TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 9, September 2014, Pages 4733–4753
S 0002-9947(2014)05962-X
Article electronically published on May 5, 2014

A WEAK* SEPARABLE $C(K)^*$ SPACE WHOSE UNIT BALL IS NOT WEAK* SEPARABLE

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Abstract. We provide a ZFC example of a compact space K such that $C(K)^*$ is w^*-separable but its closed unit ball $B_{C(K)^*}$ is not w^*-separable. All previous examples of such kind had been constructed under CH. We also discuss the measurability of the supremum norm on that $C(K)$ equipped with its weak Baire σ-algebra.

1. Introduction

Let K be a compact space (all our topological spaces are assumed to be Hausdorff) and let $C(K)$ be the Banach space of all continuous real-valued functions on K (equipped with the supremum norm). One can consider the following list of properties related to the separability in K and $C(K)^*$:

(a) K is separable;
(b) K carries a strictly positive measure of countable type;
(c) $P(K)$ (the set of all Radon probability measures on K) is w^*-separable;
(d) $B_{C(K)^*}$ (the closed unit ball of $C(K)^*$) is w^*-separable;
(e) $C(K)^*$ is w^*-separable.

It is known that the following implications hold:

$$\text{(a)} \Longrightarrow \text{(b)} \Longrightarrow \text{(c)} \Longleftrightarrow \text{(d)} \Longrightarrow \text{(e)}$$

and cannot be reversed in general. Indeed, the Stone space of the measure algebra of the Lebesgue measure on $[0,1]$ satisfies (b) but not (a), while Talagrand [17] constructed under CH two examples showing that (c)⇒(b) and (e)⇒(c) do not hold. Recently, Džamonja and Plebanek [2] gave a ZFC counterexample to (c)⇒(b).

In this paper we provide a ZFC example of a compact space K witnessing that the implication (e)⇒(d) does not hold, i.e. $C(K)^*$ is w^*-separable but $B_{C(K)^*}$ is not. The construction is given in Section 2 (see Theorem 2.1) and uses techniques which are similar to those of [2].

Section 3 is devoted to a discussion of further properties of that example which are relevant within the topic of measurability in Banach spaces. In every Banach space X one can consider the Baire σ-algebra of the weak topology, denoted by

Received by the editors December 23, 2011 and, in revised form, September 19, 2012.
2010 *Mathematics Subject Classification.* Primary 28E15, 46E15, 46E27, 54G20.
Key words and phrases. Weak*-separability, Baire σ-algebra, measurability, Kunen cardinal.
The first and third authors were supported by MEC and FEDER (Project MTM2008-05396) and Fundación Séneca (Project 08848/PI/08). The first author was supported by *Ramon y Cajal* contract (RYC-2008-02051) and an FP7-PEOPLE-ERG-2008 action. The second author was supported by MNiSW Grant N N201 418939 (2010–2013).

$\mathrm{Ba}(X,w)$, which coincides with the σ-algebra on X generated by X^* (see [3, Theorem 2.3]). While $\mathrm{Ba}(X,w)$ coincides with the Borel σ-algebra of the norm topology if X is separable, the two σ-algebras may be different if X is nonseparable. When X has a w^*-separable dual, then we can represent it as a subset of $\mathbb{R}^{\mathbb{N}}$. Namely, if (x_n^*) is a w^*-dense sequence in X^*, then the function $X \to \mathbb{R}^{\mathbb{N}}$ given by $x \mapsto (x_n^*(x))$ is one-to-one. In particular, given any equivalent norm $\|\cdot\|$ on X, the function $\|\cdot\| : X \to \mathbb{R}$ can be identified with a partial function on $\mathbb{R}^{\mathbb{N}}$. If (x_n^*) was dense in the ball B_{X^*}, then this norm function would be just the sup function, $\|x\| = \sup_n |x_n^*(x)|$. But if we have a space X with w^*-separable dual and B_{X^*} is not w^*-separable, how good can the norm function be? Could it be a $\mathrm{Ba}(X,w)$-measurable function? Note that $\|\cdot\|$ is $\mathrm{Ba}(X,w)$-measurable if and only if the closed unit ball B_X belongs to $\mathrm{Ba}(X,w)$. It is easy to check that:

$$B_{X^*} \text{ is } w^*\text{-separable} \Longrightarrow \|\cdot\| \text{ is } \mathrm{Ba}(X,w)\text{-measurable} \Longrightarrow X^* \text{ is } w^*\text{-separable}.$$

None of the reverse implications holds in general [13].

It seems to be an open problem whether the weak Baire measurability of the *supremum norm* of a $C(K)$ space is equivalent to either $C(K)^*$ being w^*-separable or to $B_{C(K)^*}$ being w^*-separable. Our compact space K of Section 2 would settle one of the two questions in the negative. In Section 3 we shall discuss whether the supremum norm of our space $C(K)$ can be represented as a measurable function on a separating family of functionals of $C(K)^*$, so that the norm would become $\mathrm{Ba}(C(K),w)$-measurable. Although we have been unable to determine the degree of measurability of the supremum norm on that $C(K)$, we shall show, however, that at least there exists a norm dense set $E \subseteq C(K)$ where the restriction of the supremum norm is relatively $\mathrm{Ba}(C(K),w)$-measurable (Theorem 3.10). This set E can be taken to be the linear span of the characteristic functions of clopen subsets of K under the set-theoretic assumption that $\mathfrak{c}$ is a Kunen cardinal (Theorem 3.22).

Terminology. We write $\mathcal{P}(S)$ to denote the power set of any set S. The cardinality of S is denoted by $|S|$. The letter $\mathfrak{c}$ stands for the cardinality of the continuum. A probability measure ν is said to be of countable type if the space $L^1(\nu)$ is separable. For a compact space K, we usually identify the dual space $C(K)^*$ with the space of all Radon measures on K. Given a Boolean algebra $\mathfrak{A}$, we write $\mathfrak{A}^+$ to denote the set of all nonzero elements of $\mathfrak{A}$. For the Boolean operations we use the usual symbols $\cup$, $\cap$, etc. and we write $\mathbf{0}$ and $\mathbf{1}$ for the least and the greatest element. The Stone space of all ultrafilters on $\mathfrak{A}$ is denoted by $\mathrm{ULT}(\mathfrak{A})$. Recall that the Stone isomorphism between $\mathfrak{A}$ and the algebra $\mathrm{Clop}(\mathrm{ULT}(\mathfrak{A}))$ of clopen subsets of $\mathrm{ULT}(\mathfrak{A})$ is given by

$$\mathfrak{A} \to \mathrm{Clop}(\mathrm{ULT}(\mathfrak{A})), \quad a \mapsto \widehat{a} = \{\mathcal{F} \in \mathrm{ULT}(\mathfrak{A}) : a \in \mathcal{F}\}.$$

Every measure μ on $\mathfrak{A}$ induces a measure $\widehat{a} \mapsto \mu(a)$ on $\mathrm{Clop}(\mathrm{ULT}(\mathfrak{A}))$ which can be uniquely extended to a Radon measure on $\mathrm{ULT}(\mathfrak{A})$ (see e.g. [15, Chapter 5]); such a Radon measure is still denoted by the same letter μ.

Given a set S and a family $\mathcal{D}$ of subsets of S, the σ-algebra on S generated by $\mathcal{D}$ is denoted by $\sigma(\mathcal{D})$. If (Ω,Σ) is any measurable space and $n \in \mathbb{N}$, we write $\bigotimes_n \Sigma$ to denote the usual product σ-algebra on Ω^n, that is,

$$\bigotimes_n \Sigma = \sigma\big(\{A_1 \times \cdots \times A_n : A_i \in \Sigma \text{ for all } i = 1,\ldots,n\}\big).$$

A cardinal κ is called a *Kunen cardinal* if the equality

$$\mathcal{P}(\kappa \times \kappa) = \mathcal{P}(\kappa) \otimes \mathcal{P}(\kappa)$$

holds true. Of course, in this case we have $\mathcal{P}(\kappa^n) = \bigotimes_n \mathcal{P}(\kappa)$ for every $n \in \mathbb{N}$. This notion was investigated by Kunen in his doctoral dissertation [9]. It is known that:

(i) any Kunen cardinal is less than or equal to $\mathfrak{c}$;
(ii) ω_1 is a Kunen cardinal;
(iii) $\mathfrak{c}$ is a Kunen cardinal under Martin's axiom, while $\mathfrak{c}$ is not a Kunen cardinal in the Cohen model after adding $\aleph_2$ reals to a model of GCH or when $\mathfrak{c}$ is real-valued measurable.

Kunen cardinals have been considered by Talagrand [16], Fremlin [6] and the authors [1] in connection with measurability properties of Banach spaces, and also by Todorcevic [18] in connection with universality properties of ℓ_∞/c_0.

2. The example

Fix any cardinal κ such that $\omega_1 \leq \kappa \leq \mathfrak{c}$. Let λ be the usual product probability measure on the Baire σ-algebra of 2^κ and let $\mathfrak{B}$ be its measure algebra. Note that $\mathfrak{B}$ has cardinality $\mathfrak{c}$ since every Baire subset of 2^κ is determined by countably many coordinates (see e.g. [8, 254M]). The letter λ will also stand for the corresponding probability measure on $\mathfrak{B}$. We shall work in the countable simple product $\mathfrak{C} := \mathfrak{B}^{\mathbb{N}}$ of $\mathfrak{B}$ so that every $c \in \mathfrak{C}$ is a sequence $c = (c(n))_n$ where $c(n) \in \mathfrak{B}$ for all $n \in \mathbb{N}$. On the Boolean algebra $\mathfrak{C}$ we consider the sequence of probability measures $\{\mu_n : n \in \mathbb{N}\}$ defined by

$$\mu_n(c) := \lambda(c(n)) \quad \text{for all } c \in \mathfrak{C}.$$

Let $\{N_b : b \in \mathfrak{B}^+\}$ be a fixed *independent* family of subsets of $\mathbb{N}$, i.e.

$$\bigcap_{b \in s} N_b \setminus \bigcup_{b' \in t} N_{b'} \neq \emptyset$$

whenever $s, t \subseteq \mathfrak{B}^+$ are finite and disjoint (see e.g. [4, p. 180, Exercise 3.6.F]); note that $\bigcap_{b \in s} N_b = \mathbb{N}$ if $s = \emptyset$ and that $\bigcup_{b' \in t} N_{b'} = \emptyset$ if $t = \emptyset$. For each $b \in \mathfrak{B}^+$, define an element $G_b \in \mathfrak{C}$ by declaring

$$G_b(n) := \begin{cases} b & \text{if } n \in N_b, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{C}$ generated by $\{G_b : b \in \mathfrak{B}^+\}$ and write $K := \mathrm{ULT}(\mathfrak{A})$. This section is devoted to proving the following:

Theorem 2.1. *$B_{C(K)^*}$ is not w^*-separable, while $\{\mu_n : n \in \mathbb{N}\}$ separates the points of $C(K)$, so $C(K)^*$ is w^*-separable.*

In order to prove Theorem 2.1 we need some preliminary work. For any finite sets $s, t \subseteq \mathfrak{B}^+$ we consider the following elements of $\mathfrak{A}$:

$$J(s) := \bigcap_{b \in s} G_b, \quad U(t) := \bigcup_{b \in t} G_b \quad \text{and} \quad W(s,t) := J(s) \setminus U(t).$$

The Boolean algebra $\mathfrak{A}$ is completely determined up to isomorphism when given its set of generators $\{G_b : b \in \mathfrak{B}^+\}$ and which elements $W(s,t)$ are zero. In this sense, Lemma 2.2 below is interpreted as the fact that $\mathfrak{A}$ is isomorphic to the free Boolean algebra generated by $\{G_b : b \in \mathfrak{B}^+\}$ modulo the relations that $W(s,t) = \mathbf{0}$ if and only if $s \cap t \neq \emptyset$ or $\bigcap_{b \in s} b = \mathbf{0}$.

Lemma 2.2. *For two finite sets $s, t \subseteq \mathfrak{B}^+$, we have $W(s,t) = \mathbf{0}$ if and only if $s \cap t \neq \emptyset$ or $\bigcap_{b\in s} b = \mathbf{0}$. In particular, for a finite $s \subseteq \mathfrak{B}^+$ the following are equivalent:*

(1) $\bigcap_{b\in s} b = \mathbf{0}$;
(2) $J(s) = \mathbf{0}$;
(3) $W(s,t) = \mathbf{0}$ *for all finite* $t \subseteq \mathfrak{B}^+ \setminus s$;
(4) $W(s,t) = \mathbf{0}$ *for some finite* $t \subseteq \mathfrak{B}^+ \setminus s$.

Proof. Let us observe that for every $n \in \mathbb{N}$ we have

$$J(s)(n) = \bigcap_{b\in s} G_b(n) = \begin{cases} \bigcap_{b\in s} b & \text{if } n \in \bigcap_{b\in s} N_b, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{2.1}$$

It follows from (2.1) that if $\bigcap_{b\in s} b = \mathbf{0}$, then $W(s,t) \subseteq J(s) = \mathbf{0}$ as well. The same immediately follows when $s \cap t \neq \emptyset$.

For the converse, suppose that $s \cap t = \emptyset$ and $\bigcap_{b\in s} b \neq \mathbf{0}$, and pick

$$n_0 \in \bigcap_{b\in s} N_b \setminus \bigcup_{b'\in t} N_{b'}.$$

Then

$$W(s,t)(n_0) = \bigcap_{b\in s} G_b(n_0) \setminus \bigcup_{b'\in t} G_{b'}(n_0) = \bigcap_{b\in s} b \neq \mathbf{0},$$

and so $W(s,t) \neq \mathbf{0}$. The second part of the lemma, with the list of equivalences, follows from the first statement and the arguments above. □

We next describe $K = \mathrm{ULT}(\mathfrak{A})$. Let us consider the family of all subsets of $\mathfrak{B}$ with the finite intersection property, that is,

$$\mathrm{FIP}(\mathfrak{B}) = \left\{ X \subseteq \mathfrak{B}^+ : b_1 \cap \ldots \cap b_n \neq \mathbf{0} \text{ for every } b_1, \ldots, b_n \in X \right\}.$$

Given $X \in \mathrm{FIP}(\mathfrak{B})$, let $\mathcal{F}_X$ be the filter on $\mathfrak{A}$ generated by

$$\{W(s,t) : s \subseteq X \text{ finite}, t \subseteq \mathfrak{B}^+ \setminus X \text{ finite}\}$$

(notice that this set has the finite intersection property by Lemma 2.2). In particular, when X is empty, then $\mathcal{F}_X$ is simply generated by the complements of all G_b's.

Lemma 2.3. $K = \{\mathcal{F}_X : X \in \mathrm{FIP}(\mathfrak{B})\}$.

Proof. Every filter of the form $\mathcal{F}_X$ is an ultrafilter on $\mathfrak{A}$, because $\{G_b : b \in \mathfrak{B}^+\}$ is a set of generators of $\mathfrak{A}$, and, for each $b \in \mathfrak{B}^+$, we have either $G_b = W(\{b\}, \emptyset) \in \mathcal{F}_X$ (if $b \in X$) or $\mathbf{1} \setminus G_b = W(\emptyset, \{b\}) \in \mathcal{F}_X$ (if $b \notin X$). Conversely, let $\mathcal{F}$ be any ultrafilter on $\mathfrak{A}$ and consider $X := \{b \in \mathfrak{B}^+ : G_b \in \mathcal{F}\}$. Notice that $X \in \mathrm{FIP}(\mathfrak{B})$ and that, for each $b \in \mathfrak{B}^+$, we have

$$G_b \in \mathcal{F}_X \iff G_b \in \mathcal{F}.$$

Since $\{G_b : b \in \mathfrak{B}^+\}$ is a set of generators of $\mathfrak{A}$, it follows that $\mathcal{F} = \mathcal{F}_X$. □

Let $\mathrm{FIP}_0(\mathfrak{B})$ be the family of all $s \in \mathrm{FIP}(\mathfrak{B})$ which are finite. The next lemma says that the clopens of the form $\widehat{W(s,t)}$ are a basis of the topology of K and also that $\{\mathcal{F}_s : s \in \mathrm{FIP}_0(\mathfrak{B})\}$ is dense in K.

Lemma 2.4. *Let $a \in \mathfrak{A}^+$ and $\mathcal{F} \in \widehat{a} \subseteq K$.*

(i) *There exist $s \in \mathrm{FIP}_0(\mathfrak{B})$ and a finite set $t \subseteq \mathfrak{B}^+ \setminus s$ such that $\mathcal{F} \in \widehat{W(s,t)} \subseteq \widehat{a}$.*

(ii) *If $\mathcal{F} = \mathcal{F}_{s_0}$ for some $s_0 \in \mathrm{FIP}_0(\mathfrak{B})$, then we can choose $s = s_0$ in (i).*

Proof. (i) Take $I \subseteq \mathfrak{B}^+$ finite such that a belongs to the subalgebra of $\mathfrak{A}$ generated by $\{G_b : b \in I\}$. The family $\mathfrak{A}_0$ of all elements of $\mathfrak{A}$ which can be written as the union of finitely many elements of the form $W(s,t)$, where $s \cup t = I$ and $s \cap t = \emptyset$, is a subalgebra of $\mathfrak{A}$ containing $\{G_b : b \in I\}$. Therefore $a \in \mathfrak{A}_0$. Since $a \in \mathcal{F}$, there exist s and t as before such that $W(s,t) \in \mathcal{F}$, hence $s \in \mathrm{FIP}_0(\mathfrak{B})$ (by Lemma 2.2) and $\mathcal{F} \in \widehat{W(s,t)} \subseteq \widehat{a}$.

(ii) Since $W(s,t) \in \mathcal{F}_{s_0}$, we have $G_b \in \mathcal{F}_{s_0}$ for all $b \in s$ and $G_b \notin \mathcal{F}_{s_0}$ for all $b \in t$. Hence $s \subseteq s_0$ and $s_0 \cap t = \emptyset$. Therefore, $\mathcal{F}_{s_0} \in \widehat{W(s_0,t)} \subseteq \widehat{W(s,t)} \subseteq \widehat{a}$. □

Another ingredient to prove Theorem 2.1 is the result isolated in Lemma 2.6 below, which is a consequence of the following characterization of w^*-separability in spaces of measures due to Mägerl and Namioka [11].

Fact 2.5. Let L be a compact space. Then the space $P(L)$ is w^*-separable if and only if there is a sequence $\{\nu_n : n \in \mathbb{N}\}$ in $P(L)$ such that for every nonempty open set $V \subseteq L$ there is $n \in \mathbb{N}$ with $\nu_n(V) > 1/2$.

Lemma 2.6. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Boolean algebras such that there is another Boolean algebra $\mathfrak{A}_3$ containing $\mathfrak{A}_1$ and $\mathfrak{A}_2$ as subalgebras and the following holds:*

$$(\star) \quad \text{for every } b \in \mathfrak{A}_2^+ \text{ there is } a \in \mathfrak{A}_1^+ \text{ such that } a \subseteq b.$$

If $P(\mathrm{ULT}(\mathfrak{A}_2))$ is not w^*-separable, then $P(\mathrm{ULT}(\mathfrak{A}_1))$ is not w^*-separable either.

Proof. Let $\{\nu_n : n \in \mathbb{N}\}$ be a sequence of probability measures on $\mathfrak{A}_1$. For each $n \in \mathbb{N}$, we can extend ν_n to a probability measure ν_n' on $\mathfrak{A}_3$ (see [10] or [12]) and we denote by ν_n the restriction of ν_n' to $\mathfrak{A}_2$. Since $P(\mathrm{ULT}(\mathfrak{A}_2))$ is not w^*-separable, by Fact 2.5 there is $b \in \mathfrak{A}_2^+$ such that $\nu_n(b) \leq 1/2$ for every $n \in \mathbb{N}$. Property $(\star)$ allows us to take $a \in \mathfrak{A}_1^+$ such that $a \subseteq b$. Then

$$\nu_n(a) = \nu_n'(a) \leq \nu_n'(b) = \nu_n(b) \leq 1/2 \quad \text{for every } n \in \mathbb{N}.$$

Another appeal to Fact 2.5 ensures that $P(\mathrm{ULT}(\mathfrak{A}_1))$ is not w^*-separable. □

According to a result of Rosenthal (see [14, Theorem 3.1]), the space $L^\infty(\nu)^*$ is not w^*-separable whenever ν is a probability measure of uncountable type. Since $C(\mathrm{ULT}(\mathfrak{B}))$ is isomorphic to $L^\infty(\lambda)$, it follows that $P(\mathrm{ULT}(\mathfrak{B}))$ is not w^*-separable. We are now ready to deal with Theorem 2.1.

Proof of Theorem 2.1. Let $\mathfrak{B}^*$ be the subalgebra of $\mathfrak{C}$ consisting of all constant sequences. Since $\mathfrak{B}^*$ is isomorphic to $\mathfrak{B}$, $P(\mathrm{ULT}(\mathfrak{B}^*))$ is not w^*-separable. On the other hand, property $(\star)$ of Lemma 2.6 holds for $\mathfrak{A}_1 = \mathfrak{A}$ and $\mathfrak{A}_2 = \mathfrak{B}^*$, hence $P(K)$ is not w^*-separable and so $B_{C(K)^*}$ is not w^*-separable either.

We now prove that $\{\mu_n : n \in \mathbb{N}\}$ separates the points of $C(K)$. Fix $h \in C(K) \setminus \{0\}$.

Step 1. Since $\{\mathcal{F}_s : s \in \mathrm{FIP}_0(\mathfrak{B})\}$ is dense in K (by Lemma 2.4(i)), there is $s \in \mathrm{FIP}_0(\mathfrak{B})$ such that $h(\mathcal{F}_s) \neq 0$. Moreover, we can assume that

$$h(\mathcal{F}_{s'}) = 0 \quad \text{whenever } s' \in \mathrm{FIP}_0(\mathfrak{B}) \text{ satisfies } |s'| < |s| \tag{2.2}$$

and that $C := h(\mathcal{F}_s) > 0$. By the continuity of h and Lemma 2.4(ii), there is a finite set $t \subseteq \mathfrak{B}^+ \setminus s$ such that, writing $a := W(s,t)$, we have

$$h(\mathcal{F}) \geq \frac{C}{2} \quad \text{for all } \mathcal{F} \in \widehat{a}. \tag{2.3}$$

Set $\delta := \frac{C}{4}\lambda(\bigcap_{b\in s} b) > 0$ and define $\mathcal{R} := \{r \subseteq s : r \neq s\} \subseteq \mathrm{FIP}_0(\mathfrak{B})$.

Step 2. Fix $r \in \mathcal{R}$. Since h is continuous and $h(\mathcal{F}_r) = 0$ (by (2.2)), we can apply Lemma 2.4(ii) to find a finite set $t'_r \subseteq \mathfrak{B}^+ \setminus r$ such that $|h(\mathcal{F})| \leq \delta$ for all $\mathcal{F} \in \widehat{W(r,t'_r)}$. Writing $t_r := t'_r \setminus s$, we have

$$a_r := W(r, t_r \cup (s \setminus r)) \subseteq W(r, t'_r) \quad \text{in } \mathfrak{A},$$

and therefore

$$|h(\mathcal{F})| \leq \delta \quad \text{for all } \mathcal{F} \in \widehat{a_r}. \tag{2.4}$$

Step 3. Define

$$t^* := t \cup \bigcup_{r\in\mathcal{R}} t_r \subseteq \mathfrak{B}^+ \setminus s$$

and choose $n \in \bigcap_{b\in s} N_b \setminus \bigcup_{b'\in t^*} N_{b'}$ (which is nonempty by independence). Hence

$$a(n) = \bigcap_{b\in s} b, \qquad a_r(n) = \bigcap_{b\in r} b \setminus \bigcup_{b'\in s\setminus r} b' \quad \text{for every } r \in \mathcal{R},$$

and therefore

$$\mathbf{1} \setminus a(n) \subseteq \bigcup_{r\in\mathcal{R}} a_r(n) \quad \text{in } \mathfrak{B}. \tag{2.5}$$

Step 4. Observe that

$$\mu_n\left(\left(K \setminus \widehat{a}\right) \setminus \bigcup_{r\in\mathcal{R}} \widehat{a_r}\right) = \mu_n\left(\left(\mathbf{1} \setminus a\right) \setminus \bigcup_{r\in\mathcal{R}} a_r\right) = \lambda\left(\left(\mathbf{1} \setminus a(n)\right) \setminus \bigcup_{r\in\mathcal{R}} a_r(n)\right) \overset{(2.5)}{=} 0.$$

Therefore

$$\left|\int_{K\setminus\widehat{a}} h\, d\mu_n\right| \leq \int_{\bigcup_{r\in\mathcal{R}} \widehat{a_r}} |h|\, d\mu_n \overset{(2.4)}{\leq} \delta.$$

It follows that

$$\begin{aligned}\mu_n(h) &= \int_{\widehat{a}} h\, d\mu_n + \int_{K\setminus\widehat{a}} h\, d\mu_n \\ &\geq \int_{\widehat{a}} h\, d\mu_n - \delta \overset{(2.3)}{\geq} \frac{C}{2}\mu_n(\widehat{a}) - \delta = \frac{C}{2}\lambda(a(n)) - \delta = \frac{C}{2}\lambda\left(\bigcap_{b\in s} b\right) - \delta = \frac{C}{4}\lambda\left(\bigcap_{b\in s} b\right) > 0.\end{aligned}$$

Thus $\mu_n(h) \neq 0$. This finishes the proof. □

3. Weak Baire measurability of the norm

In this section we address the question of whether the supremum norm is weakly Baire measurable on $C(K)$, where K is the compact space constructed in Section 2. As we have pointed out in the introduction, measurability of the norm is naturally related to w^*-separability. Throughout this section we follow the notation introduced in Section 2. The supremum norm on $C(K)$ is denoted by $\|\cdot\|$.

We first show that the family $\{\mu_n : n \in \mathbb{N}\} \subseteq P(K)$, though separating the elements of $C(K)$, is not rich enough to "measure" $B_{C(K)}$.

Proposition 3.1. *The supremum norm on $C(K)$ does not have to be measurable with respect to the σ-algebra generated by $\{\mu_n : n \in \mathbb{N}\}$.*

Proof. We identify $\mathcal{P}(\mathbb{N})$ and $\{0,1\}^{\mathbb{N}}$ in the usual way. Let $\Omega \subseteq 2^{\mathbb{N}}$ be an independent family of subsets of $\mathbb{N}$ with $|\Omega| = \mathfrak{c}$ and let Σ denote the trace of $\mathrm{Borel}(2^{\mathbb{N}})$ on Ω. Then $|\Sigma| = \mathfrak{c}$, and so we can find $A \subseteq \Omega$ such that $A \notin \Sigma$ and $|A| = |\Omega \setminus A| = \mathfrak{c}$. Now we can choose an enumeration $\Omega = \{N_b : b \in \mathfrak{B}^+\}$ such that

$$\lambda(b) = 1/2 \iff N_b \in A. \tag{3.1}$$

Define a function $f : \Omega \to C(K)$ by

$$f(N_b) := \frac{1}{2\lambda(b)} 1_{\widehat{G_b}}.$$

We claim that f is measurable with respect to Σ and the σ-algebra $\mathcal{C}$ on $C(K)$ generated by $\{\mu_n : n \in \mathbb{N}\}$. Indeed, for fixed $n \in \mathbb{N}$, we have

$$(\mu_n \circ f)(N_b) = \frac{1}{2\lambda(b)} \mu_n(G_b) = \frac{1}{2\lambda(b)} \lambda(G_b(n)) = \begin{cases} 1/2 & \text{if } n \in N_b, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\mu_n \circ f = (1/2)\pi_n$, where $\pi_n : \Omega \subseteq 2^{\mathbb{N}} \to \{0,1\}$ denotes the n-th coordinate projection, hence $\mu_n \circ f$ is Σ-measurable.

On the other hand, the composition $\|f(\cdot)\| : \Omega \to \mathbb{R}$ is not Σ-measurable because

$$\|f(N_b)\| = 1 \overset{(3.1)}{\iff} N_b \in A$$

and $A \notin \Sigma$. This fact and the Σ-$\mathcal{C}$-measurability of f imply that $\|\cdot\|$ is not $\mathcal{C}$-measurable. □

3.1. **Measurability on a norm dense set.** Let $S(K)$ denote the dense subspace of $C(K)$ consisting of all linear combinations of characteristic functions of clopen subsets of K. The next simple lemma provides a useful representation of the elements of $S(K)$.

Lemma 3.2. *Let $g \in S(K)$. Then there exist $s \subseteq \mathfrak{B}^+$ finite and a collection of real numbers $\{y_r : r \subseteq s\}$ such that*

$$g = \sum_{r \subseteq s} y_r 1_{\widehat{J(r)}}.$$

Proof. We can write g as

$$g = \sum_{r' \subseteq s} z_{r'} 1_{W\widehat{(r', s \setminus r')}}$$

for some $s \subseteq \mathfrak{B}^+$ finite and a certain collection of real numbers $\{z_{r'} : r' \subseteq s\}$. Note that there is a (unique) collection of real numbers $\{y_r : r \subseteq s\}$ such that

$$z_{r'} = \sum_{r \subseteq r'} y_r \quad \text{for every } r' \subseteq s. \tag{3.2}$$

Indeed, the y_r's can be obtained recurrently as follows:

$$\begin{aligned} y_\emptyset &:= z_\emptyset, \\ y_{\{b\}} &:= z_{\{b\}} - y_\emptyset \quad \text{for every } b \in s, \\ y_{\{b,b'\}} &:= z_{\{b,b'\}} - y_\emptyset - y_{\{b\}} - y_{\{b'\}} \quad \text{for every } b, b' \in s \text{ with } b \neq b', \end{aligned}$$

and so on. Since

$$J(r) = \bigcup_{r \subseteq r' \subseteq s} W(r', s \setminus r') \quad \text{for every } r \subseteq s$$

we conclude that

$$\begin{aligned} g &= \sum_{r' \subseteq s} z_{r'} 1_{\widehat{W(r', s \setminus r')}} \overset{(3.2)}{=} \sum_{r' \subseteq s} \sum_{r \subseteq r'} y_r 1_{\widehat{W(r', s \setminus r')}} = \sum_{r \subseteq s} y_r \sum_{r \subseteq r' \subseteq s} 1_{\widehat{W(r', s \setminus r')}} \\ &= \sum_{r \subseteq s} y_r 1_{\widehat{J(r)}}, \end{aligned}$$

and the proof is over. □

We denote by $S'(K)$ the set of all $g \in S(K)$ which can be written as

$$g = \sum_{r \subseteq s} y_r 1_{\widehat{J(r)}}$$

for some finite set $s \subseteq \mathfrak{B}^+$ and some collection of *nonzero* real numbers $\{y_r : r \subseteq s\}$. It is easy to check (via Lemma 3.2) that $S'(K)$ is norm dense in $S(K)$, and so $S'(K)$ is norm dense in $C(K)$. In Theorem 3.10 we shall prove that the restriction of the supremum norm to $S'(K)$ is measurable with respect to the trace of $\mathrm{Ba}(C(K), w)$ on $S'(K)$. The proof requires some work, and the first step is to find a substantially larger family of measures to deal with.

Lemma 3.3. *For each $T \subseteq \mathfrak{B}^+$ and each $k \in \mathbb{N}$ there is a probability measure μ_T^k on $\mathfrak{A}$ such that, for every finite set $r \subseteq \mathfrak{B}^+$, we have*

$$\mu_T^k(J(r)) = \begin{cases} \lambda\left(\bigcap_{b \in r} b\right)^k & \text{if } r \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for every finite disjoint sets $r, s \subseteq \mathfrak{B}^+$ we have

$$\mu_T^1(W(r,s)) = \begin{cases} \lambda\left(\bigcap_{b \in r} b \setminus \bigcup_{b' \in s \cap T} b'\right) & \text{if } r \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

Sometimes we shall write $\mu_T := \mu_T^1$.

Proof. Let $\mathfrak{B}_k := \mathfrak{B} \otimes \cdots \otimes \mathfrak{B}$ denote the free product of k many copies of $\mathfrak{B}$ and let λ_k denote the product measure on $\mathfrak{B}_k$ (see e.g. [7, 2.25]). Consider the function $\varphi_T^k : \mathfrak{B}^+ \to \mathfrak{B}_k$ defined by

$$\varphi_T^k(b) := \begin{cases} b \otimes \cdots \otimes b & \text{if } b \in T, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then φ_T^k preserves disjointness, so there is a Boolean homomorphism $\tilde{\varphi}_T^k : \mathfrak{A} \to \mathfrak{B}_k$ such that $\tilde{\varphi}_T^k(G_b) = \varphi_T^k(b)$ for all $b \in \mathfrak{B}^+$ (bear in mind that $\mathfrak{A}$ is isomorphic to the Boolean algebra freely generated by the G_b's modulo the relations that $W(s,t) = \mathbf{0}$ if and only if $s \cap t \neq \emptyset$ or $\bigcap_{b \in s} b = \mathbf{0}$; see Lemma 2.2 and the preceding comments). Now, it is not difficult to check that the formula

$$\mu_T^k(a) := \lambda_k(\tilde{\varphi}_T^k(a))$$

defines a probability measure μ_T^k on $\mathfrak{A}$ satisfying the required property. □

From the technical point of view, the following subsets of $S(K)$ will play a relevant role in our proof of Theorem 3.10.

Definition 3.4. Let D be a finite partition of $\mathfrak{B}^+$. We denote by $S_D(K)$ (resp. $S'_D(K)$) the set of all $g \in S(K)$ which can be written as

$$g = \sum_{r \subseteq s} y_r 1_{\widehat{J(r)}}$$

for some finite set $s \subseteq \mathfrak{B}^+$ such that $|T \cap s| \leq 1$ for every $T \in D$ and some collection of real numbers (resp. nonzero real numbers) $\{y_r : r \subseteq s\}$.

Definition 3.5. Let D be a finite partition of $\mathfrak{B}^+$ and $C \subseteq D$. We define:

(i) $T_C := \bigcup_{T \in C} T \subseteq \mathfrak{B}^+$;
(ii) a signed measure ν_C^k on $\mathfrak{A}$ (for $k = 1, 2$) by

$$\nu_C^k := \sum_{B \subseteq C} (-1)^{|C \setminus B|} \mu_{T_B}^k;$$

(iii) a function $\theta_C : C(K) \to \mathbb{R}$ by

$$\theta_C(g) := \frac{(\nu_C^1(g))^2}{\nu_C^2(g)}$$

if $\nu_C^2(g) \neq 0$ and $\theta_C(g) := 0$ otherwise;
(iv) a function $\eta_C : C(K) \to \mathbb{R}$ by

$$\eta_C(g) := \frac{\nu_C^1(g)}{\theta_C(g)}$$

if $\theta_C(g) \neq 0$ and $\eta_C(g) := 0$ otherwise.

Clearly, the functions θ_C and η_C defined above are $\mathrm{Ba}(C(K), w)$-measurable. The following lemma collects several useful properties of such functions.

Lemma 3.6. *Let D be a finite partition of $\mathfrak{B}^+$ and let $g \in S_D(K)$ be as in Definition 3.4. Fix $r \subseteq s$ and $k \in \{1, 2\}$. Writing*

$$C_r := \{T \in D : T \cap r \neq 0\},$$

the following statements hold:

(i) *If $C \subseteq C_r$, then $\nu_C^k(J(r)) = \mu_{T_C}^k(J(r))$.*
(ii) *If $C_r \subsetneq C \subseteq D$, then $\nu_C^k(J(r)) = 0$.*
(iii) *$\nu_{C_r}^k(g) = y_r(\lambda(\bigcap_{b \in r} b))^k$.*
(iv) *If $J(r) \neq \mathbf{0}$, then $\theta_{C_r}(g) = y_r$.*
(v) *If $J(r) \neq \mathbf{0}$ and $y_r \neq 0$, then $\eta_{C_r}(g) = \lambda(\bigcap_{b \in r} b)$.*

Proof. (i) For each $B \subsetneq C$ we have $r \not\subseteq T_B$ (because $C \subseteq C_r$), and so $\mu_{T_B}^k(J(r)) = 0$. Hence

$$\nu_C^k(J(r)) = \sum_{B \subseteq C} (-1)^{|C \setminus B|} \mu_{T_B}^k(J(r)) = \mu_{T_C}^k(J(r)).$$

(ii) For each $B \subseteq C$ with $C_r \not\subseteq B$ we have $r \not\subseteq T_B$, and so $\mu^k_{T_B}(J(r)) = 0$. On the other hand, given any $C_r \subseteq B \subseteq C$ we have $r \subseteq T_{C_r} \subseteq T_B$, and therefore $\mu^k_{T_B}(J(r)) = (\lambda(\bigcap_{b\in r} b))^k$. Hence

$$\nu^k_C(J(r)) = \sum_{B\subseteq C} (-1)^{|C\setminus B|}\mu^k_{T_B}(J(r)) = \sum_{C_r\subseteq B\subseteq C} (-1)^{|C\setminus B|}\mu^k_{T_B}(J(r))$$
$$= \left(\lambda\left(\bigcap_{b\in r} b\right)\right)^k \cdot \sum_{C_r\subseteq B\subseteq C} (-1)^{|C\setminus B|}.$$

The assumption $C \neq C_r$ implies

$$\sum_{C_r\subseteq B\subseteq C} (-1)^{|C\setminus B|} = \sum_{A\subseteq C\setminus C_r} (-1)^{|A|} = 0,$$

as can be easily checked by induction on $|C \setminus C_r|$. Therefore, $\nu^k_C(J(r)) = 0$.

(iii) Take any $r' \subseteq s$. Note first that if $r' \not\subseteq r$, then $r' \not\subseteq T_B$ for every $B \subseteq C_r$, hence

$$\nu^k_{C_r}(J(r')) = \sum_{B\subseteq C_r} (-1)^{|C_r\setminus B|}\mu^k_{T_B}(J(r')) = 0.$$

On the other hand, if $r' \subsetneq r$, then $C_{r'} \subsetneq C_r$ and (ii) implies $\nu^k_{C_r}(J(r')) = 0$. Observe also that $\nu^k_{C_r}(J(r)) = (\lambda(\bigcap_{b\in r} b))^k$ (by (i)). It follows that

$$\nu^k_{C_r}(g) = \sum_{r'\subseteq s} y_{r'}\nu^k_{C_r}(J(r')) = y_r\nu^k_{C_r}(J(r)) = y_r\left(\lambda\left(\bigcap_{b\in r} b\right)\right)^k.$$

(iv) Bearing in mind (iii) and the fact that $J(r) \neq \mathbf{0}$, we have $y_r = 0$ if and only if $\nu^2_{C_r}(g) = 0$. Thus, by the very definition of θ_{C_r}, the equality $\theta_{C_r}(g) = y_r$ holds whenever $y_r = 0$. On the other hand, if $y_r \neq 0$, then

$$\theta_{C_r}(g) = \frac{(\nu^1_{C_r}(g))^2}{\nu^2_{C_r}(g)} \overset{\text{(iii)}}{=} y_r$$

and

$$\eta_{C_r}(g) = \frac{\nu^1_{C_r}(g)}{\theta_{C_r}(g)} \overset{\text{(iii)}}{=} \lambda\left(\bigcap_{b\in r} b\right),$$

which proves (v). □

Our next step is to prove that, for any $Z \subseteq \mathfrak{B}^+$ and $p \in \mathbb{N}$, the mapping $g \mapsto \mu_Z(g^p)$ is measurable with respect to the trace of $\mathrm{Ba}(C(K), w)$ on $S'(K)$ (see Lemma 3.9 below). We begin by checking the measurability on subsets of the form $S'_D(K)$.

Lemma 3.7. *Let $Z \subseteq \mathfrak{B}^+$ be a set, D a finite partition of $\mathfrak{B}^+$ finer than $\{Z, \mathfrak{B}^+\setminus Z\}$ and $p \in \mathbb{N}$. Then the mapping*

$$\phi_{Z,p} : S'_D(K) \to \mathbb{R}, \quad \phi_{Z,p}(g) := \mu_Z(g^p),$$

is measurable with respect to the trace of $\mathrm{Ba}(C(K), w)$ *on* $S'_D(K)$.

Proof. Write $D(Z) := \{T \in D : T \subseteq Z\}$ and let Λ be the set of all $\beta = (\beta_C)_{C \subseteq D(Z)}$ such that $\beta_C \in \mathbb{N} \cup \{0\}$ for all $C \subseteq D(Z)$ and $\sum_{C \subseteq D(Z)} \beta_C = p$. We write

$$\binom{p}{\beta} := \frac{p!}{\prod_{C \subseteq D(Z)} \beta_C!} \quad \text{and} \quad C(\beta) := \bigcup_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} C.$$

To prove the measurability of $\phi_{Z,p}$ with respect to the trace of $\mathrm{Ba}(C(K), w)$ on $S'_D(K)$, it is sufficient to check that, for each $g \in S'_D(K)$, the following equality holds:

$$\phi_{Z,p}(g) = \sum_{\beta \in \Lambda} \binom{p}{\beta} \left(\prod_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} \theta_C^{\beta_C}(g) \right) \eta_{C(\beta)}(g). \tag{3.3}$$

Step 1. Write

$$g = \sum_{r \subseteq s} y_r 1_{\widehat{J(r)}},$$

where $s \subseteq \mathfrak{B}^+$ is finite, $|T \cap s| \leq 1$ for every $T \in D$ and $y_r \in \mathbb{R} \setminus \{0\}$ for all $r \subseteq s$. Let Δ be the set of all $\delta = (\delta_r)_{r \subseteq s}$ such that $\delta_r \in \mathbb{N} \cup \{0\}$ for all $r \subseteq s$ and $\sum_{r \subseteq s} \delta_r = p$. Writing

$$\binom{p}{\delta} := \frac{p!}{\prod_{r \subseteq s} \delta_r!} \quad \text{and} \quad r(\delta) := \bigcup_{\substack{r \subseteq s \\ \delta_r > 0}} r,$$

we have

$$g^p = \sum_{\delta \in \Delta} \binom{p}{\delta} \left(\prod_{\substack{r \subseteq s \\ \delta_r > 0}} y_r^{\delta_r} \right) 1_{\widehat{J(r(\delta))}},$$

and so

$$\phi_{Z,p}(g) = \mu_Z(g^p) = \sum_{\delta \in \Delta} \binom{p}{\delta} \left(\prod_{\substack{r \subseteq s \\ \delta_r > 0}} y_r^{\delta_r} \right) \mu_Z\big(J(r(\delta))\big). \tag{3.4}$$

Step 2. Let $\delta \in \Delta$ such that $\mu_Z(J(r(\delta))) \neq 0$. For any $r \subseteq s$ with $\delta_r > 0$ we have $J(r) \supseteq J(r(\delta))$. Hence $\mu_Z(J(r)) \neq 0$ (in particular, $J(r) \neq \mathbf{0}$), and so $r \subseteq Z$, which implies that $C_r = \{T \in D : T \cap r \neq \emptyset\} \subseteq D(Z)$. Set $\beta = (\beta_C)_{C \subseteq D(Z)}$ by declaring

$$\beta_C := \begin{cases} \delta_r & \text{if } C = C_r \text{ for some } r \subseteq s \text{ with } \delta_r > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Then $\sum_{C \subseteq D(Z)} \beta_C = \sum_{r \subseteq s} \delta_r = p$, so that $\beta \in \Lambda$. Moreover, we have

$$C(\beta) = \bigcup_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} C = \bigcup_{\substack{r \subseteq s \\ \delta_r > 0}} C_r = C_{r(\delta)}.$$

Since $\mu_Z(J(r(\delta))) \neq 0$, we have $J(r(\delta)) \neq \mathbf{0}$ and $\mu_Z(J(r(\delta))) = \lambda(\bigcap_{b \in r(\delta)} b)$. Thus, from Lemma 3.6(v) it follows that $\eta_{C(\beta)}(g) = \eta_{C_{r(\delta)}}(g) = \mu_Z(J(r(\delta)))$. On the

other hand, for each $r \subseteq s$ with $\delta_r > 0$ we have $y_r = \theta_{C_r}(g)$ by Lemma 3.6(iv), hence

$$\prod_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} \theta_C^{\beta_C}(g) = \prod_{\substack{r \subseteq s \\ \delta_r > 0}} \theta_{C_r}^{\delta_r}(g) = \prod_{\substack{r \subseteq s \\ \delta_r > 0}} y_r^{\delta_r}.$$

Therefore

$$\binom{p}{\beta} \left(\prod_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} \theta_C^{\beta_C}(g) \right) \eta_{C(\beta)}(g) = \binom{p}{\delta} \left(\prod_{\substack{r \subseteq s \\ \delta_r > 0}} y_r^{\delta_r} \right) \mu_Z\big(J(r(\delta))\big).$$

This shows that each nonzero summand of (3.4) can be written as a summand of (3.3). Note also that if $\delta' \in \Delta$ satisfies $\mu_Z(J(r(\delta'))) \neq 0$ and we define $\beta' = (\beta'_C)_{C \subseteq D(Z)} \in \Lambda$ as in (3.5) (with δ replaced by δ'), then $\beta \neq \beta'$ whenever $\delta \neq \delta'$.

Step 3. Let $\beta \in \Lambda$ such that

$$\left(\prod_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} \theta_C^{\beta_C}(g) \right) \eta_{C(\beta)}(g) \neq 0. \tag{3.6}$$

Fix $C \subseteq D(Z)$ with $\beta_C > 0$. We claim that $r_C := T_C \cap s$ satisfies $C_{r_C} = C$. Indeed, the inclusion $C_{r_C} \subseteq C$ is clear. To prove the reverse inclusion, we argue by contradiction. Suppose $C_{r_C} \subsetneq C$. By Lemma 3.6(ii), we have $\nu_C^1(J(r')) = 0$ whenever $r' \subseteq T_C$. Since we also have $\nu_C^1(J(r')) = 0$ for every $r' \subseteq s$ with $r' \not\subseteq T_C$ (by the very definition of ν_C^1), it follows that $\nu_C^1(g) = \sum_{r' \subseteq s} y_{r'} \nu_C^1(J(r')) = 0$, hence $\theta_C(g) = 0$, which contradicts (3.6). This shows that $C_{r_C} = C$, as claimed. Now Lemma 3.6(iii) ensures that

$$\nu_C^1(g) = \nu_{C_{r_C}}^1(g) = y_{r_C} \lambda \left(\bigcap_{b \in r_C} b \right).$$

Since $\nu_C^1(g) \neq 0$, the previous equality implies that $J(r_C) \neq \mathbf{0}$. From Lemma 3.6(iv) it follows that $y_{r_C} = \theta_{C_{r_C}}(g) = \theta_C(g)$.

Set $\delta = (\delta_r)_{r \subseteq s}$ by declaring

$$\delta_r := \begin{cases} \beta_C & \text{if } r = r_C \text{ for some } C \subseteq D(Z) \text{ with } \beta_C > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Then $\sum_{r \subseteq s} \delta_r = \sum_{C \subseteq D(Z)} \beta_C = p$, and hence $\delta \in \Delta$. From our previous considerations we deduce that

$$\prod_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} \theta_C^{\beta_C}(g) = \prod_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} y_{r_C}^{\beta_C} = \prod_{\substack{r \subseteq s \\ \delta_r > 0}} y_r^{\delta_r}.$$

Moreover, we claim that $\eta_{C(\beta)}(g) = \mu_Z(J(r(\delta)))$. Indeed, since

$$C(\beta) = \bigcup_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} C = \bigcup_{\substack{C \subseteq D(Z) \\ \beta_C > 0}} C_{r_C} = \bigcup_{\substack{r \subseteq s \\ \delta_r > 0}} C_r = C_{r(\delta)},$$

we have $\eta_{C(\beta)}(g) = \eta_{C_{r(\delta)}}(g)$, and so (3.6) implies that $\nu_{C_{r(\delta)}}^1(g) \neq 0$. Bearing in mind Lemma 3.6(iii) and the fact that $y_{r(\delta)} \neq 0$, we infer that $J(r(\delta)) \neq \mathbf{0}$. An

appeal to Lemma 3.6(v) now yields $\eta_{C(\beta)}(g) = \lambda(\bigcap_{b\in r(\delta)} b)$. On the other hand, the fact that

$$r(\delta) = \bigcup_{\substack{r\subseteq s\\ \delta_r>0}} r = \bigcup_{\substack{C\subseteq D(Z)\\ \beta_C>0}} r_C = \bigcup_{\substack{C\subseteq D(Z)\\ \beta_C>0}} T_C \cap s \subseteq Z$$

ensures that $\mu_Z(J(r(\delta))) = \lambda(\bigcap_{b\in r(\delta)} b)$. It follows that $\eta_{C(\beta)}(g) = \mu_Z(J(r(\delta)))$.

Therefore

$$\binom{p}{\beta}\left(\prod_{\substack{C\subseteq D(Z)\\ \beta_C>0}} \theta_C^{\beta_C}(g)\right)\eta_{C(\beta)}(g) = \binom{p}{\delta}\left(\prod_{\substack{r\subseteq s\\ \delta_r>0}} y_r^{\delta_r}\right)\mu_Z(J(r(\delta))).$$

This shows that each nonzero summand of (3.3) can be written as a summand of (3.4). Note that if $\beta' \in \Lambda$ satisfies (3.6) (with β replaced by β') and we define $\delta' = (\delta'_r)_{r\subseteq s} \in \Delta$ as in (3.7) (with β replaced by β'), then $\delta \neq \delta'$ whenever $\beta \neq \beta'$.

Thus, equality (3.3) holds true and the proof is over. □

The following folklore fact will allow us to prove Lemma 3.9 as an easy consequence of Lemma 3.7 above.

Remark 3.8. Let (Ω, Σ) be a measurable space. Write $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$, where $\Omega_j \subseteq \Omega_{j+1}$ for all $j \in \mathbb{N}$. Let $A \subseteq \Omega$ be a set such that, for each $j \in \mathbb{N}$, the intersection $A \cap \Omega_j$ belongs to the trace of Σ on Ω_j. Then $A \in \Sigma$.

Proof. For each $j \in \mathbb{N}$ we have $A \cap \Omega_j = E_j \cap \Omega_j$ for some $E_j \in \Sigma$. We claim that $A = \bigcup_{k\in\mathbb{N}} \bigcap_{j\geq k} E_j$. Indeed, we have

$$A = \bigcup_{k\in\mathbb{N}} A \cap \Omega_k = \bigcup_{k\in\mathbb{N}} \bigcap_{j\geq k} A \cap \Omega_j = \bigcup_{k\in\mathbb{N}} \bigcap_{j\geq k} E_j \cap \Omega_j \subseteq \bigcup_{k\in\mathbb{N}} \bigcap_{j\geq k} E_j.$$

On the other hand, for each $k \in \mathbb{N}$ we have

$$\bigcap_{j\geq k} E_j = \bigcup_{n\geq k} \bigcap_{j\geq k} E_j \cap \Omega_n \subseteq \bigcup_{n\geq k} E_n \cap \Omega_n = \bigcup_{n\geq k} A \cap \Omega_n = A.$$

It follows that $A = \bigcup_{k\in\mathbb{N}} \bigcap_{j\geq k} E_j \in \Sigma$. □

Lemma 3.9. *Let $Z \subseteq \mathfrak{B}^+$ be a set and $p \in \mathbb{N}$. Then the mapping*

$$\phi_{Z,p} : S'(K) \to \mathbb{R}, \quad \phi_{Z,p}(g) := \mu_Z(g^p)$$

is measurable with respect to the trace of $\mathrm{Ba}(C(K), w)$ *on* $S'(K)$.

Proof. Since $|\mathfrak{B}^+| = \mathfrak{c}$, there is a sequence $D(1), D(2), \ldots$ of finite partitions of $\mathfrak{B}^+$, each one being finer than $\{Z, \mathfrak{B}^+ \setminus Z\}$, such that:

- $D(j+1)$ is finer than $D(j)$ for all $j \in \mathbb{N}$;
- for every $s \subseteq \mathfrak{B}^+$ finite there is $j \in \mathbb{N}$ such that $|T \cap s| \leq 1$ for all $T \in D(j)$.

Indeed, let $\xi : \{0,1\}^{\mathbb{N}} \to \mathfrak{B}^+$ be any bijection and, for each $j \in \mathbb{N}$ and $\sigma \in \{0,1\}^j$, set

$$E_j^\sigma := \{x \in \{0,1\}^{\mathbb{N}} : x(i) = \sigma(i) \text{ for every } i = 1, \ldots, j\}.$$

Then the partitions

$$D(j) := \left\{\xi(E_j^\sigma) \cap Z : \sigma \in \{0,1\}^j\right\} \cup \left\{\xi(E_j^\sigma) \setminus Z : \sigma \in \{0,1\}^j\right\}, \quad j \in \mathbb{N},$$

fulfill the required properties.

Clearly, $S'(K) = \bigcup_{j\in\mathbb{N}} S'_{D(j)}(K)$ and $S'_{D(j)}(K) \subseteq S'_{D(j+1)}(K)$ for all $j \in \mathbb{N}$. The measurability of $\phi_{Z,p}$ with respect to the trace of $\mathrm{Ba}(C(K), w)$ on $S'(K)$ now follows from Lemma 3.7 and Remark 3.8. □

We have already gathered all the tools needed to prove the main result of this subsection:

Theorem 3.10. *The restriction of the supremum norm to $S'(K)$ is measurable with respect to the trace of* $\mathrm{Ba}(C(K), w)$ *on $S'(K)$.*

Proof. We fix a countable algebra $\mathcal{Z}$ on $\mathfrak{B}^+$ which separates the points of $\mathfrak{B}^+$ (the algebra of clopen subsets of $\{0,1\}^{\mathbb{N}}$ can be transferred to $\mathfrak{B}^+$ via any bijection between $\{0,1\}^{\mathbb{N}}$ and $\mathfrak{B}^+$). We claim that

$$\|g\| = \sup_{Z\in\mathcal{Z}} \limsup_{p\to\infty} \big(\mu_Z(g^{2p})\big)^{\frac{1}{2p}} \quad \text{for every } g \in C(K). \tag{3.8}$$

Indeed, the inequality "$\geq$" is obvious (each μ_Z is a probability measure). To verify the reverse inequality, fix $g \in C(K)$ and take $\varepsilon > 0$. By Lemma 2.4(i) there exist finite disjoint sets $r, s \subseteq \mathfrak{B}^+$ such that $|g(\mathcal{F})| \geq \|g\| - \varepsilon$ for every $\mathcal{F} \in \widehat{W(r,s)} \neq \emptyset$. Since $\mathcal{Z}$ separates the points of $\mathfrak{B}^+$, we can find $Z \in \mathcal{Z}$ such that $r \subseteq Z$ and $s \cap Z = \emptyset$, hence $\mu_Z(W(r,s)) = \lambda(\bigcap_{b\in r} b) > 0$. Since

$$\big(\mu_Z(g^{2p})\big)^{\frac{1}{2p}} \geq (\|g\| - \varepsilon)\big(\mu_Z(W(r,s))\big)^{\frac{1}{2p}} \quad \text{for every } p \in \mathbb{N},$$

we have

$$\limsup_{p\to\infty} \big(\mu_Z(g^{2p})\big)^{\frac{1}{2p}} \geq (\|g\| - \varepsilon) \lim_{p\to\infty} \big(\mu_Z(W(r,s))\big)^{\frac{1}{2p}} = \|g\| - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, equality (3.8) holds true.

Once we know that $\|\cdot\|$ is expressed by formula (3.8), the assertion follows from Lemma 3.9. □

3.2. **Measurability on the set of simple functions.** Any element of $S(K)$ admits a representation which cannot be simplified in a sense, as the following lemma shows.

Lemma 3.11. *Let $g \in S(K)$. Then there exist a finite set $s \subseteq \mathfrak{B}^+$ and a collection of real numbers $\{z_r : r \subseteq s\}$ such that:*

(i) $g = \sum_{r\subseteq s} z_r 1_{\widehat{W(r,s\setminus r)}}$;
(ii) *there is no $s' \subsetneq s$ such that*

$$z_r = z_{r'} \quad \textit{whenever} \quad r \cap s' = r' \cap s',\ \bigcap_{b\in r} b \neq \mathbf{0} \textit{ and } \bigcap_{b\in r'} b \neq \mathbf{0}.$$

Proof. Of course, we can write g as in (i). To get a representation satisfying (ii), we proceed by induction on $|s|$. The case $s = \emptyset$ being obvious, we assume that $s \neq \emptyset$ and that the induction hypothesis holds. Assume that (ii) fails and fix $s' \subsetneq s$ such that

$$z_r = z_{r'} \quad \text{whenever} \quad r \cap s' = r' \cap s',\ \bigcap_{b\in r} b \neq \mathbf{0} \text{ and } \bigcap_{b\in r'} b \neq \mathbf{0}.$$

For any $t \subseteq s'$ with $\bigcap_{b\in t} b \neq \mathbf{0}$, let $\mathcal{A}_t$ be the collection of all $r \subseteq s$ such that $r \cap s' = t$ and $\bigcap_{b\in r} b \neq \mathbf{0}$. Then $z_r = z_t$ for every $r \in \mathcal{A}_t$ and

$$W(t, s' \setminus t) = \bigcup_{r\in\mathcal{A}_t} W(r, s \setminus r),$$

as can be easily checked. Hence

$$g = \sum_{r\subseteq s} z_r 1_{\widehat{W(r,s\setminus r)}} = \sum_{t\subseteq s'} \sum_{\substack{r\subseteq s\\ r\cap s'=t}} z_r 1_{\widehat{W(r,s\setminus r)}} = \sum_{\substack{t\subseteq s'\\ \bigcap_{b\in t} b\neq\mathbf{0}}} \sum_{r\in\mathcal{A}_t} z_r 1_{\widehat{W(r,s\setminus r)}}$$

$$= \sum_{\substack{t\subseteq s'\\ \bigcap_{b\in t} b\neq\mathbf{0}}} \sum_{r\in\mathcal{A}_t} z_t 1_{\widehat{W(r,s\setminus r)}} = \sum_{\substack{t\subseteq s'\\ \bigcap_{b\in t} b\neq\mathbf{0}}} z_t 1_{\widehat{W(t,s'\setminus t)}} = \sum_{t\subseteq s'} z_t 1_{\widehat{W(t,s'\setminus t)}}.$$

Since $|s'| < |s|$, the induction hypothesis now ensures that g admits a representation satisfying both (i) and (ii). □

Definition 3.12. Let D be a finite partition of $\mathfrak{B}^+$. We denote by $A_D(K)$ the set of all $g \in S(K)$ which can be written as

$$g = \sum_{r\subseteq s} z_r 1_{\widehat{W(r,s\setminus r)}}$$

for some finite set $s \subseteq \mathfrak{B}^+$ and some collection of real numbers $\{z_r : r \subseteq s\}$ such that:

(i) $|T \cap s| = 1$ for every $T \in D$;
(ii) there is no $s' \subsetneq s$ such that

$$z_r = z_{r'} \quad \text{whenever} \quad r \cap s' = r' \cap s',\ \bigcap_{b\in r} b \neq \mathbf{0} \text{ and } \bigcap_{b\in r'} b \neq \mathbf{0}.$$

Our next step is to prove that the sets $A_D(K)$ defined above belong to the trace of $\mathrm{Ba}(C(K), w)$ on $S(K)$ (see Corollary 3.16 below). From now on we fix a *countable* algebra $\mathcal{Z}$ on $\mathfrak{B}^+$ which separates the points of $\mathfrak{B}^+$ (as in the proof of Theorem 3.10).

Lemma 3.13. *Let D be a finite partition of $\mathfrak{B}^+$ with $D \subseteq \mathcal{Z}$ and let $g \in A_D(K)$. Then for each $T_0 \in D$ there is $T \in \mathcal{Z}$ such that $\mu_{T\setminus T_0}(g) \neq \mu_{T\cup T_0}(g)$.*

Proof. Our proof is by contradiction. Suppose there is $T_0 \in D$ such that

$$\mu_{T\setminus T_0}(g) = \mu_{T\cup T_0}(g) \quad \text{for every } T \in \mathcal{Z}. \tag{3.9}$$

Write $g = \sum_{r\subseteq s} z_r 1_{\widehat{W(r,s\setminus r)}}$ as in Definition 3.12. Let $s' := s \setminus T_0 \subsetneq s$. In order to reach a contradiction, we claim that

$$z_r = z_{r'} \quad \text{whenever} \quad r \cap s' = r' \cap s',\ \bigcap_{b\in r} b \neq \mathbf{0} \text{ and } \bigcap_{b\in r'} b \neq \mathbf{0}.$$

Indeed, assume that $r \neq r'$ and proceed by induction on $|r \cap s'| = |r' \cap s'|$.

Suppose first that $r \cap s' = r' \cap s' = \emptyset$. Then we have $r = \emptyset$ and $r' = T_0 \cap s$ (or vice versa). By (3.9) we have $\mu_\emptyset(g) = \mu_{T_0}(g)$. Note that $\mu_\emptyset(g) = z_\emptyset$ and, writing $T_0 \cap s = \{b_0\}$, we have $\mu_{T_0}(g) = z_\emptyset(1 - \lambda(b_0)) + z_{T_0\cap s}\lambda(b_0)$. It follows that

$$z_\emptyset = z_\emptyset(1 - \lambda(b_0)) + z_{T_0\cap s}\lambda(b_0),$$

hence $z_\emptyset = z_{T_0\cap s}$, as required.

Suppose now that $r \cap s' = r' \cap s' \neq \emptyset$, together with $\bigcap_{b\in r} b \neq \mathbf{0} \neq \bigcap_{b\in r'} b$ and the inductive hypothesis. Since $r \neq r'$, we have either $b_0 \in r$ and $b_0 \notin r'$ or vice versa. We assume for instance that $b_0 \in r$ and $b_0 \notin r'$. Then

$$r = \{b_0\} \cup (r \cap s') \quad \text{and} \quad r' = r \cap s'.$$

By the inductive hypothesis,

$$z_{r_0} = z_{r_0 \cup \{b_0\}} \quad \text{for every } r_0 \subsetneq r'. \tag{3.10}$$

Set

$$T_1 := T_0 \cup \bigcup \{T \in D : T \cap s \subseteq r'\} \in \mathcal{Z}$$

and observe that $T_1 \cap s = r$. Writing

$$w(t, t') := \bigcap_{b\in t} b \setminus \bigcup_{b'\in t'} b' \in \mathfrak{B}$$

for any pair of finite sets $t, t' \subseteq \mathfrak{B}^+$, we have

(3.11)

$$\begin{aligned}\mu_{T_1}(g) = \sum_{r_0\subseteq r} z_{r_0}\lambda\big(w(r_0, r\setminus r_0)\big) &= \sum_{r_0\subseteq r'} z_{r_0}\lambda\big(w(r_0, r\setminus r_0)\big) + \sum_{\substack{r_0\subseteq r\\ b_0\in r_0}} z_{r_0}\lambda\big(w(r_0, r'\setminus r_0)\big)\\ &= z_{r'}\lambda\big(w(r', \{b_0\})\big) + \sum_{r_0\subsetneq r'} z_{r_0}\lambda\big(w(r_0, r\setminus r_0)\big)\\ &\quad + z_r\lambda\big(w(r, \emptyset)\big) + \sum_{r_0\subsetneq r'} z_{r_0\cup\{b_0\}}\lambda\big(w(r_0\cup\{b_0\}, r'\setminus r_0)\big).\end{aligned}$$

For each $r_0 \subsetneq r'$, the elements $w(r_0, r \setminus r_0)$ and $w(r_0 \cup \{b_0\}, r' \setminus r_0)$ are disjoint and their union is $w(r_0, r' \setminus r_0)$, hence (3.10) yields

$$z_{r_0}\lambda\big(w(r_0, r'\setminus r_0)\big) = z_{r_0}\lambda\big(w(r_0, r\setminus r_0)\big) + z_{r_0\cup\{b_0\}}\lambda\big(w(r_0\cup\{b_0\}, r'\setminus r_0)\big).$$

From (3.11) it follows that

$$\mu_{T_1}(g) = z_{r'}\lambda\big(w(r', \{b_0\})\big) + z_r\lambda\big(w(r, \emptyset)\big) + \sum_{r_0\subsetneq r'} z_{r_0}\lambda\big(w(r_0, r'\setminus r_0)\big). \tag{3.12}$$

Bearing in mind that $(T_1 \setminus T_0) \cap s = r'$, we also have

(3.13)

$$\mu_{T_1\setminus T_0}(g) = \sum_{r_0\subseteq r'} z_{r_0}\lambda\big(w(r_0, r'\setminus r_0)\big) = z_{r'}\lambda\big(w(r', \emptyset)\big) + \sum_{r_0\subsetneq r'} z_{r_0}\lambda\big(w(r_0, r'\setminus r_0)\big).$$

Since $\mu_{T_1\setminus T_0}(g) = \mu_{T_1}(g)$ (by (3.9)), equalities (3.12) and (3.13) yield

$$z_{r'}\lambda\big(w(r', \{b_0\})\big) + z_r\lambda\big(w(r, \emptyset)\big) = z_{r'}\lambda\big(w(r', \emptyset)\big),$$

therefore $z_r\lambda(\bigcap_{b\in r} b) = z_{r'}\lambda(\bigcap_{b\in r} b)$, and so $z_r = z_{r'}$. This finishes the proof. □

Remark 3.14. Let $g \in S(K)$ be written as $g = \sum_{r\subseteq s} z_r 1_{\widehat{W(r,s\setminus r)}}$ for some finite set $s \subseteq \mathfrak{B}^+$ and $z_r \in \mathbb{R}$. If $T, T' \subseteq \mathfrak{B}^+$ satisfy $T \cap s = T' \cap s$, then $\mu_T(g) = \mu_{T'}(g)$.

Proof. For every $r \subseteq s$ we have $r \subseteq T$ if and only if $r \subseteq T'$. In this case,

$$\mu_T(W(r, s)) = \lambda\left(\bigcap_{b\in r} b \setminus \bigcap_{b\in s\cap T} b\right) = \mu_{T'}(W(r, s)).$$

Hence $\mu_T(g) = \sum_{r\subseteq T} z_r\mu_T(W(r, s)) = \sum_{r\subseteq T'} z_r\mu_{T'}(W(r, s)) = \mu_{T'}(g)$. □

Lemma 3.15. *Let D be a finite partition of $\mathfrak{B}^+$ with $D \subseteq \mathcal{Z}$ and let $g \in S(K)$. Then $g \in A_D(K)$ if and only if the following two statements hold:*

($\star$) *for each $T_0 \in D$ there is $T \in \mathcal{Z}$ such that $\mu_{T\setminus T_0}(g) \neq \mu_{T\cup T_0}(g)$;*

($\star\star$) *for each $T_0 \in D$ and each finite partition D_0 of T_0 with $D_0 \subseteq \mathcal{Z}$, there is $Z_0 \in D_0$ such that $\mu_{T\setminus Z}(g) = \mu_{T\cup Z}(g)$ for every $Z \in D_0 \setminus \{Z_0\}$ and $T \in \mathcal{Z}$.*

Proof. "Only if" part. Suppose $g \in A_D(K)$ and write $g = \sum_{r\subseteq s} z_r 1_{\widehat{W(r,s\setminus r)}}$ as in Definition 3.12. Statement ($\star$) holds by Lemma 3.13. To check ($\star\star$), take $T_0 \in D$ and fix a finite partition D_0 of T_0 with $D_0 \subseteq \mathcal{Z}$. Since $T_0 \cap s$ is a singleton, there is $Z_0 \in D_0$ such that $Z \cap s = \emptyset$ for every $Z \in D_0 \setminus \{Z_0\}$, and so for any $T \in \mathcal{Z}$ we have $(T \setminus Z) \cap s = (T \cup Z) \cap s$, hence $\mu_{T\setminus Z}(g) = \mu_{T\cup Z}(g)$ (Remark 3.14).

"If" part. Write $g = \sum_{r\subseteq s} z_r 1_{\widehat{W(r,s\setminus r)}}$ for some finite set $s \subseteq \mathfrak{B}^+$ and some collection of real numbers $\{z_r : r \subseteq s\}$. By Lemma 3.11, this representation can be chosen in such a way that there is no $s' \subsetneq s$ such that

$$z_r = z_{r'} \quad \text{whenever} \quad r \cap s' = r' \cap s', \ \bigcap_{b\in r} b \neq \mathbf{0} \text{ and } \bigcap_{b\in r'} b \neq \mathbf{0}.$$

In order to prove that $g \in A_D(K)$ we only have to check that $|T_0 \cap s| = 1$ for every $T_0 \in D$. Observe first that, for each $T_0 \in D$, condition ($\star$) tells us that there is $T \in \mathcal{Z}$ such that $\mu_{T\setminus T_0}(g) \neq \mu_{T\cup T_0}(g)$, hence $(T \setminus T_0) \cap s \neq (T \cup T_0) \cap s$ (Remark 3.14) and so $T_0 \cap s \neq \emptyset$. Thus, we can find a finite partition $D' \subseteq \mathcal{Z}$ of $\mathfrak{B}^+$ finer than D such that $|T' \cap s| = 1$ for every $T' \in D'$. Therefore, $g \in A_{D'}(K)$.

Fix $T_0 \in D$ and set $D_0 := \{T' \in D' : T' \subseteq T_0\}$. By Lemma 3.13 applied to g and D', for each $T' \in D_0$ there is $T'' \in \mathcal{Z}$ such that $\mu_{T''\setminus T'}(g) \neq \mu_{T''\cup T'}(g)$. This fact and condition ($\star\star$) yield $|D_0| = 1$, that is, $D_0 = \{T_0\}$, and so $|T_0 \cap s| = 1$. As $T_0 \in D$ is arbitrary, $g \in A_D(K)$ and the proof is over. □

Corollary 3.16. *Let D be a finite partition of $\mathfrak{B}^+$. Then $A_D(K)$ belongs to the trace of* $\mathrm{Ba}(C(K), w)$ *on $S(K)$.*

Proof. We can assume without loss of generality (by enlarging $\mathcal{Z}$ if necessary) that $D \subseteq \mathcal{Z}$. Since $\mathcal{Z}$ is countable, Lemma 3.15 gives the result. □

Our next task is to prove that, under the assumption that $\mathfrak{c}$ is a Kunen cardinal, the restriction of the supremum norm to any set of the form $A_D(K)$ is relatively $\mathrm{Ba}(C(K), w)$-measurable (Lemma 3.21).

Lemma 3.17. *Let $\Phi : \mathfrak{B}^+ \to 2^{\mathcal{Z}}$ be the mapping defined by $\Phi(b) := (1_T(b))_{T\in\mathcal{Z}}$. Set $\Omega := \Phi(\mathfrak{B}^+)$ and let Σ be the trace of* $\mathrm{Borel}(2^{\mathcal{Z}})$ *on Ω. Then for each $n \in \mathbb{N}$ the mapping*

$$\Phi^n : ((\mathfrak{B}^+)^n, \otimes_n \sigma(\mathcal{Z})) \to (\Omega^n, \otimes_n \Sigma), \quad \Phi^n(b_1, \dots, b_n) := (\Phi(b_1), \dots, \Phi(b_n))$$

is an isomorphism of measurable spaces.

Proof. It suffices to prove the case $n = 1$. Clearly, Φ is one-to-one (because $\mathcal{Z}$ separates the points of $\mathfrak{B}^+$) and $\sigma(\mathcal{Z})$-Σ-measurable. On the other hand, for each $T_0 \in \mathcal{Z}$ we have

$$\Phi(T_0) = \{(1_T(b))_{T\in\mathcal{Z}} : b \in T_0\} = \Omega \cap \{(x_T)_{T\in\mathcal{Z}} \in 2^{\mathcal{Z}} : x_{T_0} = 1\} \in \Sigma,$$

hence Φ^{-1} is Σ-$\sigma(\mathcal{Z})$-measurable. □

Lemma 3.18. *Let D be a finite partition of $\mathfrak{B}^+$ with $D \subseteq \mathcal{Z}$, and let $T_0 \in D$ and $T \in \mathcal{Z}$. Let $g \in A_D(K)$. Write $g = \sum_{r \subseteq s} z_r 1_{\widehat{W(r,s\setminus r)}}$ as in Definition 3.12 and $T_0 \cap s = \{b_0\}$.*

(1) *The following statements are equivalent:*
   (i) $b_0 \in T$;
   (ii) $\mu_{\bar{T}}(g) = \mu_{\bar{T}\cup T_0}(g)$ *for every* $\bar{T} \in \mathcal{Z}$ *such that* $\bar{T} \cap T_0 = T \cap T_0$.

(2) *The following statements are equivalent:*
   (i') $b_0 \notin T$;
   (ii') $\mu_{\bar{T}}(g) = \mu_{\bar{T}\setminus T_0}(g)$ *for every* $\bar{T} \in \mathcal{Z}$ *such that* $\bar{T} \cap T_0 = T \cap T_0$.

Proof. (i)⇒(ii) Let $\bar{T} \in \mathcal{Z}$ be such that $\bar{T}\cap T_0 = T\cap T_0$. Since $b_0 \in T$ by assumption, we have $b_0 \in T\cap T_0 \subseteq \bar{T}$, and so $(\bar{T}\cup T_0)\cap s = \bar{T}\cap s$. Bearing in mind Remark 3.14, we get $\mu_{\bar{T}}(g) = \mu_{\bar{T}\cup T_0}(g)$. A similar argument yields (i')⇒(ii').

Now, in order to prove (ii)⇒(i) and (ii')⇒(i'), it is enough to check that statements (ii) and (ii') cannot hold simultaneously. To this end, pick $T^* \in \mathcal{Z}$ such that $\mu_{T^*\setminus T_0}(g) \neq \mu_{T^*\cup T_0}(g)$ (we apply Lemma 3.13) and set

$$\bar{T} := (T^* \setminus T_0) \cup (T \cap T_0) \in \mathcal{Z}.$$

Clearly, $\bar{T}\cup T_0 = T^*\cup T_0$ and $\bar{T}\setminus T_0 = T^*\setminus T_0$, hence we have either $\mu_{\bar{T}\cup T_0}(g) \neq \mu_{\bar{T}}(g)$ or $\mu_{\bar{T}\setminus T_0}(g) \neq \mu_{\bar{T}}(g)$. Since $\bar{T} \cap T_0 = T \cap T_0$, this shows that either (ii) or (ii') fails. □

Remark 3.19. Let $D = \{T_1, \dots, T_n\}$ be a finite partition of $\mathfrak{B}^+$ with $D \subseteq \mathcal{Z}$.

(1) Let $i \in \{1, \dots, n\}$. Given $T \in \mathcal{Z}$, since statements (ii) and (ii') in Lemma 3.18 (applied to T_i) are independent of the representation of $g \in A_D(K)$, there is a mapping $\psi_{D,T_i,T} : A_D(K) \to \{0,1\}$ such that, for any $g = \sum_{r\subseteq s} z_r 1_{\widehat{W(r,s\setminus r)}}$ as in Definition 3.12 and writing $T_i \cap s = \{b_i\}$, we have

$$\psi_{D,T_i,T}(g) := \begin{cases} 1 & \text{if } b_i \in T, \\ 0 & \text{if } b_i \notin T. \end{cases}$$

$\psi_{D,T_i,T}$ is measurable with respect to the trace of $\mathrm{Ba}(C(K), w)$ on $A_D(K)$, thanks to Lemma 3.18 (applied to T_i). Define $\psi_{D,T_i} : A_D(K) \to 2^{\mathcal{Z}}$ by $\psi_{D,T_i}(g) := (\psi_{D,T_i,T}(g))_{T\in\mathcal{Z}}$. Observe that

$$\psi_{D,T_i}(A_D(K)) \subseteq \Omega,$$

because for any $g \in A_D(K)$ as above we have $\psi_{D,T_i}(g) = \Phi(b_i)$.

(2) Thus, we can consider the mapping

$$\psi_D : A_D(K) \to \Omega^n, \quad \psi_D(g) := (\psi_{D,T_1}(g), \dots, \psi_{D,T_n}(g)).$$

Clearly, ψ_D is measurable with respect to $\otimes_n \Sigma$ and the trace of $\mathrm{Ba}(C(K), w)$ on $A_D(K)$.

(3) Let $P \subseteq \{1, \dots, n\}$. Define $\zeta_{n,P} : (\mathfrak{B}^+)^n \to \mathbb{R}$ by

$$\zeta_{n,P}(b'_1, \dots, b'_n) := \lambda\left(\bigcap_{i\in P} b'_i\right).$$

Then the mapping $L_{D,P} : A_D(K) \to \mathbb{R}$ given by $L_{D,P} := \zeta_{n,P} \circ (\Phi^n)^{-1} \circ \psi_D$ satisfies $L_{D,P}(g) = \lambda(\bigcap_{i\in P} b_i)$ for every $g \in A_D(K)$ as above.

From now on we deal with the additional assumption that $\mathfrak{c}$ is a Kunen cardinal.

Lemma 3.20. *Suppose $\mathfrak{c}$ is a Kunen cardinal. Then there is a countable algebra $\mathcal{Z}_0$ on $\mathfrak{B}^+$ separating the points of $\mathfrak{B}^+$ such that, for each $n \in \mathbb{N}$ and $P \subseteq \{1, \dots, n\}$, the mapping $\zeta_{n,P}$ is $\otimes_n \sigma(\mathcal{Z}_0)$-measurable.*

Proof. Since $|\mathfrak{B}^+| = \mathfrak{c}$ is a Kunen cardinal, each $\zeta_{n,P}$ is $\otimes_n \mathcal{P}(\mathfrak{B}^+)$-measurable. Thus, we can find a countable family $\mathcal{C}$ of subsets of $\mathfrak{B}^+$ such that $\zeta_{n,P}$ is $\bigotimes_n \sigma(\mathcal{C})$-measurable for every $n \in \mathbb{N}$ and every $P \subseteq \{1, \dots, n\}$. Now, it is enough to choose any countable algebra $\mathcal{Z}_0$ on $\mathfrak{B}^+$ which separates the points of $\mathfrak{B}^+$ and contains $\mathcal{C}$. □

Lemma 3.21. *Suppose $\mathfrak{c}$ is a Kunen cardinal. Let D be a finite partition of $\mathfrak{B}^+$. Then the restriction of the supremum norm to $A_D(K)$ is measurable with respect to the trace of* $\mathrm{Ba}(C(K), w)$ *on $A_D(K)$.*

Proof. Write $D = \{T_1, \dots, T_n\}$. We can suppose without loss of generality (by enlarging $\mathcal{Z}$ if necessary) that $D \subseteq \mathcal{Z}$ and that all functions $\zeta_{n,P}$ are $\bigotimes_n \sigma(\mathcal{Z})$-measurable (see Lemma 3.20). For each $P \subseteq \{1, \dots, n\}$, define

$$N_{D,P} : A_D(K) \to \{0, 1\}, \quad N_{D,P}(g) := \begin{cases} 1 & \text{if } L_{D,P}(g) \neq 0, \\ 0 & \text{if } L_{D,P}(g) = 0. \end{cases}$$

Since $L_{D,P}$ is measurable with respect to the trace of $\mathrm{Ba}(C(K), w)$ on $A_D(K)$ (combine Lemma 3.17 and Remark 3.19), the same holds for $N_{D,P}$.

Fix $g \in A_D(K)$ and write

$$g = \sum_{r \subseteq s} y_r 1_{\widehat{J(r)}}$$

for some $s \subseteq \mathfrak{B}^+$ finite with $|T_i \cap s| = 1$ for every $i \in \{1, \dots, n\}$ and some collection of real numbers $\{y_r : r \subseteq s\}$ (see the proof of Lemma 3.2). Lemma 3.6(iv) ensures that

$$y_r = \theta_{C_r}(g) \quad \text{for every } r \subseteq s \text{ with } J(r) \neq \mathbf{0}.$$

Hence $g = \sum_{r \subseteq s} \theta_{C_r}(g) 1_{\widehat{J(r)}}$, and therefore

$$g = \sum_{r' \subseteq s} \left(\sum_{r \subseteq r'} \theta_{C_r}(g) \right) 1_{W\widehat{(r', s \setminus r')}} \tag{3.14}$$

(see again the proof of Lemma 3.2). Write $T_i \cap s = \{b_i\}$ for every $i \in \{1, \dots, n\}$ and observe that for each $P \subseteq \{1, \dots, n\}$ we have

$$N_{D,P}(g) = \begin{cases} 1 & \text{if } \bigcap_{i \in P} b_i \neq \mathbf{0}, \\ 0 & \text{if } \bigcap_{i \in P} b_i = \mathbf{0}. \end{cases}$$

Define $C(Q) := \{T_i : i \in Q\}$ for every $Q \subseteq \{1, \dots, n\}$. From (3.14) it follows that

$$\|g\| = \sup_{\substack{r' \subseteq s \\ J(r') \neq \mathbf{0}}} \left| \sum_{r \subseteq r'} \theta_{C_r}(g) \right| = \sup_{P \subseteq \{1, \dots, n\}} \left| \sum_{Q \subseteq P} \theta_{C(Q)}(g) \right| \cdot N_{D,P}(g).$$

As $g \in A_D(K)$ is arbitrary, the norm function $\| \cdot \|$ coincides with

$$\sup_{P \subseteq \{1, \dots, n\}} \left| \sum_{Q \subseteq P} \theta_{C(Q)}(\cdot) \right| \cdot N_{D,P}(\cdot)$$

on $A_D(K)$, and so $\|\cdot\|$ is measurable with respect to the trace of $\mathrm{Ba}(C(K), w)$ on $A_D(K)$. The proof is over. □

Finally, we arrive at the main result of this subsection:

Theorem 3.22. *Suppose $\mathfrak{c}$ is a Kunen cardinal. Then the restriction of the supremum norm to $S(K)$ is measurable with respect to the trace of* $\mathrm{Ba}(C(K), w)$ *on $S(K)$.*

Proof. Let Π be the collection of all partitions of $\mathfrak{B}^+$ into finitely many elements of $\mathcal{Z}$. By Lemma 3.11, we can write $S(K) = \bigcup_{D\in\Pi} A_D(K)$. Since Π is countable and each $A_D(K)$ belongs to the trace of $\mathrm{Ba}(C(K), w)$ on $S(K)$ (see Corollary 3.16), the result follows from Lemma 3.21. □

3.3. **Some open problems.**

(A) Let L be a compact space. Is the $\mathrm{Ba}(C(L), w)$-measurability of the supremum norm on $C(L)$ equivalent to the w^*-separability of $B_{C(L)^*}$ or $C(L)^*$? What about the compact space K considered in this paper?

(B) Let $(X, \|\cdot\|)$ be a Banach space and suppose $A \subseteq X$ is a norm dense set (or linear subspace) such that $\|\cdot\|$ is relatively $\mathrm{Ba}(X, w)$-measurable on A. Does this imply that $\|\cdot\|$ is $\mathrm{Ba}(X, w)$-measurable on X?

Note that the analogous question for the property "B_{X^*} is w^*-separable" has a positive answer (just apply the Hahn-Banach theorem), while for the property "X^* is w^*-separable" it has a negative answer; see [5, Example 1.1].

(C) Let L be a compact space and consider the 'square' mapping

$$S : C(L) \to C(L), \quad S(g) := g^2.$$

Which conditions on L ensure that S is $\mathrm{Ba}(C(L), w)$-measurable?

Note that if L carries a strictly positive measure, say μ, then the supremum norm on $C(L)$ can be computed as

$$\|g\| = \lim_{n\to\infty} \left(\int_L f^{2n}\, d\mu \right)^{\frac{1}{2n}},$$

hence $\|\cdot\|$ is $\mathrm{Ba}(C(L), w)$-measurable whenever S is $\mathrm{Ba}(C(L), w)$-measurable. What about the compact space K considered in this paper?

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