in I,\\ n\geq 1}}$ *such that* $x_n^{(i)} - y_n^{(i)} \in \mathbf{k}[y_m^{(i)} \mid 1 \leq m < n] \cap V_n$.

Then, in $\mathbf{k}^\times/\{a\tau(a) \mid a \in \mathscr{A}^\times\}$ *we have*

$$\mathsf{Gd}_{\mathbf{k},\tau,\mathscr{A}}(V_d, \langle,\rangle_S, V_d^{\mathscr{A}})/\mathsf{Gd}_{\mathbf{k},\tau,\mathscr{A}}(V_d, \langle,\rangle_K, V_d^{\mathscr{A}}) = \Delta_d(A)$$

for any $d \geq 0$, *where* $V_d^{\mathscr{A}} = V_d \cap \mathscr{A}[x_n^{(i)} \mid i \in I, n \geq 1]$ *and*

$$\Delta_d(A) = \prod_{s=1}^{d} (\det A^{(s)})^{\sum_{\lambda\in \mathsf{Par}(d)} \frac{m_s(\lambda)}{|I|} \prod_{u\geq 1} \binom{m_u(\lambda)+|I|-1}{m_u(\lambda)}}.$$

The following existence result follows from the argument in [DcKK, Corollary 2.1].

Proposition 2.4. *Let* $\mathbf{k}$ *be a field. We regard a polynomial ring* $V = \mathbf{k}[y_1, y_2, \cdots]$ *as a graded* $\mathbf{k}$*-algebra via* $\mathsf{deg}\, y_n = n$ *and denote by* V_d *the* $\mathbf{k}$*-vector subspace of* V *consisting of homogeneous elements of degree* d *for* $d \geq 0$. *Assume* $\operatorname{char} \mathbf{k} = 0$ *and take a ring involution* $\tau : \mathbf{k} \xrightarrow{\sim} \mathbf{k}$.

(a) *There exists a unique bi-additive non-degenerate map* $\langle,\rangle_K : V \times V \to \mathbf{k}$ *such that*

(i) $\langle af, g\rangle_K = \tau(a)\langle f, g\rangle_K, \langle f, ag\rangle_K = a\langle f, g\rangle_K$ *and* $\langle f, g\rangle_K = \tau(\langle g, f\rangle_K)$,

(ii) $\langle 1,1\rangle_K = 1$ *and* $\langle my_m f, g\rangle_K = \langle f, \frac{\partial g}{\partial y_m}\rangle_K$,

for any $f, g \in V$ *and* $a \in \mathbf{k}$.

(b) *Let* $\mathscr{A}$ *be a subring of* $\mathbf{k}$ *such that* $\tau(\mathscr{A}) \subseteq \mathscr{A}$. *We choose an* $\mathscr{A}$*-lattice* $V_d^{\mathscr{A}}$ *of* V_d *by* $V_d^{\mathscr{A}} = V_d \cap \mathscr{A}[x_n \mid n \geq 1]$ *for* $d \geq 0$, *where* $(x_n)_{n\geq 1}$ *is defined in terms of* $(y_n)_{n\geq 1}$ *via*

$$1 + \sum_{n\geq 1} x_n z^n = \exp(\sum_{n\geq 1} y_n z^n).$$

Then, for all $d \geq 0$ *we have* $\mathsf{Gd}_{\mathbf{k},\tau,\mathscr{A}}(V_d, \langle,\rangle_K, V_d^{\mathscr{A}}) = 1$ *in* $\mathbf{k}^\times/\{a\tau(a) \mid a \in \mathscr{A}^\times\}$.

Corollary 2.5. *In the setting of Proposition* 2.3, *we may take* $x_n^{(i)}$ *for* $i \in I$ *and* $n \geq 1$ *by*

$$1 + \sum_{n\geq 1} x_n^{(i)} z^n = \exp(\sum_{n\geq 1} y_n^{(i)} z^n).$$

Then, we have $\mathsf{Gd}_{\mathbf{k},\tau,\mathscr{A}}(V_d, \langle,\rangle_S, V_d^{\mathscr{A}}) = \Delta_d(A)$.

3. Quantum groups and Shapovalov forms

Let v be an indeterminate. In the rest of the paper, we work in the field $\mathbf{k} = \mathbb{Q}(v)$ and its subring $\mathscr{A} = \mathbb{Z}[v, v^{-1}]$. We consider two ring involutions of $\mathbf{k}$. One is the identity map $\mathsf{id}_{\mathbf{k}}$ and the other is a $\mathbb{Q}$-algebra involution $\tau : \mathbf{k} \to \mathbf{k}$ which maps v to v^{-1}.

3.1. **Quantum groups.** Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable GCM and take the symmetrization $d = (d_i)_{i\in I}$ of A, i.e., a unique $d \in \mathbb{N}_+^I$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$ and $\gcd(d_i)_{i\in I} = 1$. We take a Cartan data $(\mathcal{P}, \mathcal{P}^\vee, \Pi, \Pi^\vee)$ in the following sense.

(a) $\mathcal{P}^\vee$ is a free $\mathbb{Z}$-module of rank $(2|I| - \operatorname{rank} A)$ and $\mathcal{P} = \operatorname{Hom}_{\mathbb{Z}}(\mathcal{P}^\vee, \mathbb{Z})$,
(b) $\Pi^\vee = \{h_i \mid i \in I\}$ is a $\mathbb{Z}$-linearly independent subset of $\mathcal{P}^\vee$,
(c) $\Pi = \{\alpha_i \mid i \in I\}$ is a $\mathbb{Z}$-linearly independent subset of $\mathcal{P}$,
(d) $\alpha_j(h_i) = a_{ij}$ for all $i, j \in I$.

We denote by $Q^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ the postive root lattice and denote by $\mathcal{P}^+$ the set of dominant integral weights $\{\lambda \in \mathcal{P} \mid \forall i \in I, \lambda(h_i) \in \mathbb{N}\}$. For each $i \in I$, $\Lambda_i \in \mathcal{P}^+$ is a dominant integral weight determined modulo the subgroup $\{\lambda \in \mathcal{P} \mid \forall i \in I, \lambda(h_i) = 0\}(\subseteq \mathcal{P})$ by the condition $\Lambda_i(h_j) = \delta_{ij}$ for all $j \in I$.

Recall that the Weyl group $W = W(A)$ is a subgroup of $\operatorname{Aut}(\mathcal{P})$ generated by $\{s_i : \mathcal{P} \xrightarrow{\sim} \mathcal{P}, \lambda \longmapsto \lambda - \lambda(h_i)\alpha_i \mid i \in I\}$.

In the following, we will use the usual abbreviations such as $v_i = v^{d_i}$, $[n]_\ell = \sum_{k=1}^n v^{(n+1-2k)\ell}$, $[n]_\ell! = \prod_{m=1}^n [m]_\ell$ and $\begin{bmatrix} n \\ m \end{bmatrix}_i = \frac{[n]_i!}{[m]_i![n-m]_i!}$ for $i \in I$ and $n \geq m \geq 0$.

Definition 3.1. The quantum group $U_v = U_v(A)$ is an associative $\mathbf{k}$-algebra with 1 generated by $\{e_i, f_i \mid i \in I\} \cup \{v^h \mid h \in \mathcal{P}^\vee\}$ with the following definition relations:

(a) $v^0 = 1$ and $v^h v^{h'} = v^{h+h'}$ for any $h, h' \in \mathcal{P}^\vee$,
(b) $v^{-h} e_i v^h = v^{\alpha_i(h)} e_i, v^{-h} f_i v^h = v^{-\alpha_i(h)} f_i$ for any $i \in I$ and $h \in \mathcal{P}^\vee$,
(c) $e_i f_j - f_j e_i = \delta_{ij}(K_i - K_i^{-1})/(v_i - v_i^{-1})$ for any $i, j \in I$,
(d) $\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = 0 = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)}$ for any $i \neq j \in I$,

where $K_i = v^{d_i h_i}, v_i = v^{d_i}$ and $e_i^{(n)} = e_i^n/[n]_{d_i}!, f_i^{(n)} = f_i^n/[n]_{d_i}!$.

Remark 3.2. U_v has $\mathbf{k}$-antiinvolution σ, $\mathbb{Q}$-antiinvolution Ω and $\mathbb{Q}$-antiautomorphism Υ defined by

$$\begin{aligned}
&\sigma(e_i) = f_i, \quad \sigma(f_i) = e_i, \quad \sigma(v^h) = v^h, \\
&\Omega(e_i) = f_i, \quad \Omega(f_i) = e_i, \quad \Omega(v^h) = v^{-h}, \quad \Omega(v) = v^{-1}, \\
&\Upsilon(e_i) = v_i f_i K_i^{-1}, \quad \Upsilon(f_i) = v_i^{-1} K_i e_i, \quad \Upsilon(v^h) = v^{-h}, \quad \Upsilon(v) = v^{-1}.
\end{aligned}$$

Proposition 3.3 ([Lus, §3.2])**.** *Define the three* $\mathbf{k}$*-subalgebras* $U_v^\pm, U_v^0$ *of* U_v *by*

$$U_v^+ = \langle e_i \mid i \in I\rangle, \quad U_v^- = \langle f_i \mid i \in I\rangle, \quad U_v^0 = \langle v^h \mid h \in \mathcal{P}^\vee\rangle.$$

Thus

(a) *the canonical map* $U_v^- \otimes_{\mathbf{k}} U_v^0 \otimes_{\mathbf{k}} U_v^+ \to U_v$ *is a* $\mathbf{k}$*-vector space isomorphism,*
(b) U_v^0 *is canonically isomorphic to the group* $\mathbf{k}$*-algebra* $\mathbf{k}[\mathcal{P}^\vee]$.

Definition 3.4. We define the following two maps both of which are assured by Proposition 3.3:

(a) The Harish-Chandra projection $\mathsf{HC} : U_v \to U_v^0$ which is a $\mathbf{k}$-linear projector from $U_v = U_v^0 \oplus ((\sum_{i\in I} f_i U_v) + (\sum_{i\in I} U_v e_i))$ to U_v^0.
(b) The evaluation map $\mathsf{ev}_\lambda : U_v^0 \to \mathbf{k}$ which is a $\mathbf{k}$-algebra homomorphism determined by the assignment $\mathsf{ev}_\lambda(v^h) = v^{\lambda(h)}$ for each $h \in \mathcal{P}^\vee$.

3.2. **Shapovalov forms.**

Definition 3.5 ([Lus, §3.4.5])**.** For $\lambda \in \mathcal{P}$, the Verma module is a left U_v-module

$$M(\lambda) = U_v/(\sum_{h\in\mathcal{P}} U_v(v^h - v^{\lambda(h)}) + \sum_{i\in I} U_v e_i).$$

Remark 3.6. By virtue of Proposition 3.3, as a U_v^--module $M(\lambda)$ is free of rank 1, we see that $M(\lambda)$ has a proper maximum U_v-submodule $K(\lambda)$. We reserve the symbol φ_λ for the U_v^--linear isomorphism $\varphi_\lambda : U_v^- \xrightarrow{\sim} M(\lambda), u \mapsto u + \sum_{h\in\mathcal{P}} U_v(v^h - v^{\lambda(h)}) + \sum_{i\in I} U_v e_i$.

Definition 3.7. For $\lambda \in \mathcal{P}$, we define an irreducible U_v-module $V(\lambda) = M(\lambda)/K(\lambda)$.

Proposition 3.8. *Let us take $\lambda \in \mathcal{P}$.*

(a) *There exists a unique non-degenerate symmetric $\mathbf{k}$-bilinear form $\langle , \rangle_{\mathsf{Sh}} : V(\lambda) \times V(\lambda) \to \mathbf{k}$ (called the Shapovalov form) such that $\langle 1_\lambda, 1_\lambda\rangle_{\mathsf{Sh}} = 1$ and $\langle uv, w\rangle_{\mathsf{Sh}} = \langle v, \sigma(u)w\rangle_{\mathsf{Sh}}$ for all $v, w \in V(\lambda)$ and $u \in U_v$,*

(b) *There exist unique bi-additive non-degenerate maps $\langle , \rangle_{\mathsf{QSh}} : V(\lambda) \times V(\lambda) \to \mathbf{k}$ and $\langle , \rangle_{\mathsf{RSh}} : V(\lambda) \times V(\lambda) \to \mathbf{k}$ such that for all $X \in \{\mathsf{QSh}, \mathsf{RSh}\}$*

  (i) *$\langle aw_1, w_2\rangle_X = \sigma(a)\langle w_1, w_2\rangle_X$, $\langle w_1, aw_2\rangle_X = a\langle w_1, w_2\rangle_X$ and $\langle w_1, w_2\rangle_X = \sigma(\langle w_2, w_1\rangle_X)$,*

  (ii) *$\langle 1_\lambda, 1_\lambda\rangle_X = 1$ and $\langle uw_1, w_2\rangle_{\mathsf{QSh}} = \langle w_1, \Omega(u)w_2\rangle_{\mathsf{QSh}}$, $\langle uw_1, w_2\rangle_{\mathsf{QSh}} = \langle w_1, \Upsilon(u)w_2\rangle_{\mathsf{RSh}}$.*

  for all $w_1, w_2 \in V(\lambda)$, $u \in U_v$ and $a \in \mathbf{k}$.

Proof. All cases are similar and standard. Let $X \in \{\mathsf{Sh}, \mathsf{QSh}, \mathsf{RSh}\}$ and put $\Xi = \sigma, \Omega, \Upsilon$ according to $X = \mathsf{Sh}, \mathsf{QSh}, \mathsf{RSh}$ respectively. We just sketch the constructions of $\langle , \rangle_X$ since they are needed in the proof of Proposition 3.16.

Define $\langle , \rangle'_X : M(\lambda) \times M(\lambda) \to \mathbf{k}$ by $\langle w_1, w_2\rangle'_X = \mathsf{ev}_\lambda(\mathsf{HC}(\Xi(\varphi_\lambda^{-1}(w_1))\varphi_\lambda^{-1}(w_2)))$. Then, we see that $\mathsf{Rad}(\langle , \rangle'_X) = K(\lambda)$ and $\langle , \rangle'_X$ decent to $\langle , \rangle_X : V(\lambda) \times V(\lambda) \to \mathbf{k}$, which satisfy the desired conditions. □

3.3. **Lusztig lattices.**

Theorem 3.9 ([Lus, Theorem 14.4.3], [Lus2, Theorem 4.5])**.** *Let $U_v^{\mathscr{A}}$ be an $\mathscr{A}$-subalgebra generated by $\{e_i^{(n)}, f_i^{(n)}, K_i^{\pm 1} \mid i \in I, n \geq 0\}$. Then, $U_v^{\mathscr{A}}$ is an $\mathscr{A}$-lattice of U_v.*

Definition 3.10 ([Lus, §3.5])**.** A U_v-module M is called integrable when

(i) $M = \bigoplus_{\nu\in\mathcal{P}} M_\nu$ with $\dim M_\nu < +\infty$, where $M_\nu = \{m \in M \mid \forall h \in \mathcal{P}^\vee, v^h m = v^{\nu(h)} m\}$,

(ii) for any $m \in M$ and $i \in I$, there exists some $n > 0$ such that $f_i^n m = e_i^n m = 0$.

Remark 3.11 ([Lus, Proposition 3.5.8, Proposition 5.2.7]). It is well known that

(a) for $\lambda \in \mathcal{P}$, $V(\lambda)$ is integrable if and only if $\lambda \in \mathcal{P}^+$,

(b) for $\lambda \in \mathcal{P}^+$, the set of weights $P(\lambda) := \{\mu \in \mathcal{P} \mid V(\lambda)_\mu \neq 0\}$ of $V(\lambda)$ is W-invariant.

Theorem 3.12 ([Lus, Theorem 14.4.11])**.** *Assume $\lambda \in \mathcal{P}^+$ and put $V(\lambda)^{\mathscr{A}} := U_v^{\mathscr{A}} 1_\lambda$. For all $\nu \in P(\lambda)$, $V(\lambda)_\nu^{\mathscr{A}} := V(\lambda)_\nu \cap V(\lambda)^{\mathscr{A}}$ is an $\mathscr{A}$-lattice of $V(\lambda)_\nu$.*

Definition 3.13. For $\lambda \in \mathcal{P}^+$ and $\mu \in P(\lambda)$, we define $\mathsf{Sh}_{\lambda,\mu} = \det \mathsf{Sh}^{\mathsf{M}}_{\lambda,\mu} \in \mathbf{k}^\times / v^{2\mathbb{Z}}$ and

$$\mathsf{Sh}^{\mathsf{M}}_{\lambda,\mu} = \mathsf{Sh}^{\mathsf{M}}_{\lambda,\mu}(A) = \mathsf{GM}_{\mathbf{k},\mathrm{id}_{\mathbf{k}},\mathscr{A}}(V(\lambda)_\mu, \mathsf{Sh}, V(\lambda)^{\mathscr{A}}_\mu),$$
$$\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu} = \mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}(A) = \mathsf{GM}_{\mathbf{k},\tau,\mathscr{A}}(V(\lambda)_\mu, \mathsf{QSh}, V(\lambda)^{\mathscr{A}}_\mu),$$
$$\mathsf{RSh}^{\mathsf{M}}_{\lambda,\mu} = \mathsf{RSh}^{\mathsf{M}}_{\lambda,\mu}(A) = \mathsf{GM}_{\mathbf{k},\tau,\mathscr{A}}(V(\lambda)_\mu, \mathsf{RSh}, V(\lambda)^{\mathscr{A}}_\mu).$$

Remark 3.14. Theorem 3.9 and Theorem 3.12 allow us to specialize the integrable highest weight modules. For any $\xi \in \mathbb{C}^\times$ and $\lambda \in \mathcal{P}^+$, the specialized module $V(\lambda)|_{v=\xi} := \mathbb{C}_\xi \otimes_{\mathscr{A}} V(\lambda)^{\mathscr{A}}$ is a module over $U_v|_{v=\xi} := \mathbb{C}_\xi \otimes_{\mathscr{A}} U^{\mathscr{A}}_v$ where $\mathbb{C}_\xi = \mathbb{C}$ is an $\mathscr{A}$-module and where v acts as the multiplication of ξ.

Remark 3.15. For $\lambda \in \mathcal{P}^+$ and $\xi \in \mathbb{C}^\times$, $V(\lambda)|_{v=\xi}$ is irreducible if and only if $\mathsf{Sh}_{\lambda,\mu}(\xi) \neq 0$ for all $\mu \in P(\lambda)$. Thus, for a generic $\xi \in \mathbb{C}^\times$, $V(\lambda)|_{v=\xi}$ is irreducible. It is expected that $V(\lambda)|_{v=\xi}$ is irreducible when ξ is not a root of unity [Kas, Problem 2].

Proposition 3.16. *Let $\lambda \in \mathcal{P}^+$ and take $\mu \in P(\lambda)$. We may assume $\mathsf{Sh}^{\mathsf{M}}_{\lambda,\mu} = \mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}$ and $D\mathsf{QSh}^{M}_{\lambda,\mu} = \mathsf{RSh}^{M}_{\lambda,\mu}$ for some diagonal matrix D all of whose diagonal entries belong to $v^{\mathbb{Z}}$.*

Proof. By the constructions mentioned in the proof of Proposition 3.8, it is enough to show that we can choose an $\mathscr{A}$-basis of $V(\lambda)^{\mathscr{A}}_\mu$ as a subset of $\{f^{(n_1)}_{i_1} \cdots f^{(n_s)}_{i_s} 1_\lambda \mid {}^{s\geq 0}_{1\leq \forall j\leq s, n_j\geq 1, i_j\in I}\}$. By virtue of [Lak, Theorem 6.5], it is possible. □

Corollary 3.17. *For $\lambda \in \mathcal{P}^+$ and $\mu \in P(\lambda)$, we have $\det \mathsf{QSh}^{M}_{\lambda,\mu} = \det \mathsf{RSh}^{M}_{\lambda,\mu}$ in $\mathbf{k}^\times$.*

Proof. We may assume $D\mathsf{QSh}^{M}_{\lambda,\mu} = \mathsf{RSh}^{M}_{\lambda,\mu}$ for some D with $\det D \in v^{\mathbb{Z}}$. Since ${}^{\mathrm{tr}}\mathsf{QSh}^{M}_{\lambda,\mu} = \tau(\mathsf{QSh}^{M}_{\lambda,\mu})$ and ${}^{\mathrm{tr}}\mathsf{RSh}^{M}_{\lambda,\mu} = \tau(\mathsf{RSh}^{M}_{\lambda,\mu})$, we easily see that $\det D = \tau(\det D)$. □

Proposition 3.18. *For $\lambda \in \mathcal{P}^+$, $i \in I$ and $\mu \in P(\lambda)$, we may assume $\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu} = \mathsf{QSh}^{\mathsf{M}}_{\lambda,s_i(\mu)}$.*

Proof. Since $V(\lambda)$ is integrable, we have for each $i \in I$ mutually inverse linear operators $T'_{i,-1} = \sum_{\nu\in P(\lambda)} T'^{(\nu)}_{i,-1}$ and $T''_{i,+1} = \sum_{\nu\in P(\lambda)} T''^{(\nu)}_{i,+1}$ on $V(\lambda)$ (for definitions, see [Lus, §5.2]), and we have $\langle T'_{i,-1}(v), T'_{i,-1}(w)\rangle_{\mathsf{QSh}} = \langle v, T''_{i,+1}(T'_{i,-1}(w))\rangle_{\mathsf{QSh}} = \langle v, w\rangle_{\mathsf{QSh}}$. Since $T'_{i,-1}$ (and $T''_{i,+1}$) preserves $V(\lambda)^{\mathscr{A}}$ [Lus, Proposition 41.2.2], we are done. □

4. Shapovalov determinants of the basic representations

4.1. **Untwisted affine A,D,E case.** Let $X = (a_{ij})_{i,j\in I}$ be a Cartan matrix of type A,D,E and let $\widehat{X} = X^{(1)}$ be the extended Cartan matrix of X indexed by $\widehat{I} = \{0\} \sqcup I$ as in Figure 1. Let $(a_i)_{i\in\widehat{I}}$ be the numerical labels of $\widehat{X}$ in Figure 1 and let $\delta = \sum_{i\in\widehat{I}} a_i\alpha_i$.

Remark 4.1 ([Kac, §12.6]). Let $W = W(\widehat{X})$. We have

$$P(\Lambda_0) = \{w\Lambda_0 - d\delta \mid w \in W, d \geq 0\} (= \{w\Lambda_0 - d\delta \mid w \in W/W_0, d \geq 0\}). \tag{4.1}$$

Note that $W/W_0 \cong W(X)$, where $W_0 = \{w \in W \mid w\Lambda_0 = \Lambda_0\}$.

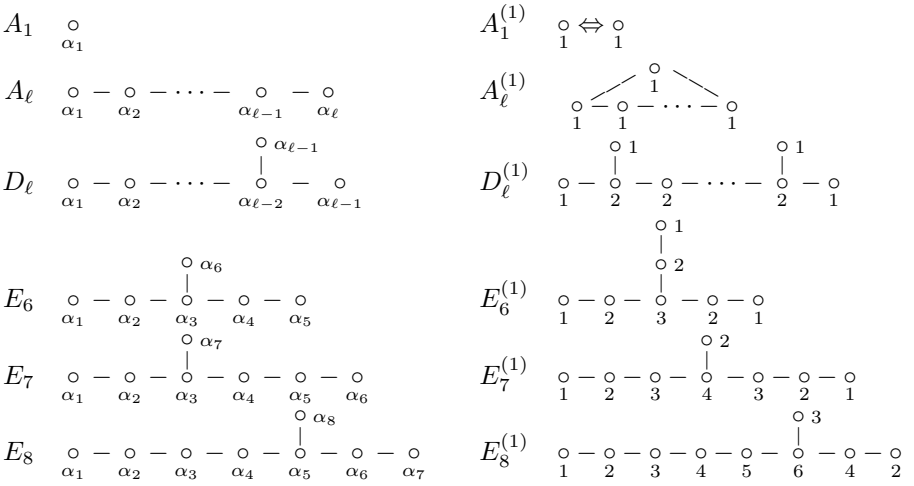

FIGURE 1. Finite and untwisted affine Dynkin diagrams of A,D,E.

We shall very briefly review the vertex operator construction of $V(\Lambda_0)$, i.e., an explicit realization of $V(\Lambda_0)$ as the tensor product of a polynomial algebra and a (twisted) group algebra [FJ] (however, we will mainly refer to [CJ] instead).

The starting point is the Drinfeld new realization of $U_v(\widehat{X})$ which claims that as a $\mathbf{k}$-algebra, $U_v(\widehat{X})$ is isomorphic to the $\mathbf{k}$-algebra generated by $\{x_{i,s}^{\pm} \mid i \in I, s \in \mathbb{Z}\} \cup \{h_{i,r} \mid i \in I, r \in \mathbb{Z} \setminus \{0\}\} \cup \{C^{\pm 1}, D^{\pm 1}, K_i^{\pm 1} \mid i \in I\}$ with the relations in [CJ, Theorem 1].

Actually, as shown by Beck [Bec] (see also [BCP, Theorem 1]), we may choose the generators in the Drinfeld new realization as particular elements in $U_v(\widehat{X})$. Some of them are constructed utilizing Lusztig's braid group action on $U_v(\widehat{X})$ from the Chevalley generators of $U_v(\widehat{X})$. Although we omit the construction, in the rest of this paper we assume this special choice for each generator in the Drinfeld new realization. The choice is used essentially in that

(a) an $\mathscr{A}$-subalgebra generated by $\{(x_{i,n}^{\pm})^r/[r]! \mid i \in I, r \geq 0\}$ coincides with $U_v^{\pm,\mathscr{A}}$ [BCP, Corollary 2.2], where $U_v^{+,\mathscr{A}} = \langle e_i^{(r)} \mid i \in I, r \geq 0 \rangle_{\mathscr{A}\text{-alg}}$ and $U_v^{-,\mathscr{A}} = \langle f_i^{(r)} \mid i \in I, r \geq 0 \rangle_{\mathscr{A}\text{-alg}}$. This is used in the proof of Theorem 4.3 (ii),
(b) we have $\Omega(h_{i,\pm r}) = h_{i,\mp r}$ for $i \in I$ and $r \geq 1$ [BCP, Lemma 3]. This will be used in the proof of Theorem 4.4.

Remark 4.2. Let $U^-(0)$ be the $\mathbf{k}$-subalgebra of $U_v(\widehat{X})$ generated by $\{h_{i,-r} \mid i \in I, r \geq 1\}$. Clearly, it is a commutative algebra. By virtue of [BCP, Proposition 1.3], $\{h_{i,-r} \mid i \in I, r \geq 1\}$ is algebraically independent over $\mathbf{k}$.

Theorem 4.3 ([CJ, Theorem 2, Theorem 3])**.** *Let $Q = \bigoplus_{p \in I} \mathbb{Z}\alpha_p$ be the root lattice of X. We can put a $U_v(\widehat{X})$-module structure on a $\mathbf{k}$-vector space $V :=$*

$U^-(0)\otimes_{\mathbf{k}}\mathbf{k}[Q]$ by extending the following assignment of action $\odot$ on V:

$$h_{i,-s}\odot(h_{i_1,-r_1}\cdots h_{i_j,-r_j}\otimes e^\beta)=h_{i,-s}h_{i_1,-r_1}\cdots h_{i_j,-r_j}\otimes e^\beta,$$

$$h_{i,s}\odot(h_{i_1,-r_1}\cdots h_{i_j,-r_j}\otimes e^\beta)=\sum_{k=1}^{j}h_{i_1,-r_1}\cdots h_{i_{k-1},-r_{k-1}}\frac{\delta_{s,r_k}[sa_{i,i_k}][s]}{s}h_{i_{k+1},-r_{k+1}}\cdots h_{i_j,-r_j}\otimes e^\beta,$$

to all the generators in the Drinfeld new realization and we can obtain a $U_v(\widehat{X})$-module isomorphism $\varphi: V\xrightarrow{\sim}V(\Lambda_0)$ such that

(i) $\varphi(h_{i_1,-r_1}\cdots h_{i_j,-r_j}\otimes e^\beta)\in V(\Lambda_0)_{\Lambda_0-\beta-(\sum_{k=1}^j r_k+(\beta,\beta)/2)\delta}$,

(ii) $\varphi(\mathscr{A}[\widetilde{P}^-_{i,r}\mid i\in I,r\geq 1]\otimes_{\mathscr{A}}\mathscr{A}[Q])=V(\Lambda_0)^{\mathscr{A}}$, *where* $1+\sum_{r\geq1}\widetilde{P}^-_{i,r}z^r=\exp(\sum_{r\geq1}\frac{h_{i,-r}}{[r]}z^r)$.

Here $(i_1,\cdots,i_j)\in I^j,(r_1,\cdots,r_j)\in\mathbb{N}_+^j,\beta=\sum_{p\in I}b_p\alpha_p\in Q$, $i\in I$, $s\geq1$, $(\beta,\beta)=\sum_{p,q\in I}b_pa_{pq}b_q$, and we regard Q as an additive subgroup of the weight lattice $\mathcal{P}$ of $\widehat{X}$.

Theorem 4.4. *Let $X=(a_{ij})_{i,j\in I}$ be a Cartan matrix of type A,D,E and let $\widehat{X}$ be the extended Cartan matrix of X indexed by $\widehat{I}=\{0\}\sqcup I$ as in Figure 1. We have*

$$\mathsf{Sh}_{\Lambda_0,w\Lambda_0-d\delta}(\widehat{X})=\prod_{s=1}^{d}(\det[X]_s)^{\sum_{\lambda\in\mathsf{Par}(d)}\frac{m_s(\lambda)}{|I|}N_\lambda}$$

in $\mathbf{k}^\times/v^{2\mathbb{Z}}$ for all $w\in W$ and $d\geq0$ where $N_\lambda=\prod_{u\geq1}\binom{m_u(\lambda)+|I|-1}{m_u(\lambda)}$ and $[X]_s=([a_{ij}]_s)_{i,j\in I}$.

Proof. By Propositions 3.16 and 3.18, we shall calculate $\det\mathsf{QSh}^{\mathsf{M}}_{\Lambda_0,\Lambda_0-d\delta}(\widehat{X})$ instead of $\mathsf{Sh}_{\Lambda_0,w\Lambda_0-d\delta}(\widehat{X})$. By Theorem 4.3, we see

(i) $V(\Lambda_0)_{\Lambda_0-d\delta}$ has a $\mathbf{k}$-basis
$\{h'_{i_1,-r_1}\cdots h'_{i_j,-r_j}\otimes e^0\mid(i_\ell)\in I^j,(r_\ell)\in\mathbb{N}_+^j \text{ with } \sum_{k=1}^j r_k=d\}$,

(ii) $V(\Lambda_0)^{\mathscr{A}}_{\Lambda_0-d\delta}$ has an $\mathscr{A}$-basis
$\{\widetilde{P}^-_{i_1,r_1}\cdots\widetilde{P}^-_{i_j,r_j}\otimes e^0\mid(i_\ell)\in I^j,(r_\ell)\in\mathbb{N}_+^j \text{ with } \sum_{k=1}^j r_k=d\}$,

where $h'_{i,\pm r}=h_{i,\pm r}/[r]$ for $i\in I$ and $r\geq1$. Note that $1+\sum_{r\geq1}\widetilde{P}^-_{i,r}z^r=\exp(\sum_{r\geq1}h'_{i,-r}z^r)$.

Since $\Omega(h'_{i,-r})=h'_{i,r}$ and

$$(sh'_{i,s})\odot(h'_{i_1,-r_1}\cdots h'_{i_j,-r_j}\otimes e^0)=\sum_{k=1}^{j}\delta_{s,r_k}[a_{i,i_k}]_sh'_{i_1,-r_1}\cdots h'_{i_{k-1},-r_{k-1}}h'_{i_{k+1},-r_{k+1}}\cdots h'_{i_j,-r_j}\otimes e^0,$$

we can apply Corollary 2.5 and get

$$\det\mathsf{QSh}^{\mathsf{M}}_{\Lambda_0,\Lambda_0-d\delta}(A)=\prod_{s=1}^{d}(\det[A]_s)^{\sum_{\lambda\in\mathsf{Par}(d)}\frac{m_s(\lambda)}{|I|}N_\lambda}.$$

□

Corollary 4.5. *Let $\mathfrak{g}_0$ be a finite-dimensional simple simply-laced complex Lie algebra.*

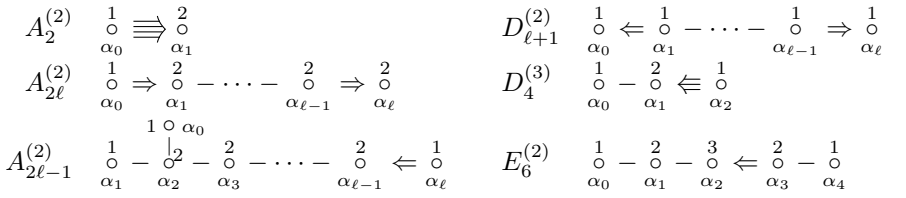

FIGURE 2. Twisted affine Dynkin diagrams of A,D,E.

(a) *If $\xi \in \mathbb{C}^\times$ is not a root of unity, then $V(\Lambda_0)|_{v=\xi}$ is irreducible.*
(b) *Let $\mathfrak{g}_0 = \mathfrak{g}(X)$ and let $\ell \geq 1$. $V(\Lambda_0)|_{v=\exp(\frac{2\pi\sqrt{-1}}{\ell})}$ is irreducible if and only if*
  (i) $\gcd(\ell, 2n) = 1, 2$ *when $X = A_{n-1}$ for $n \geq 2$,*
  (ii) *$\ell \notin 4\mathbb{Z}$ (resp. $\notin 3\mathbb{Z}, \notin 4\mathbb{Z}, \notin 60\mathbb{Z}$) when $X = D_n$ for $n \geq 4$ (resp. E_6, E_7, E_8).*

Proof. The specialized module is irreducible if and only if $(\det[X]_k)|_{v=\xi} \neq 0$ for all $k \geq 1$. Type by type, $\det([X]_1)$ is given by $\det([A_{n-1}]_1) = [n]$, $\det([D_m]_1) = v^{-m}\Phi_4(v)\Phi_4(v^{m-1})$, $\det([E_6]_1) = v^{-6}\Phi_3(v^2)\Phi_{24}(v)$, $\det([E_7]_1) = v^{-7}\Phi_4(v)\Phi_{36}(v)$, and $\det([E_8]_1) = v^{-8}\Phi_{60}(v)$ for $n \geq 2$, $m \geq 4$, where $\Phi_j(x) \in \mathbb{Z}[x]$ is the j-th cyclotomic polynomial. □

Remark 4.6. Let $\mathbf{k}$ be a field of characteristic $p \geq 2$. DeConcini-Kac-Kazhdan proved that the modulo-p reduction $\mathbf{k} \otimes_{\mathbb{Z}} V(\Lambda_0)^{\mathbb{Z}}$ of the basic $\widehat{\mathfrak{g}}_0$-module $V(\Lambda_0)$ where $V(\Lambda_0)^{\mathbb{Z}}$ is the Kostant $\mathbb{Z}$-form of $V(\Lambda_0)$ remains irreducible if and only if $\det X \neq 0$ in $\mathbf{k}$ [DcKK, Corollary 3.1]. Corollary 4.5 can be seen as a quantum analog of DeConcini-Kac-Kazhdan's result. Note that when $X = E_8$ the modulo-p reduction preserves irreduciblity for any prime $p \geq 2$ because $\det E_8 = 1$. Since the quantum characteristic $\min\{k \geq 1 \mid [k]|_{v=\xi} = 0\}$ takes arbitrary value ≥ 2, the quantum case is more subtle than the classical case.

4.2. **Twisted affine A,D,E case.** Let $\widehat{X} = (a_{ij})_{i,j\in\widehat{I}=\{\varepsilon\}\sqcup I}$ be a twisted affine GCM as in Figure 2 where $\varepsilon = \ell$ if $\widehat{X} = A_{2\ell}^{(2)}$ and otherwise $\varepsilon = 0$. Let $\delta = \sum_{i\in\widehat{I}} a_i\alpha_i$ where $(a_i)_{i\in\widehat{I}}$ are the numerical labels of $\widehat{X}$. As in the untwisted cases (see Remark 4.1), it is known that $P(\Lambda_\varepsilon) = \{w\Lambda_\varepsilon - d\delta \mid w \in W(\widehat{X}), d \geq 0\}$ [Kac, §12.6].

We discuss $\mathsf{Sh}_{\Lambda_\varepsilon, w\Lambda_\varepsilon - d\delta}(\widehat{X})$ and give a conjectural formula for it. Our motivation comes from the fact that the Cartan determinant of a faithful p-block of the Schur double covers of the symmetric groups coincides with $\mathsf{Sh}_{\Lambda_\ell, w\Lambda_\ell - d\delta}(A_{p-1}^{(2)})|_{v=1}$ for suitable $w \in W(A_{p-1}^{(2)})$ and $d \geq 0$ when $p = 2\ell + 1$ is an odd prime [BO2, Theorem 5.6].

Definition 4.7 ([BKM, §2.2])**.** We define $\{n\}_s = \frac{v^{ns} + (-1)^s v^{-ns}}{v^s + (-1)^s v^{-s}} (\in \mathscr{A})$ for $n, s \geq 1$ with $n \notin 2\mathbb{Z}$. Note that $\{n\}_s|_{v=1} = n$ if s is odd, but $\{n\}_s|_{v=1} = 1$ if s is even.

Recall Theorem 4.4 which states that for $p \geq 2$, we have

$$\mathsf{Sh}_{\Lambda_0, w\Lambda_0 - d\delta}(A_{p-1}^{(1)}) = \prod_{s=1}^{d} [p]_s^{N_{d,s,p}}$$

for all $w \in W(A^{(1)}_{p-1})$ and $d \geq 0$ where

$$N_{p,d,s} = \sum_{\lambda \in \mathsf{Par}(d)} \frac{m_s(\lambda)}{p-1} \prod_{u \geq 1} \binom{m_u(\lambda)+p-2}{m_u(\lambda)}.$$

Conjecture 4.8. *Let $p \geq 3$ be an odd integer and put $\ell = (p-1)/2$. We have $\mathsf{Sh}_{\Lambda_\ell, w\Lambda_\ell - d\delta}(A^{(2)}_{p-1}) = \prod_{s=1}^{d} \{p\}_s^{N_{\ell,d,s}}$ for all $w \in W(A^{(2)}_{p-1})$ and $d \geq 0$.*

Remark 4.9. Conjecture 4.8 should be proven similarly to Theorem 4.4 if we have

(a) a proof of Drinfeld's new realization of $U_v(A^{(2)}_{p-1})$ (see [Dri]),
(b) a proof of the vertex operator construction of the $U_v(A^{(2)}_{p-1})$-module $V(\Lambda_{(p-1)/2})$ (see [Jin, JM]),
(c) a good choice of the Drinfeld loop-like generators which affords us the Lusztig lattice in terms of the Drinfeld generators (a candidate is the PBW generators in [BeNa]).

However, (a) is not achieved so far in general (thus, logically, neither (b) nor (c)). See also the introduction of [Her].

Remark 4.10. By virtue of [Aka, Appendix B] and [Jin, JM], we can prove Conjecture 4.8 when $p = 3$ by a similar argument of [CJ, Theorem 3] and Theorem 4.4.

Conjecture 4.11. *For a twisted affine A,D,E diagram $X = X^{(r)}_N$ (where $r \geq 2$), we assign the quantities $n, k \in \mathbb{N}_+$ and $\alpha, \beta \in \mathscr{A}$ as follows:*

$X^{(r)}_N$	$A^{(2)}_{2n-1}(n \geq 3)$	$A^{(2)}_{2n}(n \geq 1)$	$D^{(2)}_{n+1}(n \geq 1)$	$E^{(2)}_6$	$D^{(3)}_4$
n	n	n	n	4	2
k	$n-1$	n	1	2	1
α	$[2]_n$	$[2n+1]^{\mathsf{su}}$	$[2]_n$	$\{3\}_2$	$[3]^{\mathsf{su}}$
β	$[n]$	$[2n+1]$	$[2]$	$[3]$	$[2]$

Here for an odd integer $p \geq 1$, we define $[p]^{\mathsf{su}} = (v^p + v^{-p})/(v + v^{-1})$. Note that for an even integer $m \geq 2$, we have $\{p\}_m = [p]^{\mathsf{su}}|_{v=v^m}$. Put

$$\gamma_{X,s} = \begin{cases} \alpha|_{v=v^s} & (s \in r\mathbb{Z}), \\ \beta|_{v=v^s} & (s \notin r\mathbb{Z}), \end{cases} \qquad f_{X,s} = \begin{cases} n & (s \in r\mathbb{Z}), \\ k & (s \notin r\mathbb{Z}). \end{cases}$$

Then, for all $w \in W(\widehat{X})$ and $d \geq 0$ we have $\mathsf{Sh}_{\Lambda_\varepsilon, w\Lambda_\varepsilon - d\delta}(\widehat{X}) = \prod_{s=1}^{d} \gamma_{X,s}^{N_{X,d,s}}$, where

$$N_{X,d,s} = \sum_{\lambda \in \mathsf{Par}(d)} \frac{m_s(\lambda)}{f_{X,s}} \prod_{i \geq 1} \binom{f_{X,i} - 1 + m_i(\lambda)}{m_i(\lambda)}.$$

Remark 4.12. Assume $X^{(r)}_N = (B_{ij})_{i,j \in \widehat{I}} \neq A^{(2)}_{2n}$ for simplicity. Let $X_N = (A_{i,j})_{i,j \in J}$ with $|J| = N$ and let $\mu : X_N \xrightarrow{\sim} X_N$ be a Dynkin diagram automorphism of order r such that the folding procedure to (X_N, μ) gives $X^{(r)}_N$ (see [Kac, §7.9, §8]). In the procedure, I is identified with the set $\{\{\mu^k(j) \mid k \in \mathbb{Z}/r\mathbb{Z}\} \mid j \in J\}$ of μ-orbits of J. We take a map $s : I \to J$ such that i corresponds to $\{\mu^k(s(i)) \mid k \in \mathbb{Z}/r\mathbb{Z}\}$ for all $i \in I$.

For $i \in I$, we put $d_i = a_i^\vee/a_i$, where $a_i^\vee$ is the numerical label of the Langlands dual of $X_N^{(r)}$. Note that we have $d_i = r/|\{\mu^k(s(i)) \mid k \in \mathbb{Z}/r\mathbb{Z}\}|$ for $i \in I$. For $t \geq 1$, we define $I(t) = \{i \in I \mid t \in d_i\mathbb{Z}\}$. Then, the quantities in the above table are designed to satisfy (see also [BK4, §5]) $f_{X,t} = |I(t)|$ and $\det Y^{(t)} = \gamma_{X,t}$, where $Y^{(t)} = (Y_{ij}^{(t)})_{i,j\in I(t)}$ and

$$Y_{ij}^{(t)} = \frac{\sum_{k\in\mathbb{Z}/r\mathbb{Z}} \exp(\frac{2\pi\sqrt{-1}}{r}tk)[A_{s(i),\mu^k(s(j))}]_t}{d_i} = \begin{cases} [B_{ij}]_t & (i = j), \\ B_{ij} & (i \neq j). \end{cases}$$

Remark 4.13. Let $\mathbf{k}$ be a field of characteristic $p \geq 2$ and let $X = X_N^{(r)}$ be a twisted affine A,D,E diagram. Under the assumption that $r \neq 0$ in $\mathbf{k}$, DeConcini-Kac-Kazhdan proved that the modulo-p reduction of the $\mathfrak{g}(X)$-module $V(\Lambda_\varepsilon)$ remains irreducible if and only if $\det X \neq 0$ in $\mathbf{k}$ [DcKK, Remark 3.1]. Conjecture 4.11 predicts an answer for Kashiwara's problem (see §1.1) when $\mathfrak{g} = \mathfrak{g}(X)$ and $\lambda(c) = 1$, and it can be seen as a quantum analog of DeConcini-Kac-Kazhdan's result.

5. Graded Cartan determinants of the symmetric groups

Recall that, in this paper, we work in the field $\mathbf{k} = \mathbb{Q}(v)$ and its subring $\mathscr{A} = \mathbb{Z}[v, v^{-1}]$.

Definition 5.1. Let A be a finite-dimensional algebra over a field $\mathbb{F}$.

(a) We denote by $\mathsf{Mod}(A)$ that the abelian category consists of finite-dimensional left A-modules and the A-homomorphisms between them.

(b) We define the Cartan matrix C_A of A to be the matrix

$$([\mathsf{PC}(D) : D'])_{D,D'\in\mathsf{Irr}(\mathsf{Mod}(A))} \in \mathsf{Mat}_{|\mathsf{Irr}(\mathsf{Mod}(A))|}(\mathbb{Z}),$$

where $\mathsf{PC}(D)$ is a projective cover of D.

Definition 5.2. Let A be a graded finite-dimensional algebra over a field $\mathbb{F}$, i.e., A has a decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$ into $\mathbb{F}$-vector spaces such that $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$.

(a) We denote by $\mathsf{Mod}_{\mathsf{gr}}(A)$ that the abelian category consists of finite-dimensional left graded A-modules and the degree preserving A-homomorphisms between them.

(b) We define the graded Cartan matrix C_A^v of A to be the matrix

$$\big(\textstyle\sum_{k\in\mathbb{Z}}[\mathsf{PC}(D) : D'\langle -k\rangle]v^k\big)_{D,D'\in\mathsf{Irr}(\mathsf{Mod}_{\mathsf{gr}}(A))/\sim} \in \mathsf{Mat}_{|\mathsf{Irr}(\mathsf{Mod}_{\mathsf{gr}}(A))/\sim|}(\mathscr{A}),$$

where $\mathsf{PC}(D)$ is a projective cover of D in $\mathsf{Mod}_{\mathsf{gr}}(A)$ and $M \sim N$ if and only if there exists $k \in \mathbb{Z}$ such that $M\langle k\rangle \cong N$ in $\mathsf{Mod}_{\mathsf{gr}}(A)$.

Remark 5.3. Let C_1^v and C_2^v be the graded Cartan matrices of A according to the different choice of representatives of $\mathsf{Irr}(\mathsf{Mod}_{\mathsf{gr}}(A))/\sim$. Then, there exists a diagonal matrix D all of whose diagonal entries belong to $v^{\mathbb{Z}}$ such that $C_2^v = \tau({}^{\mathrm{tr}}D)C_1^vD$. Thus, the graded Cartan determinant $\det C_A^v$ is well defined.

Remark 5.4. It is routine to prove that the natural map which forgets the grading gives a bijection $\mathsf{Irr}(\mathsf{Mod}_{\mathsf{gr}}(A))/\sim \xrightarrow{\sim} \mathsf{Irr}(\mathsf{Mod}(A))$ and $C_A^v|_{v=1} = C_A$.

Definition 5.5. Let $\ell \geq 2$ be an integer. Recall that $A_{\ell-1}^{(1)} = (2\delta_{i,j} - \delta_{i+1,j} - \delta_{i-1,j})_{i,j\in I:=\mathbb{Z}/\ell\mathbb{Z}}$.

(a) We define $C^v_{\ell,d} = \mathsf{QSh}^{\mathsf{M}}_{\Lambda_0, w\Lambda_0 - d\delta}(A^{(1)}_{\ell-1})$ for $w \in W(A^{(1)}_{\ell-1})$ and $d \geq 0$. $C^v_{\ell,d}$ does not depend on the choice of w as in Proposition 3.18 and $C^v_{\ell,d}$ is related to $\mathsf{RSh}^{\mathsf{M}}_{\Lambda_0, w\Lambda_0 - d\delta}(A^{(1)}_{\ell-1})$ in the sense of Proposition 3.16 and Corollary 3.17.
(b) For each $n \geq 0$, we denote by $\mathsf{Bl}_\ell(n)$ the set of tuples (ρ, d), where ρ is an ℓ-core and $d \geq 0$ is an integer such that $|\rho| + \ell d = n$.
(c) For $(\rho, d) \in \mathsf{Bl}_\ell(n)$ we define $\beta_{\rho,d} = \sum_{x \in \rho} \alpha_{\mathsf{res}(x)} + d \cdot \sum_{y \in \mathbb{Z}/\ell\mathbb{Z}} \alpha_y \in Q^+$, where $\sum_{x \in \rho}$ means that x runs all the boxes of ρ and $\mathsf{res}(x) = j - i + \ell\mathbb{Z}$ when x is located at the i-th row and j-th column.
(d) We fix a field k_ℓ which has an ℓ-th primitive root of unity η_ℓ.

It is well known that elements of $\mathsf{Bl}_\ell(n)$ label the blocks of $\mathbb{F}\mathfrak{S}_n$ for a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} = \ell > 0$, and those of the Iwahori-Hecke algebra of type A $\mathcal{H}_n(k_\ell; \eta_\ell)$ at the quantum parameter $q = \eta_\ell$ over the field k_ℓ for $\ell \geq 2$ [DJ, §5]. By abuse of notation, let $e_{\rho,d}$ be the primitive central idempotent corresponding to $(\rho, d) \in \mathsf{Bl}_\ell(n)$ of either $\mathbb{F}\mathfrak{S}_n$ (in which case $\ell = \operatorname{char}\mathbb{F}$ is prime) or $\mathcal{H}_n(k_\ell; \eta_\ell)$ for $\ell \geq 2$.

For $e \geq 2$, we denote by $R^\lambda_\beta(A^{(1)}_{e-1}; k)$ the cyclotomic KLR algebra of type $A^{(1)}_{e-1}$ for $\lambda \in \mathcal{P}^+$ and $\beta \in Q^+$ over a field k (for the defining relations, see [BK2, Theorem 1]).

For any field $\mathbb{F}$ of characteristic $\ell > 0$ (resp. k_ℓ for $\ell \geq 2$), we have as $\mathbb{F}$-algebras (resp. k_ℓ-algebras) $e_{\rho,d}\mathbb{F}\mathfrak{S}_n e_{\rho,d} \cong R^{\Lambda_0}_{\beta_{\rho,d}}(A^{(1)}_{\ell-1}; \mathbb{F})$ (resp. $e_{\rho,d}\mathcal{H}_n(k_\ell; \eta_\ell)e_{\rho,d} \cong R^{\Lambda_0}_{\beta_{\rho,d}}(A^{(1)}_{\ell-1}; k_\ell)$) for $(\rho, d) \in \mathsf{Bl}_\ell(n)$ [Rou, BK2].

We know the grading that comes from $R^{\Lambda_0}_{\beta_{\rho,d}}(A^{(1)}_{\ell-1}; \mathbb{F})$ (resp. $R^{\Lambda_0}_{\beta_{\rho,d}}(A^{(1)}_{\ell-1}; k_\ell)$) quantizes Ariki's categorification $\bigoplus_{n \geq 0} \mathsf{K}_0(\mathsf{Proj}(\mathbb{F}\mathfrak{S}_n)) \cong V(\Lambda_0)^{\mathbb{Z}}$ (resp. $\bigoplus_{n \geq 0} \mathsf{K}_0(\mathsf{Proj}(\mathcal{H}_n(k_\ell; \eta_\ell))) \cong V(\Lambda_0)^{\mathbb{Z}}$) [BK1]. Moreover, the graded Cartan pairing and the form RSh are compatible [BK3, Theorem 4.18]. By combining Proposition 3.16, Corollary 3.17 and Theorem 4.4, we get the graded Cartan determinants of the symmetric groups and its Iwahori-Hecke algebras.

Theorem 5.6. *Assume $\mathbb{F}$ is a field of characteristic $\ell > 0$ or $\mathbb{F} = k_\ell$ for $\ell \geq 2$. For each $(\rho, d) \in \mathsf{Bl}_\ell(n)$, we have* $\det C^v_{R^\lambda_{\beta_{\rho,d}}(A^{(1)}_{\ell-1}; \mathbb{F})} = \det C^v_{\ell,d} = \prod_{s=1}^d [\ell]_s^{N_{\ell,d,s}}$ *where*

$$N_{\ell,d,s} = \sum_{\lambda \in \mathsf{Par}(d)} \frac{m_s(\lambda)}{\ell - 1} \prod_{u \geq 1} \binom{m_u(\lambda) + \ell - 2}{m_u(\lambda)} = \sum_{(\lambda_i)_{i=1}^{\ell-1} \in \mathsf{Par}_{\ell-1}(d)} m_s(\lambda_1).$$

Remark 5.7. Let $\ell \geq 2$ and assume $\operatorname{char} k_\ell = 0$. By virtue of the graded version of Ariki's theorem [BK3, Corollary 5.15] and the graded Brauer-Humphreys reciprocity [HM, Theorem 2.17], we have $C^v_{R^\lambda_{\beta_{\rho,d}}(A^{(1)}_{\ell-1}; k_\ell)} = {}^{\mathrm{tr}}D_{\rho,d}D_{\rho,d}$ for any ℓ-core ρ and $d \geq 0$ where $D_{\rho,d} = (d_{\lambda,\mu}(v))_{\lambda \in B_{\rho,d}, \mu \in B_{\rho,d} \cap \mathsf{RP}_\ell(d)}$. Here $B_{\rho,d}$ is the set of partitions λ of $\ell d + |\rho|$ whose ℓ-core is ρ and $\mathsf{RP}_\ell(d)$ is the set of ℓ-regular partitions of d. Recall that we have the Fock space representation $\mathcal{F} = \bigoplus_{\lambda \in \mathsf{Par}} \mathbf{k}|\lambda\rangle$ of $U_v(A^{(1)}_{\ell-1})$ due to Hayashi, and $\bigsqcup_{n \geq 0}\{G(\mu) := \sum_{\lambda \in \mathsf{Par}(n)} d_{\lambda,\mu}(v)|\lambda\rangle \mid \mu \in \mathsf{RP}_\ell(n)\}$ is Lusztig's canonical basis (also known as Kashiwara's lower global basis) of $V(\Lambda_0) \cong U_v(A^{(1)}_{\ell-1})|\phi\rangle$ (see [LLT, §6]).

6. Conjectures on the graded Cartan invariants of the symmetric groups

6.1. **Conjectures of Külshammer-Olsson-Robinson and Hill.** Let G be a finite group and let $\mathscr{C} \subseteq G$ be a subset of G invariant under conjugation.

In [KOR, §1], they define that a $\mathscr{C}$-block of G is a certain equivalence class of the ordinary characters of G. For $G_{p'} := \{g \in G \mid \mathsf{ord}_G(g) \notin p\mathbb{Z}\}$ where $\mathsf{ord}_G(g)$ is the order of g in G, the notion of $G_{p'}$-blocks coincides with the usual notion of p-blocks.

Definition 6.1. Let $\ell \geq 2$ be an integer.

(a) A partition λ is called ℓ-class regular if no parts of λ are divisible by ℓ (i.e., $m_{k\ell}(\lambda) = 0$ for all $k \geq 1$). We denote by $\mathsf{CRP}_\ell(n)$ the set of ℓ-class regular partitions of n.

(b) We let $\mathscr{C}_{\ell'} = \bigsqcup_{\lambda \in \mathsf{CRP}_\ell(n)} C_\lambda (\subseteq \mathfrak{S}_n)$, where C_λ is a conjugacy class of $\mathfrak{S}_n$ that consists of elements whose cycle type is λ.

Külshammer-Olsson-Robinson assign the Cartan group $\mathsf{Cart}_G(\mathscr{C})$ which is a finite abelian group [KOR, §1] under the assumption that $\mathscr{C}$ is closed (i.e., $\langle x \rangle = \langle y \rangle$ implies $y \in \mathscr{C}$ for any $x \in \mathscr{C}$ and $y \in G$). We call the invariant factors of $\mathsf{Cart}_G(\mathscr{C})$ the generalized Cartan invariants. When $\ell = p$ is a prime and $\mathscr{C} = G_{p'}$, the generalized Cartan invariants are the usual Cartan invariants (i.e., the elementary divisors of the Cartan matrix $C_{\overline{\mathbb{F}_p}G}$). Recall that in p-modular representation theory, Cartan invariants carry information of certain centralizer groups.

Theorem 6.2 ([BrNe, Part III, §16])**.** *Let G be a finite group and let $p \geq 2$ be a prime. Take a set of representatives $g_1, \cdots, g_k$ of the p-regular conjugacy classes $C_1, \cdots C_k$ of G. The Cartan invariants of $\overline{\mathbb{F}_p}G$ are given by the multiset $\{|\mathsf{Syl}_p(C_G(g_i))| \mid 1 \leq i \leq k\}$.*

In the case of $G = \mathfrak{S}_n$ and $\mathscr{C} = \mathscr{C}_{\ell'}$ for $\ell \geq 2$, we may identify the Cartan group as $\mathsf{Coker}(\mathbb{Z}^{\oplus|\mathsf{RP}_\ell(n)|} \to \mathbb{Z}^{\oplus|\mathsf{RP}_\ell(n)|}, \boldsymbol{x} \mapsto \boldsymbol{x}C_{\mathcal{H}_n(k_\ell;\eta_\ell)})$ by virtue of [Don, §2.2].

Definition 6.3. Let R be a commutative ring with 1.

(a) We say that two $m \times m$ matrices $X, Y \in \mathsf{Mat}_m(R)$ are unimodular equivalent over R (and denote by $X \equiv_R Y$) if there exist $P, Q \in \mathrm{GL}_m(R)$ such that $X = PYQ$.

(b) For a matrix X and a multiset S both consisting of elements of R, we abbreviate $X \equiv_R \mathsf{diag}(S)$ to $X \equiv_R S$, where $\mathsf{diag}(S) = (s_{ij})_{i,j \in I}$ is a diagonal matrix with a multiset identity $\{s_{ii} \mid i \in I\} = S$.

Definition 6.4. Let $n \geq 1$ and let Π be a subset of the set of all prime numbers.

(a) We define $\mathsf{pdiv}(n)$ to be the set of prime divisors of n (when $n = 1$, $\mathsf{pdiv}(n) = \emptyset$).

(b) We define n_Π to be the Π-part of n, i.e., the unique positive integer n_Π such that $n \in n_\Pi\mathbb{Z}$ and $\mathsf{pdiv}(n_\Pi) \subseteq \Pi, \mathsf{pdiv}(n/n_\Pi) \cap \Pi = \emptyset$.

For $\ell \geq 2$ and $(\rho, d) \in \mathsf{Bl}_\ell(n)$, we put $C_{\ell,d} := C^v_{\ell,d}|_{v=1}$ (see Definition 5.5).

Conjecture 6.5 ([KOR, Conjecture 6.4])**.** *Let $\ell \geq 2$ be an integer. We define*

$$r_\ell(\lambda) = \prod_{k \in \mathbb{N}_+ \setminus \ell\mathbb{Z}} (\ell/\mathsf{gcd}(\ell,k))^{\lfloor \frac{m_k(\lambda)}{\ell} \rfloor} \cdot \lfloor \frac{m_k(\lambda)}{\ell} \rfloor !_{\mathsf{pdiv}(\ell/\mathsf{gcd}(\ell,k))}$$

for a partition λ. Then, for $n \geq 0$, we have

$$\bigoplus_{(\rho,d)\in \mathsf{Bl}_\ell(n)} C_{\ell,d} \equiv_{\mathbb{Z}} \{r_\ell(\lambda) \mid \lambda \in \mathsf{CRP}_\ell(n)\}.$$

Definition 6.6. Let $n \geq 1$ and let $\ell \geq 2$. We define $a_\ell(n) \geq 1$ and $\nu_\ell(n) \geq 0$ to be the unique integers such that $a_\ell(n)\ell^{\nu_\ell(n)} = n$ and $a_\ell(n) \notin \ell\mathbb{Z}$.

Let $\ell = \prod_{p\in \mathsf{pdiv}(\ell)} p^r$ be a prime factorization of an integer $\ell \geq 2$. Hill settled affirmatively the KOR conjecture when $r \leq p$ for each $p \in \mathsf{pdiv}(\ell)$ [Hil, Theorem 1.3]. In the course of the proof, Hill proposed a following refinement of the KOR conjecture into each ℓ-block for $|\mathsf{pdiv}(\ell)| = 1$.

Conjecture 6.7 ([Hil, Conjecture 10.5])**.** *Let $p \geq 2$ be a prime and let $r \geq 1$ be an integer. For a partition λ we define a power of p by*

$$\log_p I_{p,r}(\lambda) = \sum_{n\in \mathbb{N}_+\setminus p^r\mathbb{Z}} ((r - \nu_p(n))m_n(\lambda) + \sum_{t\geq 1} \lfloor m_n(\lambda)/p^t \rfloor).$$

Put $\ell = p^r$. Then, for each $d \geq 0$ we have (for the definition of u, see the Notation in §1)

$$C_{\ell,d} \equiv_{\mathbb{Z}} \bigsqcup_{s=0}^{d} \bigsqcup_{\lambda\in \mathsf{Par}(s)} \{I_{p,r}(\lambda)\}^{u(\ell-2,d-s)}.$$

Hill proved that his conjecture implies the KOR conjecture. In fact, they are equivalent.

6.2. **Gradation of Hill's conjecture.** For integers $a \geq 0$ and $b \geq 1$, we denote by $a\%b$ the remainder of a by b, namely the unique integer $0 \leq c < b$ such that $a - c \in b\mathbb{Z}$.

Conjecture 6.8. *Let $p \geq 2$ be a prime and let $r \geq 1$ be an integer. We define*

$$I^v_{p,r}(\lambda) = \prod_{n\in \mathbb{N}_+\setminus p^r\mathbb{Z}} \prod_{k=1}^{m_n(\lambda)} [p^{r+\nu_p(k)-\nu_p(n)}]_{a_p(k)p^{\nu_p(n)}}$$

for a partition λ. Put $\ell = p^r$. Then, for each $d \geq 0$ we have

$$C^v_{\ell,d} \equiv_{\mathscr{A}} \bigsqcup_{s=0}^{d} \bigsqcup_{\lambda\in \mathsf{Par}(s)} \{I^v_{p,r}(\lambda)\}^{u(\ell-2,d-s)}.$$

Remark 6.9. We have $I^v_{p,r}(\lambda)|_{v=1} = I_{p,r}(\lambda)$ for any $\lambda \in \mathsf{Par}$. Thus, Conjecture 6.8 implies Conjecture 6.7. Note that we do not necessarily have $I^v_{p,r}(\lambda_1) \in \mathscr{A} I^v_{p,r}(\lambda_2)$ or $I^v_{p,r}(\lambda_2) \in \mathscr{A} I^v_{p,r}(\lambda_1)$ for $\lambda_1, \lambda_2 \in \mathsf{Par}(n)$, unlike $v = 1$.

Remark 6.10. In the setting of Conjecture 6.8, Conjecture 6.8 implies the following.

$$C^v_{\ell,d} \equiv_{\mathbb{Q}[v,v^{-1}]} \bigsqcup_{s=0}^{d} \bigsqcup_{\lambda\in \mathsf{Par}(s)} \{I^v_{p,r}(\lambda)\}^{u(\ell-2,d-s)}. \tag{6.1}$$

When $r = 1$, (6.1) is the same as [ASY, Conjecture 8.2 (i)] (see Remark 5.7). Although we could not prove (6.1), it seems tractable by an elementary method since we have $C^v_{\ell,d} \equiv_{\mathbb{Q}[v,v^{-1}]} \bigsqcup_{s=1}^{d} \bigsqcup_{\lambda\in \mathsf{Par}(s)} \{\prod_{i\geq 1} [\ell]_i^{m_i(\lambda)}\}^{u(\ell-2,d-s)}$ which follows from the proof of Proposition 2.3.

As support to Conjecture 6.8, we prove that Conjecture 6.8 gives the graded Cartan determinants of Theorem 4.4.

Theorem 6.11. *Let $p \geq 2$ be a prime and let $r \geq 1$ be an integer. Put $\ell = p^r$. Then, for each $d \geq 0$ we have the equality*

$$\prod_{s=1}^{d} [\ell]_s^{N_{\ell,d,s}} = \prod_{s=1}^{d} \prod_{\lambda \in \mathsf{Par}(s)} I_{p,r}^{v}(\lambda)^{u(\ell-2,d-s)}.$$

Remark 6.12. Ando-Suzuki-Yamada proved in [ASY, Theorem 4.3] that for any $\ell \geq 2$

$$\prod_{s=1}^{d} [\ell]_s^{N_{\ell,d,s}} = \prod_{s=1}^{d} \prod_{\lambda \in \mathsf{Par}(s)} Q_{\ell}(\lambda)^{u(\ell-2,d-s)}, \tag{6.2}$$

where $Q_\ell(\lambda) := \prod_{n \in \mathbb{N}_+ \setminus \ell\mathbb{Z}} \prod_{k=1}^{m_n(\lambda)} [\ell^{1+\nu_\ell(k)}]_{a_\ell(k)}$. Note that $I_{p,r}^{v}(\lambda) = Q_{p^r}(\lambda)$ when $r = 1$. Thus, Theorem 6.11 gives another product expansion of the graded Cartan determinants of the symmetric groups that conjecturally comes from nice representatives of the unimodular equivalence classes of the graded Cartan matrices (but only when $|\,\mathsf{pdiv}(\ell)| = 1$).

Proof. By virtue of (6.2), it is enough to prove

$$\prod_{s=1}^{d} \prod_{\lambda \in \mathsf{Par}(s)} Q_{p^r}(\lambda)^{u(\ell-2,d-s)} = \prod_{s=1}^{d} \prod_{\lambda \in \mathsf{Par}(s)} I_{p,r}^{v}(\lambda)^{u(\ell-2,d-s)}. \tag{6.3}$$

Since $[a^b]_c = \prod_{i=1}^{b} [a]_{ca^{i-1}}$ for $a, b, c \geq 1$ and $\nu_{p^r}(n) = \lfloor \nu_p(n)/r \rfloor, a_{p^r}(n) = a_p(n) p^{\nu_p(n)\%r}$ for $n \geq 1$, it is enough to show the following multiset identity in order to prove (6.3):

$$\begin{aligned} &\bigsqcup_{s=1}^{d} \bigsqcup_{\lambda \in \mathsf{Par}(s)} \bigsqcup_{n \in \mathbb{N}_+ \setminus p^r\mathbb{Z}} \bigsqcup_{k=1}^{m_n(\lambda)} \{a_p(k) p^t \mid \nu_p(k)\%r \leq t < r + \nu_p(k)\}^{u(\ell-2,d-s)} \\ &= \bigsqcup_{s=1}^{d} \bigsqcup_{\lambda \in \mathsf{Par}(s)} \bigsqcup_{n \in \mathbb{N}_+ \setminus p^r\mathbb{Z}} \bigsqcup_{k=1}^{m_n(\lambda)} \{a_p(k) p^t \mid \nu_p(n) \leq t < r + \nu_p(k)\}^{u(\ell-2,d-s)}. \end{aligned} \tag{6.4}$$

Clearly, (6.4) follows from the following multiset identity:

$$\begin{aligned} &\bigsqcup_{\lambda \in \mathsf{Par}(s)} \bigsqcup_{n \in \mathbb{N}_+ \setminus p^r\mathbb{Z}} \bigsqcup_{k=1}^{m_n(\lambda)} \{a_p(k) p^t \mid \nu_p(n) \leq t < r + \nu_p(k)\} \\ &= \bigsqcup_{\lambda \in \mathsf{Par}(s)} \bigsqcup_{n \in \mathbb{N}_+ \setminus p^r\mathbb{Z}} \bigsqcup_{k=1}^{m_n(\lambda)} \{a_p(k) p^t \mid \nu_p(k)\%r \leq t < r + \nu_p(k)\} \end{aligned}$$

for $s \geq 1$, which follows from Proposition 6.13. □

Proposition 6.13. *Let $p \geq 2$ be a prime and let $r \geq 1$ be an integer. We have the following multiset identity for each $d \geq 1$ and $u \in \mathbb{N}_+ \setminus p\mathbb{Z}$:*

$$\bigsqcup_{\lambda \in \mathsf{Par}(d)} \bigsqcup_{n \geq 1} \bigsqcup_{\substack{1 \leq k \leq m_n(\lambda) \\ a_p(k) = u}} \{\nu_p(n)\} = \bigsqcup_{\lambda \in \mathsf{Par}(d)} \bigsqcup_{n \geq 1} \bigsqcup_{\substack{1 \leq k \leq m_n(\lambda) \\ a_p(k) = u}} \{\nu_p(k)\%r\}.$$

Before giving a proof, we need preparatory definitions.

Definition 6.14. For $d \geq 1$, we define maps $\mathsf{cut}_d, \mathsf{Infl}_d : \mathsf{Par} \to \mathsf{Par}$ by the following:

(a) for $k \geq 1$, $m_k(\mathsf{cut}_d(\lambda)) = m_k(\lambda)$ if $k \notin d\mathbb{Z}$; otherwise $m_k(\mathsf{cut}_d(\lambda)) = 0$.
(b) $\ell(\lambda) = \ell(\mathsf{Infl}_d(\lambda))$ and $(\mathsf{Infl}_d(\lambda))_i = d\lambda_i$ for $1 \leq i \leq \ell(\lambda)$.

Definition 6.15. Let $p \geq 2$ be a prime and let $r \geq 1$ be an integer. Note that any integer $a \geq 1$ can be uniquely expressed as $a = \sum_{i=0}^{r-1} a_i p^i$ with $a_{r-1} \geq 0$ and $0 \leq a_j < p$ for $0 \leq j \leq r-2$. Define the subset $\Psi_a^{p,r}$ of $\mathsf{Par}(a)$ which consists of all the partitions ν such that

(i) for all $0 \leq k < r$, we have $\sum_{h=k}^{r-1} a_h p^{h-k} \geq \sum_{h=k}^{r-1} m_{p^h}(\nu) p^{h-k}$,
(ii) we have $\sum_{h=0}^{r-1} a_h p^h = \sum_{h=0}^{r-1} m_{p^h}(\nu) p^h$.

Remark 6.16. $\Psi_a^{p,r}$ is characterized as the smallest subset of $\mathsf{Par}(a)$ such that

(a) Define $\lambda(a) \in \mathsf{Par}(a)$ by $m_{p^i}(\lambda(a)) = a_i$ for $0 \leq i < r$. Then, we have $\lambda(a) \in \Psi_a^{p,r}$.
(b) For any $\lambda \in \Psi_a^{p,r}$ and any $0 \leq i < r$ with $m_i(\lambda) > 0$, define $\mu \in \mathsf{Par}(a)$ to be

$$m_j(\mu) = \begin{cases} m_i(\lambda) - 1 & (j = i), \\ m_{i-1}(\lambda) + p & (j = i-1), \\ m_j(\lambda) & (\text{otherwise}) \end{cases}$$

for $0 \leq j < r$. Then, we have $\mu \in \Psi_a^{p,r}$.

Thus, we have $\Psi_a^{p,r} = \{\lambda \in \mathsf{CRP}_{p^r}(a) \mid \forall f \in \mathbb{N}_+ \setminus p^{\mathbb{N}}, m_f(\lambda) = 0\}$.

Proof. Because of Lemma 6.17, it is enough to prove the following multiset identity for each $a \geq 1$ and $u \in \mathbb{N}_+ \setminus p\mathbb{Z}$:

$$\bigsqcup_{\lambda \in \Psi_a^{p,r}} \bigsqcup_{n \geq 1} \bigsqcup_{\substack{1 \leq k \leq m_n(\lambda) \\ a_p(k) = u}} \{\nu_p(n)\} = \bigsqcup_{\lambda \in \Psi_a^{p,r}} \bigsqcup_{n \geq 1} \bigsqcup_{\substack{1 \leq k \leq m_n(\lambda) \\ a_p(k) = u}} \{\nu_p(k) \% r\}. \tag{6.5}$$

For any $0 \leq s < r$, we define sets $U_1(s)$ and $U_2(s)$ as follows:

$$U_1(s) = \{(\lambda, j) \in \Psi_a^{p,r} \times \mathbb{N} \mid up^j \leq m_{p^s}(\lambda)\},$$
$$U_2(s) = \{(\lambda, j, t) \in \Psi_a^{p,r} \times \mathbb{N} \times \{0, \cdots, r-1\} \mid up^j \leq m_{p^t}(\lambda), j - s \in r\mathbb{Z}\}.$$

Note that (6.5) is the same as $\bigsqcup_{0 \leq s < r} \{s\}^{|U_1(s)|} = \bigsqcup_{0 \leq s < r} \{s\}^{|U_2(s)|}$. Thus, it is enough to construct a bijection between $U_1(s)$ and $U_2(s)$ for any $0 \leq s < r$.

We claim that the map $U_2(s) \to U_1(s), (\lambda, j, t) \mapsto (\mu, i)$ is given by

$$i = t + j - s, \quad m_{p^w}(\mu) = \begin{cases} m_{p^t}(\lambda) - up^j & (s \neq t \text{ and } w = t), \\ m_{p^s}(\lambda) + up^i & (s \neq t \text{ and } w = s), \\ m_{p^w}(\lambda) & (\text{otherwise}), \end{cases}$$

where $0 \leq w < r$ is well defined. To see this, we only need a non-trivial check to prove $\mu \in \Psi_a^{p,r}$ when $s > t$. In order to prove this, it is enough to check (i)

in Definition 6.15 for $\nu = \mu$ and $h = s$. Since $\lambda \in \Psi_a^{p,r}$, we have $\sum_{h=t}^{r-1} a_h p^{h-t} \geq \sum_{h=t}^{r-1} m_{p^h}(\lambda) p^{h-t}$. Thus,

$$\begin{aligned}\sum_{h=s}^{r-1} a_h p^{h-s} + p^{t-s} \sum_{h=t}^{s-1} a_h p^{h-t} &\geq \sum_{h=s}^{r-1} m_{p^h}(\lambda) p^{h-s} + p^{t-s} \sum_{h=t}^{s-1} m_{p^h}(\lambda) p^{h-t} \\ &\geq \sum_{h=s}^{r-1} m_{p^h}(\lambda) p^{h-s} + m_{p^t}(\lambda) p^{h-t} \geq \sum_{h=s}^{r-1} m_{p^h}(\mu) p^{h-s}.\end{aligned}$$

Since $0 \leq p^{t-s} \sum_{h=t}^{s-1} a_h p^{h-t} \leq p^{t-s} \sum_{h=t}^{s-1} (p-1) \cdot p^{h-t} = 1 - p^{t-s} < 1$, we are done.

Similarly, we see that the map $U_1(s) \to U_2(s), (\mu, i) \mapsto (\lambda, j, t)$ is given by

$$t = i\%r, \quad j = i + s - t, \quad m_{p^w}(\lambda) = \begin{cases} m_{p^t}(\mu) + up^j & (s \neq t \text{ and } w = t), \\ m_{p^s}(\mu) - up^i & (s \neq t \text{ and } w = s), \\ m_{p^w}(\mu) & (\text{otherwise}) \end{cases}$$

is well defined. Clearly, it is a converse of the map $U_2(s) \to U_1(s)$ above. □

Lemma 6.17. *Let $p \geq 2$ be a prime and let $r \geq 1$ be an integer. For each $d \geq 1$, the multiset $S_d^{p,r} := \{\mathsf{cut}_{p^r}(\lambda) \mid \lambda \in \mathsf{Par}(d) \text{ such that } \mathsf{cut}_{p^r}(\lambda) \neq \phi\}$ has a decomposition of the form $S_d^{p,r} = \bigsqcup_{i \in I} \mathsf{Sort}(\prod_{j \in J_i} \mathsf{Infl}_{d_{i,j}}(\Psi_{a_{i,j}}^{p,r}))$ such that*

(a) *I is a finite non-empty set and $\{J_i \mid i \in I\}$ is a family of finite non-empty sets,*
(b) *$a_{i,j} \geq 1$ and $d_{i,j} \in \mathbb{N}_+ \setminus p\mathbb{Z}$ for $i \in I$ and $j \in J_i$,*
(c) *for any $i \in I$ and $j \neq j' \in J_i$, take $\lambda \in \mathsf{Infl}_{d_{i,j}}(\Psi_{a_{i,j}}^{p,r})$ and $\mu \in \mathsf{Infl}_{d_{i,j'}}(\Psi_{a_{i,j'}}^{p,r})$. Then, we have either $m_f(\lambda) = 0$ or $m_f(\mu) = 0$ for all $f \geq 1$,*
(d) *$\mathsf{Sort} : \mathsf{Par}^* \to \mathsf{Par}$ is a map that maps a finite sequence of partitions $(\lambda^{(g)})_{g \in G}$ to a partition μ such that $\{\mu_f \mid 1 \leq f \leq \ell(\mu)\} = \bigsqcup_{g \in G} \{\lambda_f^{(g)} \mid 1 \leq f \leq \ell(\lambda^{(g)})\}$ as multisets.*

Proof. Let λ be a non-empty partition. Clearly, $L = \{1, \cdots, \ell(\lambda)\}$ has a decomposition of the form $L = \bigsqcup_{k \in K} L_k$ such that

(i) for any $k \in K$, we have $L_k \neq \emptyset$ and there exists a unique $d_k \in \mathbb{N}_+ \setminus p\mathbb{Z}$ such that $\lambda_j \in d_k\mathbb{Z}$ and $\lambda_j/d_k \in p^{\mathbb{N}}$ for all $j \in L_k$,
(ii) for any $k \neq k' \in K$, we have $d_k \neq d_{k'}$.

Put $\lambda^{(k)} = \mathsf{Sort}((\lambda_j/d_k))_{j \in L_k})$ for $k \in K$. As in Remark 6.16, there exists a unique $a_k \geq 1$ such that $\lambda^{(k)} \in \Psi_{a_k}^{p,r}$. Thus, we have $\lambda \in \mathsf{Sort}(\prod_{k \in K} \mathsf{Infl}_{d_k}(\Psi_{a_k}^{p,r}))$.

From this construction, we see that $\mathsf{CRP}_{p^r}(d)$ has a decomposition of the form $\mathsf{CRP}_{p^r}(d) = \bigsqcup_{i \in I} \mathsf{Sort}(\prod_{j \in J_i} \mathsf{Infl}_{d_{i,j}}(\Psi_{a_{i,j}}^{p,r}))$ that satisfies (a), (b) and (c) for each $d \geq 1$.

Because we have $S_d^{p,r} = \bigsqcup_{0 \leq e < d/p^r} \mathsf{CRP}_{p^r}(d - p^r e)^{|\mathsf{Par}(e)|}$, we are done. □

6.3. **Gradation of Külshammer-Olsson-Robinson's conjecture when $\ell = p^r$.** By using the same reasoning that we used to consider a gradation of Hill's conjecture (see §1.3), we want to consider a correct gradation of the KOR conjecture. Though we could not find it in general, we propose a gradation of the KOR conjecture only when $|\,\mathsf{pdiv}(\ell)| = 1$.

Conjecture 6.18. *Let $p \geq 2$ be a prime and let $r \geq 1$ be an integer. Put $\ell = p^r$. Then, for each $n \geq 0$, we have $\bigoplus_{(\rho,d)\in \mathsf{Bl}_\ell(n)} C^v_{\ell,d} \equiv_{\mathscr{A}} \{r^v_{p,r}(\lambda) \mid \lambda \in \mathsf{CRP}_\ell(n)\}$ where*

$$r^v_{p,r}(\lambda) = \prod_{k\geq 1} \prod_{t=1}^{\lfloor \frac{m_k(\lambda)}{\ell} \rfloor} [p^{r-\nu_p(k)}]_{tp^{\nu_p(k)}} \cdot [p^{\nu_p(t)}]_{a_p(t)p^{\nu_p(k)}}$$
$$= \prod_{k\geq 1} \prod_{t=1}^{\lfloor \frac{m_k(\lambda)}{\ell} \rfloor} [p^{r-\nu_p(k)+\nu_p(t)}]_{a_p(t)p^{\nu_p(k)}}.$$

Remark 6.19. When $r = 1$, Conjecture 6.18 implies [ASY, Conjecture 8.2 (ii)], which predicts $\bigoplus_{(\rho,d)\in \mathsf{Bl}_p(n)} C^v_{p,d} \equiv_{\mathbb{Q}[v,v^{-1}]} \{r^v_{p,1}(\lambda) \mid \lambda \in \mathsf{CRP}_p(n)\}$ for $n \geq 0$ and a prime $p \geq 2$.

Proposition 6.20. *Conjecture* 6.8 *implies Conjecture* 6.18. *More precisely, we have the following multiset identity for each $r, n \geq 1$ and a prime $p \geq 2$ where $\ell = p^r$:*

$$\bigsqcup_{(\rho,d)\in \mathsf{Bl}_\ell(n)} \bigsqcup_{s=0}^{d} \bigsqcup_{\lambda\in \mathsf{Par}(s)} \{I^v_{p,r}(\lambda)\}^{u(\ell-2,d-s)} = \{r^v_{p,r}(\lambda) \mid \lambda \in \mathsf{CRP}_\ell(n)\}.$$

Proof. Since $r^v_{p,r}(\lambda) = I^v_{p,r}(\mathsf{red}_\ell(\lambda))$ and $I^v_{p,r}(\lambda) = I^v_{p,r}(\mathsf{cut}_\ell(\lambda))$ for any partition λ where $m_k(\mathsf{red}_\ell(\lambda)) = \lfloor m_k(\lambda)/\ell \rfloor$ for all $k \geq 1$, Proposition 6.20 follows from Lemma 6.21. □

Lemma 6.21 ([BH, Lemma 5.5])**.** *For any $\ell \geq 2$ and $n \geq 0$, we have the multiset identity*

$$\bigsqcup_{(\rho,d)\in \mathsf{Bl}_\ell(n)} \bigsqcup_{s=0}^{d} \bigsqcup_{\lambda\in \mathsf{Par}(s)} \{\mathsf{cut}_\ell(\lambda)\}^{u(\ell-2,d-s)} = \{\mathsf{red}_\ell(\lambda) \mid \lambda \in \mathsf{CRP}_\ell(n)\}.$$

Acknowledgements

The content of §7 was benefited from a discussion with Yasuhide Numata. The author would like to thank him and also would like to thank Hiraku Nakajima and Yoshiyuki Kimura for discussions on quantum groups. The author is grateful to Hiro-Fumi Yamada. Without his encouragement and their paper [ASY], this paper would not have been written. Finally, the author wishes to thank the anonymous referee for valuable comments and suggestions.

References

[Aka] Tatsuya Akasaka, *An integral PBW basis of the quantum affine algebra of type $A_2^{(2)}$*, Publ. Res. Inst. Math. Sci. **38** (2002), no. 4, 803–894. MR1917164 (2003h:17017)

[ASY] M. Ando, T. Suzuki and H-F. Yamada, *Combinatorics for graded Cartan matrices of the Iwahori-Hecke algebra of type A*, Ann. Comb. **17** (2013), no. 3, 427–442. MR3090170

[APW] Henning Haahr Andersen, Patrick Polo, and Ke Xin Wen, *Representations of quantum algebras*, Invent. Math. **104** (1991), no. 1, 1–59, DOI 10.1007/BF01245066. MR1094046 (92e:17011)

[Bec] Jonathan Beck, *Braid group action and quantum affine algebras*, Comm. Math. Phys. **165** (1994), no. 3, 555–568. MR1301623 (95i:17011)

[BCP] Jonathan Beck, Vyjayanthi Chari, and Andrew Pressley, *An algebraic characterization of the affine canonical basis*, Duke Math. J. **99** (1999), no. 3, 455–487, DOI 10.1215/S0012-7094-99-09915-5. MR1712630 (2000g:17013)

[BH] Christine Bessenrodt and David Hill, *Cartan invariants of symmetric groups and Iwahori-Hecke algebras*, J. Lond. Math. Soc. (2) **81** (2010), no. 1, 113–128, DOI 10.1112/jlms/jdp060. MR2580456 (2011b:20013)

[BK1] Jonathan Brundan and Alexander Kleshchev, *The degenerate analogue of Ariki's categorification theorem*, Math. Z. **266** (2010), no. 4, 877–919, DOI 10.1007/s00209-009-0603-y. MR2729296 (2011m:17024)

[BK2] Jonathan Brundan and Alexander Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math. **178** (2009), no. 3, 451–484, DOI 10.1007/s00222-009-0204-8. MR2551762 (2010k:20010)

[BK3] Jonathan Brundan and Alexander Kleshchev, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math. **222** (2009), no. 6, 1883–1942, DOI 10.1016/j.aim.2009.06.018. MR2562768 (2011i:20002)

[BK4] Jonathan Brundan and Alexander Kleshchev, *Cartan determinants and Shapovalov forms*, Math. Ann. **324** (2002), no. 3, 431–449, DOI 10.1007/s00208-002-0346-0. MR1938453 (2003j:17033)

[BK5] Jonathan Brundan and Alexander Kleshchev, *Hecke-Clifford superalgebras, crystals of type* $A^{(2)}_{2l}$ *and modular branching rules for* $\hat{S}_n$, Represent. Theory **5** (2001), 317–403, DOI 10.1090/S1088-4165-01-00123-6. MR1870595 (2002j:17024)

[BKM] Georgia Benkart, Seok-Jin Kang, and Duncan Melville, *Quantized enveloping algebras for Borcherds superalgebras*, Trans. Amer. Math. Soc. **350** (1998), no. 8, 3297–3319, DOI 10.1090/S0002-9947-98-02058-3. MR1451594 (99f:17014)

[BeNa] Jonathan Beck and Hiraku Nakajima, *Crystal bases and two-sided cells of quantum affine algebras*, Duke Math. J. **123** (2004), no. 2, 335–402. MR2066942 (2005e:17020)

[BO1] Christine Bessenrodt and Jørn B. Olsson, *A note on Cartan matrices for symmetric groups*, Arch. Math. (Basel) **81** (2003), no. 5, 497–504, DOI 10.1007/s00013-003-0799-8. MR2029709 (2004h:20014)

[BO2] Christine Bessenrodt and Jørn B. Olsson, *On Cartan matrices and lower defect groups for covering groups of symmetric groups*, Trans. Amer. Math. Soc. **358** (2006), no. 8, 3623–3635 (electronic), DOI 10.1090/S0002-9947-05-03809-2. MR2218991 (2007g:20013)

[Bra] Richard. Brauer, *On the Cartan invariants of groups of finite order*, Ann. of Math. (2) **42** (1941), 53–61. MR0003416 (2,215e)

[BrNe] R. Brauer and C. Nesbitt, *On the modular characters of groups*, Ann. of Math. (2) **42** (1941), 556–590. MR0004042 (2,309c)

[CJ] Vyjayanthi Chari and Naihuan Jing, *Realization of level one representations of* $U_q(\hat{\mathfrak{g}})$ *at a root of unity*, Duke Math. J. **108** (2001), no. 1, 183–197, DOI 10.1215/S0012-7094-01-10816-8. MR1831824 (2002m:17009)

[DcKK] C. De Concini, V. G. Kac, and D. A. Kazhdan, *Boson-fermion correspondence over* $\mathbf{Z}$, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), Adv. Ser. Math. Phys., vol. 7, World Sci. Publ., Teaneck, NJ, 1989, pp. 124–137. MR1026951 (91a:17031)

[DJ] Richard Dipper and Gordon James, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. (3) **54** (1987), no. 1, 57–82, DOI 10.1112/plms/s3-54.1.57. MR872250 (88m:20084)

[Don] Stephen Donkin, *Representations of Hecke algebras and characters of symmetric groups*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, pp. 49–67. MR1985722 (2004m:20013)

[Dri] V. G. Drinfel′d, *A new realization of Yangians and of quantum affine algebras*, Dokl. Akad. Nauk SSSR **296** (1987), no. 1, 13–17 (Russian); English transl., Soviet Math. Dokl. **36** (1988), no. 2, 212–216. MR914215 (88j:17020)

[FJ] Igor B. Frenkel and Nai Huan Jing, *Vertex representations of quantum affine algebras*, Proc. Nat. Acad. Sci. U.S.A. **85** (1988), no. 24, 9373–9377, DOI 10.1073/pnas.85.24.9373. MR973376 (90e:17028)

[Her] David Hernandez, *Kirillov-Reshetikhin conjecture: the general case*, Int. Math. Res. Not. IMRN **1** (2010), 149–193, DOI 10.1093/imrn/rnp121. MR2576287 (2011c:17029)

[Hil] David Hill, *Elementary divisors of the Shapovalov form on the basic representation of Kac-Moody algebras*, J. Algebra **319** (2008), no. 12, 5208–5246, DOI 10.1016/j.jalgebra.2007.11.009. MR2423824 (2009f:17044)

[HM] Jun Hu and Andrew Mathas, *Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A*, Adv. Math. **225** (2010), no. 2, 598–642, DOI 10.1016/j.aim.2010.03.002. MR2671176 (2011g:20006)

[HW] D. Hill and W. Wang, *Categorification of quantum Kac-Moody superalgebras*, to appear in Trans. Amer. Math. Soc.

[Jin] Nai Huan Jing, *Twisted vertex representations of quantum affine algebras*, Invent. Math. **102** (1990), no. 3, 663–690, DOI 10.1007/BF01233443. MR1074490 (92a:17019)

[JM] Naihuan Jing and Kailash C. Misra, *Vertex operators for twisted quantum affine algebras*, Trans. Amer. Math. Soc. **351** (1999), no. 4, 1663–1690, DOI 10.1090/S0002-9947-99-02098-X. MR1458306 (99i:17027)

[Kac] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219 (92k:17038)

[Kas] Masaki Kashiwara, *On crystal bases*, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 155–197. MR1357199 (97a:17016)

[KKT] S-J. Kang, M. Kashiwara and S. Tsuchioka, *Quiver Hecke superalgebras*, available at http://www.degruyter.com/view/j/crelle.ahead-of-print/crelle-2013-0089/crelle-2013-0089.xml.

[KL1] Mikhail Khovanov and Aaron D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory **13** (2009), 309–347, DOI 10.1090/S1088-4165-09-00346-X. MR2525917 (2010i:17023)

[KL2] Mikhail Khovanov and Aaron D. Lauda, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2685–2700, DOI 10.1090/S0002-9947-2010-05210-9. MR2763732 (2012a:17021)

[KOR] Burkhard Külshammer, Jørn B. Olsson, and Geoffrey R. Robinson, *Generalized blocks for symmetric groups*, Invent. Math. **151** (2003), no. 3, 513–552, DOI 10.1007/s00222-002-0258-3. MR1961337 (2004c:20019)

[Lak] V. Lakshmibai, *Bases for quantum Demazure modules*, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 199–216. MR1357200 (96i:17014)

[LLT] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm. Math. Phys. **181** (1996), no. 1, 205–263. MR1410572 (97k:17019)

[Lus] George Lusztig, *Introduction to quantum groups*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition. MR2759715 (2011j:17028)

[Lus2] George Lusztig, *Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra*, J. Amer. Math. Soc. **3** (1990), no. 1, 257–296, DOI 10.2307/1990988. MR1013053 (91e:17009)

[Ols] Jørn B. Olsson, *Lower defect groups in symmetric groups*, J. Algebra **104** (1986), no. 1, 37–56, DOI 10.1016/0021-8693(86)90232-2. MR865885 (87m:20039)

[Rou] R. Rouquier, *2-Kac-Moody algebras*, arXiv:0812.5023

[Tsu] Shunsuke Tsuchioka, *Hecke-Clifford superalgebras and crystals of type* $D_l^{(2)}$, Publ. Res. Inst. Math. Sci. **46** (2010), no. 2, 423–471, DOI 10.2977/PRIMS/13. MR2722783 (2011g:20008)

Kavli Institute for the Physics and Mathematics of the Universe (WPI), Todai Institutes for Advanced Study, the University of Tokyo, Kashiwa, Japan 277-8583

E-mail address: tshun@kurims.kyoto-u.ac.jp