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ON THE CONSISTENCY OF THE COMBINATORIAL CODIFFERENTIAL

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Abstract. In 1976, Dodziuk and Patodi employed Whitney forms to define a combinatorial codifferential operator on cochains, and they raised the question whether it is consistent in the sense that for a smooth enough differential form the combinatorial codifferential of the associated cochain converges to the exterior codifferential of the form as the triangulation is refined. In 1991, Smits proved this to be the case for the combinatorial codifferential applied to 1-forms in two dimensions under the additional assumption that the initial triangulation is refined in a completely regular fashion, by dividing each triangle into four similar triangles. In this paper we extend the result of Smits to arbitrary dimensions, showing that the combinatorial codifferential on 1-forms is consistent if the triangulations are uniform or piecewise uniform in a certain precise sense. We also show that this restriction on the triangulations is needed, giving a counterexample in which a different regular refinement procedure, namely Whitney's standard subdivision, is used. Further, we show by numerical example that for 2-forms in three dimensions, the combinatorial codifferential is not consistent, even for the most regular subdivision process.

1. Introduction

Let M be an n-dimensional polytope in $\mathbb{R}^n$, triangulated by a simplicial complex $\mathcal{T}_h$ of maximal simplex diameter h, which we orient by fixing an order for the vertices. (Although we restrict ourselves to polytopes for simplicity, several of the results below can easily be extended to triangulated Riemannian manifolds.) We denote by $\Lambda^k = \Lambda^k(M)$ the space of smooth differential k-forms on M. The Euclidean inner product restricted to M determines the Hodge star operator $\Lambda^k \to \Lambda^{n-k}$, and the inner product on Λ^k given by $\langle u, v\rangle = \int u \wedge \star v$. The space $L^2\Lambda^k$ is the completion of Λ^k with respect to this norm, i.e., the space of differential k-forms with coefficients in L^2. We then define $H\Lambda^k$ to be the space of forms u in $L^2\Lambda^k$ whose exterior derivative du, which may be understood in the sense of distributions, belongs to $L^2\Lambda^{k+1}$. These spaces combine to form the L^2 de Rham complex

$$0 \to H\Lambda^0 \xrightarrow{d} H\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n \to 0.$$

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Viewing the exterior derivative d as an unbounded operator $L^2\Lambda^k$ to $L^2\Lambda^{k+1}$ with domain $H\Lambda^k$, we may define its adjoint d^*. Thus a differential k-form u belongs to the domain of d^* if the operator $v \mapsto \langle u, dv\rangle_{L^2\Lambda^k}$ is bounded on $L^2\Lambda^{k-1}$, and then

$$\langle d^*u, v\rangle_{L^2\Lambda^{k-1}} = \langle u, dv\rangle_{L^2\Lambda^k}, \quad v \in H\Lambda^{k-1}.$$

In particular, every u which is smooth and supported in the interior of M belongs to the domain of d^* and $d^*u = (-1)^{k(n-k+1)} \star d \star u$.

Let $\Delta_k(\mathcal{T}_h)$ denote the set of k-dimensional simplices of $\mathcal{T}_h$. We denote by $C_k(\mathcal{T}_h)$ the space of formal linear combinations of elements of $\Delta_k(\mathcal{T}_h)$ with real coefficients, the space of k-chains, and by $C^k(\mathcal{T}_h) = C_k(\mathcal{T}_h)^*$ the space of k-cochains. The coboundary maps $d^c : C^k(\mathcal{T}_h) \to C^{k+1}(\mathcal{T}_h)$ then determine the cochain complex. The *de Rham map* R_h maps Λ^k onto $C^k(\mathcal{T}_h)$ taking a differential k-form u to the cochain

$$R_h u : C_k(\mathcal{T}_h) \to \mathbb{R}, \quad c \mapsto \int_c u. \tag{1.1}$$

The canonical basis for $C^k(\mathcal{T}_h)$ consists of the cochains a_τ, $\tau \in \Delta_k(\mathcal{T}_h)$, where a_τ takes the value 1 on τ and zero on the other elements of $\Delta_k(\mathcal{T}_h)$. The associated *Whitney form* is given by

$$W_h a_\tau = k! \sum_{i=0}^{k} (-1)^i \lambda_i \, d\lambda_0 \wedge \cdots \wedge \widehat{d\lambda_i} \wedge \cdots \wedge d\lambda_k,$$

where $\lambda_0, \ldots, \lambda_k$ are the piecewise linear basis functions associated to the vertices of the simplex listed, i.e., λ_i is the continuous piecewise linear function equal to 1 at the ith vertex of τ and vanishing at all the other vertices of the triangulation. The span of $W_h a_\tau$, $\tau \in \Delta_k(\mathcal{T}_h)$, defines the space of Λ_h^k of Whitney k-forms. Its elements are piecewise affine differential k-forms which belong to $H\Lambda^k$ and satisfy $d\Lambda_h^k \subset \Lambda_h^{k+1}$. Thus the Whitney forms comprise a finite-dimensional subcomplex of the L^2 de Rham complex called the Whitney complex:

$$0 \to \Lambda_h^0 \xrightarrow{d} \Lambda_h^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda_h^n \to 0.$$

The *Whitney map* W_h maps $C^k(\mathcal{T}_h)$ isomorphically onto Λ_h^k and satisfies

$$W_h d^c c = dW_h c, \quad c \in C^k(\mathcal{T}_h), \tag{1.2}$$

i.e., is a cochain isomorphism of the cochain complex onto the Whitney complex. Although Whitney k-forms need not be continuous, each has a well-defined trace on the simplices in $\Delta_k(\mathcal{T}_h)$, so the de Rham map (1.1) is defined for $u \in \Lambda_h^k$. The Whitney map is a one-sided inverse of the de Rham map: $R_h W_h c = c$ for $c \in C^k(\mathcal{T}_h)$. The reverse composition $\pi_h = W_h R_h : \Lambda^k \to \Lambda_h^k$ defines the *canonical projection* into Λ_h^k.

In [3] and [4], Dodziuk and Patodi defined an inner product on cochains by declaring the Whitney map to be an isometry:

$$\langle a, b\rangle = \langle W_h a, W_h b\rangle_{L^2\Lambda^k}, \quad a, b \in C^k(\mathcal{T}_h). \tag{1.3}$$

They then used this inner product to define the adjoint δ^c of the coboundary:

$$\langle \delta^c a, b\rangle = \langle a, d^c b\rangle, \quad a, b \in C^k(\mathcal{T}_h). \tag{1.4}$$

Since the coboundary operator d^c may be viewed as a combinatorial version of the differential operator of the de Rham complex, its adjoint δ^c may be viewed as a

combinatorial codifferential, and together they define the combinatorial Laplacian on cochains given by

$$\Delta^c = d^c\delta^c + \delta^c d^c : C^k(\mathcal{T}_h) \to C^k(\mathcal{T}_h).$$

The work of Dodziuk and Patodi concerned the relation between the eigenvalues of this combinatorial Laplacian and those of the Hodge Laplacian.

Dodziuk and Patodi asked whether the combinatorial codifferential δ^c is a consistent approximation of d^* in the sense that if we have a sequence of triangulations $\mathcal{T}_h$ with maximum simplex diameter tending to zero and satisfying some regularity restrictions, then

$$\lim_h \|W_h\delta^c R_h u - d^* u\| = 0, \tag{1.5}$$

for sufficiently smooth $u \in \Lambda^k$ belonging to the domain of d^*. Here and henceforth the norm $\|\cdot\|$ denotes the L^2 norm.

Since $C^k(\mathcal{T}_h)$ and Λ_h^k are isometric, we may state this question in terms of Whitney forms, without invoking cochains. Define the Whitney codifferential $d_h^* : \Lambda_h^k \to \Lambda_h^{k-1}$ by

$$\langle d_h^* u, v\rangle_{L^2\Lambda^{k-1}} = \langle u, dv\rangle_{L^2\Lambda^k}, \quad u \in \Lambda_h^k,\ v \in \Lambda_h^{k-1}. \tag{1.6}$$

Combining (1.2), (1.3), and (1.4), we see that $d_h^* = W_h\delta^c W_h^{-1}$. Therefore, $W_h\delta^c R_h = d_h^*\pi_h$, and the question of consistency becomes whether

$$\lim_h \|d_h^*\pi_h u - d^* u\| = 0, \tag{1.7}$$

for smooth u in the domain of d^*.

In Appendix II of [4], the authors suggest a counterexample to (1.7) for 1-forms (i.e., $k = 1$) on a two-dimensional manifold, but, as pointed out by Smits [7], the example is not valid, and the question has remained open. Smits himself considered the question, remaining in the specific case of 1-forms on a two-dimensional manifold, and restricting himself to a sequence of triangulations obtained by *regular standard subdivision*, meaning that the triangulation is refined by dividing each triangle into four similar triangles by connecting the midpoints of the edges, resulting in a piecewise uniform sequence of triangluations. See Figure 5 for an example. In this case, Smits proved that (1.5) or, equivalently, (1.7) holds.

Smits's result leaves open various questions. Does the consistency of the 1-form codifferential on regular meshes in two dimensions extend to

- Mesh sequences which are not obtained by regular standard subdivision?
- More than two dimensions?
- The combinatorial codifferential on k-forms with $k > 1$?

In this paper we show that the answer to the second question is affirmative, but the answers to the first and third are negative. More precisely, in Section 2 we present a simple counterexample to consistency for a quadratic 1-form on the sequence of triangulations shown in Figure 1. While these meshes are not obtained by regular standard subdivision, they may be obtained by another systematic subdivision process, *standard subdivision*, as defined by Whitney in [8, Appendix II, § 4]. Next, in Section 3, we recall a definition of *uniform* triangulations in n-dimensions which was formulated in the study of superconvergence of finite element methods, and we use the superconvergence theory to extend Smits's result on the consistency of the combinatorial codifferential on 1-forms to n-dimensions, for triangulations that are

uniform or piecewise uniform. In Section 4, we provide computational confirmation of these results, both positive and negative. Finally, in Section 5, we numerically explore the case of 2-forms in three dimensions and find that the combinatorial codifferential is inconsistent, even for completely uniform mesh sequences.

2. A counterexample to consistency

We take as our domain M the square $(-1,1)\times(-1,1)\subset\mathbb{R}^2$, and as initial triangulation the division into four triangles obtained via drawing the two diagonals. We refine a triangulation by subdividing each triangle into four using standard subdivision. In this way we obtain the sequence of *crisscross triangulations* shown in Figure 1, with the mth triangulation consisting of 4^m isoceles right triangles. We index the triangulation by the diameter of its elements, so we denote the mth triangulation by $\mathcal{T}_h$ where $h=4/2^m$. Using this triangulation, the authors of [5] showed that superconvergence does not hold for piecewise linear Lagrange elements.

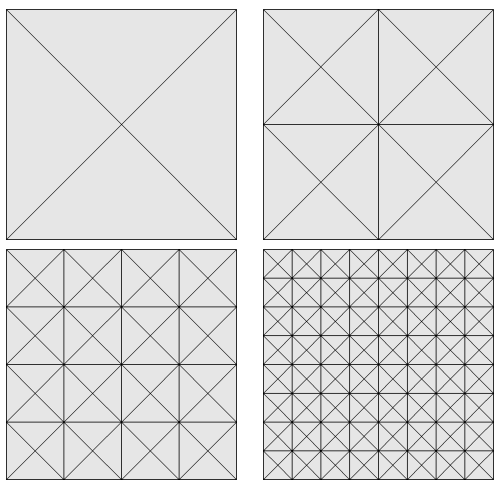

Figure 1. $\mathcal{T}_2$, $\mathcal{T}_1$, $\mathcal{T}_{1/2}$, $\mathcal{T}_{1/4}$, the first four crisscross triangulations.

Define $p:M\to\mathbb{R}$ by $p(x,y)=x-x^3/3$ and let $u=dp=(1-x^2)dx\in\Lambda^1(M)$. Now for $q\in H\Lambda^0(M)$ (i.e., the Sobolev space $H^1(M)$), we have

$$\star dq=\star\left(\frac{\partial q}{\partial x}dx+\frac{\partial q}{\partial y}dy\right)=\frac{\partial q}{\partial x}dy-\frac{\partial q}{\partial y}dx,$$

so

$$\langle u,dq\rangle_{L^2\Lambda^1}=\int_M u\wedge\star dq=\int_M(1-x^2)\frac{\partial q}{\partial x}\,dx\,dy=\int_M 2xq\,dx\,dy=\langle 2x,q\rangle_{L^2\Lambda^0}.$$

Thus u belongs to the domain of d^* and $d^*u=2x$. As an alternative verification, we may identify 1-forms and vector fields. Then u corresponds to the vector field $(1-x^2,0)$ which has vanishing normal component on ∂M, and so belongs to the domain of $d^*=-\operatorname{div}$ and $d^*u=-\operatorname{div}(1-x^2,0)=2x$.

Set $w_h = d_h^* \pi_h u$. Now $w_h \in \Lambda_h^0$, i.e., it is a continuous piecewise linear function. The projections π_h into the Whitney forms form a cochain map, so $\pi_h u = \pi_h dp = d\pi_h p = \operatorname{grad} \pi_h p$, where $\pi_h p$ is piecewise linear interpolant of p. Thus $w_h \in \Lambda_h^0$ is determined by the equations

$$\int_M w_h q \, dx \, dy = \int_M \operatorname{grad} \pi_h p \cdot \operatorname{grad} q \, dx \, dy, \quad q \in \Lambda_h^0. \tag{2.1}$$

It turns out that we can give the solution to this problem explicitly. Since w_h is a continuous piecewise linear function, it is determined by its values at the vertices of the triangulation $\mathcal{T}_h$. The coordinates of the vertices are integer multiples of $h/2$. In fact the value of w_h at a vertex (x, y) depends only on x and for $h \leq 1$ is given by

$$w_h(x,y) = \begin{cases} -h, & x = -1, \\ 0, & -1 < x < 1, \ x \text{ a multiple of } h, \\ h, & x = 1, \\ -6 + 2h, & x = -1 + h/2, \\ 6x, & -1 + h/2 < x < 1 - h/2, \ x \text{ an odd multiple of } h/2, \\ 6 - 2h, & x = 1 - h/2. \end{cases}$$

A plot of the piecewise linear function w_h is shown in Figure 3 for $h = 1/2$. To verify the formula it suffices to check (2.1) for all piecewise linear functions q that vanish on all vertices except one. There are several cases depending on how close the vertex is to the boundary, and the computation is tedious, but elementary. Here we only give the details when the vertex is (x, y) with $-1 + h/2 < x < 1 - h/2$ and x is an odd multiple of $h/2$. To this end, let q be the piecewise linear function that is one on vertex (x, y) and vanishes on all the remaining vertices. In this case, the support of q is the union of the four triangles T_1, T_2, T_3, T_4 that have (x, y) as a vertex (see Figure 2). According to the formula, in the support of q, one has $w_h = 6\,x\,q$. A simple calculation then shows that the left-hand side of (2.1) is

$$\int_M w_h q \, dx \, dy = 6\,x \sum_{i=1}^{4} \int_{T_i} q^2 dx \, dy = 4\,x\,m,$$

where $m = h^2/4 = |T_i|$ for any i.

To calculate the right-hand side of (2.1) for this q, we calculate that

$$\operatorname{grad} q = \frac{2}{h} \begin{cases} (1,0), & \text{on } T_1, \\ (0,1), & \text{on } T_2, \\ (-1,0), & \text{on } T_3, \\ (0,-1), & \text{on } T_4, \end{cases}$$

and

$$\operatorname{grad} \pi_h p = \frac{2}{h} \begin{cases} (p(x) - p(x - \frac{h}{2}), 0), & \text{on } T_1, \\ (\frac{1}{2}[p(x + \frac{h}{2}) - p(x - \frac{h}{2})], p(x) - \frac{1}{2}[p(x + \frac{h}{2}) + p(x - \frac{h}{2})]), & \text{on } T_2, \\ (p(x + \frac{h}{2}) - p(x), 0), & \text{on } T_3, \\ (\frac{1}{2}[p(x + \frac{h}{2}) - p(x - \frac{h}{2})], \frac{1}{2}[p(x + \frac{h}{2}) + p(x - \frac{h}{2})] - p(x)), & \text{on } T_4. \end{cases}$$

Hence,

$$\int_M \operatorname{grad} \pi_h p \cdot \operatorname{grad} q \, dx \, dy = \sum_{i=1}^{4} \int_{T_i} \pi_h p \cdot \operatorname{grad} q \, dx \, dy$$
$$= \frac{16}{h^2}(p(x) - \frac{1}{2}[p(x - \frac{h}{2}) + p(x + \frac{h}{2})])m = 4\,xm.$$

This verifies (2.1) for this piecewise linear function q.

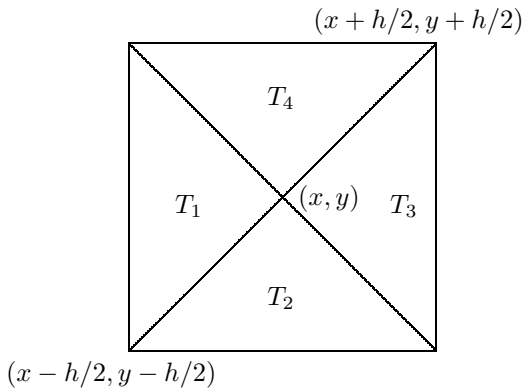

FIGURE 2. The support of the piecewise linear function q.

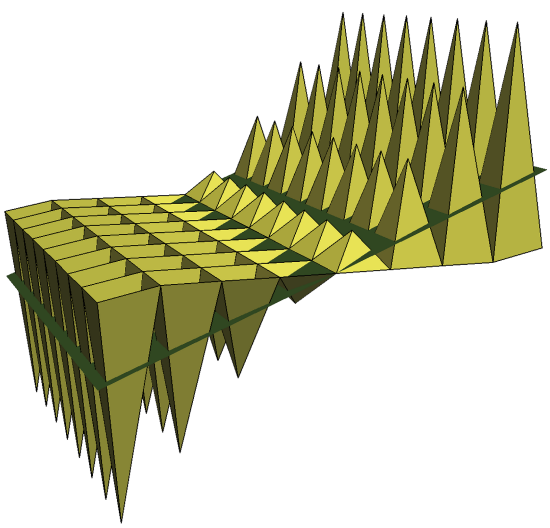

FIGURE 3. The spiked surface is the graph of the piecewise linear function $w_h = d_h^* \pi_h f$ for $h = 1/4$. The plane is the graph of the linear function $d^* u$. (Color available online.)

Finally, we note that, since w_h essentially oscillates between $6x$ and 0, it does not converge in L^2 to $d^* u$ (or to anything else) as h tends to zero.

3. Consistency for 1-forms on piecewise uniform meshes

We continue to consider a sequence of triangulations $\mathcal{T}_h$ indexed by a positive parameter h tending to 0. We take h to be equivalent to the maximal simplex diameter

$$ch \le \max_{T\in\Delta_n(\mathcal{T}_h)} \operatorname{diam} T \le Ch,$$

for some positive constants C, c independent of h (throughout we denote by C and c generic constants, not necessarily the same in different occurrences). We also assume that the sequence of triangulations is *shape regular* in the sense that there exists $c > 0$ such that

$$\rho(T) \ge c \operatorname{diam} T,$$

for all $T \in \mathcal{T}_h$ and all h, where $\rho(T)$ is the diameter of the ball inscribed in T.

We begin with some estimates for the approximation of a k-form by an element of Λ_h^k. For this we need to introduce the spaces of differential forms with coefficients in a Sobolev space. Let m be a non-negative integer and u a k-form defined on a domain $M \subset \mathbb{R}^n$, which we may expand as

$$u = \sum_{1\le i_1<\cdots<i_k\le n} u_{i_1\cdots i_k}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}. \tag{3.1}$$

Using multi-index notation for partial derivatives of the coefficients $u_{i_1\cdots i_k}$, we define the mth Sobolev norm and seminorm by

$$\|u\|^2_{H^m\Lambda^k} = \sum_{1\le i_1<\cdots<i_k\le n}\ \sum_{|\alpha|\le m} \|D^\alpha u_{i_1\cdots i_k}\|^2_{L^2(M)},$$
$$|u|^2_{H^m\Lambda^k} = \sum_{1\le i_1<\cdots<i_k\le n}\ \sum_{|\alpha|= m} \|D^\alpha u_{i_1\cdots i_k}\|^2_{L^2(M)},$$

and define the space $H^m\Lambda^k(M)$ to consist of all k-forms in M for which the Sobolev norm $\|u\|_{H^m\Lambda^k}$ is finite.

With this notation, we can state the basic approximation result that for any shape regular sequence of triangulations there is a constant C such that

$$\inf_{v\in\Lambda_h^k} \|u-v\| \le Ch\|u\|_{H^1\Lambda^k}, \quad u\in H^1\Lambda^k(M). \tag{3.2}$$

For a proof, see [1, Theorem 5.8]. Since $H^1\Lambda^k$ is dense in $L^2\Lambda^k$, this implies that

$$\operatorname{dist}(f,\Lambda_h^k) := \inf_{v\in\Lambda_h^k} \|f-v\| \to 0 \text{ as } h\to 0, \quad f\in L^2\Lambda^k(M). \tag{3.3}$$

In addition to the best approximation estimate (3.2), we also need an $O(h)$ estimate on the projection error $\|u-\pi_h u\|$. For this we require more regularity of u, since $\pi_h u$ is defined in terms of traces of u on k-dimensional faces, which need not be defined on $H^1\Lambda^k$.

Lemma 3.1. *Let $\{\mathcal{T}_h\}$ be a shape regular sequence of triangulations of $M\subset\mathbb{R}^n$ and k an integer between 0 and n. Let ℓ be the smallest integer so that $\ell > (n-k)/2$. Then there exists a constant C, depending only on n and the shape regularity constant, such that*

$$\|\pi_h u - u\|_{L^2\Lambda^k} \le C\sum_{m=1}^{\ell} h^m |u|_{H^m\Lambda^k}, \quad u\in H^\ell\Lambda^k(M). \tag{3.4}$$

Proof. First we note that the canonical projection is defined simplex by simplex, as

$$(\pi_h u)|_T = \pi_T(u|_T),$$

where, for v a k-form on T, $\pi_T v$ is its interpolant into the space of Whitney forms on the single simplex T. Therefore, it is enough to prove that

$$\|u - \pi_T u\|_{L^2\Lambda^k(T)} \le C \sum_{m=1}^{\ell} h^m |u|_{H^m\Lambda^k(T)}, \quad u \in H^\ell\Lambda^k(T), \tag{3.5}$$

with the constant C depending on T only through its shape constant. We prove this first for the unit right simplex in $\mathbb{R}^n$, $\hat{T}$, with vertices at the origin and the n points $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots)$, ... Since $\ell > (n-k)/2$, we obtain, by the Sobolev embedding theorem, that $\|\pi_{\hat{T}} u\|_{L^2\Lambda^k(\hat{T})} \le C\|u\|_{H^\ell\Lambda^k(\hat{T})}$, and so, by the triangle inequality,

$$\|u - \pi_{\hat{T}} u\|_{L^2\Lambda^k(\hat{T})} \le C\|u\|_{H^\ell\Lambda^k(\hat{T})}.$$

Now let $\bar{u} = n! \int_{\hat{T}} u$, a constant k-form on $\hat{T}$ equal to the average of u. Then $\pi_{\hat{T}}\bar{u} = \bar{u}$, so

$$\begin{aligned}\|u - \pi_{\hat{T}} u\|_{L^2\Lambda^k(\hat{T})} &= \|(u - \bar{u}) - \pi_{\hat{T}}(u - \bar{u})\|_{L^2\Lambda^k(\hat{T})} \\ &\le C\|u - \bar{u}\|_{H^\ell\Lambda^k(\hat{T})} \le C(\|u - \bar{u}\|_{L^2\Lambda^k(\hat{T})} + \sum_{m=1}^{\ell} |u|_{H^m\Lambda^k(\hat{T})}),\end{aligned}$$

where we have used the fact that $\bar{u}$ is a constant form, so its mth Sobolev seminorm vanishes for $m \ge 1$. Now we invoke Poincaré's inequality

$$\|u - \bar{u}\|_{L^2\Lambda^k(\hat{T})} \le C|u|_{H^1\Lambda^k(\hat{T})}.$$

Putting things together, and writing $\hat{u}$ instead of u, we have shown that

$$\|\hat{u} - \pi_{\hat{T}}\hat{u}\|_{L^2\Lambda^k(\hat{T})} \le C \sum_{m=1}^{\ell} |\hat{u}|_{H^m\Lambda^k(\hat{T})}, \quad \hat{u} \in H^\ell\Lambda^k(\hat{T}). \tag{3.6}$$

This is the desired result (3.5) in the case $T = \hat{T}$.

To obtain the result for a general simplex, we scale via an affine diffeomorphism $F : \hat{T} \to T$. If u is the k-form on T given by (3.1), then

$$F^*u = \sum_{\{1 \le i_1 < \cdots < i_k \le n\}} \sum_{j_1,\ldots,j_k=1}^{n} (u_{i_1\cdots i_k} \circ F) \frac{\partial F^{i_1}}{\partial \hat{x}^{j_1}} \cdots \frac{\partial F^{i_k}}{\partial \hat{x}^{j_k}}\, d\hat{x}^{j_1} \wedge \cdots \wedge d\hat{x}^{j_k}. \tag{3.7}$$

Each of the partial derivatives $\partial F^{i_p}/\partial \hat{x}^{j_q}$ is a constant bounded by h. Using the chain rule and change of variables in the integration, we find that

$$c|F^*u|_{H^m\Lambda^k(\hat{T})} \le (\operatorname{vol} T)^{-1/2}\, h^{m+k} |u|_{H^m\Lambda^k(T)} \le C|F^*u|_{H^m\Lambda^k(\hat{T})}, \tag{3.8}$$

where the constants c and C depend only on m and n and the shape regularity constant of T. Combining (3.6) and (3.8) we get

$$\begin{aligned}\|u - \pi_T u\|_{L^2\Lambda^k(T)} &\le C(\operatorname{vol} T)^{1/2} h^{-k} \|\hat{u} - \pi_{\hat{T}}\hat{u}\|_{L^2\Lambda^k(\hat{T})} \\ &\le C(\operatorname{vol} T)^{1/2} h^{-k} \sum_{m=1}^{\ell} |\hat{u}|_{H^m\Lambda^k(\hat{T})} \le C \sum_{m=1}^{\ell} h^m |u|_{H^m\Lambda^k(T)},\end{aligned}$$

which establishes (3.5). □

Our approach to bounding the norm of the consistency error is to relate it to another quantity which has been studied in the finite element literature, namely

$$A_h(u) := \sup_{v_h \in \Lambda_h^{k-1}} \frac{\langle u - \pi_h u, dv_h \rangle}{\|v_h\|}. \tag{3.9}$$

Theorem 3.2. *Assume the approximation property* (3.3)*. Then, for any smooth* $u \in L^2\Lambda^k$ *belonging to the domain of* d^* *we have*

$$\lim_h \|d^* u - d_h^* \pi_h u\| = 0 \iff \lim_h A_h(u) = 0.$$

This follows immediately from Lemma 3.3.

Lemma 3.3. *Let* $1 \le k \le n$*, and let* $u \in L^2\Lambda^k$ *be smooth and in the domain of* d^**. Then*

$$A_h(u) \le \|d^* u - d_h^* \pi_h u\| \le \operatorname{dist}(d^* u, \Lambda_h^{k-1}) + A_h(u). \tag{3.10}$$

Proof. The first inequality is straightforward. For any $v_h \in \Lambda_h^{k-1}$,

$$\frac{\langle u - \pi_h u, dv_h \rangle}{\|v_h\|} = \frac{\langle d^* u - d_h^* \pi_h u, v_h \rangle}{\|v_h\|} \le \|d^* u - d_h^* \pi_h u\|.$$

For the second inequality, we introduce the L^2-orthogonal projection $P_h : L^2\Lambda^{k-1} \to \Lambda_h^{k-1}$ and invoke the triangle inequality to get

$$\|d^* u - d_h^* \pi_h u\| \le \|d^* u - P_h d^* u\| + \|P_h d^* u - d_h^* \pi_h u\| = \operatorname{dist}(d^* u, \Lambda_h^{k-1}) + \|w\|, \tag{3.11}$$

where $w = P_h d^* u - d_h^* \pi_h u \in \Lambda_h^k$. Now

$$\|w\|^2 = \langle P_h d^* u - d_h^* \pi_h u, w \rangle = \langle u - \pi_h u, dw \rangle, \tag{3.12}$$

and hence

$$\|w\| = \frac{\langle u - \pi_h u, dw \rangle}{\|w\|} \le \sup_{v_h \in \Lambda_h^{k-1}} \frac{\langle u - \pi_h u, dv_h \rangle}{\|v_h\|} = A_h(u), \tag{3.13}$$

which completes the proof. □

Thus we wish to bound $\langle u - \pi_h u, dv_h \rangle / \|v_h\|$ for smooth u in the domain of d^* and $v_h \in \Lambda_h^k$. An obvious approach is to apply the Cauchy–Schwarz inequality and then use the approximation estimate (3.4) to obtain

$$|\langle u - \pi_h u, dv_h \rangle| \le \|u - \pi_h u\| \|dv_h\| \le Ch \|u\|_{H^\ell \Lambda^k} \|dv_h\|. \tag{3.14}$$

To continue, we need to bound $\|dv_h\| / \|v_h\|$ for v_h an arbitrary non-zero element of Λ_h^k. Because Λ_h^k consists of piecewise polynomials, it is possible to bound its derivative in terms of its value using a Bernstein type inequality or inverse estimate. This gives that

$$\|dv_h\| \le C \underline{h}^{-1} \|v_h\|, \quad v_h \in \Lambda_h^k, \tag{3.15}$$

where $\underline{h} = \min_{T \in \Delta_n(\mathcal{T}_h)} \operatorname{diam} T$. Unfortunately, even if we assume that our triangulations are *quasiuniform*, i.e., that $\underline{h} \ge ch$ for some fixed $c > 0$, this just leads to the bound

$$A_h(u) \le C \|u\|_{H^\ell \Lambda^k},$$

which does not tend to zero with h. In fact, we cannot hope to get a bound which tends to zero without further hypotheses, since, as we have seen, even for the nice mesh sequence and form u considered in the previous section, d_h^* is not consistent, and so $A_h(u)$ does not tend to zero.

Nonetheless, for very special mesh sequences it is possible to improve the bound (3.14) from first to second order in h. This was established by Brandts and Křížek in their work on gradient superconvergence [2]. The mesh condition is embodied by the following concept.

Definition 3.4 ([2])**.** A triangulation $\mathcal{T}$ on M is called *uniform* if there exist n linearly independent vectors $e_1, \dots, e_n$, such that

(1) Every simplex in $\mathcal{T}$ contains an edge parallel to each e_j.
(2) If an edge e is parallel to one of the e_j and is not contained in ∂M, then the union P_e of simplices containing e is invariant under reflection through the midpoint m_e of e, i.e., $2m_e - x \in P_e$ for all $x \in P_e$.

The crisscross triangulations shown in Figure 1 satisfy the first condition of the definition, but not the second, and so are not uniform. On the other hand, the mesh sequence that is obtained by starting from a single triangle, or from a division of a square into two triangles and applying regular standard subdivision, is uniform. See the first two rows of Figure 4. A uniform triangulation of the cube in n dimensions is obtained by subdividing it into m^n subcubes, and dividing each of these into $n!$ simplices sharing a common diagonal, with all the diagonals of the subcubes chosen to be parallel. The 3D case is shown in Figure 4. We refer to [2] for more details.

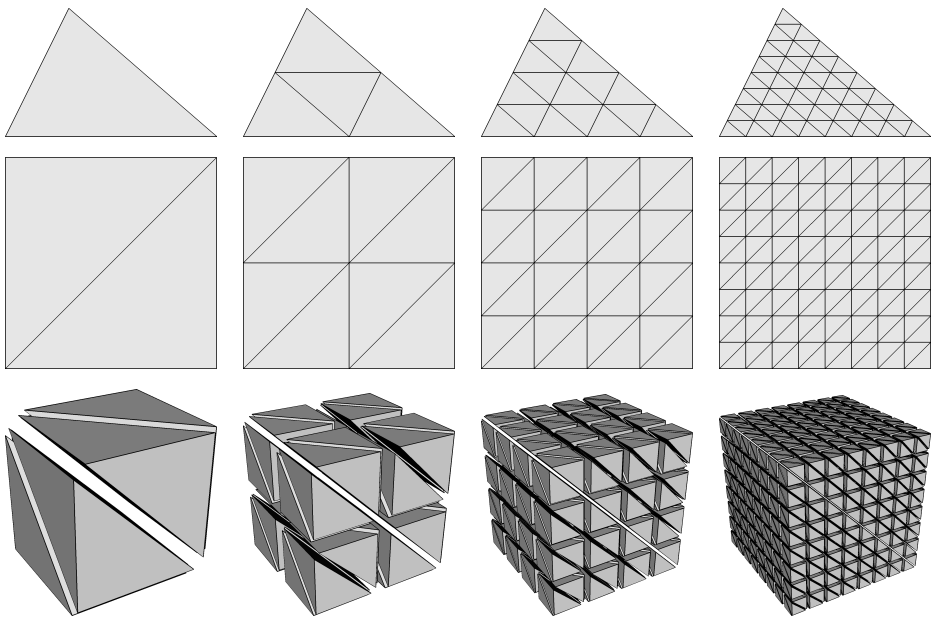

FIGURE 4. Uniform triangulations.

Theorem 3.4 of [2] claims that if $\{\mathcal{T}_h\}$ is a shape regular family of uniform triangulations of M, and if u is a smooth 1-form, then there exists a constant $C > 0$ such that

$$|\langle \pi_h u - u, dv_h \rangle| \le C h^2 \|u\|_{H^2\Lambda^1} \|dv_h\|,$$

for all $v_h \in \Lambda_h^0 \cap \mathring{H}^1(M)$ and $h > 0$. Here $\mathring{H}^1(M)$ denotes the space of $H^1(M)$ functions with vanishing trace on ∂M. However, their proof uses the inequality (cf.

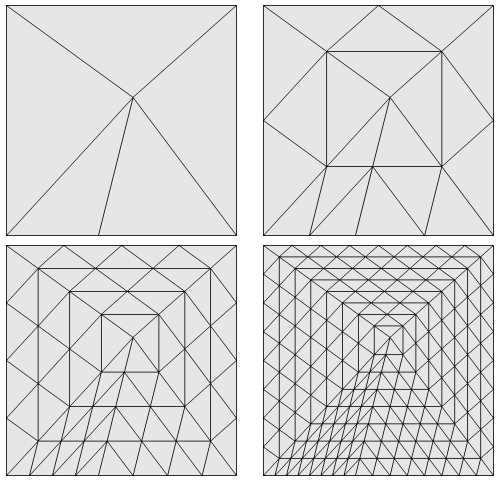

FIGURE 5. A piecewise uniform sequence of triangulations.

(1.5) of [2])

$$\|\pi_h u - u\|_{L^2\Lambda^1} \leq C\, h \|u\|_{H^1\Lambda^1},$$

where C is a constant independent of u. This would imply that π_h can be continuously extended to $H^1\Lambda^1$, which is impossible for $n \geq 3$. Fortunately, the proof in [2] works verbatim if the above inequality is replaced by (3.4). Hence, the following result is essentially proved in [2].

Theorem 3.5. *Let $\{\mathcal{T}_h\}$ be a shape regular family of uniform triangulations of M, and let u be a smooth 1-form. Furthermore, let ℓ be the smallest integer so that $\ell > (n-1)/2$. Then there exists a constant $C > 0$ such that*

$$|\langle \pi_h u - u, dv_h \rangle| \leq C h^2 \|u\|_{H^\ell\Lambda^1} \|dv_h\|, \tag{3.16}$$

for all $v_h \in \Lambda_h^0 \cap \mathring{H}^1(M)$ and $1 > h > 0$.

Next we consider piecewise uniform sequences of triangulations.

Definition 3.6. A family $\mathcal{T}_h$ of triangulations of the polytope M is called *piecewise uniform* if there is a triangulation $\mathcal{T}$ of M such that for each h, $\mathcal{T}_h$ is a refinement of $\mathcal{T}$ and for each $T \in \mathcal{T}$ and each h, the restriction of $\mathcal{T}_h$ to $T \in \Delta_n(T)$ is uniform.

If, as in [7], we start with an arbitrary triangulation of a polygon and refine it by standard regular subdivision, the resulting sequence of triangulations is piecewise uniform. This is illustrated in Figure 5. The following theorem shows that d^* is consistent for 1-forms on piecewise uniform meshes, thus generalizing the main result of [7] from 2 to n dimensions.

Theorem 3.7. *Assume that the family of triangulations $\{\mathcal{T}_h\}$ is a shape regular, quasiuniform, and piecewise uniform. Let $u \in H^\ell\Lambda^1(M)$ be a 1-form in the domain of d^*, where ℓ is the smallest integer satisfying $\ell > (n-1)/2$. Then we have*

$$\lim_{h\to 0} \|d^* u - d_h^* \pi_h u\| = 0. \tag{3.17}$$

Proof. Let $\mathcal{T}$ denote the triangulation of M with respect to which the triangulations $\mathcal{T}_h$ are uniform. We will apply Theorem 3.5 to the uniform mesh sequences obtained

by restricting $\mathcal{T}_h$ to each $T \in \mathcal{T}$. To this end, let $K = \bigcup_{T\in\mathcal{T}} \partial T$ denote the skeleton of $\mathcal{T}$, and set

$$\Sigma_h = \bigcup \{ T \in \mathcal{T}_h \,|\, T \cap K \neq \emptyset \}.$$

We can decompose an arbitrary function $v_h \in \Lambda_h^0$ as

$$v_h = w_h + \sum_{T\in\mathcal{T}} v_h^T, \tag{3.18}$$

where $w_h \in \Lambda_h^0$ is supported in Σ_h and $v_h^T \in \Lambda_h^0$ is supported in T. Indeed, we just take w_h to coincide with v at the vertices of the triangulation contained in K and to vanish at the other vertices, while $v_h^T = v$ at the vertices in the interior of T and vanishes at the other vertices. Because the mesh family is shape regular and quasiuniform, there exist positive constants C, c such that

$$c\|v\|^2 \leq h \sum_{x\in\Delta_0(\mathcal{T}_h)} |v(x)|^2 \leq C\|v\|^2, \quad v \in \Lambda_h^0,$$

from which we obtain the stability bound

$$\|w_h\| + \sum_{T\in\mathcal{T}} \|v_h^T\| \leq C\|v_h\|. \tag{3.19}$$

Using the decomposition (3.18) of v_h we get

$$\begin{aligned} |\langle \pi_h u - u, dv_h\rangle| &\leq |\langle \pi_h u - u, dw_h\rangle| + \sum_{T\in\mathcal{T}} |\langle \pi_h u - u, dv_h^T\rangle| \\ &\leq Ch\|u\|_{H^\ell\Lambda^1(\Sigma_h)}\|dw_h\| + Ch^2 \sum_{T\in\mathcal{T}} \|u\|_{H^\ell\Lambda^1(M)}\|dv_h^T\| \\ &\leq C\|u\|_{H^\ell\Lambda^1(\Sigma_h)}\|w_h\|_{L^2(\Sigma_h)} + Ch \sum_{T\in\mathcal{T}} \|u\|_{H^\ell\Lambda^1(M)}\|v_h^T\| \\ &\leq C\left(\|u\|_{H^\ell\Lambda^1(\Sigma_h)} + h\|u\|_{H^\ell\Lambda^1(M)}\right)\|v_h\|, \end{aligned} \tag{3.20}$$

where we have used the Cauchy–Schwarz inequality, the projection error estimate (3.4), the second order estimate (3.16) (which holds on the uniform meshes on each T), the inverse estimate of (3.15), and the L^2-stability bound (3.19). Since the volume of Σ_h goes to 0 as $h \to 0$, so does $\|u\|_{H^\ell\Lambda^1(\Sigma_h)}$. Thus $A_h(u)$ vanishes with h, and the desired result is a consequence of Theorem 3.2. □

Remark 3.8. The preceding proof shows that as long as the triangulation is mostly uniform, in the sense that the volume of the defective region goes to 0 as $h \to 0$, we obtain consistency. One can also extract information on the convergence rate. For instance, using the fact that Σ_h is $O(h)$, we obtain $\|u\|_{H^\ell\Lambda^1(\Sigma_h)} \leq C\sqrt{h}\|u\|_{C^\ell\Lambda^1}$ for $u \in C^\ell\Lambda^1(M)$.

4. Computational experiments for 1-forms

In this section, we present numerical computations confirming the consistency of d_h^* for 1-forms on uniform and piecewise uniform meshes in 2 and 3 dimensions, and other computations confirming its inconsistency on more general meshes. The four tables in this section display the results of computations with various mesh sequences. In each case we show the maximal simplex diameter h, the number of simplices in the mesh, the consistency error $\|d_h^*\pi_h f - d^* f\|$, and the apparent order

inferred from the ratio of consecutive errors. All computations were performed using the FEniCS finite element software library [6].

The first two tables concern the problem on the square described in Section 2, i.e., the approximation of d^*u where $u = (1-x^2)dx$. Table 1 shows the results when the piecewise uniform mesh sequence shown in Figure 5 is used for the discretization. Notice that the consistency error clearly tends to zero as $O(h)$.

TABLE 1. When computed using the 2-dimensional piecewise uniform mesh sequence of Figure 5, the consistency error tends to 0.

n	h	triangles	error	order
1	5.00e−01	20	6.25e−01	
2	2.50e−01	80	3.08e−01	1.02
3	1.25e−01	320	1.56e−01	0.98
4	6.25e−02	1,280	7.85e−02	0.99
5	3.12e−02	5,120	3.94e−02	1.00
6	1.56e−02	20,480	1.97e−02	1.00

By contrast, Table 2 shows the counterexample described analytically in Section 2, using the mesh sequence of Figure 1, obtained by standard subdivision. In this case, the consistency error does not converge to zero, as is clear from the computations.

TABLE 2. With the mesh sequence of Figure 1, the consistency error does not tend to 0.

n	h	triangles	error	order
1	5.00e−01	16	1.15	
2	2.50e−01	64	1.50	−0.38
3	1.25e−01	256	1.60	−0.09
4	6.25e−02	1,024	1.62	−0.02
5	3.12e−02	4,096	1.63	−0.01
6	1.56e−02	16,384	1.63	−0.00

Similar results hold in 3 dimensions. We computed the error in d_h^*u on the cube $(-1,1)^3$, where again u is given by $(1-x^2)dx$. We calculated with two mesh sequences, both starting from a partition of the cube into six congruent tetrahedra, all sharing a common edge along the diagonal from $(-1,-1,-1)$ to $(1,1,1)$. We constructed the first mesh sequence by regular subdivision, yielding the meshes shown in Figure 6. These are uniform meshes, and the numerical results given in Table 3 clearly demonstrate consistency. For the second mesh sequence we applied standard subdivision, obtaining the sequence of structured but non-uniform triangulations shown in Figure 7. In this case d_h^* is inconsistent. See Table 4.

5. Inconsistency for 2-forms in 3 dimensions

We have seen that for 1-forms, d_h^* is consistent if computed using piecewise uniform mesh sequences, but not with general mesh sequences. It is also easy to see that consistency holds for n-forms in n-dimensions for *any* mesh sequence.

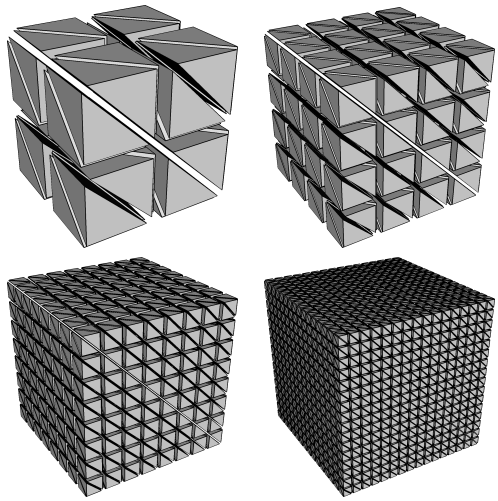

FIGURE 6. Uniform mesh sequence in 3D, obtained by regular subdivision.

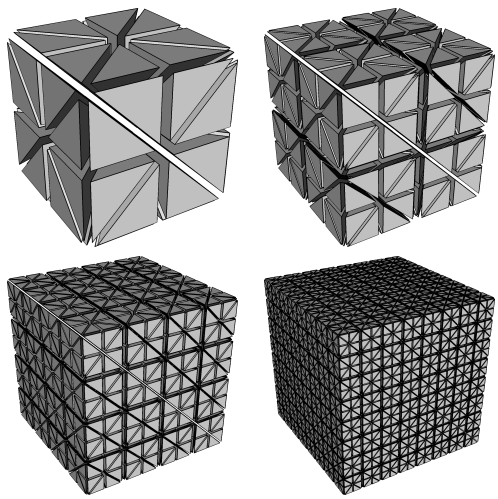

FIGURE 7. As in 2D, the mesh sequence in 3D obtained by standard subdivision is not uniform.

This is because the canonical projection π_h onto the Whitney n-forms (which are just the piecewise constant forms) is the L^2 orthogonal projection. Now if v_h is a Whitney $(n-1)$-form, then dv_h is a Whitney n-form, so the inner product $\langle u - \pi_h u, dv_h \rangle = 0$. Thus $A_h(u)$, defined in (3.9), vanishes identically, and so d_h^* is consistent by Theorem 3.2. Having understood the situation for 1-forms and n-forms, this leaves open the question of whether consistency holds for k-forms with k strictly between 1 and n. In this section we study 2-forms in 3 dimensions and give numerical results indicating that d_h^* is not consistent, even for uniform meshes.

Let $u = (1-x^2)(1-y^2)dx \wedge dy$, a 2-form on the cube $M = (-1,1)^3$. The corresponding vector field is $(0, 0, (1-x^2)(1-y^2))$ which has vanishing tangential

TABLE 3. The consistency error for d_h^* on 1-forms in 3D tends to zero when using the uniform mesh sequence of Figure 6.

n	h	tetrahedra	error	order
1	1.00e+00	48	1.69e+00	
2	5.00e−01	384	9.70e−01	0.80
3	2.50e−01	3,072	5.13e−01	0.92
4	1.25e−01	24,576	2.63e−01	0.96
5	6.25e−02	196,608	1.33e−01	0.98
6	3.12e−02	1,572,864	6.69e−02	0.99

TABLE 4. The consistency error for d_h^* on 1-forms in 3D, using the non-uniform mesh sequence of Figure 7, does not tend to zero.

n	h	tetrahedra	error	order
0	1.00e+00	48	1.81e+00	
1	5.00e−01	384	2.71e+00	−0.58
2	2.50e−01	3,072	3.02e+00	−0.16
3	1.25e−01	24,576	3.11e+00	−0.04
4	6.25e−02	196,608	3.13e+00	−0.01

components on ∂M. Therefore u belongs to the domain of d^* and d^*u is the 1-form corresponding to $\operatorname{curl} u$, i.e., $d^*u = -2(1-x^2)ydx + 2x(1-y^2)dy$. Table 5 shows the consistency error $\|d_h^*\pi_h u - d^*u\|_{L^2\Lambda^1}$ computed using the sequences of uniform meshes displayed in Figure 6. This mesh sequence yields a consistent approximation of d^*h for 1-forms, but the experiments clearly indicate that this is not so for 2-forms.

TABLE 5. The consistency error does not tend to zero for 2-forms, even on a uniform mesh sequence.

n	h	triangles	error	order
1	1.00e+00	48	1.59e+00	
2	5.00e−01	384	1.18e+00	0.43
3	2.50e−01	3072	1.00e+00	0.24
4	1.25e−01	24576	9.47e−01	0.08
5	6.25e−02	196608	3.37e+00	−1.83

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