

ON SIMPLICIAL RESOLUTIONS OF FRAMED LINKS

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ABSTRACT. In this paper, we investigate the simplicial groups obtained from the link groups of naive cablings on any given framed link. Our main result states that the resulting simplicial groups have the homotopy type of the loop space of a wedge of 3-spheres. This gives simplicial group models for some loop spaces using link groups.

1. INTRODUCTION

The purpose of this article is to give canonical simplicial resolutions of framed links. Framed links have been studied by various people with fruitful results such as the classical Kirby calculus on framed links [17] and quantum groups constructed from framed links [23]. In this article, we consider naive cablings on framed links, which will be described in detail below. By taking link groups, the naive cabling process on framed links induces a canonical simplicial group associated to each framed link. This gives a simplicial resolution for the link group of any link with a frame. The cabling construction in this article is the link analogue of the naive cabling on braids considered in [7, 8], where some interesting connections between Vassiliev invariants and the Adams spectral sequence were discovered. In comparison with braids, the naive cablings on framed links seem to have richer structures, as we will see from our main result in this article. The resulting simplicial group has the homotopy type of the loop space of a wedge of 3-spheres.

A *framed link*, denoted by L^f , means a smooth link L in S^3 equipped with a smooth nonzero normal vector field f defined on (a small neighborhood of) L . An alternative way to consider framed links is as twisted-annulus links. A twisted-annulus link in S^3 is a finite collection of disjoint images of smooth embeddings of line bundles over the circle in S^3 , equipped with an orientation. The correspondence between these notions is given in a natural way. Namely, for each twisted-annulus link, there is an associated framed link given by the collection of core curves framed by the positive unit normal vector field of the twisted-annulus. Conversely, for any framed link L^f , let t_L be the tangent vector field on L . There is an associated

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twisted-annulus link consisting of a collection of twisted-annuli A obtained by pushing the link L along the direction given by the cross product $X = f \times t_L$ so that $f/|f|$ becomes an orientation of A . The resulting twisted-annulus link is denoted by $(L; X, A)$ with the exponential map

$$\exp: \{(P, tX(P)) \mid -1 < t < 1, P \in L\} \longrightarrow A$$

by projecting the normal vector fields in the direction of X to the twisted-annuli A with $\exp(P, 0) = P$ for $P \in L$. In order to avoid ambiguity, we restrict ourselves to framed links whose exponential map is injective for $t < 1$ in this paper. For each $k \geq 0$, let

$$L_k = \exp \left\{ \left(P, \left(1 - \frac{1}{k+1} \right) X(P) \right) \mid P \in L \right\}.$$

Note that $L_0 = L$ and L_k is a copy of L being pushed away from L by the vector field X . The *naive cabling* on the twisted-annulus link $(L; X, A)$ is a sequence of links $\mathbb{L}_n, 0 \leq n < \infty$, given by

$$\mathbb{L}_n = \{L_0, L_1, \dots, L_n\}.$$

(**Note.** If L is a framed q -link, then \mathbb{L}_n is a framed $(n+1)q$ -link.) By taking link group $G(\mathbb{L}_n) = \pi_1(S^3 \setminus \mathbb{L}_n)$, there is a canonical simplicial group structure on the sequence of link groups

$$\mathbb{G}(L; X) = \{G(\mathbb{L}_n)\}_{n \geq 0}$$

described as follows.

Recall that a *simplicial group* [11, 15] means a sequence of groups $G_* = \{G_n\}_{n \geq 0}$ together with face homomorphisms $d_i: G_n \rightarrow G_{n-1}$ and degeneracy homomorphisms $s_i: G_n \rightarrow G_{n+1}$ for $0 \leq i \leq n$ such that the following *simplicial identities* hold:

- (1) $d_i d_j = d_j d_{i+1}$ for $i \geq j$.
- (2) $s_j s_i = s_{i+1} s_j$ for $j \leq i$.
- (3) $d_j s_i = \begin{cases} s_{i-1} d_j, & j < i, \\ \text{id}, & j = i, i + 1, \\ s_i d_{j-1}, & j > i + 1. \end{cases}$

The i -th face homomorphism and the i -th degeneracy homomorphism on $\mathbb{G}(L; X) = \{G(\mathbb{L}_n)\}_{n \geq 0}$ are given by removing the i -th copy of L and doubling the i -th copy of L , respectively. Note that the framing is defined on a small neighborhood of L and the removing and doubling operations can be applied inside the neighborhood. Thus the basepoint of the link group can be chosen freely outside the neighborhood of L , and we just abbreviate the link group as $G(\mathbb{L}_n)$. More precisely, the group homomorphism $d_i: G(\mathbb{L}_n) \rightarrow G(\mathbb{L}_{n-1})$ is induced by the continuous map

$$S^3 \setminus \mathbb{L}_n \hookrightarrow S^3 \setminus \tilde{d}_i \mathbb{L}_n \xrightarrow{\phi_i} S^3 \setminus \mathbb{L}_{n-1},$$

where $\tilde{d}_i \mathbb{L}_n = \{L_0, \dots, L_{i-1}, L_{i+1}, \dots, L_n\}$ and ϕ_i is the homeomorphism induced by the flow associated with the vector field X that pushes the links $L_{i+1}, L_{i+2}, \dots, L_n$ to $L_i, L_{i+1}, \dots, L_{n-1}$ in the twisted-annuli A , respectively, while fixing

L_0, \dots, L_{i-1} . Let A_i be the twisted-annuli given by

$$A_i = \exp \left\{ \left(P, \left((1-t) \left(1 - \frac{1}{i+1} \right) + t \left(1 - \frac{1}{i+2} \right) \right) X(P) \right) \mid P \in L, 0 \leq t \leq 1/2 \right\}.$$

Namely, A_i is the twisted-annuli bounded by L_i and the middle link

$$L_{i+1/2} = \exp \left\{ \left(P, \left(\frac{1}{2} \left(1 - \frac{1}{i+1} \right) + \frac{1}{2} \left(1 - \frac{1}{i+2} \right) \right) X(P) \right) \mid P \in L \right\}$$

between L_i and L_{i+1} in A . Then the group homomorphism $s_i: G(\mathbb{L}_n) \rightarrow G(\mathbb{L}_{n+1})$ is induced by the composite of continuous maps

$$S^3 \setminus \mathbb{L}_n \xleftarrow{\iota} S^3 \setminus (\mathbb{L}_n \cup A_i) \hookrightarrow S^3 \setminus (\mathbb{L}_n \cup L_{i+1/2}) \xrightarrow{\psi_i} S^3 \setminus \mathbb{L}_{n+1}.$$

Here the first map in the above composition is given by a homotopy inverse r of ι . Since the subspace $S^3 \setminus (\mathbb{L}_n \cup A_i)$ is a strong deformation retract of $S^3 \setminus \mathbb{L}_n$, $\iota_*: \pi_1(S^3 \setminus (\mathbb{L}_n \cup A_i)) \rightarrow \pi_1(S^3 \setminus \mathbb{L}_n)$ is an isomorphism, and so $r_* = \iota_*^{-1}$. The last map ψ_i is the homeomorphism similar to ϕ_i , where the flow pushes $L_{i+1/2}, L_{i+1}, \dots, L_n$ to $L_{i+1}, L_{i+2}, \dots, L_{n+1}$, respectively. From the construction of d_i and s_i , it is routine to show that the simplicial identities hold, and so $\mathbb{G}(L; X)$ is a simplicial group.

A natural question is to determine the homotopy type of the geometric realization of the simplicial group $\mathbb{G}(L; X)$. To answer this question, observe that any link L has a unique (up to ambient isotopy) splitting decomposition

$$L \cong L^{[1]} \sqcup L^{[2]} \sqcup \dots \sqcup L^{[p]},$$

where each $L^{[j]}$ is a nonsplittable sublink of L and $L^{[1]}, \dots, L^{[p]}$ are mutually separated by embedded 2-spheres. For a framed link L^f , each nonsplittable factor $L^{[j]}$ of L has the canonical frame $f^{[j]}$ given by the restriction $f|_{L^{[j]}}$. Thus the associated twisted-annulus link $(L; X, A)$ admits the canonical splitting decomposition

$$(L; X, A) \cong (L^{[1]}; X^{[1]}, A^{[1]}) \sqcup (L^{[2]}; X^{[2]}, A^{[2]}) \sqcup \dots \sqcup (L^{[p]}; X^{[p]}, A^{[p]}).$$

Our main theorem is as follows.

Theorem 1.1. *Let L^f be a framed link in S^3 with the associated twisted-annulus link $(L; X, A)$ and the canonical splitting decomposition*

$$(L; X, A) \cong (L^{[1]}; X^{[1]}, A^{[1]}) \sqcup (L^{[2]}; X^{[2]}, A^{[2]}) \sqcup \dots \sqcup (L^{[p]}; X^{[p]}, A^{[p]})$$

such that $(L^{[1]}; X^{[1]}, A^{[1]}), \dots, (L^{[k]}; X^{[k]}, A^{[k]})$ are nontrivial nonsplittable twisted-annulus links and $(L^{[k+1]}; X^{[k+1]}, A^{[k+1]}), \dots, (L^{[p]}; X^{[p]}, A^{[p]})$ are trivial twisted-annulus knots. Let $\mathbb{G}(L; X)$ be the simplicial group obtained from the naive cabling on L^f . Then the geometric realization $|\mathbb{G}(L; X)|$ is homotopy equivalent to $\Omega(\bigvee^k S^3)$.

By the above result, the homotopy type of $\mathbb{G}(L; X)$ only depends on the number of the nontrivial nonsplittable factors of the framed link L^f . However the group structures on $\mathbb{G}(L; X)$ are of course sharply distinct for different framed links. Observe that, for any map $f: S^3 \rightarrow S^3$ which sends (a small neighborhood of) a twisted-annulus link A homeomorphically onto (a small neighborhood of) another twisted-annulus link \tilde{A} as twisted-annuli such that $f(S^3 \setminus A) \subseteq S^3 \setminus \tilde{A}$, the restriction of f to link complements induces a simplicial group homomorphism from

$\mathbb{G}(L; X, A)$ to $\mathbb{G}(\tilde{L}; \tilde{X}, \tilde{A})$. Under such a setting for the morphisms on framed links, $\mathbb{G}(-; -)$ gives a co-functor from the category of framed links to the category of simplicial groups.

The simplicial group models for $\Omega(\bigvee^k S^3)$ can be given by Kan’s free group construction on reduced simplicial sets [16] with its geometric realization having the homotopy type of $\bigvee^k S^3$. The simplicial groups given by free products have been studied by Quillen [22] and Carlsson [6]. A braid group construction of simplicial groups has been given in [8]. The objects $\mathbb{G}(L; X)$ provide simplicial group models using link groups. In particular, for any nontrivial nonsplittable framed link L^f , the associated simplicial group $\mathbb{G}(L; X)$ always has the homotopy type of ΩS^3 .

A relation between Kan’s construction and $\mathbb{G}(L; X)$ can be described as follows. Choose Moore cycles $\alpha_1, \dots, \alpha_k$ in $\mathbb{G}(L; X)_2 = G(L_2)$ that represents a basis for $\pi_2(\mathbb{G}(L; X)) \cong \pi_2\left(\Omega\left(\bigvee^k S^3\right)\right) \cong \mathbb{Z}^{\oplus k}$. Let $f_{\alpha_i} : S^2 \rightarrow \mathbb{G}(L; X)$ be the representing simplicial map of α_i , where S^2 is the simplicial 2-sphere. This gives a simplicial map

$$f = \bigvee_{i=1}^k f_{\alpha_i} : \bigvee_{i=1}^k S^2 \longrightarrow \mathbb{G}(L; X).$$

The mapping f extends uniquely to a simplicial homomorphism

$$\tilde{f} : F\left[\bigvee_{i=1}^k S^2\right] \longrightarrow \mathbb{G}(L; X),$$

where $F[K] \cong G\Sigma K$ is Milnor’s construction [18]. The mapping \tilde{f} is a homotopy equivalence because it is a loop map between the loop space of a wedge of 3-spheres inducing isomorphisms on the second homotopy groups, where one can apply the Hurewicz theorem and the homology version of the Whitehead Theorem for simply connected spaces [14, Corollary 4.33] to the delooping of \tilde{f} . In other words, there exists a simplicial map $g : \mathbb{G}(L; X) \rightarrow F\left[\bigvee_{i=1}^k S^2\right]$ as the homotopy inverse of \tilde{f} . We should point out that there may not exist a simplicial homomorphism g as a homotopy inverse of \tilde{f} . (**Note.** If $k = 1$ and L^f is a nontrivial framed knot, it is a routine exercise to check that there does not exist a simplicial homomorphism as a homotopy inverse of \tilde{f} .)

In geometry, there is a canonical co-multiplication $\mu' : S^3 \rightarrow S^3 \vee S^3$ obtained by pinching the equator to a point. Recall that the Hopf invariant $\Omega S^3 \rightarrow \Omega S^{2n+1}$ can be obtained from the mapping $\Omega\mu' : \Omega S^3 \rightarrow \Omega(S^3 \vee S^3)$ by projecting to a factor ΩS^{2n+1} in $\Omega(S^3 \vee S^3)$ using the Hilton-Milnor Theorem. A construction of the mapping $\Omega\mu'$ using the $\mathbb{G}(L; X)$ model can be given as follows: Choose a nonsplittable framed link L^f . Make a copy \tilde{L}^f of L^f located separately from L^f . Then choose a framed knot K in the link complement $S^3 \setminus (L^f \cup \tilde{L}^f)$ such that K links both L^f and \tilde{L}^f . The resulting link $\bar{L}^f = L^f \cup K \cup \tilde{L}^f$ is a nonsplittable framed link. The inclusion

$$S^3 \setminus \bar{L} \subseteq S^3 \setminus (L \cup \tilde{L})$$

induces a simplicial homomorphism

$$\phi : \mathbb{G}(\bar{L}; \bar{X}) \simeq \Omega S^3 \longrightarrow \mathbb{G}(L \cup \tilde{L}; X \cup \tilde{X}) \simeq \Omega(S^3 \vee S^3),$$

which is homotopic to $\Omega\mu'$ because ϕ is a loop map and the composites

$$\begin{aligned} \mathbb{G}(\tilde{L}; \tilde{X}) &\xrightarrow{\phi} \mathbb{G}(L \cup \tilde{L}; X \cup \tilde{X}) \xrightarrow{p_L} \mathbb{G}(L; X), \\ \mathbb{G}(\tilde{L}; \tilde{X}) &\xrightarrow{\phi} \mathbb{G}(L \cup \tilde{L}; X \cup \tilde{X}) \xrightarrow{p_{\tilde{L}}} \mathbb{G}(\tilde{L}; \tilde{X}) \end{aligned}$$

are homotopy equivalences, where p_L and $p_{\tilde{L}}$ are induced by canonical inclusions of link complements. On the other hand, the loop of the folding map $\nabla: S^3 \vee S^3 \rightarrow S^3$ cannot be constructed directly as above because a splittable link $L \cup \tilde{L}$ does not contain a nonsplittable sublink that links with both a nontrivial component of L and a nontrivial component of \tilde{L} . However algebraically $\Omega\nabla$ can be obtained canonically from the free product $G(\mathbb{L}_n \cup \tilde{\mathbb{L}}_n) = G(\mathbb{L}_n) * G(\tilde{\mathbb{L}}_n)$ to $G(\mathbb{L}_n)$. The composite

$$\psi: \mathbb{G}(\tilde{L}; \tilde{X}) \xrightarrow{\phi} \mathbb{G}(L \cup \tilde{L}; X \cup \tilde{X}) \cong \mathbb{G}(L; X) * \mathbb{G}(\tilde{L}; \tilde{X}) \longrightarrow \mathbb{G}(L; X)$$

is a simplicial epimorphism whose geometric realization is homotopic to $\Omega[2]: \Omega S^3 \rightarrow \Omega S^3$, the looping of a degree 2 map. Thus the kernel of ψ gives a simplicial group model for $\Omega(S^3\{2\})$, where $S^3\{2\}$ is the homotopy fibre of the degree 2 map $[2]: S^3 \rightarrow S^3$. One can apply Theorem 1.1 to construct simplicial group models for other loop spaces using link groups.

Theorem 1.1 provides a possibility to study homotopy theory and homotopy groups using low dimensional techniques. Observe that the elements in link groups can be represented by knots in link complements. The study on the intersections of these represented knots and incompressible surfaces in link complements might be a geometric way to explore the elements in homotopy groups from their represented knots in link complements.

Recall that the current major computational tool for homotopy groups is the Adams spectral sequence, which can be obtained from the mod p lower central series of Kan’s free group construction on reduced simplicial sets established in various foundational works on the machinery concerning convergence, lower terms and differentials [1–3, 10]. For a group G , let $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$ be the lower central series of G . Let $\gamma_\infty(G) = \bigcap_n \gamma_n(G)$. From our unpublished preliminary explorations on the lower central series on the simplicial groups $\mathbb{G}(L; X)$, the simplicial group $\mathbb{G}(L; X)/\gamma_\infty\mathbb{G}(L; X)$ seems homotopy equivalent to the loop space of a wedge of 2-spheres and $K(\pi, 1)$ -spaces for some cyclic groups π depending on the behaviour of the longitudes. This indicates that the spectral sequence derived from the lower central series might converge to the homotopy groups of a wedge of 2-spheres and some $K(\pi, 1)$ spaces rather than $\pi_*(\mathbb{G}(L; X))$. On the other hand, the incompressible surfaces in link complements and representations on link groups may produce different filtrations on the simplicial group $\mathbb{G}(L; X)$ with the potential to give new spectral sequences converging to homotopy groups. The exploration of the filtrations on the simplicial groups $\mathbb{G}(L; X)$ will be given in our subsequent work on this topic.

It seems that the method of naive cabling can be applied to surgeries on framed links and algebraic invariants (such as quantum groups, chain complexes and homology) canonically constructed from link diagrams. Theorem 1.1 indicates that the derived groups of resulting simplicial resolutions may be nontrivial. (In our case, we apply link groups to the simplicial resolution and obtain the associated derived groups given by homotopy groups of a wedge of 3-spheres.)

The proof of Theorem 1.1 will be given by showing that the classifying space $B\mathbb{G}(L; X)$ of the simplicial group $\mathbb{G}(L; X)$ is homotopy equivalent to a wedge of 3-spheres. A simple combination of methods on simplicial groups and knot theory provides a proof that the space $B\mathbb{G}(L; X)$ is simply connected. Then the Quillen spectral sequence [21] is used to determine the homology of the space $B\mathbb{G}(L; X)$, which seems to be an elegant application of the Quillen spectral sequence on bi-simplicial groups.

The article is organized as follows. In section 2, we investigate some properties of the framed link. In section 3, we give a brief review on the Moore chain complex and the classifying spaces of simplicial groups. The proof of Theorem 1.1 is given in section 4.

2. THE FRAMED LINKS

Recall that for a link L in S^3 , if there is an embedding of S^2 into S^3 such that

$$L \cong L' \sqcup L'',$$

where L' and L'' are nonempty and located in different sides of $S^3 \setminus S^2$, then we say that L is *splittable*. Otherwise, L is called *nonsplittable*. From Papakyriakopoulos' Theorem [20], the complement of a nonsplittable link in S^3 is a $K(\pi, 1)$ space.

For a framed link L^f , we say that L^f is *nonsplittable* if L is nonsplittable. For a nonsplittable framed link L^f , the associated twisted-annulus link $(L; X, A)$ is called *trivial* if L is a trivial knot and the naive cabling \mathbb{L}_n is a trivial $(n + 1)$ -link. Otherwise, $(L; X, A)$ is called *nontrivial*. The following property will be used later.

Proposition 2.1. *Let L^f be a nonsplittable framed link in S^3 with the associated twisted-annulus link $(L; X, A)$ nontrivial. Then for any $n \geq 0$, the naive cabling \mathbb{L}_n is nonsplittable.*

Proof. Note that $\mathbb{L}_0 = L$ is nonsplittable; first we prove that \mathbb{L}_1 is nonsplittable.

Without loss of generality, we assume that \mathbb{L}_1 is splittable. Since L is nonsplittable, from [12, Proposition 2.2],

$$S^3 \setminus \mathbb{L}_1 \cong (S^3 \setminus \mathbb{L}_0) \vee (S^3 \setminus \mathbb{L}_0) \vee S^2.$$

After taking the fundamental group and applying the face and degeneracy maps, we have

$$(2.1) \quad \begin{array}{ccc} & G(\mathbb{L}_1) \cong G(\mathbb{L}_0) * G(\mathbb{L}_0) & \\ s_0 \nearrow & & \searrow (d_0, d_1) \\ G(\mathbb{L}_0) & \xrightarrow{\Delta} & G(\mathbb{L}_0) \times G(\mathbb{L}_0) \end{array}$$

where Δ is the canonical diagonal map. Diagram (2.1) commutes, since from the simplicial structure on $\mathbb{G}(L; X)$, $d_0 s_0 = \text{id}$, $d_1 s_0 = \text{id}$.

Let Z be the classifying space of $G(\mathbb{L}_0)$, i.e., $Z = B\mathbb{G}(\mathbb{L}_0)$. From diagram (2.1), we have the following commutative diagram up to homotopy:

$$\begin{array}{ccc} & Z \vee Z & \\ B s_0 \nearrow & & \searrow B(d_0, d_1) \\ Z & \xrightarrow{\Delta} & Z \times Z \end{array}$$

Hence Z is a co-H-space. From [13, p. 353], $\pi_1(Z)$ is a free group. While on the other hand, $\pi_1(Z) \cong \pi_1(K(G(\mathbb{L}_0), 1)) \cong G(\mathbb{L}_0) \cong G(L)$, so $G(L)$ is a free group.

Now suppose the link L has q components; we divide the discussion into the following two cases.

Case 1: $q = 1$. $G(L)$ is free if and only if L is a trivial knot, and from the splittability of \mathbb{L}_1 , \mathbb{L}_1 is a trivial 2-link, which implies that $(L; X, A)$ is trivial, a contradiction.

Case 2: $q > 1$. Since $G(L)$ is free, $H_2(G(L)) = 0$. But L is nonsplittable; from [20], $S^3 \setminus L$ is a $K(G(L), 1)$ space, thus $H_2(G(L)) \cong H_2(K(G(L), 1)) \cong H_2(S^3 \setminus L) \cong \mathbb{Z}^{\oplus(q-1)}$, again a contradiction.

Hence under the hypothesis of the proposition, \mathbb{L}_1 is nonsplittable. Now if for some $n > 1$, \mathbb{L}_n is splittable, there is an embedding of S^2 into S^3 which separates S^3 into two parts A and B such that

$$\mathbb{L}_n = L' \sqcup L''$$

with $L' \subseteq A$ and $L'' \subseteq B$. Suppose $L_0 \subseteq A$. Since $\mathbb{L}_1 = \{L_0, L_1\}$ is nonsplittable, $\mathbb{L}_1 \subseteq A$. Suppose for some $j \neq 0, 1$, the copy $L_j \subseteq B$; then $L_0 \sqcup L_j$ is splittable. But $L_0 \sqcup L_j$ is ambient isotopic to \mathbb{L}_1 , which is nonsplittable, a contradiction. This completes the proof. □

From Proposition 2.1 and Papakyriakopoulos' Theorem [20], we have the following conclusion.

Corollary 2.2. *Let L^f be a nonsplittable framed link in S^3 with the associated twisted-annulus link $(L; X, A)$ nontrivial. Then the link complement $S^3 \setminus \mathbb{L}_n$ is a $K(\pi, 1)$ -space for each $n \geq 0$.*

Proof. The assertion follows immediately from Theorem 2.1 in [12]. □

3. SIMPLICIAL GROUPS

Recall that a framed link L^f is a smooth link L in S^3 together with a smooth nonzero normal vector field f defined on (a small neighborhood of) L , and from any framed link L^f , we get a twisted-annulus link $(L; X, A)$, and hence the naive cabling on it is given by a sequence of links as $\mathbb{L}_n = \{L_0, L_1, \dots, L_n\}$, $0 \leq n < \infty$. We define two operations on the link \mathbb{L}_n as follows. Let

$$\tilde{d}_i \mathbb{L}_n = \{L_0, \dots, L_{i-1}, L_{i+1}, \dots, L_n\}$$

be the sublink of \mathbb{L}_n by removing the i -th copy of \mathbb{L}_n and

$$\tilde{s}_k \mathbb{L}_n = \{L_0, \dots, L_k, L_{k+1/2}, L_{k+1}, \dots, L_n\}$$

the link obtained by doubling the k -th copy of \mathbb{L}_n .

From the continuous maps,

$$S^3 \setminus \mathbb{L}_n \hookrightarrow S^3 \setminus \tilde{d}_i \mathbb{L}_n \xrightarrow{\phi_i} S^3 \setminus \mathbb{L}_{n-1},$$

we get a group homomorphism between link groups $d_i : G(\mathbb{L}_n) \longrightarrow G(\mathbb{L}_{n-1})$. From the composition of continuous maps

$$S^3 \setminus \mathbb{L}_n \xleftarrow{\iota} S^3 \setminus (\mathbb{L}_n \cup A_k) \hookrightarrow S^3 \setminus (\mathbb{L}_n \cup L_{k+1/2}) \xrightarrow{\psi_k} S^3 \setminus \mathbb{L}_{n+1},$$

we get a group homomorphism between link groups $s_k : G(\mathbb{L}_n) \longrightarrow G(\mathbb{L}_{n+1})$.

From the definitions of d_i, s_k , we have the following conclusion.

Proposition 3.1. *For any framed link L^f , there is an associated simplicial group $\mathbb{G}(L; X) = \{G(\mathbb{L}_n)\}_{n \geq 0}$ obtained from the naive cabling.*

Some basic properties of simplicial groups are reviewed next.

3.1. The Moore chain complex. Let $G = \{G_n\}_{n \geq 0}$ be a simplicial group. Define

$$N_n G = \bigcap_{i=1}^n \text{Ker}(d_i : G_n \rightarrow G_{n-1}).$$

For a simplicial group G , the *Moore chain complex* NG is the sequence of groups

$$\cdots \longrightarrow N_{n+1}G \xrightarrow{d_0|} N_n G \xrightarrow{d_0|} N_{n-1}G \longrightarrow \cdots$$

where the first face d_0 maps $N_n G$ into $N_{n-1}G$ and the composition

$$N_n G \xrightarrow{d_0|} N_{n-1}G \xrightarrow{d_0|} N_{n-2}G$$

is the trivial homomorphism.

A classical theorem due to J. C. Moore [19] is as follows.

Theorem 3.2 (Moore). *The homotopy groups of the geometric realization $|\Gamma|$ of a simplicial group Γ are given by*

$$\pi_n(|\Gamma|) \cong H_n(N\Gamma; d_0|)$$

for all $n \geq 0$.

The Moore complex NG has a hypercrossed complex structure (see [5]) which admits the original simplicial group G to be rebuilt. By applying the simplicial structure to the link groups of the naive cabling of framed links, we have the following normal form theorem on these link groups.

Theorem 3.3 (Normal form theorem). *Let L^f be a framed link in S^3 with the associated twisted-annulus link $(L; X, A)$. Then for any n , the link group $G(\mathbb{L}_n)$ has a semi-direct product decomposition*

$$G(\mathbb{L}_n) \cong (\dots (N_n G \rtimes_{s_{n-1}} N_{n-1} G) \rtimes \dots \rtimes_{s_{n-2}} \dots \rtimes_{s_1} N_1 G) \rtimes (\dots (s_0 N_{n-1} G \rtimes_{s_1} s_0 N_{n-2} G) \rtimes \dots \rtimes_{s_{n-1}} s_{n-2} \dots \rtimes_{s_0} N_0 G),$$

where $G = \mathbb{G}(L; X)$.

Proof. From the construction of naive cabling and Proposition 3.1, $\mathbb{G}(L; X) = \{G(\mathbb{L}_n)\}_{n \geq 0}$ is a simplicial group; the proof follows directly from [9, p. 158]. \square

3.2. Classifying spaces of simplicial groups. Given a simplicial group G , the simplicial set WG is defined by setting

$$WG_n = \{(g_n, g_{n-1}, \dots, g_0) | g_i \in G_i\} = G_n \times G_{n-1} \times \dots \times G_0$$

with faces and degeneracies given by

$$d_i(g_n, g_{n-1}, \dots, g_0) = \begin{cases} (d_i g_n, d_{i-1} g_{n-1}, \dots, d_1 g_{n-i+1}, d_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, \dots, g_0) & \text{if } i < n, \\ (d_n g_n, d_{n-1} g_{n-1}, \dots, d_1 g_1) & \text{if } i = n, \end{cases}$$

$$s_i(g_n, g_{n-1}, \dots, g_0) = (s_i g_n, s_{i-1} g_{n-1}, \dots, s_0 g_{n-i}, e, g_{n-i-1}, \dots, g_0).$$

The left action of G on WG given by

$$h \cdot (g_n, g_{n-1}, \dots, g_0) = (hg_n, g_{n-1}, \dots, g_0)$$

is simplicial for $h \in G_n$ and $(g_n, g_{n-1}, \dots, g_0) \in WG_n$. The classifying space

$$\bar{W}G = WG/G$$

is the quotient of WG by the action of G .

For each fixed n , we have simplicial sets WG_n and $\bar{W}G_n$ as above. The face operation $d_j : G_n \rightarrow G_{n-1}$ induces

$$Wd_j : WG_n \rightarrow WG_{n-1}$$

and

$$\bar{W}d_j : \bar{W}G_n \rightarrow \bar{W}G_{n-1}$$

for $0 \leq j \leq n$. The degeneracy operation $s_i : G_n \rightarrow G_{n+1}$ induces

$$Ws_i : WG_n \rightarrow WG_{n+1}$$

and

$$\bar{W}s_i : \bar{W}G_n \rightarrow \bar{W}G_{n+1}$$

for $0 \leq i \leq n$. Then we get two bi-simplicial sets WG_{**} and $\bar{W}G_{**}$. Let $\Delta(WG_{**})$ and $\Delta(\bar{W}G_{**})$ be the diagonal simplicial sets of WG_{**} and $\bar{W}G_{**}$, respectively.

The Bousfield spectral sequence on bi-simplicial sets [21] is a spectral sequence with

$$E_{p,q}^2 = \pi_p(\pi_q(WG_{**})) \Rightarrow \pi_{p+q}(\Delta(WG_{**})).$$

As a classifying space, for each n , WG_n is contractible, so $E_{p,q}^2 = 0$ for all p, q , and $\pi_n(\Delta(WG_{**})) = 0$ for all n . Hence $\Delta(WG_{**})$ is contractible. It follows that $\Delta(\bar{W}G_{**}) = \Delta(WG_{**})/G$ is also a classifying space of G , thus $\bar{W}G_* = \Delta(\bar{W}G_{**})$.

Later we will need the bi-simplicial set $\bar{W}G_{**}$ as a classifying complex of G .

4. PROOF OF THEOREM 1.1

Lemma 4.1. *Let L^f be a nonsplittable framed link in S^3 with the associated twisted-annulus link $(L; X, A)$ nontrivial and $\mathbb{G}(L; X) = \{G(\mathbb{L}_n)\}_{n \geq 0}$ the simplicial group obtained from the naive cabling. Then $\bar{W}\mathbb{G}(L; X)$ is simply connected.*

Proof. From simplicial homotopy theory, we know that

$$\pi_1(\bar{W}\mathbb{G}(L; X)) \cong \pi_0(\mathbb{G}(L; X)),$$

so it suffices to show that $\pi_0(\mathbb{G}(L; X)) = 0$.

For simplicity, we denote $\mathbb{G}(L; X)$ by G ; thus

$$G_0 = \pi_1(S^3 \setminus \mathbb{L}_0), \quad G_1 = \pi_1(S^3 \setminus \mathbb{L}_1)$$

and $N_1G = \text{Ker}(d_1 : G_1 \rightarrow G_0)$.

By definition, $\pi_0(G) = G_0/d_0(N_1G)$; our goal is just to prove that the face map $d_0 : N_1G \rightarrow G_0$ is onto. That is, for any $[\omega] \in G_0$, we can find an $[\tilde{\omega}] \in N_1G$ such that $d_0[\tilde{\omega}] = [\omega]$. We are looking for an $[\tilde{\omega}] \in \pi_1(S^3 \setminus \mathbb{L}_1)$ satisfying the following two conditions:

- (1) $d_1[\tilde{\omega}] = 1$,
- (2) $d_0[\tilde{\omega}] = [\omega]$.

Since $[\omega] \in \pi_1(S^3 \setminus \mathbb{L}_0)$, up to homotopy, ω is an embedding $\omega : S^1 \hookrightarrow S^3 \setminus \mathbb{L}_0$, so ω can be considered as a knot embedded in $S^3 \setminus \mathbb{L}_0$, that is, $\omega \subseteq (S^3 \setminus \mathbb{L}_0)$. Let $\mathbb{L}_1 = \{L_0, L_1\}$. After the operation $d_0 : S^3 \setminus \mathbb{L}_1 \hookrightarrow S^3 \setminus \mathbb{L}_0$ we can regard \mathbb{L}_0 as L_1 , so $\omega \subseteq (S^3 \setminus L_1)$ under the operation d_0 .

Since $S^3 \setminus \mathbb{L}_1 = (S^3 \setminus L_1) \setminus L_0$, we can perturb ω slightly so that it is in general position with respect to L_0 , and denote the new knot by λ . Thus in $\pi_1(S^3 \setminus L_1)$, we have

$$d_0([\lambda]) = [\lambda] = [\omega].$$

Now we consider $d_1([\lambda]) \in \pi_1(S^3 \setminus L_0)$. Since S^3 is simply connected, we can choose a projection of λ such that it has

- (1) no self crossings;
- (2) finite crossings A_1, \dots, A_p with L_0 ;
- (3) finite crossings B_1, \dots, B_q with L_1 .

In the operation d_1 , we forget the copy L_1 , and λ is a knot in $S^3 \setminus L_0$. Now we do some surgery on λ , precisely, by changing A_i to A'_i or A''_i , $1 \leq i \leq p$; see Figure 1. We can finally untangle λ and L_0 , and we denote the resulting knot by $\tilde{\omega}$. Then $[\tilde{\omega}] = 1$ in $\pi_1(S^3 \setminus L_0)$, which means that

$$d_1([\tilde{\omega}]) = 1.$$

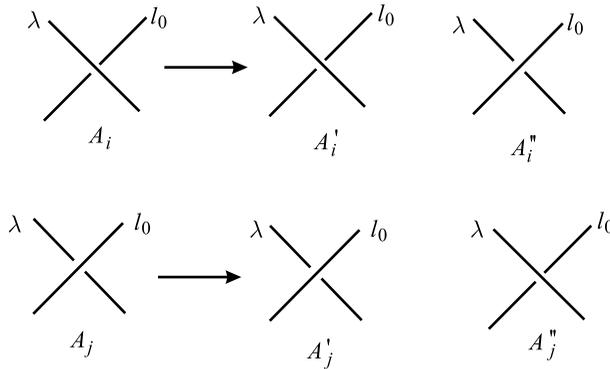


FIGURE 1. The surgery on λ

Note that the surgery is occurring on λ in a small neighborhood of its crossing points with L_0 , hence $\tilde{\omega} = \lambda$ outside a small neighborhood of L_0 and $\tilde{\omega} \neq \lambda$ inside a small neighborhood of L_0 . In fact, the only difference between $\tilde{\omega}$ and λ are the crossing points with L_0 , but the operation d_0 is deleting L_0 from \mathbb{L}_1 , so the differences between $\tilde{\omega}$ and λ disappear after the operation of d_0 . Thus we have

$$d_0[\tilde{\omega}] = d_0[\lambda].$$

Hence we get $[\tilde{\omega}] \in \pi_1(S^3 \setminus \mathbb{L}_1)$ satisfying

- $d_0[\tilde{\omega}] = d_0[\lambda] = [\omega]$,
- $d_1[\tilde{\omega}] = 1$.

This completes the proof. □

For a pointed simplicial set K , $\mathbb{Z}[K]_n = \mathbb{Z}(K_n)/\mathbb{Z}(*).$ Let $\Delta[1]$ be the simplicial 1-simplex.

Lemma 4.2. *Let L^f be a nonsplittable framed q -link in S^3 with the associated twisted-annulus link $(L; X, A)$ nontrivial. Then $H_1(\bar{W}\mathbb{G}(L; X)_{**}) = \{H_1(G(\mathbb{L}_n))\}_{n \geq 0}$ forms a simplicial group. Furthermore, there is an isomorphism of simplicial groups*

$$H_1(\bar{W}\mathbb{G}(L; X)_{**}) \cong \mathbb{Z}[\Delta[1]]^{\oplus q}$$

and $H_p(H_1(\bar{W}\mathbb{G}(L; X)_{**})) = 0$ for each p .

Proof. Since L^f is nonsplittable and $(L; X, A)$ is nontrivial, from Corollary 2.2, $S^3 \setminus \mathbb{L}_n$ is a $K(G(\mathbb{L}_n), 1)$ space; thus

$$H_1(G(\mathbb{L}_n)) \cong H_1(K(G(\mathbb{L}_n), 1)) \cong H_1(S^3 \setminus \mathbb{L}_n).$$

Since $H_1(S^3 \setminus \mathbb{L}_n) \cong \pi_1(S^3 \setminus \mathbb{L}_n)^{\text{ab}}$, the maps d_i and s_k defined in Proposition 3.1 can naturally extend to $\{H_1(G(\mathbb{L}_n))\}_{n \geq 0}$.

Since \mathbb{L}_n has $q(n+1)$ components, $H_1(G(\mathbb{L}_n)) = \mathbb{Z}^{\oplus q(n+1)}$ has $q(n+1)$ generators consisting of the meridians of components of \mathbb{L}_n , i.e., the generators of $H_1(G(\mathbb{L}_n))$ are $\{\alpha_1^{(0)}, \dots, \alpha_q^{(0)}, \dots, \alpha_1^{(n)}, \dots, \alpha_q^{(n)}\}$ with each $\alpha_s^{(t)}$ representing the meridian of the component $l_s^{(t)}$.

Since $\tilde{d}_i \mathbb{L}_n = \{L_0, \dots, L_{i-1}, L_{i+1}, \dots, L_n\}$, the effect of the face map d_i on the generators of $H_1(G(\mathbb{L}_n))$ is as follows:

$$(4.1) \quad d_i(\alpha_m^{(t)}) = \begin{cases} \alpha_m^{(t)}, & 1 \leq m \leq q, \quad 0 \leq t \leq i-1, \\ 0, & 1 \leq m \leq q, \quad t = i, \\ \alpha_m^{(t-1)}, & 1 \leq m \leq q, \quad i+1 \leq t \leq n, \end{cases}$$

which induces the face maps $d_i : H_1(G(\mathbb{L}_n)) \rightarrow H_1(G(\mathbb{L}_{n-1}))$ for $0 \leq i \leq n$.

Since $\tilde{s}_k \mathbb{L}_n = \mathbb{L}_n \cup L_{k+1/2}$, the effect of the degeneracy map s_k on the generators of $H_1(G(\mathbb{L}_n))$ is as follows:

$$(4.2) \quad s_k(\alpha_m^{(t)}) = \begin{cases} \alpha_m^{(t)}, & 1 \leq m \leq q, \quad 0 \leq t \leq k-1, \\ \alpha_m^{(k)} + \alpha_m^{(k+1)}, & 1 \leq m \leq q, \quad t = k, \\ \alpha_m^{(t+1)}, & 1 \leq m \leq q, \quad k+1 \leq t \leq n, \end{cases}$$

which induces the degeneracy maps $s_k : H_1(G(\mathbb{L}_n)) \rightarrow H_1(G(\mathbb{L}_{n+1}))$ for $0 \leq k \leq n$.

Since this is naturally induced from the operations \tilde{d}_i and \tilde{s}_k , we know that $\{H_1(G(\mathbb{L}_n))\}_{n \geq 0}$ with the faces d_i and degeneracies s_k defined as equations (4.1) and (4.2) forms a simplicial group.

Now $\alpha_i^{(0)}$ is a generator of $H_1(G(\mathbb{L}_1))$ and the representing map of $\alpha_i^{(0)}$ is

$$f_i : \Delta[1] \longrightarrow H_1(\bar{W}\mathbb{G}(L; X)_{**})$$

such that $f_i(0, 1) = \alpha_i^{(0)}$. Thus there is a unique simplicial map

$$f = f_1 \sqcup \dots \sqcup f_q : \underbrace{\Delta[1] \sqcup \dots \sqcup \Delta[1]}_q \longrightarrow H_1(\bar{W}\mathbb{G}(L; X)_{**})$$

such that $f((0, 1)_i) = \alpha_i^{(0)}$ in $H_1(G(\mathbb{L}_1))$. Then f induces a unique simplicial homomorphism

$$\varphi : \underbrace{\mathbb{Z}(\Delta[1]) \oplus \dots \oplus \mathbb{Z}(\Delta[1])}_q \longrightarrow H_1(\bar{W}\mathbb{G}(L; X)_{**})$$

given by the linear extension.

Observe that, in 0-dimension, $\varphi((1)_i) = \varphi(d_0(0, 1)_i) = d_0\varphi((0, 1)_i) = d_0f((0, 1)_i) = d_0\alpha_i^{(0)} = 0$ in $H_1(G(\mathbb{L}_0))$. Hence the simplicial homomorphism φ factors through the simplicial quotient group $\underbrace{\mathbb{Z}[\Delta[1]] \oplus \cdots \oplus \mathbb{Z}[\Delta[1]]}_q = \mathbb{Z}[\Delta[1]]^{\oplus q}$.

Next we show that the induced simplicial homomorphism

$$\bar{\varphi} : \mathbb{Z}[\Delta[1]]^{\oplus q} \longrightarrow H_1(\bar{W}\mathbb{G}(L; X)_{**})$$

is a simplicial isomorphism in each dimension.

Note that in dimension n ,

$$\begin{aligned} \bar{\varphi}(\underbrace{(0, \dots, 0)}_k, \underbrace{1, \dots, 1}_{n+1-k})_i &= \bar{\varphi}(s_{n-1} \cdots s_k s_{k-2} \cdots s_0(0, 1)_i) \\ &= s_{n-1} \cdots s_k s_{k-2} \cdots s_0(\alpha_i^{(0)}) \\ &= \alpha_i^{(0)} + \cdots + \alpha_i^{(k-1)}, \end{aligned}$$

so we have a table of correspondence as follows:

$$\begin{aligned} (0, 1, \dots, 1)_i &\mapsto \alpha_i^{(0)}, \\ (0, 0, 1, \dots, 1)_i &\mapsto \alpha_i^{(0)} + \alpha_i^{(1)} \\ &\dots \\ (\underbrace{0, \dots, 0}_k, \underbrace{1, \dots, 1}_{n+1-k})_i &\mapsto \alpha_i^{(0)} + \alpha_i^{(1)} + \cdots + \alpha_i^{(k-1)} \\ &\dots \\ (0, \dots, 0, 1)_i &\mapsto \alpha_i^{(0)} + \alpha_i^{(1)} + \cdots + \alpha_i^{(n-1)}, \\ (0, 0, \dots, 0)_i &\mapsto \alpha_i^{(0)} + \alpha_i^{(1)} + \cdots + \alpha_i^{(n-1)} + \alpha_i^{(n)}. \end{aligned}$$

From the above table, it is easy to check that $\bar{\varphi}$ is a bijective simplicial map; hence $H_1(\bar{W}\mathbb{G}(L; X)_{**})$ is simplicial isomorphic to $\mathbb{Z}[\Delta[1]]^{\oplus q}$ and its homology is $H_p(H_1(\bar{W}\mathbb{G}(L; X)_{**})) \cong \tilde{H}_p(\bigvee^q \Delta[1]) = 0$, for each p . □

Next we discuss the simplicial abelian group

$$H_2(\bar{W}\mathbb{G}(L; X)_{**}) = \{H_2(G(\mathbb{L}_n))\}_{n \geq 0}.$$

Recall that for a framed q -link L^f in S^3 , we get the naive cabling

$$\mathbb{L}_n = \{L_0, L_1, \dots, L_n\}$$

on the twisted-annulus link $(L; X, A)$. Let $Y(\mathbb{L}_n) = \partial\bar{V}(\mathbb{L}_n)$ be the boundary of the tubular neighborhood of \mathbb{L}_n . $Y(\mathbb{L}_n)$ is a collection of tori, and we denote it by $Y(\mathbb{L}_n) = \{T_1^{(0)}, \dots, T_q^{(0)}, \dots, T_1^{(n)}, \dots, T_q^{(n)}\}$ with each $T_s^{(t)}$ corresponding to the component $l_s^{(t)}$ of \mathbb{L}_n , and the group $H_2(Y(\mathbb{L}_n))$ is generated by the fundamental classes $\{[T_1^{(0)}], \dots, [T_q^{(0)}], \dots, [T_1^{(n)}], \dots, [T_q^{(n)}]\}$ of $Y(\mathbb{L}_n)$. When n varies, we get a sequence of groups $\Gamma = \{H_2(Y(\mathbb{L}_n))\}_{n \geq 0}$.

Now the two operations \tilde{d}_i and \tilde{s}_k defined on \mathbb{L}_n will induce maps between homology groups. Since $\tilde{d}_i\mathbb{L}_n = \{L_0, \dots, L_{i-1}, L_{i+1}, \dots, L_n\}$, the effect of the face

map d_i on the generators of $H_2(Y(\mathbb{L}_n))$ is as follows:

$$(4.3) \quad d_i([T_m^{(t)}]) = \begin{cases} [T_m^{(t)}], & 1 \leq m \leq q, \quad 0 \leq t \leq i - 1, \\ 0, & 1 \leq m \leq q, \quad t = i, \\ [T_m^{(t-1)}], & 1 \leq m \leq q, \quad i + 1 \leq t \leq n, \end{cases}$$

which induces the face maps $d_i : H_2(Y(\mathbb{L}_n)) \rightarrow H_2(Y(\mathbb{L}_{n-1}))$ for $0 \leq i \leq n$.

Since $\tilde{s}_k \mathbb{L}_n = \mathbb{L}_n \cup L_{k+1/2}$, the effect of the degeneracy map s_k on the generators of $H_2(Y(\mathbb{L}_n))$ is as follows:

$$(4.4) \quad s_k([T_m^{(t)}]) = \begin{cases} [T_m^{(t)}], & 1 \leq m \leq q, \quad 0 \leq t \leq k - 1, \\ [T_m^{(k)}] + [T_m^{(k+1)}], & 1 \leq m \leq q, \quad t = k, \\ [T_m^{(t+1)}], & 1 \leq m \leq q, \quad k + 1 \leq t \leq n, \end{cases}$$

which induces the degeneracy maps $s_k : H_2(Y(\mathbb{L}_n)) \rightarrow H_2(Y(\mathbb{L}_{n+1}))$ for $0 \leq k \leq n$. Following the lines of the proof of Lemma 4.2, we have

Proposition 4.3. *Let L^f be a framed q -link in S^3 with the associated naive cabling \mathbb{L}_n . Then the sequence of groups $\Gamma = \{H_2(Y(\mathbb{L}_n))\}_{n \geq 0}$ forms a simplicial group. Furthermore, Γ is isomorphic to $\mathbb{Z}[\Delta[1]]^{\oplus q}$ as simplicial groups.*

Suppose $\tilde{L} = \{\tilde{l}_1, \dots, \tilde{l}_m\}$ is any nonsplittable m -link in S^3 . $S^3 \setminus \tilde{L}$ is a $K(G(\tilde{L}), 1)$ space, where $G(\tilde{L}) = \pi_1(S^3 \setminus \tilde{L})$. By definition, we have $H_2(G(\tilde{L})) \cong H_2(K(G(\tilde{L}), 1)) \cong H_2(S^3 \setminus \tilde{L}) \cong H_2(S^3 \setminus V(\tilde{L}))$, where $V(\tilde{L})$ is a tubular neighborhood of \tilde{L} , and $\bar{V}(\tilde{L}) = \bigsqcup_{i=1}^m \bar{V}(\tilde{l}_i) \simeq \bigsqcup_{i=1}^m S^1$. Denote $Y = \partial \bar{V}(\tilde{L}) = \bigsqcup_{i=1}^m T_i$ as a collection of mutually disjoint tori, and $M = S^3 \setminus V(\tilde{L})$. Now $M \cup \bar{V}(\tilde{L}) = S^3$, $M \cap \bar{V}(\tilde{L}) = Y$, and the Mayer-Vietoris sequence is as follows:

$$\begin{aligned} H_3(Y) &\longrightarrow H_3(M) \oplus H_3(\bar{V}(\tilde{L})) \longrightarrow H_3(S^3) \longrightarrow H_2(Y) \\ &\longrightarrow H_2(M) \oplus H_2(\bar{V}(\tilde{L})) \longrightarrow H_2(S^3) \longrightarrow H_1(Y) \\ &\longrightarrow H_1(M) \oplus H_1(\bar{V}(\tilde{L})) \longrightarrow H_1(S^3) \longrightarrow H_0(Y) \\ &\longrightarrow H_0(M) \oplus H_0(\bar{V}(\tilde{L})) \longrightarrow H_0(S^3) \longrightarrow 0. \end{aligned}$$

Since M and $\bar{V}(\tilde{L})$ are both 3-manifolds with boundary and $\bar{V}(\tilde{L}) \simeq \bigsqcup_{i=1}^m S^1$, $H_3(M) = 0$, $H_3(\bar{V}(\tilde{L})) = 0$, and $H_2(\bar{V}(\tilde{L})) = 0$. As we know, $H_2(S^3) = 0$; hence we get an exact sequence as follows:

$$(4.5) \quad 0 \longrightarrow H_3(S^3) \longrightarrow H_2(Y) \longrightarrow H_2(M) \longrightarrow 0,$$

where $H_2(Y) \cong \mathbb{Z}^{\oplus m}$ is generated by the fundamental classes $\{[T_1], \dots, [T_m]\}$ of the components of Y .

Suppose L^f is a nonsplittable framed link in S^3 with the associated twisted-annulus link $(L; X, A)$ nontrivial. From Proposition 2.1, the naive cabling \mathbb{L}_n is nonsplittable. Let $\tilde{L} = \mathbb{L}_n$, $Y(\mathbb{L}_n) = \partial \bar{V}(\mathbb{L}_n)$, and $M(\mathbb{L}_n) = S^3 \setminus V(\mathbb{L}_n)$. From the arguments above, we get a collection of exact sequences:

$$(4.6) \quad 0 \longrightarrow H_3(S^3) \longrightarrow H_2(Y(\mathbb{L}_n)) \longrightarrow H_2(G(\mathbb{L}_n)) \longrightarrow 0.$$

Proposition 4.4. *Let L^f be a nonsplittable framed link in S^3 with the associated twisted-annulus link $(L; X, A)$ nontrivial. Then there is a short exact sequence of*

Δ -groups:

$$0 \longrightarrow \{H_3(S^3) = \mathbb{Z}\}_{n \geq 0} \longrightarrow \{H_2(Y(\mathbb{L}_n))\}_{n \geq 0} \longrightarrow \{H_2(G(\mathbb{L}_n))\}_{n \geq 0} \longrightarrow 0.$$

Proof. It is easy to see that $\{H_3(S^3) = \mathbb{Z}\}_{n \geq 0}$ with face map $d_i = \text{id}$ forms a Δ -group.

Recall that $M(\mathbb{L}_n) = S^3 \setminus V(\mathbb{L}_n)$ and $Y(\mathbb{L}_n) = \partial \bar{V}(\mathbb{L}_n) = \partial M(\mathbb{L}_n)$. Let W be the collection of solid tori bounded by the components of $Y(\mathbb{L}_n)$ that are not in $Y(\mathbb{L}_{n-1})$. Clearly W consists of q mutually disjoint solid tori and so $H_2(W) = 0$. Then we have the following commutative diagram:

$$\begin{array}{ccc} Y(\mathbb{L}_n) \hookrightarrow & M(\mathbb{L}_n) \\ \downarrow g & \downarrow \tilde{d}_i \\ Y(\mathbb{L}_{n-1}) \sqcup W \hookrightarrow & M(\mathbb{L}_{n-1}) \end{array}$$

By taking the second homology, we have the commutative diagram

$$(4.7) \quad \begin{array}{ccc} H_2(Y(\mathbb{L}_n)) & \longrightarrow & H_2(M(\mathbb{L}_n)) \\ \downarrow g_* & & \downarrow d_i \\ H_2(Y(\mathbb{L}_{n-1})) \oplus H_2(W) & \longrightarrow & H_2(M(\mathbb{L}_{n-1})) \end{array}$$

From the commutativity of diagram (4.7), the effect of g_* on the generators of $H_2(Y(\mathbb{L}_n))$ is exactly the face map d_i defined in equation (4.3). Since \mathbb{L}_n is non-splittable, by Corollary 2.2, $H_2(M(\mathbb{L}_n)) \cong H_2(G(\mathbb{L}_n))$; then the following diagram commutes:

$$(4.8) \quad \begin{array}{ccc} H_2(Y(\mathbb{L}_n)) & \longrightarrow & H_2(G(\mathbb{L}_n)) \\ \downarrow d_i & & \downarrow d_i \\ H_2(Y(\mathbb{L}_{n-1})) & \longrightarrow & H_2(G(\mathbb{L}_{n-1})) \end{array}$$

This implies that $\{H_2(Y(\mathbb{L}_n))\}_{n \geq 0} \longrightarrow \{H_2(G(\mathbb{L}_n))\}_{n \geq 0}$ is a Δ -map.

From the definition of the framed link, $\tilde{d}_i \mathbb{L}_n$ is ambient isotopic to \mathbb{L}_{n-1} , i.e., there exists an ambient isotopy $h_t : S^3 \longrightarrow S^3$ such that $h_0 = \text{id}$, $h_1(\tilde{d}_i \mathbb{L}_n) = \mathbb{L}_{n-1}$, and moreover, the restriction $h_1| : S^3 \setminus \tilde{d}_i \mathbb{L}_n \longrightarrow S^3 \setminus \mathbb{L}_{n-1}$ is a homeomorphism.

Now we have two distinct Mayer-Vietoris pairs for S^3 as $(M(\mathbb{L}_n), \bar{V}(\mathbb{L}_n))$ and $(M(\mathbb{L}_{n-1}), \bar{V}(\mathbb{L}_{n-1}) \sqcup W)$. Then the following diagram induced by inclusions and isotopy is clearly commutative:

$$\begin{array}{ccccc} Y(\mathbb{L}_n) \hookrightarrow & & M(\mathbb{L}_n) & & \\ \downarrow & \searrow g & \downarrow & \searrow h_1 \circ \tilde{d}_i & \\ & & Y(\mathbb{L}_{n-1}) \sqcup W \hookrightarrow & & M(\mathbb{L}_{n-1}) \\ \downarrow & & \downarrow & & \downarrow \\ \bar{V}(\mathbb{L}_n) \hookrightarrow & & S^3 & & \\ \downarrow \text{id} & & \downarrow h_1 & & \downarrow \\ \bar{V}(\mathbb{L}_{n-1}) \sqcup W \hookrightarrow & & S^3 & & \end{array}$$

Hence we get the commutative diagram between MV sequences as

$$\begin{CD}
 H_3(S^3) @>>> H_2(Y(\mathbb{L}_n)) @>>> H_2(M(\mathbb{L}_n)) \oplus H_2(\bar{V}(\mathbb{L}_n)) \\
 @V h_{1*} \simeq \text{id} VV @V g_* VV @V d_i \oplus \text{id} VV \\
 H_3(S^3) @>>> H_2(Y(\mathbb{L}_{n-1})) \oplus H_2(W) @>>> H_2(M(\mathbb{L}_{n-1})) \oplus H_2(\bar{V}(\mathbb{L}_n))
 \end{CD}$$

where the effect of g_* on the generators of $H_2(Y(\mathbb{L}_n))$ is exactly the face map d_i defined in equation (4.3). Thus we get a commutative diagram up to homotopy

$$(4.9) \quad \begin{CD}
 H_3(S^3) @>>> H_2(Y(\mathbb{L}_n)) \\
 @V \text{id} VV @V d_i VV \\
 H_3(S^3) @>>> H_2(Y(\mathbb{L}_{n-1}))
 \end{CD}$$

This implies that $\{H_3(S^3) = \mathbb{Z}\}_{n \geq 0} \rightarrow \{H_2(Y(\mathbb{L}_n))\}_{n \geq 0}$ is a Δ -map.

From commutative diagrams (4.8) and (4.9), the statement that

$$\{H_3(S^3) = \mathbb{Z}\}_{n \geq 0} \longrightarrow \{H_2(Y(\mathbb{L}_n))\}_{n \geq 0} \twoheadrightarrow \{H_2(G(\mathbb{L}_n))\}_{n \geq 0}$$

is a short exact sequence of Δ -groups holds. □

Lemma 4.5. *Let L^f be a nonsplittable framed link in S^3 with the associated twisted-annulus link $(L; X, A)$ nontrivial. Then*

$$H_p(H_2(\bar{W}\mathbb{G}(L; X)_{**})) \cong \begin{cases} \mathbb{Z}, & p = 1, \\ 0, & p \neq 1. \end{cases}$$

Proof. Let $A = \{H_3(S^3) = \mathbb{Z}\}_{n \geq 0}$, $B = \{H_2(Y(\mathbb{L}_n))\}_{n \geq 0}$ and $C = \{H_2(G(\mathbb{L}_n))\}_{n \geq 0}$. From Proposition 4.4 and the simplicial homology theory, we get a long exact sequence

$$(4.10) \quad \cdots \rightarrow H_p(A) \rightarrow H_p(B) \rightarrow H_p(C) \rightarrow H_{p-1}(A) \rightarrow \cdots$$

Now $A = \{\mathbb{Z}\}_{n \geq 0}$ is a Δ -group with $d_i = \text{id}$, so as a Δ -group, $A \cong \mathbb{Z}(\ast)$; hence

$$(4.11) \quad H_p(A) \cong \begin{cases} \mathbb{Z}, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

Next we compute $H_*(B)$. From Proposition 4.3 and Lemma 4.2, $H_p(B) = 0$, for each p .

By substituting the homology groups of A and B in the long exact sequence (4.10), we have $H_i(C) = 0$ for $i \geq 2$ and the following exact sequence:

$$0 \rightarrow H_1(C) \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow H_0(C);$$

hence $H_1(C) \cong \mathbb{Z}$ and $H_0(C) = 0$. This completes the proof. □

Lemma 4.6. *Let L^f be a nonsplittable framed link in S^3 with the associated twisted-annulus link $(L; X, A)$ nontrivial. Then the geometric realization of $\mathbb{G}(L; X)$ is homotopy equivalent to ΩS^3 .*

Proof. It suffices to show that the classifying spaces of $\mathbb{G}(L; X)$ and ΩS^3 are homotopy equivalent, that is, $\bar{W}\mathbb{G}(L; X) \simeq \bar{W}\Omega S^3$, i.e., $\bar{W}\mathbb{G}(L; X) \simeq S^3$.

By Lemma 4.1, $\bar{W}\mathbb{G}(L; X)$ is simply connected. Now we compute the homology of the simplicial set $\bar{W}\mathbb{G}(L; X)$. Recall that, for any pointed simplicial set X , the reduced homology

$$\tilde{H}_n(X; \mathbb{Z}) \cong \pi_n(\mathbb{Z}[X])$$

for any n (see [10]). We are going to compute $\pi_*(\mathbb{Z}[\Delta(\bar{W}\mathbb{G}(L; X)_{**})])$ using the bi-simplicial classifying spaces mentioned in section 3.2 and the Quillen spectral sequence [21]. Namely, for a bi-simplicial group G , there is a spectral sequence $E_{p,q}^2 = \pi_p(\pi_q(G))$ converging to $\pi_{p+q}(\Delta G)$, where ΔG is the diagonal. In our case, $\mathbb{Z}[\bar{W}\mathbb{G}(L; X)_{**}]$ is a bi-simplicial group and $\mathbb{Z}[\bar{W}\mathbb{G}(L; X)_*]$ is the diagonal. From Quillen spectral sequence [21], we have a spectral sequence $\pi_p(\pi_q(\mathbb{Z}[\bar{W}\mathbb{G}(L; X)_{**}]))$ converging to $\pi_{p+q}(\mathbb{Z}[\bar{W}\mathbb{G}(L; X)_*])$.

By Lemma 4.2 and Lemma 4.5, the E^2 -terms for $\mathbb{Z}[\bar{W}\mathbb{G}(L; X)_{**}]$ are as follows:

$$E_{p,q}^2 \cong \begin{cases} \mathbb{Z}, & p = 1, q = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\tilde{H}_i(\bar{W}\mathbb{G}(L; X); \mathbb{Z}) \cong \pi_i(\mathbb{Z}[\bar{W}\mathbb{G}(L; X)_*]) \cong \begin{cases} \mathbb{Z}, & i = 3, \\ 0, & \text{otherwise,} \end{cases}$$

thus

$$H_*(\bar{W}\mathbb{G}(L; X); \mathbb{Z}) \cong H_*(S^3; \mathbb{Z}).$$

By Lemma 4.1, $\bar{W}\mathbb{G}(L; X)$ is simply connected. By the Hurewicz theorem,

$$\pi_3(\bar{W}\mathbb{G}(L; X)) \cong H_3(\bar{W}\mathbb{G}(L; X)) \cong \mathbb{Z}.$$

The representing map of the generator of $\pi_3(\bar{W}\mathbb{G}(L; X)) \cong \mathbb{Z}$ induces a simplicial map

$$f : S^3 \longrightarrow \bar{W}\mathbb{G}(L; X)$$

such that the induced homomorphism

$$f_* : H_3(S^3; \mathbb{Z}) \longrightarrow H_3(\bar{W}\mathbb{G}(L; X); \mathbb{Z})$$

is an isomorphism, and so

$$f_* : H_q(S^3; \mathbb{Z}) \longrightarrow H_q(\bar{W}\mathbb{G}(L; X); \mathbb{Z})$$

is an isomorphism for each q . By Whitehead's theorem, we know that f is a homotopy equivalence, i.e., $\bar{W}\mathbb{G}(L; X) \simeq S^3$, and hence $\mathbb{G}(L; X) \simeq \Omega S^3$. \square

Lemma 4.7. *Let L^f be a framed knot in S^3 with the associated twisted-annulus link $(L; X, A)$ trivial. Then $\mathbb{G}(L; X)$ is isomorphic to $F[\Delta[1]]$ as simplicial groups and $\mathbb{G}(L; X)$ is contractible.*

Proof. From the assumption, the naive cabling \mathbb{L}_n is a trivial $(n+1)$ -link, hence $G(\mathbb{L}_n) = \pi_1(S^3 \setminus \mathbb{L}_n) \cong F_{n+1}$ is a free group generated by the meridians $\{\alpha_0, \dots, \alpha_n\}$ of \mathbb{L}_n .

From Proposition 3.1, $\mathbb{G}(L; X) = \{G(\mathbb{L}_n)\}_{n \geq 0}$ is a simplicial group, and the face and generacy maps on the generators of $G(\mathbb{L}_n)$ are

$$(4.12) \quad d_i(\alpha_m) = \begin{cases} \alpha_m, & 0 \leq m \leq i-1, \\ e, & m = i, \\ \alpha_{m-1}, & i+1 \leq m \leq n, \end{cases}$$

$$(4.13) \quad s_k(\alpha_m) = \begin{cases} \alpha_m, & 0 \leq m \leq k-1, \\ \alpha_k \cdot \alpha_{k+1}, & m = k, \\ \alpha_{m+1}, & k+1 \leq m \leq n. \end{cases}$$

Since α_0 is a generator of $G(\mathbb{L}_1)$, the representing map of α_0 is

$$f_{\alpha_0} : \Delta[1] \longrightarrow \mathbb{G}(L; X)$$

such that $f_{\alpha_0}(0, 1) = \alpha_0$.

Suppose $F(\Delta[1])$ is a free group generated by $\Delta[1]$. Then there exists a unique homomorphism $\tilde{f}_{\alpha_0} : F(\Delta[1]) \longrightarrow \mathbb{G}(L; X)$ such that the following diagram commutes:

$$(4.14) \quad \begin{array}{ccc} \Delta[1] & \xrightarrow{f_{\alpha_0}} & \mathbb{G}(L; X) \\ \downarrow & \nearrow \tilde{f}_{\alpha_0} & \\ F(\Delta[1]) & & \end{array}$$

Clearly, $\tilde{f}_{\alpha_0}(1) = f_{\alpha_0}(1) = f_{\alpha_0}(d_0(0, 1)) = d_0 f_{\alpha_0}((0, 1)) = d_0 \alpha_0 = e$ in $G(\mathbb{L}_0)$. For each n , the basepoint $*$ of $\Delta[1]_n$ is $(\underbrace{1, \dots, 1}_{n+1}) = \underbrace{s_0 \cdots s_0}_{n}(1)$, thus

$$\begin{aligned} \tilde{f}_{\alpha_0}(\underbrace{(1, \dots, 1)}_{n+1}) &= f_{\alpha_0}(\underbrace{s_0 \cdots s_0}_{n}(1)) \\ &= \underbrace{s_0 \cdots s_0}_n f_{\alpha_0}(1) \\ &= \underbrace{s_0 \cdots s_0}_n (e) = e \text{ in } G(\mathbb{L}_n). \end{aligned}$$

So \tilde{f}_{α_0} factors through the quotient

$$\varphi : F(\Delta[1]) / (* = 1) \longrightarrow \mathbb{G}(L; X).$$

As we know, Milnor's construction $F[\Delta[1]]_n$ is the free group generated by $\Delta[1]_n$ subject to the single relation that the basepoint $* = 1$, so in fact we get a simplicial homomorphism

$$\varphi : F[\Delta[1]] \longrightarrow \mathbb{G}(L; X).$$

Note that in dimension n ,

$$\begin{aligned} \varphi(\underbrace{(0, \dots, 0)}_k, \underbrace{1, \dots, 1}_{n+1-k}) &= \varphi(s_{n-1} \cdots s_k s_{k-2} \cdots s_0(0, 1)) \\ &= s_{n-1} \cdots s_k s_{k-2} \cdots s_0(\alpha_0) \\ &= \alpha_0 \cdots \alpha_{k-1}. \end{aligned}$$

It follows by induction that each $\alpha_i \in \text{Im} \varphi$ for $0 \leq i \leq n$. Hence all the generators of $G(\mathbb{L}_n)$ are in $\text{Im} \varphi$, which implies that $\varphi : F[\Delta[1]]_n \longrightarrow G(\mathbb{L}_n)$ is a surjective map between two free groups. By the Hopfian property, φ is an isomorphism. Hence $\mathbb{G}(L; X)$ is isomorphic to $F[\Delta[1]]$ as simplicial groups, and thus it is contractible. \square

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Since $(L; X, A)$ has the canonical splitting decomposition as

$$(L; X, A) \cong (L^{[1]}; X^{[1]}, A^{[1]}) \sqcup (L^{[2]}; X^{[2]}, A^{[2]}) \sqcup \dots \sqcup (L^{[p]}; X^{[p]}, A^{[p]})$$

with each $L^{[i]}$ nonsplittable, we have the free product decomposition

$$\mathbb{G}(L; X) \cong \bigast_{i=1}^p \mathbb{G}(L^{[i]}; X^{[i]})$$

as simplicial groups.

If $(L^{[i]}; X^{[i]}, A^{[i]})$ is nontrivial (with $L^{[i]}$ nonsplittable), then, by Lemma 4.6, $\mathbb{G}(L^{[i]}; X^{[i]})$ is homotopy equivalent to ΩS^3 . If $(L^{[i]}; X^{[i]}, A^{[i]})$ is trivial (with $L^{[i]}$ nonsplittable), by Lemma 4.7, $\mathbb{G}(L^{[i]}; X^{[i]})$ is contractible.

Then from the splitting decomposition of $(L; X, A)$, we have that $\mathbb{G}(L; X)$ is homotopy equivalent to $\Omega(\bigvee^k S^3)$. \square

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