CASTELNUOVO-MUMFORD REGULARITY AND THE DISCRETENESS OF F-JUMPING COEFFICIENTS IN GRADED RINGS

MORDECHAI KATZMAN AND WENLIANG ZHANG

ABSTRACT. In this paper we show that the sets of F-jumping coefficients of ideals form discrete sets in certain graded F-finite rings. We do so by giving a criterion based on linear bounds for the growth of the Castelnuovo-Mumford regularity of certain ideals. We further show that these linear bounds exist for one-dimensional rings and for ideals of (most) two-dimensional domains. We conclude by applying our technique to prove that all sets of F-jumping coefficients of all ideals in the determinantal ring given as the quotient by 2×2 minors in a 2×3 matrix of indeterminates form discrete sets.

1. INTRODUCTION

The aim of this paper is to establish the discreteness of the set of F-jumping coefficients of ideals in a certain class of graded F-finite rings which includes one-dimensional rings and two-dimensional domains. The remainder of this introductory section will review the prerequisite notions necessary to understand the problem at hand and the methods used to solve them.

All rings in this paper are commutative and have prime characteristic p. If S is such a ring and M is an S-module, for all $e \geq 0$ we may define an S-bisubmodule $F_*^e M$, which is identical to M as an Abelian group, and on which S acts on the right with the given action while the left action is given by $s \cdot a = s^{p^e} a$ for all $s \in S$ and $a \in M$. We shall further assume in this paper that all rings S are F-*finite*, i.e., that $F_*^e S$ are finitely generated left S-modules for all $e \geq 0$. Given a reduced ring S as above we also define a (non-commutative) graded algebra $\mathcal{C}_S = \bigoplus_{e \geq 0} \operatorname{Hom}_S(F_*^e S, S)$ where multiplication of $\phi \in \operatorname{Hom}_S(F_*^\alpha S, S)$ and $\psi \in \operatorname{Hom}_S(F_*^\beta S, S)$ is defined as $\phi \psi = \phi \circ F_*^\alpha(\psi) \in \operatorname{Hom}_S(F_*^{\alpha+\beta} S, S)$ (cf. [S11a, §3] and [Bli09, §2]). Note that $\mathcal{C}_S$ is an S-bimodule, as S acts on each $\operatorname{Hom}_S(F_*^\alpha S, S)$ on the left via its left action on $F_*^\alpha S$ and on the right via its right action on $F_*^\alpha S$.

When S is reduced we shall tacitly identify the inclusion $S \hookrightarrow F_*^e S$ given by $s \mapsto s \cdot 1 = s^{p^e}$ with the inclusion of rings $S \subset S^{1/p^e}$.

Definition 1.1. Given a ring S, an ideal $\mathfrak{a} \subseteq S$ and a positive real number t, we define the *test ideal* $\tau(\mathfrak{a}^t)$ to be the unique smallest non-zero ideal $J \subseteq R$ such that $\phi((\mathfrak{a}^{\lceil t(p^e-1) \rceil} J)^{1/p^e}) \subseteq J$ for all $e > 0$ and all $\phi \in \operatorname{Hom}_S(F_*^e S, S)$.

Our definition is inspired by the work of Karl Schwede (cf. [S11a]) and is equivalent to the original definition in [HY]. It is not immediately clear why the unique smallest non-zero ideal $J \subseteq R$ in Definition 1.1 should exist; its existence is guaranteed by [S11a, Theorem 3.18].

These generalized test ideals and their characteristic-zero counterparts, multiplier ideals, have recently attracted the attention of many algebraic geometers and algebraists. A specific direction of research aims to relate the F-jumping coefficients of ideals defined below to the geometrical properties of the varieties defined by these ideals.

Generalized test ideals satisfy the following basic properties:

Theorem 1.2 (cf. Remark 2.12 in [MTW] and Lemma 3.23 in [BSTZ]). *Let $\mathfrak{a}$ be any ideal.*

(a) *For all $0 < s < t$, $\tau(\mathfrak{a}^s) \supseteq \tau(\mathfrak{a}^t)$.*
(b) *For any $t > 0$ there exists an $\epsilon > 0$ such that $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^t)$ for all $t \leq c < t + \epsilon$.*

These two properties above suggest the following:

Definition 1.3. A positive real number t is an F-jumping coefficient of the ideal $\mathfrak{a}$ if $\tau(\mathfrak{a}^{t-\epsilon}) \supsetneq \tau(\mathfrak{a}^t)$ for all $\epsilon > 0$.

The study of the nature of the set of F-jumping coefficients of a given ideal has recently attracted intense attention. The question of the rationality and discreteness of these sets has recently been studied in a number of papers, for example [H], [BMS08], [BMS09], [KLZ], [S11b], [BSTZ], and [STZ]. In particular, the discreteness of F-jumping coefficients of an ideal in a non-$\mathbb{Q}$-Gorenstein ring remains open.

This paper links the question of the discreteness of sets of F-jumping coefficients with the notion of Castelnuovo-Mumford regularity, a classical numerical invariant. This notion has already played an important role in the study of rings of characteristic p. For example, in [K] the connection between the linear growth of Castelnuovo-Mumford regularity of Frobenius powers of ideals and tight closure was explored. Specifically, the following question was raised there:

Question 1.4 (Question 2 in [K]). Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k of characteristic $p > 0$ and let I, J be two homogeneous ideals of R. Does the Castelnuovo-Mumford regularity of $I + J^{[p^e]}$ grow linearly with p^e?

Surprisingly, even a positive answer to the above question in the case when I is principal will imply the discreteness of F-jumping coefficients, and this is one of our main results:

Theorem 3.3. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over an F-finite field $\mathbb{K}$ of prime characteristic p and $I = (g_1, \ldots, g_\nu)$ be a homogeneous ideal of R. Assume that $\mathrm{reg}(R/(I^{[p^e]} + g_i R)) \leq C p^e$ for all $e \geq 1$ and $1 \leq i \leq \nu$, where C is a constant independent of e and i. Then the sets of F-jumping coefficients of all ideals (homogeneous or not) in $S = R/I$ are discrete.*

This theorem is proved by showing that its hypothesis implies the gauge-boundedness of R/I– the discreteness of the sets of F-jumping coefficients is a corollary of this. We do not know whether the hypothesis of the theorem is equivalent to the gauge-boundedness of R/I.

In general, Question 1.4 is rather difficult, and not much progress has been made since it was first raised. In this paper, using a deep result of Eisenbud-Huneke-Ulrich in [EHU], we give a positive answer to Question 1.4 in a special case as follows:

Corollary 3.6. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over an F-finite field $\mathbb{K}$ of prime characteristic p and $I = (g_1, \ldots, g_\nu)$ be a homogeneous ideal of R. If $\dim(\operatorname{Sing}(R/(g_i)) \cap V(I)) \leq 1$ for all $1 \leq i \leq \nu$, then $\operatorname{reg}(R/(I^{[p^e]} + g_i R)) \leq Cp^e$ for all $e \geq 1$ and $1 \leq i \leq \nu$, where C is a constant independent of e and i. Consequently, the sets of F-jumping coefficients of all ideals (homogeneous or not) in $S = R/I$ are discrete.*

This corollary immediately implies the discreteness of the set of jumping coefficients in graded one-dimensional F-finite rings and (most) two-dimensional domains (Remark 3.7).

The last section in this paper shows the usefulness of our approach by applying Theorem 3.3 to a determinantal ring.

Corollary 4.6. *Let $R = \mathbb{K}[x_{ij}]$ with $1 \leq i \leq 2$, $1 \leq j \leq 3$. Let I be the ideal generated by the 2×2 minors of the matrix (x_{ij}). Then all sets of F-jumping coefficients of all ideals (homogeneous or not) in R/I are discrete.*

2. GAUGE BOUNDEDNESS AND CASTELNUOVO-MUMFORD REGULARITIES

In [Bli09], Manuel Blickle introduced a notion called *gauge boundedness*, which has particular significance to the work presented here, and which we now describe. Henceforth, let R denote the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over an F-finite field $\mathbb{K}$, let $\mathfrak{m}$ denote the ideal R generated by its variables, and for all $d \geq 0$ let R_d be the $\mathbb{K}$-vector subspace of R spanned by all monomials $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ with $0 \leq \alpha_1, \ldots, \alpha_n \leq d$. Given an R-module M generated by a finite set of elements $\{m_1, \ldots, m_k\}$, we define a filtration $\{M_d\}_{d \geq 0}$ of M by setting $M_d = R_d m_1 + \cdots + R_d m_k$ for $d \geq 0$ and $M_{-\infty} = \{0\}$. Having defined this we now obtain a *gauge* $\delta_M : M \to \mathbb{N} \cup -\infty$ defined as

$$\delta_M(m) = \begin{cases} -\infty, & \text{if } m = 0, \\ d, & \text{if } m \neq 0 \text{ and } m \in M_d \setminus M_{d-1}. \end{cases}$$

In particular any cyclic R-module has a natural gauge obtained by choosing as a set of generators the singleton consisting of the image of 1.

In this paper we shall call a quotient $S = R/I$ *gauge bounded* if there exists a set of homogeneous generators $\{\psi_\gamma \in \{\mathcal{C}_S\}_{e_\gamma}\}_{\gamma \in \Gamma}$ of $\mathcal{C}_S$ viewed as a right S-module such that for some constant K and all $r \in R$, $\delta_S(\psi_\gamma(r + I)) \leq \delta_S(r + I)p^{-e_\gamma} + K$ (cf. [Bli09, Definition 4.7]).

Theorem 2.1 (Corollary 4.16 in [Bli09]). *If $S = R/I$ is gauge bounded, the set of F-jumping coefficients of any ideal in S is discrete.*

Recall our assumption that R is F-finite; this amounts to the finiteness of the field extension $\mathbb{K} \subset \mathbb{K}^{1/p^e}$ for all (equivalently, some) $e \geq 0$. For $e \geq 0$ fix a $\mathbb{K}$-basis $\mathcal{B}_e$ of $\mathbb{K}^{1/p^e}$ and assume further that $1 \in \mathcal{B}_e$.

Recall (e.g., from section 1 of [F]) that $F_*^e R \cong \mathbb{K}^{1/p^e}[x_1^{1/p^e}, \ldots, x_n^{1/p^e}]$, and our assumption that R is F-finite implies that $F_*^e R$ is a free (left) R-module with free basis $\{bx_1^{\alpha_1/p^e} \ldots x_n^{\alpha_n/p^e} \mid b \in \mathcal{B}_e, 0 \leq \alpha_1, \ldots, \alpha_n < p^e\}$.

We introduce the following notation: for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ let x^α denote the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ in R and let x^{α/p^e} denote the monomial $x_1^{\alpha_1/p^e} \ldots x_n^{\alpha_n/p^e}$

in $F_*^e R \cong \mathbb{K}^{1/p^e}[x_1^{1/p^e}, \ldots, x_n^{1/p^e}]$. We shall also denote the condition $a \leq \alpha_1, \ldots, \alpha_n < b$ with $a \leq \alpha < b$ and the equalities $\alpha_1 = \cdots = \alpha_n = a$ with $\alpha = a$. For any $e \geq 0$ let $\Lambda_e = \{(\alpha_1, \ldots, \alpha_n) \,|\, 0 \leq \alpha_1, \ldots, \alpha_n < p^e\}$.

We define the *trace map* $T : F_*^e R \to R$ to be the projection onto the free summand $Rx_1^{(p^e-1)/p^e} \ldots x_n^{(p^e-1)/p^e}$ and recall that there is an isomorphism $F_*^e R \to \mathrm{Hom}_R(F_*^e R, R)$ sending r to the composition $T \circ \mu_{r^{1/p^e}}$, where $\mu_{r^{1/p^e}} : F_*^e R \to F_*^e R$ is given by multiplication by r^{1/p^e} on the right. For a quotient $S = R/I$ we have an explicit expression for $\mathcal{C}_S$: the results in [F, §1] imply that each $\mathrm{Hom}_S(F_*^e S, S)$ with its right S-module structure is isomorphic to $\dfrac{(I^{[p^e]} : I)}{I^{[p^e]}}\overline{T}$, where $a\overline{T}(r + I) = T(ar) + I$ for all $a \in (I^{[p^e]} : I)$.

Lemma 2.2. *The hypothesis of Theorem 2.1 will be satisfied if we can find a constant K and, for all $e \geq 0$, a set of generators $g_1, \ldots, g_{\nu_e}$ of $(I^{[p^e]} : I)$ such that $\delta_R(g_i) = \delta_S(g_i + I) \leq Kp^e$ for all $1 \leq i \leq g_{\nu_e}$.*

Proof. Fix any $e \geq 0$ and write $q = p^e$. First note that for any $r \in R$, $\delta_R(r) = \delta_{F_*^e R}(r)/q$.

Now pick $g = g_i$ for some $1 \leq i \leq \nu_e$ and write

$$g^{1/q} = \sum_{\alpha \in \Lambda_e, v \in \mathcal{B}_e} g_{\alpha,v} v x^{\alpha/q},$$

where each $g_{\alpha,v}$ is in R. For any $r^{1/q} \in \mathbb{K}^{1/p^e}[x_1^{1/p^e}, \ldots, x_n^{1/p^e}]$ write

$$r^{1/q} = \sum_{\beta \in \Lambda_e, w \in \mathcal{B}_e} r_{\beta,w} w x^{\beta/q},$$

where each $r_{\beta,w}$ is in R, and compute

$$\delta_R\left(T \circ \mu_{g^{1/q}}(r^{1/q})\right) = \delta_R\left(\sum_{\substack{\alpha, \beta \in \Lambda_e, v, w \in \mathcal{B}_e, \\ \alpha + \beta = q-1, vw \in \mathbb{K}}} g_{\alpha,v} r_{\beta,w}\right)$$

$$\leq \max\{\delta_R(g_{\alpha,v} r_{\beta,w}) \,|\, \alpha, \beta \in \Lambda_e, v, w \in \mathcal{B}_e, \alpha + \beta = q-1, vw \in \mathbb{K}\}.$$

Now each $\delta_R(g_{\alpha,v} r_{\beta,w})$ above is at most $\delta_R(g_{\alpha,v}) + \delta_R(r_{\beta,w})$ (cf. [Bli09, Lemma 4.1]). Then:

$$\delta_R(g_{\alpha,v}) = \delta_{F_*^e R}(g_{\alpha,v})/q \leq \delta_{F_*^e R}\left(g_{\alpha,v} v x^{\alpha/q}\right)/q \leq \delta_{F_*^e R}\left(g^{1/q}\right)/q \leq K$$

and

$$\delta_R(r_{\alpha,w}) = \delta_{F_*^e R}(r_{\alpha,w})/q \leq \delta_{F_*^e R}\left(r_{\alpha,w} w x^{\beta/q}\right)/q \leq \delta_{F_*^e R}\left(r^{1/q}\right)/q;$$

hence

$$\delta_R\left(T \circ \mu_{g^{1/q}}(r^{1/q})\right) \leq K + \delta_{F_*^e R}\left(r^{1/q}\right)/q.$$

$\square$

Remark 2.3. It is clear that, as the gauge of any element in R is at most its degree, the condition in Lemma 2.2 will be satisfied if there is a constant K such that the maximal degree of all minimal generators of $(I^{[p^e]} : I)$ is bounded by Kp^e. If $I \subseteq R$ is a homogeneous ideal, then the maximal degree of all minimal generators of $(I^{[p^e]} : I)$ is bounded by the Castelnuovo-Mumford regularity of $(I^{[p^e]} : I)$.

If $I \subseteq R$ is not a homogeneous ideal, we may consider the homogenization $h(I)$ of I with respect to a new variable. If the maximal degree of all minimal generators of $((h(I))^{[p^e]} : h(I))$ is bounded by Kp^e, then by de-homogenizing one can see that $(I^{[p^e]} : I)$ can be generated by polynomials whose degrees are bounded by Kp^e.

We shall assume henceforth that $I \subseteq R$ is a homogeneous ideal. Following Remark 2.3, in order to find a uniform K (independent of e) such that the maximal degree of a minimal generator of $(I^{[p^e]} : I)$ is bounded by Kp^e, we shall attempt a harder problem, namely, the bounding of the Castelnuovo-Mumford regularity of these homogenous ideals. But first very briefly we review the notion of Castelnuovo-Mumford regularity and some of its properties.

Recall that the Castelnuovo-Mumford regularity of a finitely generated graded R-module M (which we shall denote $\operatorname{reg}(M)$) is defined in terms of its minimal graded resolutions $F_\bullet$. To wit, for all $i \geq 0$ write $F_i = \bigoplus_{j=1}^{\operatorname{rank}(F_i)} R(-d_{ij})$, where $R(-d)$ denotes the degree shift $R(-d)_j = R_{d+j}$; $\operatorname{reg}(M)$ is then defined as $\max\{d_{ij} - i \,|\, i \geq 0, 1 \leq j \leq \operatorname{rank}(F_i)\}$. Note that the shifts $d_{0,j}$ are the degrees of a set of homogeneous minimal generators of M; hence the maximal degree of a homogenous minimal generator of M is bounded by $\operatorname{reg}(M)$.

Alternatively, one may define Castelnuovo-Mumford regularity in terms of local cohomology. If $M = \bigoplus M_d$ is a graded Artinian R-module, we have $\operatorname{reg} M = \max\{d \,|\, M_d \neq 0\}$, and for a general graded R-module we have

$$\operatorname{reg} M = \max_{i \geq 0}\{\operatorname{reg} \operatorname{Tor}_i(M, R/\mathfrak{m}) - i\} = \max_{j \geq 0}\{\operatorname{reg} \operatorname{H}^j_\mathfrak{m}(M) + j\}$$

(cf. [E05, Corollary 4.5]). Two immediate consequences of this characterization of regularity are the fact that if $M_1 \to M_2 \to M_3$ is a graded exact sequence of Artinian R-modules, then $\operatorname{reg} M_2 \leq \max\{\operatorname{reg} M_1, \operatorname{reg} M_3\}$, and if $0 \to N_1 \to N_2 \to N_3 \to 0$ is a graded short exact sequence of R-modules, then $\operatorname{reg} N_2 \leq \max\{\operatorname{reg} N_1, \operatorname{reg} N_3\}$ (cf. [E95, Corollary 20.19]).

The rest of this paper will explore instances of graded ideals $I \subseteq R$ for which there exists a constant K such that $\operatorname{reg}(I^{[p^e]} : I) \leq Kp^e$ for all $e \geq 0$; these ideals will satisfy the hypothesis of Lemma 2.2 and we will be able to deduce the discreteness of the sets of F-jumping coefficients of all ideals in $S = R/I$. For the sake of readability we shall abbreviate the expression of the condition above as $\operatorname{reg}(I^{[p^e]} : I) = \mathcal{O}(p^e)$. We can now claim the following.

Corollary 2.4. *If $\operatorname{reg}(I^{[p^e]} : I) = \mathcal{O}(p^e)$, then $R = S/I$ is gauge-bounded. Hence the sets of jumping coefficients of all ideals of R are discrete.*

3. The main results

In this section we establish the conditions of Corollary 2.4 on the growth of regularity in some interesting cases.

Throughout this section we fix a minimal set of homogeneous generators $\{g_1, \ldots, g_\nu\}$ for the homogenous ideal $I \subseteq R$. We denote $d_i = \deg g_i$ for all $1 \leq i \leq \nu$ and we define the graded module

$$B = \frac{\bigoplus_{i=1}^\nu R(d_i)}{R\Gamma},$$

where

$$\Gamma = \begin{pmatrix} g_1 \\ \vdots \\ g_\nu \end{pmatrix},$$

together with its graded free resolution $F_\bullet$,

$$0 \to R \xrightarrow{\Gamma} \bigoplus_{i=1}^{\nu} R(d_i) \to 0.$$

For any $e \geq 0$ we may use this graded resolution to compute $T_j^e = \mathrm{Tor}_j^R(R/I^{[p^e]}, B)$, and we obtain $T_1^e = (I^{[p^e]} : I)/I^{[p^e]}$, $T_0^e = B \otimes R/I^{[p^e]}$ and $T_j^e = 0$ for all $j > 1$. Since

$$\frac{R/I^{[p^e]}}{(I^{[p^e]} : I)/I^{[p^e]}} \cong \frac{R}{(I^{[p^e]} : I)}$$

and

$$\mathrm{reg}\, \frac{R/I^{[p^e]}}{(I^{[p^e]} : I)/I^{[p^e]}} = \mathrm{reg}(I^{[p^e]} : I)/I^{[p^e]} + 1,$$

we deduce that $\mathrm{reg}\,(I^{[p^e]} : I) = \mathrm{reg}\, \frac{R}{(I^{[p^e]}:I)} - 1 = \mathcal{O}(p^e)$ if and only if $\mathrm{reg}\, T_1^e = \mathcal{O}(p^e)$.

The one-dimensional case is straightforward.

Theorem 3.1. *If* $\dim R/I = 1$, *all sets of F-jumping coefficients of ideals in R/I are discrete.*

Proof. In view of Corollary 2.4 and the discussion above, we need to show that $\mathrm{reg}\, T_1^e = \mathcal{O}(p^e)$.

Consider the short exact sequences $0 \to T_1^e \to R/I^{[p^e]} \to C \to 0$, where C is the cokernel of the first non-zero map and $0 \to C \to \bigoplus_{i=1}^{\nu} R/I^{[p^e]}(d_i) \to T_0^e \to 0$. These induce long exact sequences

(1) $$0 \to \mathrm{H}_{\mathfrak{m}}^0(T_1^e) \to \mathrm{H}_{\mathfrak{m}}^0(R/I^{[p^e]}) \to \mathrm{H}_{\mathfrak{m}}^0(C) \to \dots$$

and

(2) $$0 \to \mathrm{H}_{\mathfrak{m}}^0(C) \to \bigoplus_{i=1}^{\nu} \mathrm{H}_{\mathfrak{m}}^0(R/I^{[p^e]})(d_i) \to \mathrm{H}_{\mathfrak{m}}^0(T_0^e) \to \dots.$$

Note that since R is regular, one may obtain a minimal graded free resolution of $R/I^{[p^e]}$ by applying the Frobenius functor to a minimal graded free resolution of R/I, and hence $\mathrm{reg}\, R/I^{[p^e]} = \mathcal{O}(p^e)$. The inclusions $\mathrm{H}_{\mathfrak{m}}^0(T_1^e) \hookrightarrow \mathrm{H}_{\mathfrak{m}}^0(R/I^{[p^e]})$ and $\mathrm{H}_{\mathfrak{m}}^0(C) \hookrightarrow \mathrm{H}_{\mathfrak{m}}^0(\bigoplus_{i=1}^{\nu} R/I^{[p^e]})(d_i)$ now imply $\mathrm{reg}\, \mathrm{H}_{\mathfrak{m}}^0(T_1^e) = \mathcal{O}(p^e)$ and $\mathrm{reg}\, \mathrm{H}_{\mathfrak{m}}^0(C) = \mathcal{O}(p^e)$. The last equality in turn implies that $\mathrm{reg}\, \mathrm{H}_{\mathfrak{m}}^1(T_1^e) = \mathcal{O}(p^e)$. If $\dim R/I = 1$, these calculations show that $\mathrm{reg}\, T_1^e = \mathcal{O}(p^e)$ and the result follows. $\square$

We now tackle the general case. We fix homogeneous elements $u_1, \dots, u_N \in R$ of degrees $d_1, \dots, d_N$. For $1 \leq k \leq N$ let

$$U_k = \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} \in R^k.$$

For all $1 \leq k \leq N$ we can define the graded module $B_k = \frac{\bigoplus_{i=1}^{\nu} R(d_i)}{RU_k}$ together with its graded free resolution

$$0 \to R \xrightarrow{U_k} \bigoplus_{i=1}^{k} R(d_i) \to 0.$$

For any $e \geq 0$ we may use this graded resolution to compute $T_{kj}^e = \operatorname{Tor}_j^R(R/I^{[p^e]}, B_k)$. Just as in the previous theorem we have two short exact sequences, $0 \to T_{k1}^e \to R/I^{[p^e]} \to C_k \to 0$ and $0 \to C_k \to \bigoplus_{i=1}^{k} R/I^{[p^e]}(d_i) \to T_{k0}^e \to 0$, which induce long exact sequences

$$(3) \qquad \cdots \to \operatorname{H}_{\mathfrak{m}}^j(T_{k1}^e) \to \operatorname{H}_{\mathfrak{m}}^j(R/I^{[p^e]}) \to \operatorname{H}_{\mathfrak{m}}^j(C_k) \to \cdots$$

and

$$(4) \qquad \cdots \to \operatorname{H}_{\mathfrak{m}}^j(C_k) \to \bigoplus_{i=1}^{k} \operatorname{H}_{\mathfrak{m}}^j(R/I^{[p^e]})(d_i) \to \operatorname{H}_{\mathfrak{m}}^j(T_{k0}^e) \to \cdots.$$

These imply that $\operatorname{reg} \operatorname{H}_{\mathfrak{m}}^0(C_k) = \mathcal{O}(p^e)$ and $\operatorname{reg} \operatorname{H}_{\mathfrak{m}}^1(T_{k1}^e) = \mathcal{O}(p^e)$. Set J_k to be the ideal in R generated by $u_1, \ldots, u_k$ for all $1 \leq k \leq N$; we have $T_{k1}^e = (I^{[p^e]} : J_k)/I^{[p^e]}$. Hence from the short exact sequence $0 \to (I^{[p^e]} : J_k)/I^{[p^e]} \to R/I^{[p^e]} \to R/(I^{[p^e]} : J_k) \to 0$ we can deduce that

$$(5) \qquad \operatorname{reg} \operatorname{H}_{\mathfrak{m}}^1\left(R/(I^{[p^e]} : J_k)\right) = \mathcal{O}(p^e).$$

Lemma 3.2. *Assume that* $\operatorname{reg}(R/(I^{[p^e]} + u_i R)) = \mathcal{O}(p^e)$ *for all* $1 \leq i \leq N$. *Then for all* $1 \leq k \leq N$ *and all* $j \geq 0$

$$\operatorname{reg} \operatorname{H}_{\mathfrak{m}}^j\left(\frac{\bigoplus_{i=1}^{k}(R/I^{[p^e]})(d_i)}{R/I^{[p^e]}U_k}\right) = \mathcal{O}(p^e).$$

Proof. We proceed by induction on j. First we prove the case when $j = 0$ which we will do by induction on $1 \leq k \leq N$, the case $k = 1$ being immediate from the hypothesis. Assume that $k > 1$ and that the lemma holds for all smaller values of k. Fix $e \geq 0$ and abbreviate $A = R/I^{[p^e]}$. We apply the induction hypothesis to the graded short exact sequence

$$(6) \quad 0 \to \frac{AU_{k-1} \oplus Au_k}{AU_k} \to \frac{\bigoplus_{i=1}^{k} A(d_i)}{AU_k} \to \left(\frac{\bigoplus_{i=1}^{k-1} A(d_i)}{AU_{k-1}}\right) \oplus \frac{A}{Au_k}(d_k) \to 0$$

and reduce the problem to the bounding of the regularity of

$$\operatorname{H}_{\mathfrak{m}}^0\left(\frac{AU_{k-1} \oplus Au_k}{AU_k}\right).$$

For all $1 \leq j \leq k$ let $J_j = u_1 R + \cdots + u_j R$. We have a graded map

$$\phi : A \to \frac{AU_{k-1} \oplus Au_k}{AU_k},$$

which sends a to the image of $aU_{k-1} \oplus 0$ in $\frac{AU_{k-1} \oplus Au_k}{AU_k}$. Note that in $\frac{AU_{k-1} \oplus Au_k}{AU_k}$ we always have

$$(u_1, \ldots, u_{k-1}, 0) + (0, \ldots, 0, u_k) = 1 \cdot (u_1, \ldots, u_k) = 1 \cdot U_k = 0.$$

Hence, for any given element $(au_1, \ldots, au_{k-1}, a'u_k) \in \frac{AU_{k-1} \oplus Au_k}{AU_k}$, we have

$$(au_1, \ldots, au_{k-1}, a'u_k) = a \cdot (u_1, \ldots, u_{k-1}, 0) + a' \cdot (0, \ldots, 0, u_k)$$
$$= a \cdot (u_1, \ldots, u_{k-1}, 0) + (-a') \cdot (u_1, \ldots, u_{k-1}, 0)$$
$$= (a - a') \cdot (u_1, \ldots, u_{k-1}, 0) = \phi(a - a'),$$

and so ϕ is surjective. Also note that

$$\ker \phi = \left\{ a \mid \begin{pmatrix} au_1 \\ \vdots \\ au_{k-1} \\ 0 \end{pmatrix} = z \begin{pmatrix} u_1 \\ \vdots \\ u_{k-1} \\ u_k \end{pmatrix} \text{ for some } z \in A \right\}$$
$$= \left((I^{[p^e]} : u_1)A \cap \cdots \cap (I^{[p^e]} : u_{k-1})A \right) + (I^{[p^e]} : u_k)A$$
$$= \left(I^{[p^e]} : J_{k-1} \right) A + (I^{[p^e]} : u_k)A.$$

Using the graded isomorphism $A/\ker \phi \cong \dfrac{AU_{k-1} \oplus Au_k}{AU_k}$ we reduce the problem to the bounding of the regularity of

$$\mathrm{H}_{\mathfrak{m}}^0 \left(\frac{A}{(I^{[p^e]} : J_{k-1}R)A + (I^{[p^e]} : u_k R)A} \right).$$

Now the short exact sequence

$$0 \to \frac{A}{(I^{[p^e]} : J_k)A} \to \frac{A}{(I^{[p^e]} : J_{k-1})A} \oplus \frac{A}{(I^{[p^e]} : u_k R)A}$$
$$\to \frac{A}{(I^{[p^e]} : J_{k-1})A + (I^{[p^e]} : u_k R)A} \to 0$$

yields the exact sequence

(7)
$$\mathrm{H}_{\mathfrak{m}}^i \left(\frac{A}{(I^{[p^e]} : J_{k-1})A} \right) \oplus \mathrm{H}_{\mathfrak{m}}^i \left(\frac{A}{(I^{[p^e]} : u_k R)A} \right)$$
$$\to \mathrm{H}_{\mathfrak{m}}^i \left(\frac{A}{(I^{[p^e]} : J_{k-1}R)A + (I^{[p^e]} : u_k R)A} \right)$$
$$\to \mathrm{H}_{\mathfrak{m}}^{i+1} \left(\frac{A}{(I^{[p^e]} : J_k R)A} \right),$$

which (with $i = 0$) establishes $\operatorname{reg} \mathrm{H}_{\mathfrak{m}}^0 \left(A/ \left(I^{[p^e]} : J_{k-1}R \right) A + (I^{[p^e]} : u_k R)A \right) = \mathcal{O}(p^e)$ because of our induction hypothesis and (5). This concludes the proof of the initial step of our induction, namely, that of $j = 0$.

Assume that we have proved that

$$\operatorname{reg} \mathrm{H}_{\mathfrak{m}}^l \left(\frac{\bigoplus_{i=1}^k (R/I^{[p^e]})(d_i)}{R/I^{[p^e]}U_k} \right) = \mathcal{O}(p^e)$$

for $0 \le l \le j - 1$ and for all $1 \le k \le N$. From the exact sequence (4), we can see that

$$\operatorname{reg} \mathrm{H}_{\mathfrak{m}}^\gamma (C_k) = \mathcal{O}(p^e)$$

for all $0 \leq \gamma \leq j$ and for all $1 \leq k \leq N$. Then from the exact sequence (3), we can see that

$$\operatorname{reg} \operatorname{H}_{\mathfrak{m}}^{\delta}(T_{k1}^e) = \mathcal{O}(p^e)$$

for all $0 \leq \delta \leq j+1$ and for all $1 \leq k \leq N$, i.e.

$$\operatorname{reg} \operatorname{H}_{\mathfrak{m}}^{\delta}\left(\frac{R}{(I^{[p^e]} : J_k)}\right) = \mathcal{O}(p^e)$$

for all $0 \leq \delta \leq j+1$.

Now we will prove that $\operatorname{H}_{\mathfrak{m}}^{j}\left(\frac{\bigoplus_{i=1}^{k}(R/I^{[p^e]})(d_i)}{R/I^{[p^e]}U_k}\right) = \mathcal{O}(p^e)$ by using induction on k.

When $k = 1$, this is precisely our assumption that $\operatorname{reg}\left(R/(I^{[p^e]} + u_i R)\right) = \mathcal{O}(p^e)$ for all $1 \leq i \leq N$. Now the induction hypothesis applied to the exact sequence (6) reduces our proof to bounding the regularity of

$$\operatorname{H}_{\mathfrak{m}}^{i}\left(\frac{AU_{k-1} \oplus Au_k}{AU_k}\right),$$

for all $0 \leq i \leq j$. The exact sequence (7) finishes the proof. $\qquad \square$

Theorem 3.3. *If* $\operatorname{reg}\left(R/(I^{[p^e]} + g_i R)\right) = \mathcal{O}(p^e)$ *for all* $1 \leq i \leq \nu$*, then all sets of F-jumping coefficients of all ideals in R/I are discrete.*

Proof. Apply Lemma 3.2 with $N = \nu$ and $u_k = g_k$ for all $1 \leq k \leq \nu$ and deduce that $\operatorname{reg} T_1^e = \mathcal{O}(p^e)$. $\qquad \square$

Remark 3.4. The non-graded case may be obtained by combining Remark 2.3 and Theorem 3.3. For an arbitrary radical ideal $J = (g_1, \ldots, g_\nu)$ in R consider the homogenizations $h(I)$ and $h(g_i)$ with respect to a new variable Z. If $\operatorname{reg}\left(R[Z]/(h(J)^{[p^e]} + h(g_i)R[Z])\right) = \mathcal{O}(p^e)$ for all $1 \leq i \leq \nu$, then R/J would be gauge bounded and thus all sets of F-jumping coefficients of all ideals in R/J would be discrete.

Lemma 3.5 (cf. section 6 in [K] and Theorem 4.2 in [C])**.** *Let T be a standard graded algebra of prime characteristic p and let M be a finitely generated T-module. Let $F_T(-) = (-) \otimes_T F_*^e T$ denote the Frobenius functor. Let $\operatorname{Sing} T$ denote the singular locus of T. If $\dim\left(\operatorname{Sing} T \cap \operatorname{Supp} M\right) \leq 1$, then $\operatorname{reg} F_T^e(M) = \mathcal{O}(p^e)$.*

Proof. This follows from Theorem 4.2 in [C]. $\qquad \square$

Corollary 3.6. *If* $\dim\left(\operatorname{Sing}(R/g_i R) \cap V(I)\right) \leq 1$ *for all* $1 \leq i \leq \nu$*, then* $\operatorname{reg}(R/(I^{[p^e]} + g_i R)) = \mathcal{O}(p^e)$ *for all* $1 \leq i \leq \nu$*. Consequently, the sets of F-jumping coefficients of all ideals (homogeneous or not) in $S = R/I$ are discrete.*

Proof. Applying Lemma 3.5 to $T = R/g_i R$ and $M = R/I$ for all $1 \leq i \leq \nu$, we have that $\operatorname{reg} R/(I^{[p^e]} + u_i R) = \mathcal{O}(p^e)$. Now Theorem 3.3 finishes the proof. $\qquad \square$

Remark 3.7. The assumption that $\dim\left(\operatorname{Sing}(R/g_i R) \cap V(I)\right) \leq 1$ for all $1 \leq i \leq \nu$ does not seem to be very restrictive. For instance, when $\dim(R/I) = 2$, if I happens to be a prime ideal and p does not divide the degree of any one of the generators $\{g_1, \ldots, g_\nu\}$, then $\dim\left(\operatorname{Sing}(R/(g_i)) \cap V(I)\right) \leq 1$ is automatically satisfied.

4. An application

Throughout this section we fix the following notation: $R = \mathbb{K}[x, y, z, u, v, w]$, $g_1 = yu - xv$, $g_2 = zu - xw$, $g_3 = zv - yw$, and $S = R/g_3R$. In this section we apply Theorem 3.3 to the determinantal ring $R/(g_1R + g_2R + g_3R)$. Our main aim is to illustrate the usefulness of this theorem with a relatively simple example; however, the fact that this determinantal ring is gauge bounded is also shown in [KSS] using direct methods. Our method is perhaps less computational and might be more generalizable. Although the argument below might seem to involve formidable computational insight, it is really the formalization of results inspired by computations with Macaulay2 [GH]. The main technical result in this section is a description of resolutions over a hypersurface (which are eventually periodic of period two (cf. [E80])), one such for each value of $e \geq 0$. In many cases, including other determinantal rings, the computation of these resolutions for small values of e reveals a pattern which can be formally described using methods similar to the those used below.

Back to our example, we fix the reverse lexicographical order on the monomials of R with $x > y > z > u > v > w$ and note that the terms of g_1, g_2 and g_3 were listed in descending order.

We now also fix $q = p^e$ for some e; the bulk of this section will be devoted to the bounding of $\operatorname{reg} R/(g_1R + g_2^qR + g_3^qR)$ and we do so by computing an explicit free resolution of the S-module $S/(g_2^qS + g_3^qS)$.

For all $0 \leq j \leq q$ define $h_j = x^j z^q u^{q-j} v^j - x^q y^j w^q$; note that $h_0 = g_2^q$ and $h_q = x^q g_3^q$.

Lemma 4.1.

(a) *For any $a, b \in R$, let $\mathcal{S}(a, b)$ denote the S-polynomial of a and b.*
    (i) *For all $0 \leq i \leq j \leq q$ we have*

$$\mathcal{S}(h_j, h_i) = u^{j-i} h_j - x^{j-i} v^{j-i} h_i \quad = \quad x^q w^q (x^{j-i} y^i v^{j-i} - y^j u^{j-i})$$

$$= \quad -x^q w^q \sum_{k=1}^{j-i} x^{k-1} y^{j-k} u^{j-i-k} v^{k-1} g_1.$$

    (ii) *For all $0 \leq j \leq q$ we have*

$$\mathcal{S}(h_j, g_3^q) = v^{q-j} h_j - x^j u^{q-j} g_3^q \quad = \quad w^q (x^j y^q u^{q-j} - x^q y^j v^{q-j})$$

$$= \quad w^q \sum_{k=1}^{q-j} x^{j-1+k} y^{q-k} u^{q-j-k} v^{k-1} g_1.$$

    (iii) *For all $0 \leq j \leq q-2$ we have $\mathcal{S}(h_j, g_1) = yh_j - x^j z^q u^{q-j-1} v^j g_1 = h_{j+1}$.*
    (iv) *$\mathcal{S}(h_{q-1}, g_1) = yh_{q-1} - x^{q-1} z^q v^{q-1} g_1 = x^q g_3^q$.*
    (v) *$\mathcal{S}(g_3^q, g_1) = yug_3^q - z^q v^q g_1$.*

(b) *Let G be the row vector $[h_0, \ldots, h_{q-1}, g_3^q, g_1]$. Then the entries in G form a Gröbner basis for the ideal $g_1R + g_2^qR + g_3^qR$.*

(c) *$G = [g_2^q, g_3^q, g_1]T$, where T is a $3 \times (q+2)$ matrix of the form*

$$\begin{bmatrix} 1 & y & y^2 & & y^{q-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & * & * & & * & 0 & 1 \end{bmatrix}.$$

(d) *For all $1 \leq \ell \leq q + 2$, let $\mathbf{e}_\ell$ denote the ℓth-standard vector in R^{q+2}. The module of syzygies of G is generated by vectors of the form*
   (i) $V_{i,j} = x^{j-i}v^{j-i}\mathbf{e}_i - u^{j-i}\mathbf{e}_j + (\star)\mathbf{e}_{q+2}$ *for all $1 \leq i < j \leq q$,*
   (ii) $V_{i,q+1} = v^{q-i+1}\mathbf{e}_i - x^i u^{q-i}\mathbf{e}_{q+1} + (\star)\mathbf{e}_{q+2}$ *for all $1 \leq i < q + 1$,*
   (iii) $V_{i,q+2} = y\mathbf{e}_i - \mathbf{e}_{i+1} + (\star)\mathbf{e}_{q+2}$ *for all $1 \leq i < q$,*
   (iv) $V_{q,q+2} = y\mathbf{e}_q - x^q\mathbf{e}_{q+1} + (\star)\mathbf{e}_{q+2}$,
   (v) $V_{q+1,q+2} = g_1\mathbf{e}_{q+1} - g_3^q\mathbf{e}_{q+2}$,
   where $(\star)$'s denote unspecified elements in R.

Proof. Statements (a)(i) and (a)(ii) follow from the telescoping sums

$$\sum_{k=1}^{j-i} x^{k-1}y^{j-k}u^{j-i-k}v^{k-1}(yu - xv) = y^j u^{j-i} - x^{q+j-i}y^i v^{j-i}$$

and

$$\sum_{k=1}^{q-j} x^{k+j-1}y^{q-k}u^{q-k-j}v^{k-1}(yu - xv) = x^j y^q u^{q-j} - x^q y^j v^{q-j},$$

respectively. The rest of the statements in (a) follow from straightforward calculations.

We now know that the S-polynomials between the elements of G reduce to zero with respect to the elements of G, and hence (b) follows from Buchberger's Theorem (cf. [AL, §1.7]).

Part (c) follows inductively from the fact that $h_{j+1} = \mathcal{S}(h_j, g_1) \equiv yh_j$ modulo g_1.

Part (d) follows [AL, Theorem 3.4.1] and the corresponding calculation in part (a). □

Lemma 4.2. *The module of syzygies of the image of (g_2^q, g_3^q) in S is generated by the vectors*

$$\begin{bmatrix} y^j v^{q-j} \\ x^j u^{q-j} \end{bmatrix}$$

with $0 \leq j \leq q$.

Proof. We apply [AL, Theorem 3.4.3] and conclude that the module of syzygies of (g_2^q, g_3^q, g_1) is generated by $TV_{i,j}$ for all $1 \leq i < j \leq q + 2$, with the vectors $V_{i,j}$ as in the previous lemma.

(i) For all $1 \leq i < j \leq q$

$$TV_{i,j} = \begin{bmatrix} y^{i-1}x^{j-i}v^{j-i} - y^{j-1}u^{j-i} \\ 0 \\ * \end{bmatrix} \equiv \begin{bmatrix} y^{i-1}y^{j-i}u^{j-i} - y^{j-1}u^{j-i} \\ 0 \\ * \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix} \pmod{g_1}.$$

(ii) For all $1 \leq i < q + 1$,

$$TV_{i,q+1} = \begin{bmatrix} y^{i-1}v^{q-i+1} \\ -x^i u^{q-i} \\ * \end{bmatrix}.$$

(iii) For all $1 \leq i < q$

$$TV_{i,q+2} = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}.$$

(iv)

$$TV_{q,q+2} = \begin{bmatrix} y^q \\ -x^q \\ * \end{bmatrix}.$$

(v)

$$TV_{q+1,q+2} = \begin{bmatrix} 0 \\ g_1 \\ * \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix} \quad (\mathrm{mod}\ g_1).$$

$\square$

Lemma 4.3. *Let*

$$M = \begin{bmatrix} v^q & yv^{q-1} & \cdots & y^j v^{q-j} & \cdots & y^q \\ -u^q & -xu^{q-1} & \cdots & -x^j u^{q-j} & \cdots & -x^q \end{bmatrix}.$$

The module of syzygies of the columns of M viewed as vectors in S^2 is generated by the vectors $W_i = x\mathbf{e}_i - u\mathbf{e}_{i+1}$ and $U_i = y\mathbf{e}_i - v\mathbf{e}_{i+1}$ for all $1 \leq i \leq q$, where $\mathbf{e}_\ell$ denote the elementary vectors in R^{q+1}.

Proof. We extend our monomial order from R to R^2 by using its term-over-position extension (cf. [AL, §3.5]). For all $0 \leq i < j \leq q$ we compute the following S-polynomials:

$$\mathcal{S}\left(\begin{bmatrix} y^j v^{q-j} \\ -x^j u^{q-j} \end{bmatrix}, \begin{bmatrix} y^i v^{q-i} \\ -x^i u^{q-i} \end{bmatrix} \right) = u^{j-i} \left(\begin{bmatrix} y^j v^{q-j} \\ -x^j u^{q-j} \end{bmatrix} \right) - x^{j-i} \left(\begin{bmatrix} y^i v^{q-i} \\ -x^i u^{q-i} \end{bmatrix} \right)$$

$$= \sum_{k=1}^{j-1} x^{k-1} y^{j-k} u^{j-i-k} v^{q-j+k-1} \begin{bmatrix} g_1 \\ 0 \end{bmatrix},$$

where the last equality follows from the telescopic series

$$\sum_{k=1}^{j-i} x^{k-1} y^{j-k} u^{j-i-k} v^{q-j+k-1} g_1 = y^j u^{j-i} v^{q-j} - x^{j-i} y^i v^{q-i}.$$

For all $0 \leq i \leq q$ we have $\mathcal{S}\left(\begin{bmatrix} y^i v^{q-i} \\ -x^i u^{q-i} \end{bmatrix}, \begin{bmatrix} g_1 \\ 0 \end{bmatrix} \right) = 0$. For all $0 \leq i < q$

$$\mathcal{S}\left(\begin{bmatrix} y^i v^{q-i} \\ -x^i u^{q-i} \end{bmatrix}, \begin{bmatrix} 0 \\ g_1 \end{bmatrix} \right) = y \begin{bmatrix} y^i v^{q-i} \\ -x^i u^{q-i} \end{bmatrix} + x^i u^{q-i-1} \begin{bmatrix} 0 \\ g_1 \end{bmatrix}$$

$$= v \begin{bmatrix} y^{i+1} v^{q-i-1} \\ -x^{i+1} u^{q-i-1} \end{bmatrix}$$

and

$$\mathcal{S}\left(\begin{bmatrix} y^q \\ -x^q \end{bmatrix}, \begin{bmatrix} 0 \\ g_1 \end{bmatrix} \right) = yu \begin{bmatrix} y^q \\ -x^q \end{bmatrix} + x^q \begin{bmatrix} 0 \\ g_1 \end{bmatrix} = xv \begin{bmatrix} y^q \\ -x^q \end{bmatrix} + y^q \begin{bmatrix} g_1 \\ 0 \end{bmatrix}.$$

We can now conclude that the columns of M viewed as vectors in R^2 together with the vectors $\begin{bmatrix} g_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ g_1 \end{bmatrix}$ form a Gröbner basis. We may now apply

([AL, Theorem 3.7.3]) and list the syzygies among the columns in M viewed as elements in S^2 as $u^{j-i}\mathbf{e}_j - x^{j-i}\mathbf{e}_i$ for all $1 \le i < j \le q$, and $y\mathbf{e}_i - v\mathbf{e}_{i+1}$.

We now note that for all $1 \le i < j \le q$,

$$u^{j-i}\mathbf{e}_j - x^{j-i}\mathbf{e}_i = -\sum_{k=1}^{j-i} x^{k-1}u^{j-i-k}W_j,$$

and the result follows. $\qquad\square$

Theorem 4.4. *The S-module $S/(g_2^q S + g_3^q S)$ has a minimal free resolution*

$$\dots \xrightarrow{\phi_5} S^{2q}(-3q-2k) \xrightarrow{\phi_4} S^{2q}(-3q-2k+1)$$

$$\xrightarrow{\phi_5} \dots \xrightarrow{\phi_5} S^{2q}(-3q-2) \xrightarrow{\phi_4} S^{2q}(-3q-1)$$

$$\xrightarrow{\phi_3} S^{q+1}(-3q) \xrightarrow{\phi_2} S^2(-2q) \xrightarrow{\phi_1} S^1 \to 0.$$

Proof. We take $\phi_1 = [g_2^q, g_3^q]$; the matrix ϕ_2 is then the one whose columns are described in the statement of Lemma 4.2 and the matrix ϕ_3 is the one whose columns consist of the first $q+1$ coordinates of the vectors U_i, W_i $(1 \le i \le q)$ described in the statement of Lemma 4.3.

The syzygies on the columns of ϕ_3 arise from the relations $xU_i - yW_i \equiv 0$ (mod g_1) and $uU_i - vW_i \equiv 0$ (mod g_1) for all $1 \le i \le q$, and so the columns of ϕ_4 are given by $x\mathbf{e}_{2j-1} - y\mathbf{e}_{2j}$ and $u\mathbf{e}_{2j-1} - v\mathbf{e}_{2j}$ for all $1 \le j \le q$, where $\mathbf{e}_\ell$ denotes the ℓth elementary vector in S^{2q}.

The syzygies on the columns of ϕ_4 arise from the relations $u(x\mathbf{e}_{2j-1} - y\mathbf{e}_{2j}) - x(u\mathbf{e}_{2j-1} - v\mathbf{e}_{2j})$ and $v(x\mathbf{e}_{2j-1} - y\mathbf{e}_{2j}) - y(u\mathbf{e}_{2j-1} - v\mathbf{e}_{2j})$ for all $1 \le j \le q$, so the columns of ϕ_5 are given by $u\mathbf{e}_{2j-1} - x\mathbf{e}_{2j}$ and $v\mathbf{e}_{2j-1} - y\mathbf{e}_{2j}$ for all $1 \le j \le q$.

The syzygies on the columns of ϕ_5 are now given by the columns of ϕ_4; hence at this point we get a period-2 linear resolution. $\qquad\square$

Remark 4.5 (Base change for Tor). If $A \to B$ is a ring homomorphism, then there is a homology spectral sequence ([W, Theorem 5.6.6])

$$E_{i,j}^2 = \operatorname{Tor}_i^B(\operatorname{Tor}_j^A(M, B), N) \Rightarrow \operatorname{Tor}_{i+j}^A(M, N)$$

for each A-module M and each B-module N.

When $B = A/fA$, where f is a non-zero-divisor in A, it is clear that $\operatorname{Tor}_i^A(M, B) = 0$ except for $i = 0, 1$. Also, $\operatorname{Tor}_0^A(M, B) = M/fM$ and $\operatorname{Tor}_1^A(M, B) = \operatorname{ann}_M f$ and the spectral sequence gives rise to a long exact sequence

(8)
$$\cdots \to \operatorname{Tor}_{i-1}^B(\operatorname{ann}_M f, N) \to \operatorname{Tor}_i^A(M, N) \to \operatorname{Tor}_i^B(M/fM, N)$$
$$\to \operatorname{Tor}_{i-2}^B(\operatorname{ann}_M f, N) \to \cdots .$$

Corollary 4.6. *All sets of jumping coefficients of ideals in $R/g_1 R + g_2 R + g_3 R$ are discrete.*

Proof. Setting $A = R$, $B = S = R/g_1 R$, $M = S/(g_2^q S + g_3^q S) = R/g_1 R + g_2^q R + g_3^q R$, and $N = \mathbb{K}$ in the long exact sequence (8) we obtain

$$\dots \to \operatorname{Tor}_{i-1}^S(S/(g_2^q S + g_3^q S), \mathbb{K})$$
$$\to \operatorname{Tor}_i^R(R/(g_1 R + g_2^q R + g_3^q R), \mathbb{K}) \to \operatorname{Tor}_i^S(S/g_2^q S + g_3^q S, \mathbb{K}) \to \dots ,$$

where here $\mathbb{K}$ denotes the quotient of S by its irrelevant ideal. The previous theorem now implies that the degrees of elements in a graded $\mathbb{K}$-basis for $\operatorname{Tor}_i^R(R/(g_1 R + g_2^q R + g_3^q R), \mathbb{K})$ are bounded by $3q$. Hence $\operatorname{reg} \operatorname{Tor}_i^R(R/(g_1 R + g_2^q R + g_3^q R), \mathbb{K}) = \mathcal{O}(q)$, and by symmetry we also have

$$\operatorname{reg} \operatorname{Tor}_i^R(R/(g_1^q R + g_2 R + g_3^q R), \mathbb{K}) = \operatorname{reg} \operatorname{Tor}_i^R(R/(g_1^q R + g_2^q R + g_3 R), \mathbb{K}) = \mathcal{O}(q).$$

The corollary now follows from Theorem 3.3. $\square$

ACKNOWLEDGMENTS

REFERENCES

[AL]   William W. Adams and Philippe Loustaunau, *An introduction to Gröbner bases*, Graduate Studies in Mathematics, vol. 3, American Mathematical Society, Providence, RI, 1994. MR1287608 (95g:13025)

[Bli09] Manuel Blickle, *Test ideals via algebras of p^{-e}-linear maps*, J. Algebraic Geom. **22** (2013), no. 1, 49–83. MR2993047

[BMS08] Manuel Blickle, Mircea Mustaţă, and Karen E. Smith, *Discreteness and rationality of F-thresholds*, Michigan Math. J. **57** (2008), 43–61, DOI 10.1307/mmj/1220879396. Special volume in honor of Melvin Hochster. MR2492440 (2010c:13003)

[BMS09] Manuel Blickle, Mircea Mustaţă, and Karen E. Smith, *F-thresholds of hypersurfaces*, Trans. Amer. Math. Soc. **361** (2009), no. 12, 6549–6565, DOI 10.1090/S0002-9947-09-04719-9. MR2538604 (2011a:13006)

[BSTZ]   Manuel Blickle, Karl Schwede, Shunsuke Takagi, and Wenliang Zhang, *Discreteness and rationality of F-jumping numbers on singular varieties*, Math. Ann. **347** (2010), no. 4, 917–949, DOI 10.1007/s00208-009-0461-2. MR2658149 (2011k:13008)

[C]   M. Chardin. *On the behavior of Castelnuovo-Mumford regularity with respect to some functors,* arXiv:0706.2731

[EHU]   David Eisenbud, Craig Huneke, and Bernd Ulrich, *The regularity of Tor and graded Betti numbers*, Amer. J. Math. **128** (2006), no. 3, 573–605. MR2230917 (2007b:13027)

[E80]   David Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64, DOI 10.2307/1999875. MR570778 (82d:13013)

[E95]   David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry. MR1322960 (97a:13001)

[E05]   David Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry. MR2103875 (2005h:13021)

[F]   Richard Fedder, *F-purity and rational singularity*, Trans. Amer. Math. Soc. **278** (1983), no. 2, 461–480, DOI 10.2307/1999165. MR701505 (84h:13031)

[GH]   D. R.Grayson and M. E. Stillman. *Macaulay2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.

[H]   Nobuo Hara, *F-pure thresholds and F-jumping exponents in dimension two*, Math. Res. Lett. **13** (2006), no. 5-6, 747–760. With an appendix by Paul Monsky. MR2280772 (2007m:14032)

[HY]   Nobuo Hara and Ken-Ichi Yoshida, *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), no. 8, 3143–3174 (electronic), DOI 10.1090/S0002-9947-03-03285-9. MR1974679 (2004i:13003)

[K]   Mordechai Katzman, *The complexity of Frobenius powers of ideals*, J. Algebra **203** (1998), no. 1, 211–225, DOI 10.1006/jabr.1997.7246. MR1620654 (99e:13003)

[KLZ]   Mordechai Katzman, Gennady Lyubeznik, and Wenliang Zhang, *On the discreteness and rationality of F-jumping coefficients*, J. Algebra **322** (2009), no. 9, 3238–3247, DOI 10.1016/j.jalgebra.2008.11.032. MR2567418 (2011c:13005)

[KSS]   M. Katzman, K. Schwede and A. K. Singh. *Rings of Frobenius operators,* in preparation.

[MTW]   Mircea Mustaţă, Shunsuke Takagi, and Kei-ichi Watanabe, *F-thresholds and Bernstein-Sato polynomials*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2005, pp. 341–364. MR2185754 (2007b:13010)

[S11a]   Karl Schwede, *Test ideals in non-$\mathbb{Q}$-Gorenstein rings*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5925–5941, DOI 10.1090/S0002-9947-2011-05297-9.   MR2817415 (2012c:13011)

[S11b]   Karl Schwede, *A note on discreteness of F-jumping numbers*, Proc. Amer. Math. Soc. **139** (2011), no. 11, 3895–3901, DOI 10.1090/S0002-9939-2011-10825-6.   MR2823035 (2012e:13008)

[STZ]   Karl Schwede, Kevin Tucker, and Wenliang Zhang, *Test ideals via a single alteration and discreteness and rationality of F-jumping numbers*, Math. Res. Lett. **19** (2012), no. 1, 191–197. MR2923185

[W]   Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001)

Department of Pure Mathematics, Hicks Building, University of Sheffield, Sheffield S3 7RH, United Kingdom
    E-mail address: M.Katzman@sheffield.ac.uk

Department of Mathematics, University of Nebraska, Lincoln, Nebraska 68588
    E-mail address: wzhang15@unl.edu