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LOCI OF COMPLEX POLYNOMIALS, PART I

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Abstract. The classical Grace theorem states that every circular domain in the complex plane $\mathbb{C}$ containing the zeros of a polynomial $p(z)$ contains a zero of any of its apolar polynomials. We say that a closed domain $\Omega \subseteq \mathbb{C}^*$ is a *locus* of $p(z)$ if it contains a zero of any of its apolar polynomials and is the smallest such domain with respect to inclusion. In this work we establish several general properties of the loci and show, in particular, that the property of a set being a locus of a polynomial is preserved under a Möbius transformation. We pose the problem of finding a locus inside the smallest disk containing the roots of $p(z)$ and solve it for polynomials of degree 3. Numerous examples are given.

1. Introduction

Denote by $\mathbb{C}$ the complex plane and let $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$. By $\mathcal{P}_n$ denote the set of all complex polynomials

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \tag{1.1}$$

of degree at most n, where $a_0, \ldots, a_n \in \mathbb{C}$ are constants. The degree of p is the highest power of z in (1.1) that has non-zero coefficient. To every polynomial $p \in \mathcal{P}_n$, we correspond a multiaffine symmetric polynomial in n complex variables:

$$P(z_1, z_2, \ldots, z_n) := \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} S_k(z_1, z_2, \ldots, z_n), \tag{1.2}$$

where

$$S_k(z_1, z_2, \ldots, z_n) := \sum_{1 \le i_1 < \cdots < i_k \le n} z_{i_1} z_{i_2} \cdots z_{i_k}$$

are the elementary symmetric polynomials of degree $k = 1, 2, \ldots, n$, with

$$S_0(z_1, z_2, \ldots, z_n) := 1.$$

Clearly, one has $p(z) = P(z, z, \ldots, z)$. We say that $P(z_1, z_2, \ldots, z_n)$ is the *n-th polarization of p* or the *symmetrization of p with n variables* or just the *symmetrization of p* for short. The n-tuple $\{z_1, z_2, \ldots, z_n\}$ is a *solution* of P (or p) if $P(z_1, z_2, \ldots, z_n) = 0$. The polarization operation is well known and investigated; see for example [4, Section 4]. For example, if we consider $z^2 - z$ as a polynomial in $\mathcal{P}_3$, then its symmetrization with three variables is $(z_1 z_2 + z_1 z_3 + z_2 z_3)/\binom{3}{2} - (z_1 + z_2 + z_3)/\binom{3}{1}$

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and the triple $\{1,1,1\}$ is a solution. The important thing to keep in mind is that a solution of p is always an n-tuple in $\mathbb{C}$ even if the real degree of p is less than n.

A polynomial $q \in \mathcal{P}_n$, given by

$$q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0,$$

is called *apolar with* p if

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} a_k b_{n-k} = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k p^{(k)}(0) q^{(n-k)}(0) = 0. \tag{1.3}$$

This definition of apolarity extends the one in [1, Definition 3.3.1, p. 102] in that it depends on n. In particular, it allows the leading coefficients of p or q to be zero. The following lemma is easy to verify, so we state it without a proof.

Lemma 1.1. *The n-tuple $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{C}$ is a solution of p if and only if the polynomial $q(z) = (z - z_1) \cdots (z - z_n)$ is apolar with p.*

Let

$$T(z) = (az + b)/(cz + d) \quad \text{with } ad - bc \neq 0 \tag{1.4}$$

be a non-degenerate Möbius transformation. For every polynomial $p \in \mathcal{P}_n$ we define

$$T[p](z) := (cz + d)^n p(T(z)).$$

For example, if $p(z) := z^2 + 1 \in \mathcal{P}_3$, then $T[p](z) := (cz + d)(az + d)^2 + (cz + d)^3$. If $U(z) = (ez + f)/(gz + h)$ is another Möbius transformation, then for every polynomial $p \in \mathcal{P}_n$ we have

$$U[T[p]](z) = (gz + h)^n (cU(z) + d)^n p(T(U(z))) = (T \circ U)[p](z).$$

We need the following known fact; see [1, Remark 3.3.4, p. 103].

Lemma 1.2. *If $p, q \in \mathcal{P}_n$ are apolar, then so are $T[p]$ and $T[q]$.*

Technically, the statement of Lemma 1.2, given in [1, Remark 3.3.4, p. 103], requires that both p and q be of degree n, as well as that both $T[p]$ and $T[p]$ be of degree n. The justification of the lemma is analogous: Every Möbius transformation T, with $ad - bc \neq 0$, is a composition of transformations of the type $1/z$ and $az + b$ for $a \neq 0$. It is not difficult to see that Lemma 1.2 holds for these two types of transformations.

The most important and useful relation between the zeros of two apolar polynomials is the classical theorem of Grace; see [1, p. 107]. A circular domain, open or closed, is the interior or exterior of a circle, or a half plane determined by a line in the complex plane.

Theorem 1.3 (Grace theorem)**.** *Let p and q be apolar polynomials of degree n. Then, every circular domain containing all the zeros of one of these contains at least one zero of the other.*

The Grace theorem says that every circular domain containing the zeros of p, with $a_n \neq 0$, contains a point from every solution $\{z_1, z_2, \ldots, z_n\}$ of p. Our long term goal is to search for tighter versions of the Grace theorem. Our main definition is next.

Definition 1.4. Let Ω be a closed subset of $\mathbb{C}^*$. We say that Ω is a locus holder of $p \in \mathcal{P}_n$ if Ω contains at least one point from every solution of p. A minimal by inclusion locus holder Ω is called a locus of p.

Rephrasing the Grace theorem, if $a_n \neq 0$, then every circular domain containing the zeros of p is a locus holder of p. Every locus of p leads to a strengthening of the Grace theorem that cannot be strengthened any further. Clearly, if α is a zero of p and Ω is a locus, then $\alpha \in \Omega$, since $\{\alpha, \alpha, \ldots, \alpha\}$ is a solution of p. The next lemma clarifies a basic fact.

Lemma 1.5. *Every locus holder of p contains a locus.*

Proof. Let Ω be a locus holder and consider all closed subsets of Ω, partially ordered by inclusion, that are also locus holders. Let $T := \{T_\alpha\}_\alpha$ be a non-empty chain (i.e. a totally ordered set) of locus holders in Ω. If we show that there is a locus holder contained in every set of the chain, then by Zorn's lemma there is a minimal locus holder, that is, a locus. Indeed, the intersection, call it Ω_T, of all sets in the chain is a closed set. Suppose there is a solution $\{z_1, \ldots, z_n\}$ of p not having an element in Ω_T. Then, for each $i = 1, \ldots, n$ there is an element T_{α_i} of the chain not containing z_i. Since the chain is totally ordered, there is a smallest set among $\{T_{\alpha_i} : i = 1, \ldots, n\}$, and it cannot be a locus holder. □

We now recall the well-known Grace-Walsh-Szegő coincidence theorem; see [1, p. 108] or [2, 3].

Theorem 1.6 (Grace-Walsh-Szegő coincidence theorem)**.** *Let $P(z_1, z_2, \ldots, z_n)$ be a multiaffine symmetric polynomial. If the degree of P is n, then every circular domain containing the points $z_1, z_2, \ldots, z_n$ contains at least one point z such that $P(z_1, z_2, \ldots, z_n) = P(z, z, \ldots, z)$. If the degree of P is less than n, then the same conclusion holds, provided the circular domain is convex.*

An application of the Grace-Walsh-Szegő coincidence theorem is the following observation. Its proof is given in Appendix A.

Theorem 1.7. *Let p be a polynomial with at least two distinct zeros and $a_n \neq 0$. If all zeros of p are on the boundary of a closed circular domain B, then B is a locus of p.*

The drawback of Theorem 1.7 is that the zeros of p may be on a circle with a very large radius. Such is the case when all the zeros of p are colinear. Thus, it is natural to ask for a locus of p that is smaller. One arrives at the idea of finding a locus inside the smallest closed disk containing the zeros of p. One of the purposes of this paper is to find a locus of p inside that smallest disk in the case when the degree of p is at most 3.

2. Elementary properties of the loci

We begin by taking care of two pesky cases. Suppose that $p \in \mathcal{P}_n$ is a constant, $p(z) = a_0$. If $a_0 \neq 0$, then the symmetrization $P(z_1, z_2, \ldots, z_n)$ has no solutions and the empty set is a locus of p. If $a_0 = 0$, then every n-tuple $\{z_1, z_2, \ldots, z_n\}$ is a solution of p, and hence $\mathbb{C}^*$ is a locus. Suppose that $p \in \mathcal{P}_n$ has only one root of multiplicity n, that is, $p(z) = a_n(z - \alpha)^n$ for $a_n \neq 0$. Then,

(2.1)
$$P(z_1, z_2, \ldots, z_n) = a_n \sum_{k=0}^{n} (-1)^{n-k} \alpha^{n-k} S_k(z_1, z_2, \ldots, z_n) = a_n(z_1 - \alpha) \cdots (z_n - \alpha)$$

shows that every solution $\{z_1, z_2, \ldots, z_n\}$ of P contains a point equal to α. In this case, $\Omega = \{\alpha\}$ is a locus and it is unique. So, from now on we are going to assume that the polynomial $p \in \mathcal{P}_n$ is neither a constant nor has a zero of multiplicity n. Those two cases can be captured simultaneously with the help of the following definition.

Definition 2.1. If $p \in \mathcal{P}_n$ has degree $m \leq n$, we say that ∞ is a zero of p of multiplicity $n - m$.

With Definition 2.1 in mind, the constant polynomial in $\mathcal{P}_n$ and the polynomial $p(z) = a_n(z - \alpha)^n$ for $a_n \neq 0$ both have a unique zero of multiplicity n. To avoid those trivial cases, we introduce the notation

$$\mathcal{P}_n^* := \{p \in \mathcal{P}_n : p \text{ has at least two distinct zeros in } \mathbb{C}^*\}.$$

For example, if we consider the polynomial z^3 to be in $\mathcal{P}_5$, then $z^3 \in \mathcal{P}_5^*$, since then it has two distinct roots in $\mathbb{C}^*$: 0 of multiplicity 3 and ∞ of multiplicity 2. But, if we consider the polynomials z^5 and z^0 to be in $\mathcal{P}_5$, then neither of them is in $\mathcal{P}_5^*$.

The rest of this section investigates how non-degenerate Möbius transformations change a locus. We begin with a simple case.

Lemma 2.2. *Let $T(z) = az + b$ with $a \neq 0$. If Ω is a locus of p, then $T^{-1}(\Omega)$ is a locus of $p(az + b)$.*

Proof. Using Lemma 1.1, $\{z_1, z_2, \ldots, z_n\}$ is a solution of p if and only if the polynomial $q(z) := (z - z_1) \cdots (z - z_n)$ is apolar with p. By Lemma 1.2, the polynomials $p(az + b)$ and $q(az + b)$ are apolar, where

$$q(az + b) = a^n(z - T^{-1}(z_1)) \cdots (z - T^{-1}(z_n)).$$

Thus, if $\{z_1, z_2, \ldots, z_n\}$ is a solution of $p(z)$, then $\{T^{-1}(z_1), T^{-1}(z_2), \ldots, T^{-1}(z_n)\}$ is a solution of $p(az + b)$. The opposite direction of the latter statement follows analogously. □

Lemma 2.3. *Let $p \in \mathcal{P}_n^*$ be a polynomial of degree $m < n$. Then, every locus Ω of p contains ∞.*

Proof. Formally, the polynomial p has ∞ as a zero. Since $p \in \mathcal{P}_n^*$, there must be a finite zero, call it α. Suppose first that $\alpha = 0$, implying that $a_0 = 0$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the zeros of $z^n + 1 = 0$. Then, we have $S_k(\omega_1, \omega_2, \ldots, \omega_n) = 0$ for $k = 1, \ldots, n-1$, $S_0(\omega_1, \omega_2, \ldots, \omega_n) = 1$, and $S_n(\omega_1, \omega_2, \ldots, \omega_n) = (-1)^n$. Thus, for every $r > 0$ the n-tuple $\{r\omega_1, r\omega_2, \ldots, r\omega_n\}$ is a solution of p. Since Ω is a closed set and contains at least one of the components of that solution for every $r > 0$, it contains ∞.

If $\alpha \neq 0$, then consider the transformation $T(z) = z + \alpha$. Apply Lemma 2.2 to see that $T^{-1}(\Omega)$ is a locus of $p(T(z))$. According to the last paragraph, $T^{-1}(\Omega)$ contains ∞, and so does Ω. □

Combining the last lemma with the discussion preceding Definition 2.1 and the comments before Theorem 1.6 gives the following observation.

Corollary 2.1. *Every locus of a non-constant polynomial p contains all its zeros.*

Theorem 2.4 (Extended Grace theorem)**.** *If $p, q \in \mathcal{P}_n$ are apolar, then every circular domain containing all the zeros of p contains at least one zero of q and vice versa.*

Proof. If $p, q \in \mathcal{P}_n$ are both of degree n, then the theorem is the classical theorem of Grace. If $p, q \in \mathcal{P}_n$ are both of degree strictly less than n, then both polynomials have ∞ as a zero; see Definition 2.1. Hence, the theorem is true. Finally, let the degree of p be $m < n$, while the degree of q is n. The rest follows immediately from the original Grace theorem and Lemma 1.2 after choosing the Möbius transformation T to be such that T^{-1} sends the roots of p and ∞ into $\mathbb{C}$. □

Rephrasing the extended Grace theorem, we have that every circular domain containing the zeros of p, in the sense of Definition 2.1, is a locus holder of p. Of course, if the degree of p is $m < n$, those circular domains are half planes or the exterior of a circle.

The next theorem says that the property that a set is a locus of a polynomial in $\mathcal{P}_n^*$ is invariant under a Möbius transformation.

Theorem 2.5. *Let T be defined by* (1.4). *The set Ω is a locus of $p \in \mathcal{P}_n^*$ if and only if $T^{-1}(\Omega)$ is a locus of $T[p]$.*

Proof. Let Ω be a locus of p. It is easy to see that we need to consider only the cases 1) $T(z) = az + b$, where $a \neq 0$; and 2) $T(z) = 1/z$. The first case was covered by Lemma 2.2. So, suppose that $T(z) = 1/z$. Let $q(z) := (z - z_1) \cdots (z - z_n)$. Consider the polynomials $p_1(z) := z^n p(1/z)$ and $q_1(z) := z^n q(1/z)$, that is,

$$q_1(z) = (-1)^n z_1 \cdots z_n (z - 1/z_1) \cdots (z - 1/z_n).$$

If $\{z_1, z_2, \ldots, z_n\}$, with non-zero components, is a solution of p, then by Lemma 1.1 p and q are apolar. Thus, by Lemma 1.2 p_1 and q_1 are apolar, implying that $\{1/z_1, 1/z_2, \ldots, 1/z_n\}$ is a solution of $p_1(z)$. (Note that the degree of p_1 may drop, but the important fact is that the degree of q_1 stays n.) To prove the opposite statement, note that $p(z) = z^n p_1(1/z)$ and that $p_1 \in \mathcal{P}_n^*$; hence we can exchange the roles of p and p_1 above. Thus, if $\{z_1, z_2, \ldots, z_n\}$, with non-zero components, is a solution of $p_1(z)$, then $\{1/z_1, 1/z_2, \ldots, 1/z_n\}$ is a solution of p.

Let Ω be a locus of p and let $\{z_1, z_2, \ldots, z_n\}$ be an arbitrary solution of p_1. To simplify the notation, define $1/\Omega := \{1/z : z \in \Omega\}$. We consider two cases.

Case 1. If $z_1 = z_2 = \cdots = z_n = 0$, then $p_1(0) = 0$ implies that the degree of p is strictly less than n. By Lemma 2.3, we have $\infty \in \Omega$ or $0 \in 1/\Omega$.

Case 2. Suppose that $z_k = 0$ for all $k = 1, \ldots, s$ and $z_k \neq 0$ for all $k = s + 1, \ldots, n$. Let $P_1(z_1, z_2, \ldots, z_n)$ be the symmetrization of p_1. For a fixed $\epsilon > 0$, let $\mu = \mu(\epsilon)$ be a zero of the equation $P_1(z_1 + \epsilon, \ldots, z_s + \epsilon, z_{s+1} + \mu, \ldots, z_n + \mu) = 0$. The zeros of a polynomial are continuous functions of the coefficients, so we may assume that $\mu(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, for all small enough ϵ, the n-tuple $\{z_1 + \epsilon, \ldots, z_s + \epsilon, z_{s+1} + \mu, \ldots, z_n + \mu\}$ is a solution of p_1 with non-zero components. Thus, $\{1/(z_1 + \epsilon), \ldots, 1/(z_s + \epsilon), 1/(z_{s+1} + \mu), \ldots, 1/(z_n + \mu)\}$ is a solution of p, implying that one of its components is in Ω. Since Ω is a closed set, taking the limit as $\epsilon \to 0$ shows that one of the components of $\{1/z_1, 1/z_2, \ldots, 1/z_n\}$ is in Ω. Equivalently, one of $\{z_1, z_2, \ldots, z_n\}$ is in $1/\Omega$.

Both cases show that $1/\Omega$ is a locus holder for p_1.

Now, let Ω_1 be a locus for p_1 contained in $1/\Omega$. Repeat the above argument exchanging the roles of p and p_1 to conclude that $1/\Omega_1$ is a locus holder for p. Since $1/\Omega_1 \subseteq \Omega$, the minimality property of Ω implies that we must have equality. We have shown that $T^{-1}(\Omega)$ is a locus of $T[p]$. The opposite direction follows from the fact that $T^{-1}[T[p]](z) = p(z)$. □

As a corollary we obtain the extended version of Theorem 1.7.

Corollary 2.2. *If all zeros of $p \in \mathcal{P}_n^*$ are on the boundary of a closed circular domain B, then B is a locus of p.*

Proof. The case $a_n \neq 0$ is Theorem 1.7. Suppose $a_n = 0$; then B is a closed half plane having the finite roots of p on its boundary. Let the finite roots of p be $\alpha_1, \alpha_2, \ldots, \alpha_m$. Let T be a Möbius transformation such that T^{-1} sends the finite roots of p, together with ∞, into $\mathbb{C}$. Since the degree of $T[p]$ is n and its roots $T^{-1}(\alpha_1), T^{-1}(\alpha_2), \ldots, T^{-1}(\alpha_m)$, and $T^{-1}(\infty)$ all lie on a circle, the result follows from Theorem 1.7. □

3. Notation

The following notation is used throughout. Context specific notation is introduced when needed.

For distinct points $a, b \in \mathbb{C}$, denote by $[a, b]$ the closed segment between a and b. Similarly, by (a, b) denote the open segment.

Denote by $C(a, r)$ the circle with centre a and radius r and by $C(a, b, c)$ the circle determined by the distinct points $a, b, c \in \mathbb{C}^*$. The *interior of* $C(a, b, c)$ is the region on the left-hand side when $C(a, b, c)$ is traversed in the direction determined by a, b, c. If the circle is defined by a centre point and a radius, then the *interior of* $C(a, r)$ is the bounded component of $\mathbb{C} \setminus C(a, r)$.

By $B[a, r]$ (resp. $B(a, r)$) denote the closed (resp. open) disk with centre a and radius r. By $B[a, b, c]$ denote the closed circular domain determined by the oriented circle $C(a, b, c)$ together with its interior. Similarly, by $B(a, b, c)$ denote the interior of $C(a, b, c)$. This notation will be used to denote a half plane; for example, $B[0, 1, \infty]$ is the closed upper half plane of $\mathbb{C}$.

For distinct, non-colinear points $a, b, c \in \mathbb{C}$, by $D[a, b, c]$ denote the closed *bounded* disk having a, b, and c on its boundary. For distinct, colinear points $a, b, c \in \mathbb{C}$, by $D[a, b, c]$ denote either one of the closed half planes having a, b, and c on its boundary. The differences between $B[a, b, c]$ and $D[a, b, c]$ are: a) $B[a, b, c]$ is always oriented, and that determines its interior; and b) $D[a, b, c]$ is always bounded, unless the points a, b, c are colinear.

For distinct points $a, b, c \in \mathbb{C}$ by $S[a, b, c]$ denote the smallest closed disk containing the points a, b, and c. Note that $D[a, b, c] = S[a, b, c]$ if and only if there is no diameter of $D[a, b, c]$ such that a, b, and c are strictly on one side of it. Thus, $D[a, b, c] \neq S[a, b, c]$ implies that one of the points a, b, or c is in the interior of $S[a, b, c]$.

Given three distinct points a, b, c, by $\operatorname{arc}(a, b, c)$ we denote the *open* arc of the circle of $C(a, b, c)$, starting from a, passing through b, and ending at c. By $\operatorname{arc}[a, b, c]$ we denote the same arc but including the endpoints a and c. Analogously, we define the half-open arcs $\operatorname{arc}(a, b, c]$ and $\operatorname{arc}[a, b, c)$.

Given a set $\Omega \subset \mathbb{C}^*$ by $\partial\Omega$ and $\operatorname{int}\Omega$ denote the boundary and the interior of Ω.

The cube roots of -1 are denoted by

$$e_1 := e^{-i\pi/3}, e_2 := e^{i\pi/3}, \text{ and } e_3 := -1.$$

4. Polynomials of degree 2

Let $p(z) = (z - \alpha_1)(z - \alpha_2) = z^2 + a_1 z + a_0$ be a polynomial in $\mathcal{P}_2^*$, that is, $\alpha_1 \neq \alpha_2$. We are interested in finding loci of $p(z)$, or equivalently of the symmetrized

polynomial

$$P(z_1, z_2) = z_1 z_2 + \frac{a_1}{2}(z_1 + z_2) + a_0.$$

Clearly, the solutions are $\{z, T(z)\}$, where $z \in \mathbb{C}$ and $T(z)$ is the symmetric Möbius transformation

$$T(z) = \frac{-a_1 z - 2a_0}{2z + a_1}. \tag{4.1}$$

The elementary properties of symmetric Möbius transformations imply that every closed circular domain having α_1 and α_2 on its boundary is a locus of $p(z)$. In fact, we may construct more loci in the following way. Let Γ be a simple (not self-intersecting) Jordanian curve connecting α_1 and α_2 and contained entirely in one of the half planes defined by the line through α_1 and α_2. Then, the domain defined by the closed Jordanian curve $\Gamma \cup T(\Gamma)$ is a locus of $p(z)$.

5. Polynomials of degree 3

Let

$$p(z) = (z - \alpha_1)(z - \alpha_2)(z - \alpha_3) = z^3 + a_2 z^2 + a_1 z + a_0 \tag{5.1}$$

be a polynomial in $\mathcal{P}_3^*$. That is, $p(z)$ has at least two distinct zeros; assume $\alpha_1 \neq \alpha_2$. (The case when all the zeros are equal is trivial and is considered in (2.1).) We are interested in finding loci of $p(z)$, or equivalently of the symmetrized polynomial

$$P(z_1, z_2, z_3) = z_1 z_2 z_3 + \frac{a_2}{3}(z_1 z_2 + z_1 z_3 + z_2 z_3) + \frac{a_1}{3}(z_1 + z_2 + z_3) + a_0. \tag{5.2}$$

Proposition 5.1. *Suppose the roots of $p(z)$ are distinct. The set Ω is a locus of $p(z)$ if and only if $W^{-1}(\Omega)$ is a locus of $z^3 + 1$, where*

$$W(z) := -\frac{z(\alpha_1\alpha_2 e_3 + \alpha_1\alpha_3 e_2 + \alpha_2\alpha_3 e_1) + (\alpha_1\alpha_2 e_3 + \alpha_1\alpha_3 e_1 + \alpha_2\alpha_3 e_2)}{z(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) + (\alpha_1 e_2 + \alpha_2 e_1 + \alpha_3 e_3)}. \tag{5.3}$$

The Möbius transformation (5.3) is the one satisfying $W(e_k) = \alpha_k$ for $k = 1, 2, 3$. The proof of Proposition 5.1 is given in Section 9. In Sections 6 and 7 we exhibit several families and particular examples of loci of $z^3 + 1$. Since our goal is to strengthen the Grace theorem, we are interested in finding a locus of $p(z)$ that is, in some sense, as small as possible.

On the one hand, Theorem 1.7 shows that $B[\alpha_1, \alpha_2, \alpha_3]$ (and in particular $D[\alpha_1, \alpha_2, \alpha_3]$) is a locus of $p(z)$. Unfortunately, that set may be very large, for example, when the three roots are colinear. On the other hand, the Grace theorem implies that $S[\alpha_1, \alpha_2, \alpha_3]$ is a locus holder for $p(z)$. Naturally, our goal is to find a locus of $p(z)$ inside the smallest closed disk containing the roots.

The requirement in Proposition 5.1 (as well as in Theorem 5.1 below) that the roots of $p(z)$ be distinct is not significant, since if $\alpha_1 = \alpha_3 \neq \alpha_2$, then the closed disk with diameter $[\alpha_1, \alpha_2]$ is both a locus and the smallest closed disk containing the roots. There is nothing more to do.

Define the rational quadratic map obtained by solving the equation $P(z_1, z, z) = 0$ for z_1:

$$Q(z) := -\frac{a_2 z^2 + 2a_1 z + 3a_0}{3z^2 + 2a_2 z + a_1}. \tag{5.4}$$

We now formulate the main result about a locus of $p(z)$ inside the smallest disk containing its roots. Part a) follows from the discussion above it. The proof of part b) is deferred to Appendix B.

Theorem 5.1. *Suppose the roots of $p(z)$ are distinct.*

a) If $S[\alpha_1, \alpha_2, \alpha_3] = D[\alpha_1, \alpha_2, \alpha_3]$, that disk is both a locus and the smallest disk containing the roots.

b) If $S[\alpha_1, \alpha_2, \alpha_3] \neq D[\alpha_1, \alpha_2, \alpha_3]$, suppose for concreteness $\alpha_3 \in int\, S[\alpha_1, \alpha_2, \alpha_3]$. Let $\Gamma \subset D[\alpha_1, \alpha_2, \alpha_3]$ be the semi-circle of $\partial S[\alpha_1, \alpha_2, \alpha_3]$ with endpoints α_1, α_2. Then, the closed domain Ω with boundary $\Gamma \cup Q(\Gamma)$ is a locus of $p(z)$ contained in $S[\alpha_1, \alpha_2, \alpha_3]$.

There is one narrow case when it is easy to find a locus inside the smallest disk containing the roots. This case is when the roots are colinear, and it is treated in Proposition 6.3, part 2 and illustrated in Figure 5(b). A big advantage of the locus constructed in Theorem 5.1, apart from being general, is the following continuity property.

Proposition 5.2. *The locus Ω in Theorem 5.1 is continuous when considered as a set-valued map in α_3, as long as α_3 is in the original disk $S[\alpha_1, \alpha_2, \alpha_3]$ and does not cross the segment $[\alpha_1, \alpha_2]$. In addition, when α_3 approaches the boundary of the original disk $S[\alpha_1, \alpha_2, \alpha_3]$, the locus Ω approaches the original $S[\alpha_1, \alpha_2, \alpha_3]$.*

The proof of Proposition 5.2 is given in Appendix B. We conclude this section with an example illustrating Theorem 5.1.

Example 5.2. a) Consider the polynomial

$$p(z) = (z^2 - 1)(z - (1/2 + i/4)) = z^3 - (1/2 + i/4)z^2 - z + (1/2 + i/4).$$

The fractional quadratic function (5.4) for this polynomial is

$$Q(z) = -\frac{(2+i)z^2 + 8z - (6+3i)}{-12z^2 + (4+2i)z + 4}.$$

Let $(\alpha_1, \alpha_2, \alpha_3) := (-1, 1, 1/2 + i/4)$. The locus Ω described in Theorem 5.1 is the shaded area depicted in Figure 1. The unit disk, $S[-1, 1, 1/2 + i/4]$, is the smallest disk containing the zeros -1, 1, and $1/2 + i/4$. The arc Γ is the semi-circle through the points $-1, -i$, and 1. The boundary of Ω in the upper half plane is the image of Γ under $Q(z)$.

The dotted circle is the boundary of $D[-1, 1, 1/2 + i/4]$. Both $B[-1, 1, 1/2 + i/4]$ and $B[1, -1, 1/2 + i/4]$ are also loci of $p(z)$, according to Theorem 1.7

One can formulate the strongest versions of the Grace theorem: Every polynomial $q(z)$ apolar with $p(z)$ (in our extended sense) has at least one root inside Ω. (Similarly, it has at least one root in $B[-1, 1, 1/2 + i/4]$, or has at least one root in $B[1, -1, 1/2 + i/4]$.) These are three instances of tightest possible versions of the Grace theorem by inclusion.

b) Consider the polynomial

$$p(z) = z^3 - z.$$

The fractional quadratic function (5.4) for this polynomial is

$$Q(z) = \frac{2z}{3z^2 - 1}.$$

Let $(\alpha_1, \alpha_2, \alpha_3) := (-1, 1, 0)$. The locus Ω described in Theorem 5.1 is the shaded

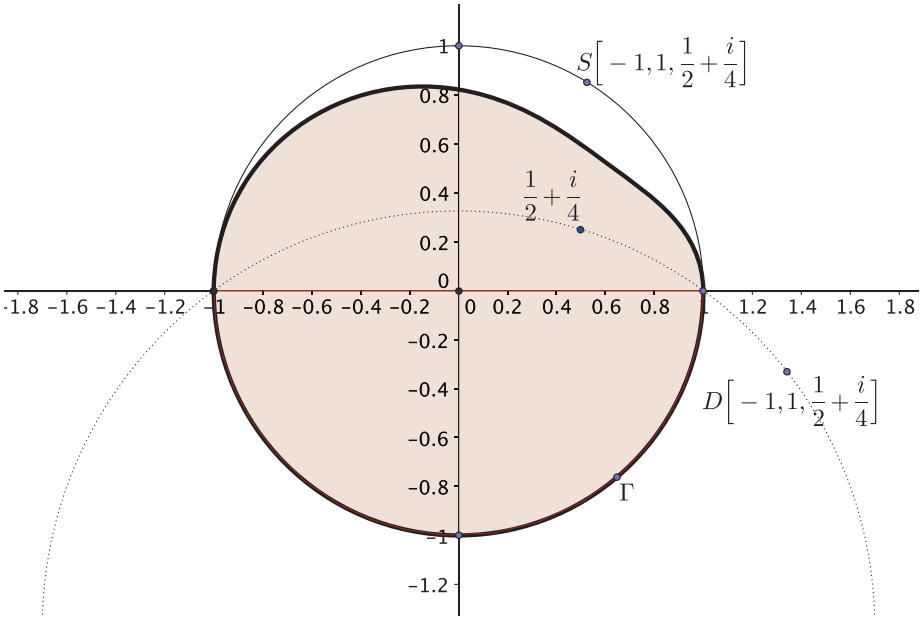


FIGURE 1. A locus of $p(z) = (z^2-1)(z-(1/2+i/4))$ in the smallest disk containing the zeros

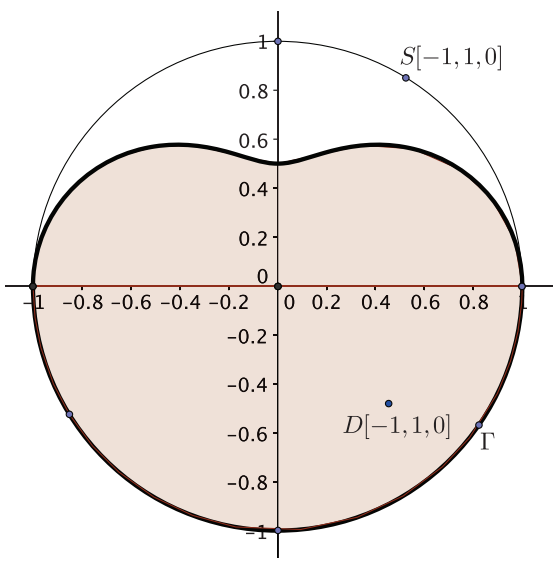


FIGURE 2. A locus of $p(z) = z^3 - z$ in the smallest disk containing the zeros

area depicted in Figure 2. The disk $D[-1,1,0]$ is chosen to be the closed lower half plane, since the roots are colinear. In Subsection 6.2 we exhibit more loci of the polynomial $z^3 - z$ and in particular a different locus contained in $S[-1,1,0]$; see Figure 4(c). $\square$

6. Examples

6.1. **Simple loci for z^3+1.** Let $r \in [0,1]$. Let $\Delta := B[e_1, 0, \infty] \cup B[0, e_2, \infty]$, that is, Δ is the union of two closed half planes. Let D_1 be the closed disk with centre 0 and radius r. Let D_2 be the closed disk with centre 0 and radius $1/r^2$. Define the set

$$\Omega_r := \big(D_1 \setminus \Delta\big) \cup \big(D_2 \cap \Delta\big) \tag{6.1}$$

depicted in Figure 3.

Proposition 6.1. *The set Ω_r is a locus of z^3+1 for every $r \in [0,1]$.*

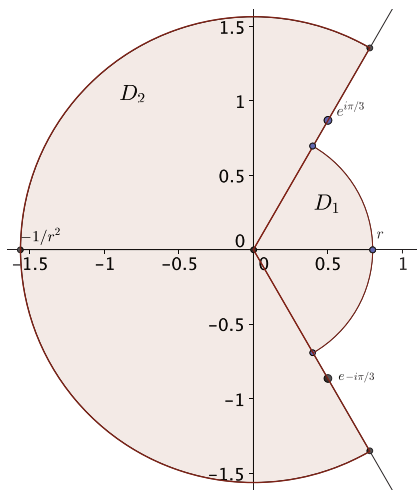


Figure 3. The locus Ω_r for $r \in [0,1]$

Proof. The proof is relatively easy to obtain directly. We note that when $r > 0$, the proposition is a particular case of Theorem 7.4, and we direct the reader to Corollary 7.1 for a formal proof. The proof in the case $r = 0$ is straightforward. □

The proposition is not true for $r > 1$, since then Ω_r does not contain -1, a root of z^3+1.

6.2. **Loci of the polynomial $z^3 - z$.** The polynomial $q(z) := z^3 - z$ is the first of a sequence of polynomials that is dealt with in the second part of this paper and connects the notion of a locus to the classical Rolle Theorem. It is appropriate to discuss it here and it is somewhat connected to the discussion in Subsection 6.3.

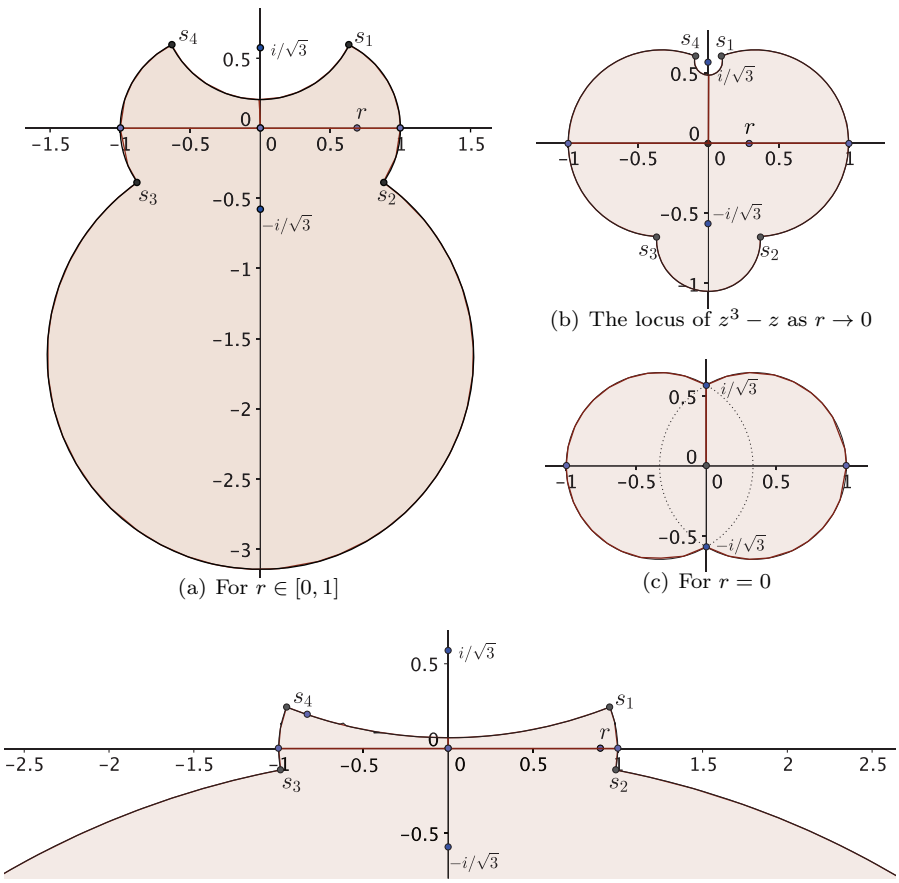
The roots of $z^3 - z$ are $(\alpha_1, \alpha_2, \alpha_3) := (-1, 1, 0)$ and the Möbius transformation (5.3) becomes

$$W_q(z) := \frac{i}{\sqrt{3}} \frac{z+1}{z-1}. \tag{6.2}$$

We now describe $W_q(\Omega_r)$; see Figure 4(a). Define $s_1 := W_q(e_2/r^2)$, $s_2 := W_q(re_2)$, $s_3 := W_q(re_1)$, and $s_4 := W_q(e_1/r^2)$. Then,

$$W_q : \begin{cases} \operatorname{arc}(e_1/r^2, e_1, re_1) \\ \operatorname{arc}(re_1, r, re_2) \\ \operatorname{arc}(re_2, e_2, e_2/r^2) \\ \operatorname{arc}(e_2/r^2, -1/r^2, e_1/r^2) \end{cases} = \begin{cases} \operatorname{arc}(s_4, -1, s_3), \\ \operatorname{arc}(s_3, i(r+1)/\sqrt{3}(r-1), s_2), \\ \operatorname{arc}(s_2, 1, s_1), \\ \operatorname{arc}(s_1, i(1-r^2)/\sqrt{3}(1+r^2), s_4). \end{cases}$$

Orienting each of these arcs with the three points on them, one can see that $W_q(\Omega_r)$ is the domain enclosed by the curve in Figure 4(a).



(a) For $r \in [0, 1]$

(b) The locus of $z^3 - z$ as $r \to 0$

(c) For $r = 0$

(d) The locus of $z^3 - z$ as $r \to 1$

FIGURE 4. The locus $W_q(\Omega_r)$ of $z^3 - z$

The arc $(s_4, -1, s_3)$ is on the circle $C(i/\sqrt{3}, -1, -i/\sqrt{3})$ and the arc $(s_2, 1, s_1)$ is on the circle $C(i/\sqrt{3}, 1, -i/\sqrt{3})$. Note that both circles do not depend on r. The other two arcs arc $(s_3, i(r+1)/\sqrt{3}(r-1), s_2)$ and arc $(s_1, i(1-r^2)/\sqrt{3}(1+r^2), s_4)$ are on circles that are symmetric with respect to the real axis. (Non-degenerate Möbius transformation preserves symmetry.)

Now consider the two limiting cases. As r approaches 0 (refer to Figure 4(b)), we have

$$\lim_{r\to 0}\begin{cases}\operatorname{arc}(s_4,-1,s_3)\\ \operatorname{arc}(s_3,i(r+1)/\sqrt{3}(r-1),s_2)\\ \operatorname{arc}(s_2,1,s_1)\\ \operatorname{arc}(s_1,i(1-r^2)/\sqrt{3}(1+r^2),s_4)\end{cases}=\begin{cases}\operatorname{arc}(i/\sqrt{3},-1,-i/\sqrt{3}),\\ -i/\sqrt{3},\\ \operatorname{arc}(-i/\sqrt{3},-1,i/\sqrt{3}),\\ i/\sqrt{3}.\end{cases}$$

In other words, $W_q(\Omega_0)=D[-i/\sqrt{3},1,i/\sqrt{3}]\cup D[i/\sqrt{3},-1,-i/\sqrt{3}]$ is the union of two disks; see Figure 4(c). As r approaches 1, we have

$$\lim_{r\to 1}\begin{cases}\operatorname{arc}(s_4,-1,s_3)\\ \operatorname{arc}(s_3,i(r+1)/\sqrt{3}(r-1),s_2)\\ \operatorname{arc}(s_2,1,s_1)\\ \operatorname{arc}(s_1,i(1-r^2)/\sqrt{3}(1+r^2),s_4)\end{cases}=\begin{cases}-1,\\ \operatorname{arc}(-1,\infty,1),\\ 1,\\ \operatorname{arc}(1,0,-1).\end{cases}$$

In other words, $W_q(\Omega_1)$ is the closed lower half plane of the complex plane; see Figure 4(d).

6.3. **Loci made of two disks.** Proposition 5.1 implies that $W(\Omega_r)$ is a locus of $p(z)$ for every $r\in[0,1]$. The case when $r=0$ is of interest. Let $B_1:=W(B[e_1,0,\infty])$ and let $B_2:=W(B[0,e_2,\infty])$. Note that $\alpha_3\in B_1\cap B_2$, since $W(e_3)=\alpha_3$. Let

$$\Omega_d:=B_1\cup B_2.$$

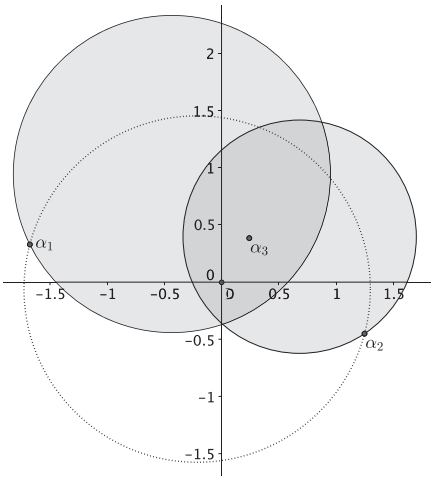
Proposition 6.2. *The set Ω_d is a locus of $p(z)$.*

Proposition 6.2 says that every polynomial of degree 3, with distinct zeros, has a locus that is a union of two closed circular domains. Figure 5 illustrates this for the polynomial (5.1). In all parts, the locus is the shaded area. The dotted circle is the one with diameter $[\alpha_1,\alpha_2]$. In the first part the locus is bounded; in the second the locus is in the interior of the dotted circle; while in the third part it is unbounded. Figure 5 shows instances of general properties that we summarize next.

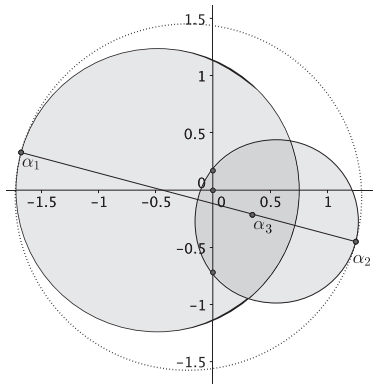
Proposition 6.3. *Suppose the numbers α_1,α_2, and α_3 are distinct.*

1. *The locus Ω_d is bounded if and only if $[\alpha_1,\alpha_2]$ is the strictly longest side of the triangle with vertices α_1,α_2, and α_3.*
2. *The locus Ω_d is contained inside the closed disk with diameter $[\alpha_1,\alpha_2]$ if and only if $\alpha_3\in(\alpha_1,\alpha_2)$.*

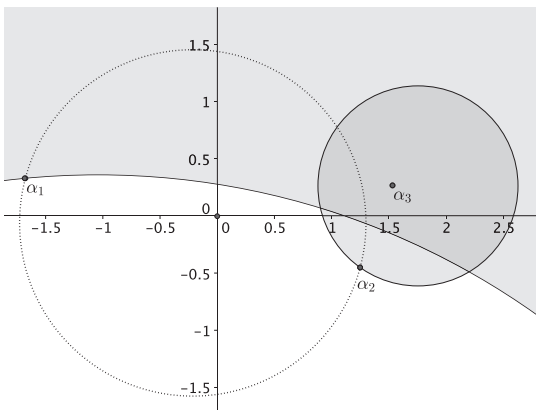
Proof. We omit the proof of part 1 since it is outside of our focus. 2. Let S be the circle with diameter $[\alpha_1,\alpha_2]$. The circle $C(e_1,e_2,e_3)$ is orthogonal to the lines $C(0,e_1,\infty)$ and $C(0,e_2,\infty)$ and so are their images under $W(z)$. Since $W(C(e_1,e_2,e_3))=C(\alpha_1,\alpha_2,\alpha_3)$, the disks B_1 and B_2 are tangent to S if and only if α_3 is on the line through α_1 and α_2. The 'only if' direction is trivial. For the other direction, without loss of generality assume $\alpha_1=-1$, $\alpha_2=1$, and suppose $\alpha_3\in(-1,1)$. Let T be the Möbius transformation that maps the three distinct points $(-1,1,\alpha_3)$ onto $(-1,1,0)$, explicitly $T(z)=(z-\alpha_3)/(-\alpha_3 z+1)$. We have $T(W(z))=W_q(z)$, where W_q is defined by (6.2), and hence $T(\Omega_d)=W_q(\Omega_0)$, where Ω_0 is defined by (6.1) with $r=0$. Since $W_q(\Omega_0)$ is in the closed unit disk (see Figure 4(c)) and since T preserves it, we conclude that Ω_d is in it as well. $\square$



(a) Bounded locus



(b) Locus in the smallest disc



(c) Unbounded locus

FIGURE 5. Illustrating Proposition 6.2

7. LOCI OF $z^3 + 1 = 0$

7.1. **The locus needed for the proof of Theorem 5.1.** Fix a point $s \in [0, 1]$ and let

$$\Gamma := \operatorname{arc}[e_2, s, e_1].$$

Define the function

$$g(z) := -1/z^2$$

and let

(7.1) Ω_3 be the closed bounded domain with boundary $\Gamma \cup g(\Gamma)$,

that is,

$$\partial\Omega_3 = \Gamma \cup g(\Gamma).$$

It can be shown that as s approaches 1, the Jordan curve $\Gamma \cup g(\Gamma)$ approaches $C(0, 1)$. This subsection is devoted to the proof of the following result.

Theorem 7.1. *For any $s \in [0,1]$ the set Ω_3 is a locus of $z^3+1=0$. Moreover, $\Omega_3 \subseteq \mathbb{C} \setminus B(e_2, s, e_1)$.*

First, let us take care of the special case $s = 1$. Then, $\Gamma = \operatorname{arc}[e_2, 1, e_1]$ and $g(\operatorname{arc}[e_2, 1, e_1]) = \operatorname{arc}[e_2, -1, e_1]$. Hence, $\Omega_3 = B[0,1]$ is trivially a locus of z^3+1. Moreover, $\Omega_3 = \mathbb{C} \setminus B(e_2, 1, e_1)$, since $B(e_2, 1, e_1)$ is the exterior of the closed unit disk. Thus, for the rest of this subsection,

$$\text{we assume } s \in [0,1).$$

The arc Γ separates the set $B[0,1] \cap B[1,1]$ into two domains. The *closed* domain containing 1 is denoted by Δ_1, while the *closure* of the set $B[0,1] \setminus \Delta_1$ is denoted by Δ_2. Note that $\Delta_1 \cap \Delta_2 = \Gamma$ and $\Delta_1 \cup \Delta_2 = B[0,1]$. Define the image $\Delta_3 := g(\Delta_1)$, and denote by Δ_4 the closure of the complement $\mathbb{C} \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$. Thus, we have $\Omega := \Delta_2 \cup \Delta_3$. For brevity in this subsection, we denote

$$C := C(e_2, s, e_1).$$

The proof of Theorem 7.1 is based on two lemmas.

Lemma 7.2. *For any $w, z \in \Gamma$ such that $|\sqrt{z}/w| < 1$, we have either $i\sqrt{z}/w \in \Delta_1$ or $-i\sqrt{z}/w \in \Delta_1$.*

Proof. Let D be the closed circular domain with border C, containing the point 1, that is, $D := B[e_2, s, e_1]$ (D may or may not be bounded). Note that $\Delta_1 = B[0,1] \cap D$. Refer to Figures 6 and 7 for illustrations. Fix $w, z \in \Gamma$. Without loss of generality, assume that

$$\operatorname{Im}(w) \geq 0.$$

(The other situation can be resolved by taking the complex conjugation of $\sqrt{z}/w$.) Let $z = z_1 + iz_2$ and let $w = w_1 + iw_2$. The fact that e_1, w, z, and e_2 are on a circle is equivalent to their cross ratio (e_1, w, z, e_2) being a real number. Expanding $\operatorname{Im}(e_1, w, z, e_2) = 0$ and grouping the terms appropriately gives

$$(1-|z|^2)(1-2w_1) = (1-|w|^2)(1-2z_1). \tag{7.2}$$

Assume, without loss of generality, that the square root $\sqrt{z}$ taken is the one with $\operatorname{sign}\operatorname{Im}(\sqrt{z}) = \operatorname{sign}\operatorname{Im}(z)$. Suppose that $|\sqrt{z}/w| < 1$ and note that this condition guarantees that $\pm i\sqrt{z}/w \in B[0,1]$. All we have to show is that either $i\sqrt{z}/w \in D$ or $-i\sqrt{z}/w \in D$. The condition $|\sqrt{z}/w| < 1$ also implies that

$$|z| < 1. \tag{7.3}$$

The situation when $z = 0$ is trivial, so assume that $z \neq 0$. Let $z^* \in \Gamma$ be such that $|z^*| = \sqrt{|z|}$ and $\operatorname{Im}(z^*) \geq 0$. Let $z^* = z_1^* + iz_2^*$. □

Claim 1. In order to have $i\sqrt{z}/w \in D$, it is sufficient to show that $i\sqrt{z}/z^* \in D$ and $i\sqrt{z}/e_2 \in D$.

Proof. Consider the Möbius map $T(u) := i\sqrt{z}/u$. It sends the circle C into another circle. Consider the image of $\operatorname{arc}(z^*, w, e_2)$. Note that $T(e_1) = \sqrt{z}e^{i(\pi/3+\pi/2)}$ is always a point in the half plane $\{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0\}$ and in addition $|T(e_1)| \leq 1$. Hence, $T(e_1) \notin D$ for every $s \in [0,1]$.

We claim that if $T(z^*) \in D$ and $T(e_2) \in D$, then the whole image of $\operatorname{arc}(z^*, w, e_2)$, under T, is in D, and in particular $T(w) = i\sqrt{z}/w$ is in D. Indeed, suppose that $T(z^*) \in D$ and $T(e_2) \in D$ but that the conclusion is false. Then, the image of

$\operatorname{arc}(z^*, w, e_2)$ must leave D and return to D. But $T(e_1) \notin D$, hence the (distinct) circles $T(C)$ and C have at least three intersection points. This is a contradiction. $\square$

The rest of our efforts are focused on showing that if $z \in \Gamma$, then $i\sqrt{z}/z^* \in D$ and $i\sqrt{z}/e_2 \in D$.

Claim 2. If $z \in \Gamma$, then $i\sqrt{z}/z^* \in D$.

Proof. Since $|i\sqrt{z}/z^*| = 1$, it is enough to show that $\arg(i\sqrt{z}/z^*)$ is in $[-\pi/3, \pi/3]$. Let $\alpha := \arg(z^*)$ and let $\beta := \arg(\sqrt{z})$; then $\arg(i\sqrt{z}/z^*) = \pi/2 + \beta - \alpha$. (The argument is assumed to take values in $(-\pi, \pi]$.) Since $\alpha \in [-\pi/2, \pi/2]$ and $\beta \in [-\pi/4, \pi/4]$, the inequality $\pi/2 + \beta - \alpha \geq -\pi/3$ is trivial to verify. It is left to verify that $\pi/6 + \beta \leq \alpha$. Since $\operatorname{Im}(z^*) \geq 0$ we have $\alpha \geq 0$. If $\pi/6 + \beta \leq 0$, there is nothing to show, and in the case when $\pi/6 + \beta \geq 0$, the inequality $\pi/6 + \beta \leq \alpha$ is equivalent to $\cos(\pi/6 + \beta) \geq \cos(\alpha)$. On the one hand, we have

$$\cos(\alpha) = \frac{z_1^*}{|z^*|} = \frac{z_1^*}{\sqrt{|z|}} = \frac{|z| + 2z_1}{2\sqrt{|z|}(1 + |z|)},$$

where for the last equality we applied (7.2) with $w := z^*$ and, using (7.3), expressed z_1^* in terms of z_1 and $|z|$. On the other hand, $2\cos^2(\beta) - 1 = 1 - 2\sin^2(\beta) = \cos(2\beta) = z_1/|z|$. Hence, $\cos(\beta) = \sqrt{(|z| + z_1)/2|z|}$ and $\sin(\beta) = \operatorname{sign}(\beta)\sqrt{(|z| - z_1)/2|z|}$. Thus, we have

$$\cos(\pi/6 + \beta) = \frac{\sqrt{3}}{2}\sqrt{\frac{|z| + z_1}{2|z|}} - \operatorname{sign}(\beta)\frac{1}{2}\sqrt{\frac{|z| - z_1}{2|z|}}.$$

A glance at these expressions reveals that it is enough to prove the inequality $\cos(\pi/6 + \beta) \geq \cos(\alpha)$ only for $\beta \geq 0$, and after trivial simplifications it becomes

$$\sqrt{3}\sqrt{|z| + z_1} - \sqrt{|z| - z_1} \geq \sqrt{2}\left(\frac{|z| + 2z_1}{1 + |z|}\right). \tag{7.4}$$

Multiplying both sides by $\sqrt{3}\sqrt{|z| + z_1} + \sqrt{|z| - z_1}$ and dividing by $\sqrt{2}(|z| + 2z_1)/(1 + |z|)$ we obtain

$$\sqrt{2}(1 + |z|) \geq \sqrt{3}\sqrt{|z| + z_1} + \sqrt{|z| - z_1}.$$

Square both sides, divide by 2, and consider the remaining terms as a quadratic function in z_2. One sees that

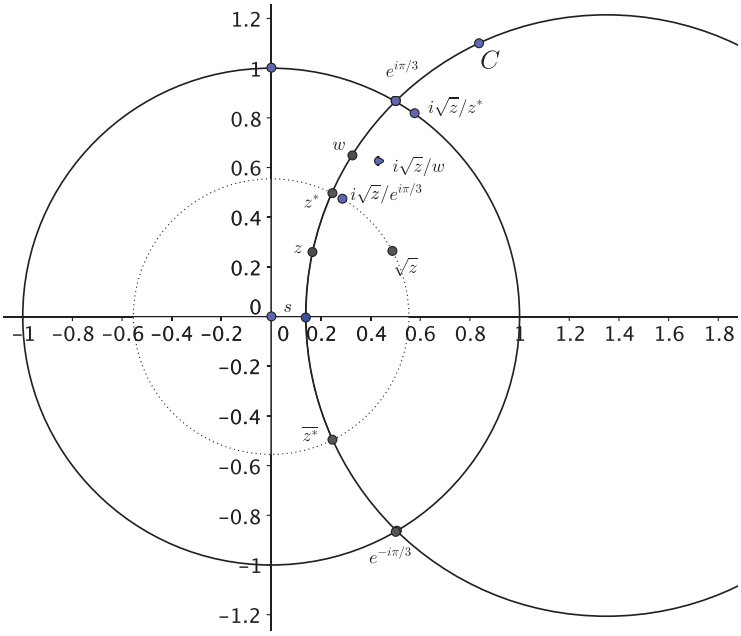
$$z_2^2 - \sqrt{3}z_2 + (1 + z_1^2 - z_1) \geq 0, \tag{7.5}$$

since the discriminant is $-(2z_1 - 1)^2 \leq 0$, and shows that (7.5) holds. The proof of Claim 2 is complete. $\square$

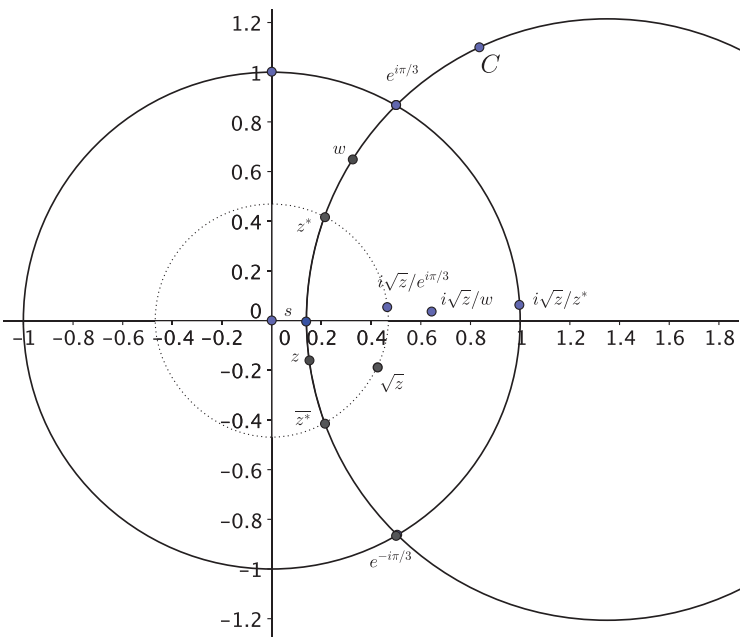
Claim 3. If $z \in \Gamma$, then $i\sqrt{z}/e_2 \in D$.

Proof. As in the proof of Claim 2, let $\alpha := \arg(z^*)$ and let $\beta := \arg(\sqrt{z})$. In that, we showed that $\pi/6 + \beta \leq \alpha$. But the point z^* does not change if we replace z by its complex conjugate $\bar{z}$. This changes β to $-\beta$ and shows that we also have $\pi/6 - \beta \leq \alpha$. Since $|i\sqrt{z}/e_2| = |z^*|$, to prove the claim, it is enough to show that $\arg(i\sqrt{z}/e_2) \in [-\alpha, \alpha]$. Since $\arg(i\sqrt{z}/e_2) = \pi/2 + \beta - \pi/3$, the fact that $\pi/2 + \beta - \pi/3 \in [-\alpha, \alpha]$ follows.

This concludes the proof of Lemma 7.2. $\square$

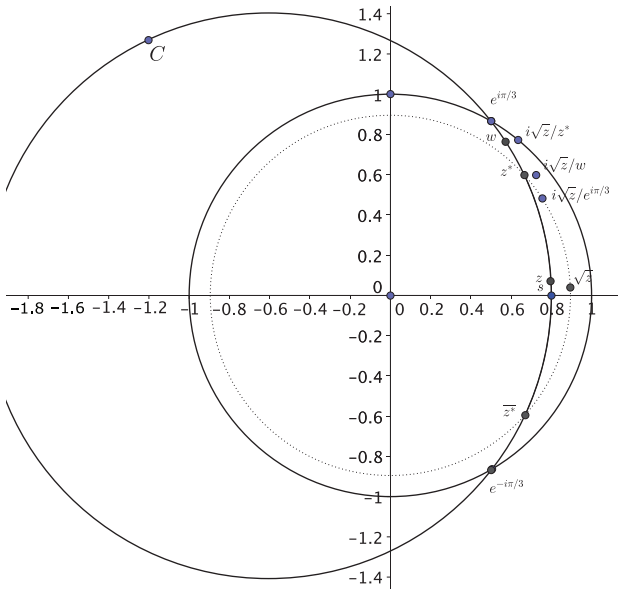


(a) $s \in [0, 1/2],\ \mathrm{Im}\,(z) \geq 0$

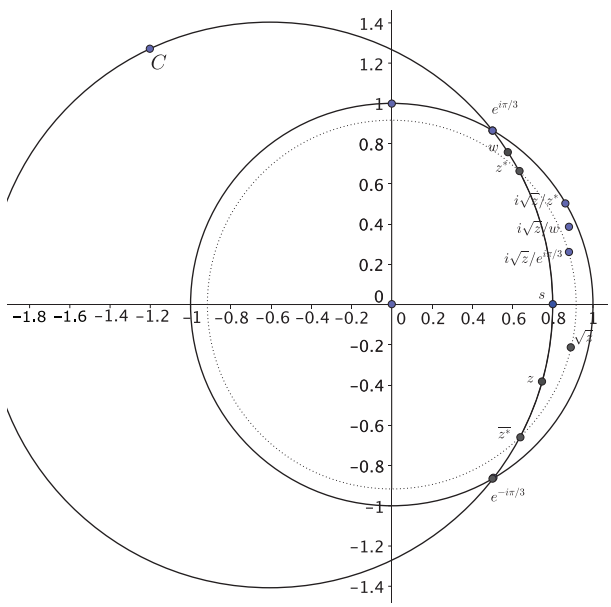


(b) $s \in [0, 1/2],\ \mathrm{Im}\,(z) \leq 0$

FIGURE 6. Illustrating Lemma 7.2



(a) $s \in [1/2, 1]$, $\operatorname{Im}(z) \geq 0$



(b) $s \in [1/2, 1]$, $\operatorname{Im}(z) \leq 0$

FIGURE 7. Illustrating Lemma 7.2

Next, we summarize several easy properties of the map g that are needed in Lemma 7.3—the second lemma needed for the proof of Theorem 7.1.

Properties of the function g. While reading the next paragraph, refer to Figure 8 for illustrations. We have

$$C(0,r)\cap(\Gamma\cup g(\Gamma))=\begin{cases}\emptyset & \text{for } r\in[0,s),\\ \text{one point} & \text{for } r=s,\\ \text{two points} & \text{for } r\in(s,1/s^2),\\ \text{one point} & \text{for } r=1/s^2,\\ \emptyset & \text{for } r\in(1/s^2,\infty).\end{cases}$$

Now fix an $r\in(s,1]$, and note that we have $g(C(0,r))=C(0,1/r^2)$. In order to describe the action of the map g on $C(0,r)$ more precisely, we need the circle $-C$, that is, the reflection of the circle of C with respect to 0. The intersection of $C(0,r)$ with $(-C)\cup C$ consists of four points. Starting from the positive orthant and moving counterclockwise, label these points as $a_1(r), a_2(r), -a_1(r)$, and $-a_2(r)$. Then, g corresponds the following pairs of arcs in a one-to-one and onto fashion:

$$g: C(0,r)\cap\Delta_1\to C(0,1/r^2)\cap\Delta_3,$$

$$g:\operatorname{arc}[a_1(r),ir,a_2(r)]\to C(0,1/r^2)\cap\Delta_4. \tag{7.6}$$

These two correspondences combined say that g maps $\operatorname{arc}(-a_2(r),a_1(r),a_2(r)]$ one-to-one and onto $C(0,1/r^2)$. The situation is analogous on $\operatorname{arc}[a_2(r),-a_1(r),-a_2(r))$, but we do not need it. □

Lemma 7.3. *The set Ω_3 is a locus.*

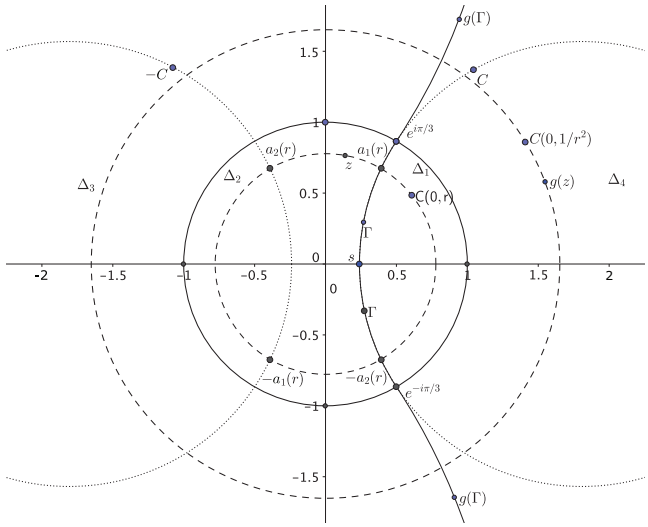
Proof. Let D be the closed circular domain with border C and containing the point 1. (Note that D may or may not be bounded and that by the definitions $\Delta_1=D\cap B[0,1]$.)

First we show that Ω_3 is a locus holder. Consider a solution $\{z_1,z_2,z_3\}$ of $z_1z_2z_3=-1$. We need to show that one of the components of the solution is in Ω_3. Since $B[0,1]$ is a locus, we may assume that $z_1\in B[0,1]=\Delta_1\cup\Delta_2$. If $z_1\in\Delta_2$, then we are done, so assume that $z_1\in\Delta_1$. Consider two cases depending on the location of z_2.

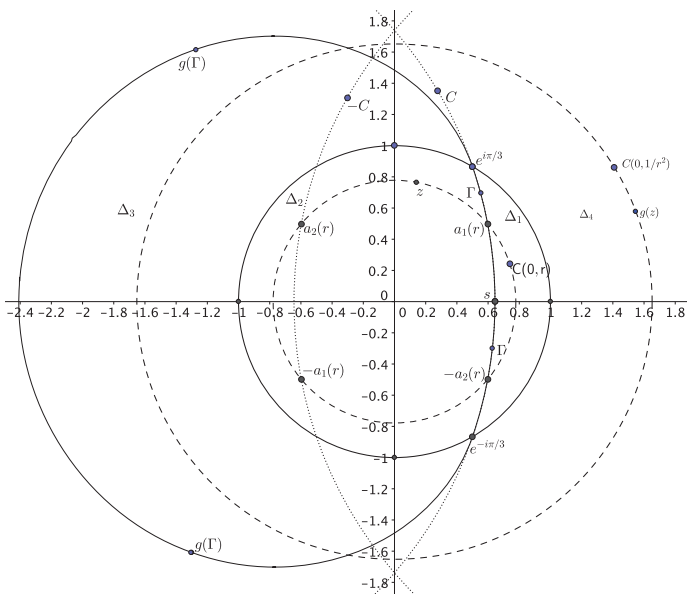
Case 1. Suppose $z_2\in\Delta_1$. Consider the bi-linear equation $u_1u_2-z_1z_2=0$ in the variables u_1,u_2. Since $(u_1,u_2):=(z_1,z_2)$ is trivially a solution, by the Grace-Walsh-Szegő coincidence theorem, a circular domain containing z_1,z_2 contains a solution, call it w_3, of the equation $w^2=z_1z_2$. Since, $z_1,z_2\in D$ we may assume that $w_3\in D$, and moreover, since $|w_3|=\sqrt{|z_1z_2|}\le 1$, we see that $w_3\in D\cap B[0,1]=\Delta_1$. Hence, $z_3=-1/(z_1z_2)=-1/w_3^2\in\Delta_3\subset\Omega_3$, since $\Delta_3=g(\Delta_1)$, and we are done.

Case 2. Suppose $z_2\notin\Delta_1$. Without loss of generality assume $z_2\notin\Omega_3=\Delta_2\cup\Delta_3$, implying that $z_2\notin B[0,1]=\Delta_1\cup\Delta_2$. If $|z_2|\ge 1/|z_1|$, then $z_3\in B[0,1]$. If $z_3\in\Delta_2$ we are done and if $z_3\in\Delta_1$, then we repeat Case 1 with z_3 in place of z_2. Thus, we may assume that $|z_2|<1/|z_1|$ or equivalently that $z_2\in B(0,1/|z_1|)$. Define $r:=|z_1|$ and $r':=1/\sqrt{|z_2|}$.

By the properties of the map g, there is a unique point, denote it by w_2, on the arc $(a_1(r'),ir',a_2(r'))$ such that $g(w_2)=-1/w_2^2=z_2$. See Figure 9. In order to complete the proof of the lemma, it is enough to show that either $i\sqrt{z_1}/w_2$ or $-i\sqrt{z_1}/w_2$ is in Δ_1. Indeed, suppose first $i\sqrt{z_1}/w_2\in\Delta_1$. Letting $w_3:=i\sqrt{z_1}/w_2$ we have that $g(w_3)\in\Delta_3$. But, $g(w_3)=-1/w_3^2=w_2^2/z_1=-1/(z_1z_2)=z_3$. Hence, $z_3\in\Omega_3$. The case when $-i\sqrt{z_1}/w_2\in\Delta_1$ is analogous.



(a) $s \in [0, 1/2]$



(b) $s \in [1/2, 1]$

FIGURE 8. Properties of the map g

The rest of our efforts are focused on showing that either $i\sqrt{z_1}/w_2$ or $-i\sqrt{z_1}/w_2$ is in Δ_1. Note first that since $|z_2| < 1/|z_1|$, we have $\sqrt{|z_1|}/|w_2| < 1$.

The number $\pm i\sqrt{z_1}/w_2$ rotates counterclockwise when z_1 rotates counterclockwise towards $a_1(r)$ and w_2 rotates clockwise towards $a_1(r')$. Conversely, the number $\pm i\sqrt{z_1}/w_2$ rotates clockwise when z_1 rotates clockwise towards $-a_2(r)$ and w_2 rotates counterclockwise towards $a_2(r')$. Thus, it is sufficient to show that either $i\sqrt{z_1}/w_2$ or $-i\sqrt{z_1}/w_2$ is in Δ_1 only when $(z_1, w_2) = (a_1(r), a_1(r'))$ or when

$(z_1, w_2) = (-a_2(r), a_2(r'))$. In the first case, since $\sqrt{|a_1(r)|}/|a_1(r')| < 1$ and $a_1(r), a_1(r') \in \Gamma$, Lemma 7.2 implies that either $i\sqrt{a_1(r)}/a_1(r')$ or $-i\sqrt{a_1(r)}/a_1(r')$ is in Δ_1. In the second case, since $\sqrt{|a_2(r)|}/|a_2(r')| < 1$ and $-a_2(r), -a_2(r') \in \Gamma$, Lemma 7.2 implies that either $i\sqrt{-a_2(r)}/(-a_2(r'))$ or $-i\sqrt{-a_2(r)}/(-a_2(r'))$ is in Δ_1. That is, we have that either $i\sqrt{-a_2(r)}/a_2(r')$ or $-i\sqrt{-a_2(r)}/a_2(r')$ is in Δ_1, and we are done showing that Ω_3 is a locus holder.

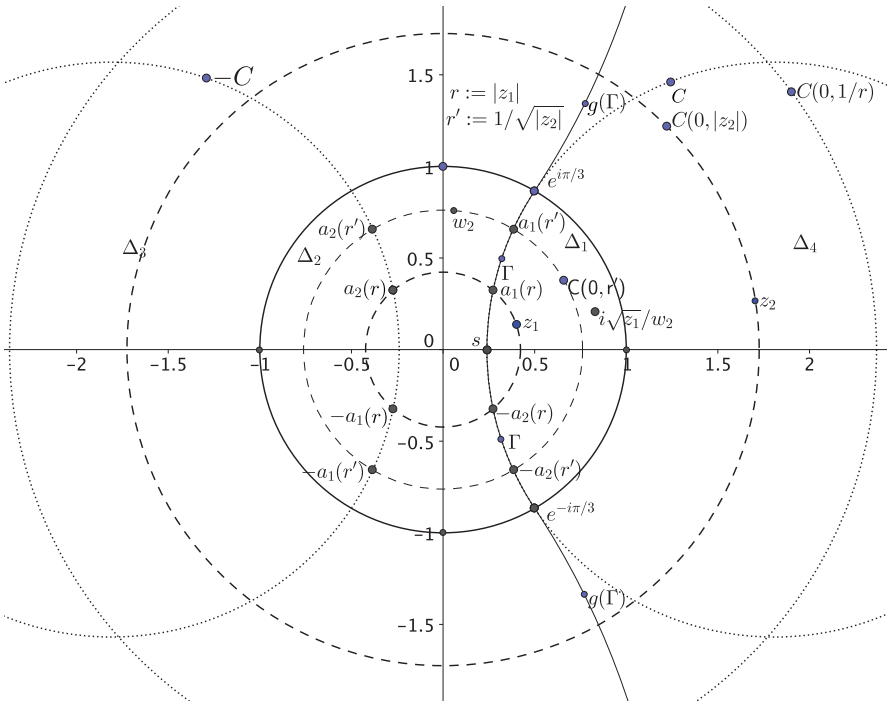


FIGURE 9. Illustrating Lemma 7.3, Case 2

Suppose Ω_3 is not a locus. Then, it contains a locus, Ω', as a proper subset, that is, there is a $u \in \operatorname{int}(\Omega_3 \setminus \Omega')$. Since $\Omega_3 = \Delta_2 \cup \Delta_3$, perturbing u slightly, if necessary, we may assume that $u \in \operatorname{int}\Delta_2 \cup \operatorname{int}\Delta_3$. If $u \in \operatorname{int}\Delta_3$, then there is a $z \in \operatorname{int}\Delta_1$ such that $u = -1/z^2$ (recall that by definition $\Delta_3 = g(\Delta_1)$). The solution $\{z, z, u\}$ of $z_1 z_2 z_3 + 1 = 0$ does not have a component in Ω', a contradiction. Suppose now that $u \in \operatorname{int}\Delta_2$. Let $r := |u| < 1$. If $u \in \operatorname{arc}[a_1(r), ir, a_2(r)]$, then by (7.6) we have $g(u) \in C(0, 1/r^2) \cap \Delta_4$ and (perturbing u slightly if necessary) we obtain a solution $\{u, u, g(u)\}$ that has no component in Ω'. The situation is similar if $u \in \operatorname{arc}[-a_1(r), -ir, -a_2(r)]$. Finally, suppose $u \in \operatorname{arc}[a_2(r), -r, -a_1(r)]$. Let $u = u_1 + iu_2$ and consider the point $v := -u_1 + iu_2$. By the symmetry with respect to the imaginary axis, we have $v \in \operatorname{arc}[a_1(r), r, -a_2(r)] \subset \Delta_1$ and $w := -1/(uv) = 1/|u|^2 > 1$. Since the ray $(1, \infty)$ on the real line does not intersect Ω_3 (perturbing u slightly if necessary), we obtain a solution $\{u, v, w\}$ that has no component in Ω'. The proof is complete. $\square$

Proof of Theorem 7.1. Lemma 7.3 shows that Ω_3, having boundary $\Gamma \cup g(\Gamma)$, is a locus. It is left to show that $\Omega_3 \subseteq \mathbb{C} \setminus B(e_2, s, e_1)$. It is sufficient to show

that $\partial\Omega_3 \cap B(e_2, s, e_1) = \emptyset$. Since $\Gamma \cap B(e_2, s, e_1) = \emptyset$, we need to show that $g(\Gamma) \cap B(e_2, s, e_1) = \emptyset$. Suppose, on the contrary, that $-1/w^2 \in B(e_2, s, e_1)$ for some $w \in \Gamma$. By continuity, there is a small $\epsilon > 0$ such that $w + \epsilon \in B(e_2, s, e_1)$ and $-1/(w+\epsilon)^2 \in B(e_2, s, e_1)$. Then, there is a closed circular domain D, a subset of $B(e_2, s, e_1)$, containing both $w + \epsilon$ and $-1/(w+\epsilon)^2$. That is, D contains the solution $\{w + \epsilon, w + \epsilon, -1/(w+\epsilon)^2\}$ of the equation $z_1 z_2 z_3 + 1 = 0$. By the Grace-Walsh-Szegő coincidence theorem, D contains a root of $z^3 + 1 = 0$. That is a contradiction. (In reaching the contradiction, it helps to visualize two cases $s \in [0, 1/2]$ and $s \in (1/2, 1)$.) □

7.2. **A general construction of loci.** This section describes a general technique for finding loci of the polynomial $z^3 + 1 = 0$. The main result is Theorem 7.4, and while examining it, the reader should note the similarities between the construction of the boundary of set (7.1) and set (7.8). We address this issue at the very end of this subsection.

Consider a function $\rho(t) : [-\pi/3, \pi/3] \to (0, 1]$ and its extension on the interval $(\pi/3, 5\pi/3)$ by

$$\rho(t) = \rho\left(\frac{\pi - t}{2}\right)^{-2} \quad \text{whenever } t \in (\pi/3, 5\pi/3), \tag{7.7}$$

and then to the whole real line by making it periodic:

$$\rho(t) = \rho(t + 2k\pi), \quad k = 0, 1, 2, \dots .$$

Theorem 7.4. *The set*

$$\Omega_\rho := \mathrm{cl}\,\{z \in \mathbb{C} : |z| \le \rho(\arg z)\} \tag{7.8}$$

is a locus of the polynomial $z^3 + 1 = 0$ *if for any* $t_1, t_2 \in [-\pi/3, \pi/3]$ *the function* $\rho(t)$ *satisfies the following two conditions:*

$$\rho(2t_1 + 2t_2 - \epsilon\pi) \ge \rho(t_1)^2 \rho(t_2)^2, \tag{7.9}$$

when $t_1 + t_2 \in (-2\pi/3, -\pi/3] \cup [\pi/3, 2\pi/3)$, *where* $\epsilon = \mathrm{sign}\,(t_1 + t_2)$, *and*

$$\rho(t_1)\rho(t_2) \ge \rho((t_1 + t_2)/2)^2. \tag{7.10}$$

Proof. Suppose conditions (7.9) and (7.10) hold. We have to prove that at least one component of every solution $\{z_1, z_2, z_3\}$ of $z_1 z_2 z_3 + 1 = 0$ is in Ω_ρ and that Ω_ρ does not contain a smaller closed set with this property. We start with the first part. Let $z_k = r_k e^{it_k}$ for $k = 1, 2, 3$ be such that $z_1 z_2 z_3 + 1 = 0$. Let $\Delta := B(e_1, 0, \infty) \cup B(0, e_2, \infty)$, that is, Δ is the union of the open half planes. Denote its complement by $\Delta^c := \mathbb{C} \setminus \Delta$. We consider four cases.

1) All three points z_1, z_2, z_3 are in Δ. At least one of these points is in the unit disk and outside of Δ^c, hence in Ω_ρ.

2) The points z_1, z_2 are in Δ and $z_3 \in \Delta^c$. Suppose that all three points are outside Ω_ρ. As Ω_ρ is star-like with respect to the origin, we may move the points z_1, z_2, z_3, without changing their arguments, in such a way that z_1 and z_2 go to the contour of Ω_ρ and z_3 stays outside Ω_ρ. Then, by definition for some $t_1', t_2' \in (\pi/3, 5\pi/3)$ and $t_3 \in [-\pi/3, \pi/3]$ we have

$$z_1 = \rho\left(\frac{\pi - t_1'}{2}\right)^{-2} e^{it_1'}, \quad z_2 = \rho\left(\frac{\pi - t_2'}{2}\right)^{-2} e^{it_2'}, \quad z_3 = r_3 e^{it_3}$$

and

$$-1 = z_1 z_2 z_3 = \rho\left(\frac{\pi - t_1}{2}\right)^{-2} \rho\left(\frac{\pi - t_2}{2}\right)^{-2} r_3 e^{i(t_1' + t_2' + t_3)}.$$

Hence, $t_1' + t_2' + t_3 = (2n-1)\pi$ for some integer n and

$$r_3 = \rho\left(\frac{\pi - t_1'}{2}\right)^{2} \rho\left(\frac{\pi - t_2'}{2}\right)^{2} =: \rho(t_1)^2 \rho(t_2)^2,$$

where we have defined $t_k := (\pi - t_k')/2 \in (-\pi/3, \pi/3)$, for $k = 1, 2$. Thus, we get $t_3 = (2n-3)\pi + 2t_1 + 2t_2$. The condition $t_3 \in [-\pi/3, \pi/3]$ implies that $t_1 + t_2$ is in $(-2\pi/3, -\pi/3] \cup [\pi/3, 2\pi/3)$ and that

$$n = \begin{cases} 1 & \text{if } t_1 + t_2 \in [\pi/3, 2\pi/3), \\ 2 & \text{if } t_1 + t_2 \in (-2\pi/3, -\pi/3]. \end{cases}$$

Hence, the premise of condition (7.9) is satisfied and we use it to conclude

$$r_3 \le \rho(2t_1 + 2t_2 - \epsilon\pi) = \rho(t_3),$$

where $\epsilon = \operatorname{sign}(t_1 + t_2)$. That shows z_3 is in Ω_ρ, a contradiction.

3) The points z_1, z_2 are in Δ^c and $z_3 \in \Delta$. Suppose that all three points are outside Ω_ρ. As in 2), we may suppose that z_1, z_3 are on the contour of Ω_ρ and z_2 is outside Ω_ρ. Then, by definition for some $t_k \in [-\pi/3, \pi/3]$, $k = 1, 2$, and $t_3' \in (\pi/3, 5\pi/3)$ we have

$$z_1 = \rho(t_1) e^{it_1}, \quad z_2 = r_2 e^{it_2}, \quad z_3 = \rho\Big(\frac{\pi - t_3'}{2}\Big)^{-2} e^{it_3'}$$

and

$$-1 = z_1 z_2 z_3 = \rho(t_1) r_2 \rho\Big(\frac{\pi - t_3'}{2}\Big)^{-2} e^{i(t_1 + t_2 + t_3')}.$$

Hence, $t_1 + t_2 + t_3' = (2n-1)\pi$ for some integer n and

$$r_2 = \rho(t_1)^{-1} \rho\Big(\frac{\pi - t_3'}{2}\Big)^{2} =: \rho(t_1)^{-1} \rho(t_3)^2,$$

where we have defined $t_3 := (\pi - t_3')/2 \in (-\pi/3, \pi/3)$. By condition (7.10), we conclude that

$$r_2 = \rho(t_1)^{-1} \rho(t_3)^2 = \rho(t_1)^{-1} \rho((t_1 + t_2)/2)^2 \le \rho(t_2),$$

showing that z_2 is in Ω_ρ, a contradiction.

4) The points z_1, z_2, z_3 are in Δ^c. Condition (7.10) says that $\rho(t)$ is log-convex on $[-\pi/3, \pi/3]$, hence necessarily, the function $\rho(t)$ is continuous on $(-\pi/3, \pi/3)$ with possible discontinuities at $\pm\pi/3$ satisfying

$$1 \ge \rho(-\pi/3) \ge \lim_{t \to -\pi/3^+} \rho(t) \text{ and } 1 \ge \rho(\pi/3) \ge \lim_{t \to \pi/3^-} \rho(t).$$

Therefore, the continuation of $\rho(t)$ satisfies

$$\lim_{t \to -\pi/3^-} \rho(t) \ge \rho(-\pi/3)^{-2} \ge 1 \text{ and } \lim_{t \to \pi/3^+} \rho(t) \ge \rho(\pi/3)^{-2} \ge 1.$$

In order to have $z_1 z_2 z_3 = -1$, it must be true that either $\arg(z_k) = \pi/3$ for all $k = 1, 2, 3$ or that $\arg(z_k) = -\pi/3$ for all $k = 1, 2, 3$. Suppose the former case holds (with the latter being analogous). Since $|z_1 z_2 z_3| = 1$, we cannot have $|z_k| > \rho(\pi/3)^{-2}$ for all $k = 1, 2, 3$. This shows that at least one of $\{z_1, z_2, z_3\}$ is in Ω_ρ.

This concludes the proof that Ω_ρ is a locus holder. Next, we show that Ω_ρ is a locus.

In order to reach a contradiction, suppose Ω_ρ is not a locus. Then, it contains a locus, Ω', as a proper subset, that is, there is a $u \in \operatorname{int}(\Omega_\rho \setminus \Omega')$. Since $\Omega_\rho = \big(\Omega_\rho \cap \Delta\big) \cup \big(\Omega_\rho \cap \Delta^c\big)$, perturbing u slightly, if necessary, we may assume that u is in either $\operatorname{int}\big(\Omega_\rho \cap \Delta\big)$ or in $\operatorname{int}\big(\Omega_\rho \cap \Delta^c\big)$.

Case 1. Suppose $u \in \operatorname{int}\big(\Omega_\rho \cap \Delta\big)$. Without loss of generality suppose that $\arg(u) \in (\pi/3, \pi]$. (The case when $\arg(u) \in (-\pi, -\pi/3)$ is analogous.) Since Ω_ρ is a star-like set with respect to the origin, let $v \in \Delta$ be the unique point where the ray from 0 through u intersects the boundary of Ω_ρ, hence $\arg(v) = \arg(u)$ and in addition $|v| > |u|$. Using equation (7.7), define the point $z := \rho(t)e^{it}$, where $t := (\pi - \arg(v))/2$. It is clear that $z \in \partial\Omega_\rho \cap \Delta^c$ and it is straightforward to verify that $\{z, z, v\}$ is a solution of $z_1 z_2 z_3 = -1$. Now, we aim to define a new solution $\{z', z', u\}$ of $z_1 z_2 z_3 = -1$ for which $z' \notin \Omega_\rho$. This implies that $z', u \notin \Omega'$ and is a contradiction with the fact that Ω' is a locus. The idea is to move v to u, thus decreasing its modulus, without changing its argument, while compensating by moving z out of Ω_ρ by increasing its modulus, without changing its argument. Formally, we let $z' := (|z| + \epsilon)e^{i \arg(z)}$, where ϵ is the strictly positive solution (such exists since $|v| > |u|$) of the quadratic equation

$$x^2 + 2|z|x + \left(\frac{1}{|v|} - \frac{1}{|u|}\right) = 0.$$

Case 2. Suppose $u \in \operatorname{int}\big(\Omega_\rho \cap \Delta^c\big)$. Without loss of generality suppose that $\arg(u) \in [0, \pi/3]$. (The case when $\arg(u) \in [-\pi/3, 0)$ is analogous.) Since Ω_ρ is a star-like set with respect to the origin, let $v \in \Delta^c$ be the unique point where the ray from 0 through u intersects the boundary of Ω_ρ, hence $\arg(v) = \arg(u)$ and in addition $|v| > |u|$. Using equation (7.7), define the point $z := \rho(t)e^{it}$, where $t := \pi - 2\arg(v)$. It is clear that $z \in \partial\Omega_\rho \cap \Delta$ and it is straightforward to verify that $\{z, v, v\}$ is a solution of $z_1 z_2 z_3 = -1$. Now, we aim to define a new solution $\{z', v', u\}$ of $z_1 z_2 z_3 = -1$ for which $z', v' \notin \Omega_\rho$. This implies that $z', v', u \notin \Omega'$ and is a contradiction with the fact that Ω' is a locus. The idea is to move one of the v's to u, thus decreasing its modulus, without changing its argument, while compensating by moving z and the other v out of Ω_ρ by increasing their moduli, without changing their argument. Formally, we let $z' := (|z| + \epsilon)e^{i \arg(z)}$ and $v' := (|v| + \epsilon)e^{i \arg(v)}$, where ϵ is the strictly positive solution (such exists since $|v| > |u|$) of the quadratic equation

$$x^2 + (|z| + |v|)x + \left(\frac{1}{|v|} - \frac{1}{|u|}\right) = 0.$$

This concludes the proof of Theorem 7.4. □

Corollary 7.1. *The set Ω_r, defined in Proposition* 6.1*, is a locus of $z^3 + 1$ for every $r \in [0, 1]$.*

Proof. We only consider the case $r > 0$. (The case $r = 0$ could be verified directly.) In that case, we have $\Omega_r = \Omega_\rho$ for the function $\rho(t) := r$ for all $t \in [-\pi/3, \pi/3]$. Since that function trivially satisfies conditions (7.9) and (7.10), we are done. □

The points on the boundary of the locus in Theorem 7.1 are a function of their argument if and only if $s \in [2 - \sqrt{3}, 1]$. (The problem is that the arc Γ cannot be

represented by a polar function if $s \in [0, 2-\sqrt{3})$.) So it seems that, in this case, Theorem 7.1 can be derived as a corollary of Theorem 7.4. But, in order to do that, one needs to show that the function $\rho(t)$ describing the circular arc Γ,

(7.11)
$$\rho(t) = \begin{cases} \frac{1-s^2}{1-2s}\cos(t) + \frac{1}{1-2s}\sqrt{\big(s(s-2)\big)^2 + \big(1-s^2\big)^2\cos^2(t)} & \text{if } s \in [2-\sqrt{3}, 0.5), \\ \frac{1}{2\cos(t)} & \text{if } s = 0.5, \\ \frac{1-s^2}{2s-1}\cos(t) + \frac{1}{2s-1}\sqrt{\big(s(s-2)\big)^2 + \big(1-s^2\big)^2\cos^2(t)} & \text{if } s \in (0.5, 1], \end{cases}$$

for $t \in [-\pi/3, \pi/3]$, satisfies conditions (7.9) and (7.10). While one can show that condition (7.10) indeed holds, we failed to find a direct verification of condition (7.9). Hence, in the proof of Theorem 7.1 we were forced to take a different path which, as an added bonus, covers the case $s \in [0, 2-\sqrt{3})$.

Problem 7.1. Are conditions (7.9) and (7.10) also necessary for Ω_3 to be a locus?

If the answer to Problem 7.1 is positive, then using Theorem 7.1 it would follow that function (7.11) satisfies conditions (7.9) and (7.10).

8. Appendix A

To simplify the notation, define the coefficients $b_m := a_m/\binom{n}{m}$. It is straightforward to verify that

$$P(z_1, \dots, z_n) = z_{n-1}z_n P_2(z_1, \dots, z_{n-2}) + (z_{n-1} + z_n)P_1(z_1, \dots, z_{n-2}) + P_0(z_1, \dots, z_{n-2}), \tag{8.1}$$

where

$$P_k(z_1, z_2, \dots, z_{n-2}) = \sum_{m=k}^{n+k-2} b_m S_{m-k}(z_1, z_2, \dots, z_{n-2}), \quad \text{for } k = 0, 1, 2. \tag{8.2}$$

Lemma 8.1. *If $p(z)$ has at least two distinct zeros and $a_n \neq 0$, then the discriminant*

$$P_1(z_3, \dots, z_n)^2 - P_2(z_3, \dots, z_n)P_0(z_3, \dots, z_n), \tag{8.3}$$

of (8.1)*, is not identically* 0.

Proof. The discriminant (8.3) is a polynomial in $z_3, \dots, z_n$ and is identically zero only if all coefficients are zero. Substitute (8.2) in the formula for the discriminant and note that the elementary symmetric functions can be treated as independent variables. That is, defining $s_k := S_k(z_3, \dots, z_n)$, the discriminant is

$$\Big(\sum_{m=1}^{n-1} b_m s_{m-1}\Big)^2 - \Big(\sum_{m=2}^{n} b_m s_{m-2}\Big)\Big(\sum_{m=0}^{n-2} b_m s_m\Big) = \sum_{i,j=0}^{n-2} s_i s_j \Big(b_{i+1}b_{j+1} - \frac{1}{2}\big(b_{i+2}b_j - b_{j+2}b_i\big)\Big).$$

If (8.3) is identically 0, then we must have

$$b_{i+1}^2 - b_{i+2}b_i = 0 \quad \text{for } i = 0, \dots, n-2,$$
$$2b_{i+1}b_{j+1} - b_{i+2}b_j - b_{j+2}b_i = 0 \quad \text{for } 0 \le i < j \le n-2.$$

The first set of equations shows that if $b_i = 0$, then $b_{i+1} = b_{i+2} = \cdots = b_{n-1} = 0$. Note that $b_0 \neq 0$, since otherwise $p(z) = a_n z^n$, contradicting the assumption that p has at least two distinct zeros. Let $i \in \{1, \ldots, n-1\}$ be the smallest index with $b_i = 0$. If $i \geq 2$, then using the relationship $b_{i-1}^2 - b_i b_{i-2} = 0$, we reach the contradiction $b_{i-1} = 0$. If $i = 1$, then from above we have $b_{n-2} = b_{n-1} = 0$, and using the relationship $2b_1 b_{n-1} - b_2 b_{n-2} - b_n b_0 = 0$, we reach a contradiction, since $b_n \neq 0$.

Assume now that all b_i's are non-zero. The first set of equations, again, shows that b_{i+1}/b_i is a constant, call it b, for all $i = 0, 1, \ldots, n-1$. One verifies that the second set of equations is redundant (divide each equation by $b_{i+1}b_{j+1}$). Hence, $a_i = b_i\binom{n}{i} = b_0\binom{n}{i}b^i$ for $i = 1, 2, \ldots, n$. In other words, we have $p(z) = b_0(z-b)^n$, contradicting the fact that $p(z)$ has at least two zeros. □

Proof of Theorem 1.7. If B is not a locus, then being a locus holder, it contains, as a proper subset, a locus Ω. Hence, there is a point $z^* \in \operatorname{int}(B \setminus \Omega)$. Consider the unique point z° such that $\{z^*, \ldots, z^*, z^\circ\}$ is a solution of p. (The point z° is uniquely determined whenever z^* is not a zero of the polynomial $z \mapsto \frac{\partial P}{\partial z_n}(z, \ldots, z)$. We can avoid the latter situation by perturbing the point z^*, while keeping it in $\operatorname{int}(B \setminus \Omega)$.) We consider two cases.

Case 1. $z^\circ \in \operatorname{int}\Omega$. Since $\operatorname{int}\Omega \subset \operatorname{int} B$, there is a circular domain $B' \subset B$ containing every point in the solution $\{z^*, \ldots, z^*, z^\circ\}$ of P. By the Grace-Walsh-Szegő coincidence theorem, $p(z)$ has a zero in B', which is a contradiction.

Case 2. $z^\circ \in \partial\Omega$. Consider the relationship between z_{n-1} and z_n determined by the requirement that $\{z^*, \ldots, z^*, z_{n-1}, z_n\}$ be a solution of p. This relationship expresses z_n as a Möbius transformation of z_{n-1}:

$$z_{n-1}z_n P_2(z^*, \ldots, z^*) + (z_{n-1} + z_n)P_1(z^*, \ldots, z^*) + P_0(z^*, \ldots, z^*) = 0. \tag{8.4}$$

Clearly, $(z_{n-1}, z_n) := (z^*, z^\circ)$ satisfies (8.4). The problem that we address next is that this Möbius transformation may be degenerate. By Lemma 8.1, there are $n-2$ points $(z_1^*, \ldots, z_{n-2}^*)$ arbitrarily close to $(z^*, \ldots, z^*)$ such that

$$P_1(z_1^*, \ldots, z_{n-2}^*)^2 - P_2(z_1^*, \ldots, z_{n-2}^*)P_0(z_1^*, \ldots, z_{n-2}^*) \neq 0.$$

It is not difficult to see that there are z'_{n-1} and z'_n converging to z^* and z°, respectively, as $(z_1^*, \ldots, z_{n-2}^*)$ converges to $(z^*, \ldots, z^*)$, such that

$$z'_{n-1}z'_n P_2(z_1^*, \ldots, z_{n-2}^*) + (z'_{n-1} + z'_n)P_1(z_1^*, \ldots, z_{n-2}^*) + P_0(z_1^*, \ldots, z_{n-2}^*) = 0. \tag{8.5}$$

Take $z_1^*, \ldots, z_{n-2}^*$ in $\operatorname{int}(B\setminus\Omega)$ close to z^* so that z'_{n-1} is in $\operatorname{int}(B\setminus\Omega)$. If $z'_n \in \operatorname{int}\Omega$, then we are in the situation of Case 1. If $z'_n \in \partial\Omega$, then (8.5) expresses z'_n as a *non-degenerate* Möbius transformation of z'_{n-1}. Such Möbius transformations send open sets into open sets. Hence, a small open neighbourhood around the point z'_{n-1} is mapped into a small open neighbourhood around the point z'_n. Choosing $z'_{n-1} \in \operatorname{int}(B\setminus\Omega)$ in such a way that $z'_n \notin \Omega$, we construct a solution $\{z_1^*, \ldots, z_{n-2}^*, z'_{n-1}, z'_n\}$ that does not have a point in Ω, contradicting the fact that Ω is a locus. □

9. Appendix B

Proof of Proposition 5.1. We show that

$$W[p](z) = (\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2(z^3 + 1),$$

and then the result follows from Theorem 2.5. Separate the numerator and the denominator of the Möbius transformation $W(z)$. That is, let $W(z) = N(z)/D(z)$, where

$$N(z) := z(\alpha_1\alpha_2 e_3 + \alpha_1\alpha_3 e_2 + \alpha_2\alpha_3 e_1) + (\alpha_1\alpha_2 e_3 + \alpha_1\alpha_3 e_1 + \alpha_2\alpha_3 e_2),$$
$$D(z) := -z(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) - (\alpha_1 e_2 + \alpha_2 e_1 + \alpha_3 e_3).$$

Then, $W[p](z) = (N(z) - \alpha_1 D(z))(N(z) - \alpha_2 D(z))(N(z) - \alpha_3 D(z))$. The first multiple in the latter product can be developed as follows, using the fact that $e_1 + e_2 + e_3 = 0$:

$$\begin{aligned} N(z) - \alpha_1 D(z) &= z(\alpha_1\alpha_2 e_3 + \alpha_1\alpha_3 e_2 + \alpha_2\alpha_3 e_1 + \alpha_1\alpha_1 e_1 + \alpha_1\alpha_2 e_2 + \alpha_1\alpha_3 e_3) \\ &\quad + (\alpha_1\alpha_2 e_3 + \alpha_1\alpha_3 e_1 + \alpha_2\alpha_3 e_2 + \alpha_1\alpha_1 e_2 + \alpha_1\alpha_2 e_1 + \alpha_1\alpha_3 e_3) \\ &= z(-\alpha_1\alpha_2 e_1 - \alpha_1\alpha_3 e_1 + \alpha_2\alpha_3 e_1 + \alpha_1\alpha_1 e_1) \\ &\quad + (-\alpha_1\alpha_2 e_2 - \alpha_1\alpha_3 e_2 + \alpha_2\alpha_3 e_2 + \alpha_1\alpha_1 e_2) \\ &= (ze_1 + e_2)(-\alpha_1\alpha_2 - \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_1) \\ &= (ze_1 + e_2)(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3). \end{aligned} \tag{9.1}$$

Similarly, we have

$$N(z) - \alpha_2 D(z) = (ze_2 + e_1)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_1), \tag{9.2}$$
$$N(z) - \alpha_3 D(z) = (ze_3 + e_3)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2). \tag{9.3}$$

Hence, using that $-1 = e_1e_2e_3$ and $1 = e_1e_2$, we get

$$\begin{aligned} W[p](z) &= (-1)(\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2(ze_1 + e_2)(ze_2 + e_1)(ze_3 + e_3) \\ &= (\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2(z + e_2^2)(z + e_1^2)(z + e_3^2) \\ &= (\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2(z - e_1)(z - e_2)(z - e_3), \end{aligned}$$

where in the last equality, we used that $e_2^2 = -e_1$, $e_1^2 = -e_2$, and $e_3^2 = -e_3$. The result follows. □

The symmetrized polynomial (5.2) depends implicitly on the roots α_1, α_2, and α_3 of $p(z)$ and can be written as

$$\begin{aligned} P(z_1, z_2, z_3, \alpha_1, \alpha_2, \alpha_3) &= z_1z_2z_3 - \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)(z_1z_2 + z_2z_3 + z_3z_1) \\ &\quad + \frac{1}{3}(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)(z_1 + z_2 + z_3) - \alpha_1\alpha_2\alpha_3. \end{aligned}$$

Solving the equation $P(z_1, z_2, z_3, \alpha_1, \alpha_2, \alpha_3) = 0$ for z_3, we define

$$F(z_1, z_2) := \frac{z_1z_2(\alpha_1 + \alpha_2 + \alpha_3) - (z_1 + z_2)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 3\alpha_1\alpha_2\alpha_3}{3z_1z_2 - (z_1 + z_2)(\alpha_1 + \alpha_2 + \alpha_3) + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)}.$$

Recall the definition of $W(z)$ in (5.3).

Lemma 9.1. *For any $z_1, z_2 \in \mathbb{C}$ we have*

$$F(W(z_1), W(z_2)) = W\Big(-\frac{1}{z_1z_2}\Big). \tag{9.4}$$

In particular,

$$F(W(z), W(z)) = Q(W(z)) = W\big(-1/z^2\big),$$

where $Q(z)$ is defined in (5.4).

Proof. As in the proof of Theorem 5.1, let $W(z) = N(z)/D(z)$, where

$$N(z) := z(\alpha_1\alpha_2 e_3 + \alpha_1\alpha_3 e_2 + \alpha_2\alpha_3 e_1) + (\alpha_1\alpha_2 e_3 + \alpha_1\alpha_3 e_1 + \alpha_2\alpha_3 e_2),$$
$$D(z) := -z(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) - (\alpha_1 e_2 + \alpha_2 e_1 + \alpha_3 e_3).$$

Similarly, separate the numerator and the denominator of the rational function $F(z_1, z_2)$. That is, let $F(z_1, z_2) = F_1(z_1, z_2)/F_2(z_1, z_2)$, where

$$F_1(z_1, z_2) := z_1 z_2(\alpha_1 + \alpha_2 + \alpha_3) - (z_1 + z_2)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 3\alpha_1\alpha_2\alpha_3,$$
$$F_2(z_1, z_2) := 3z_1 z_2 - (z_1 + z_2)(\alpha_1 + \alpha_2 + \alpha_3) + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3).$$

We need to show that

$$\frac{F_1(N(z_1)/D(z_1), N(z_2)/D(z_2))}{F_2(N(z_1)/D(z_1), N(z_2)/D(z_2))} = \frac{N(-1/z_1 z_2)}{D(-1/z_1 z_2)}. \tag{9.5}$$

Multiply the numerator and the denominator on the left-hand side of (9.5) by $D(z_1)D(z_2)$ and consider each one separately. Below we utilize identities (9.1), (9.2), and (9.3) which hold for every $z \in \mathbb{C}$:

$$\begin{aligned}
&F_1(N(z_1)/D(z_1), N(z_2)/D(z_2))D(z_1)D(z_2) = N(z_1)N(z_2)(\alpha_1 + \alpha_2 + \alpha_3) \\
&\qquad - (N(z_1)D(z_2) + N(z_2)D(z_1))(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 3\alpha_1\alpha_2\alpha_3 D(z_1)D(z_2) \\
&\quad = \alpha_3(N(z_2) - \alpha_1 D(z_2))(N(z_1) - \alpha_2 D(z_1)) \\
&\qquad + \alpha_1(N(z_2) - \alpha_2 D(z_2))(N(z_1) - \alpha_3 D(z_1)) \\
&\qquad\quad + \alpha_2(N(z_2) - \alpha_3 D(z_2))(N(z_1) - \alpha_1 D(z_1)) \\
&\quad = \alpha_3(z_2 e_1 + e_2)(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(z_1 e_2 + e_1)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_1) \\
&\qquad + \alpha_1(z_2 e_2 + e_1)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_1)(z_1 e_3 + e_3)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) \\
&\qquad\quad + \alpha_2(z_2 e_3 + e_3)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)(z_1 e_1 + e_2)(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \\
&\quad = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)\big(\alpha_3(z_2 e_1 + e_2)(z_1 e_2 + e_1)(\alpha_2 - \alpha_1) \\
&\qquad + \alpha_1(z_2 e_2 + e_1)(z_1 e_3 + e_3)(\alpha_3 - \alpha_2) + \alpha_2(z_2 e_3 + e_3)(z_1 e_1 + e_2)(\alpha_1 - \alpha_3)\big).
\end{aligned} \tag{9.6}$$

Next, using that $e_1 - e_2 = i\sqrt{3}e_3$, $e_1 - e_3 = -i\sqrt{3}e_2$, and $e_2 - e_3 = i\sqrt{3}e_1$, we have the identities

$$\begin{aligned}
(z_2 e_3 + e_3)(z_1 e_1 + e_2) - (z_2 e_2 + e_1)(z_1 e_3 + e_3) &= z_1 z_2(e_1 e_3 - e_2 e_3) + (e_2 e_3 - e_1 e_3) \\
&= -i\sqrt{3}(z_1 z_2 e_3 - e_3), \\
(z_2 e_2 + e_1)(z_1 e_3 + e_3) - (z_2 e_1 + e_2)(z_1 e_2 + e_1) &= z_1 z_2(e_2 e_3 - e_2 e_1) + (e_1 e_3 - e_1 e_2) \\
&= -i\sqrt{3}(z_1 z_2 e_1 - e_2), \\
(z_2 e_1 + e_2)(z_1 e_2 + e_1) - (z_2 e_3 + e_3)(z_1 e_1 + e_2) &= z_1 z_2(e_1 e_2 - e_1 e_3) + (e_1 e_2 - e_2 e_3) \\
&= -i\sqrt{3}(z_1 z_2 e_2 - e_1).
\end{aligned}$$

Substituting into expression (9.6) we continue:

$$\begin{aligned}
F_1(N(z_1)/D(z_1), N(z_2)/D(z_2))D(z_1)D(z_2) &= -i\sqrt{3}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) \\
&\quad \times \big(\alpha_1\alpha_2(z_1 z_2 e_3 - e_3) + \alpha_1\alpha_3(z_1 z_2 e_1 - e_2) + \alpha_2\alpha_3(z_1 z_2 e_2 - e_1)\big) \\
&= -i\sqrt{3}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)(z_1 z_2)N(-1/z_1 z_2).
\end{aligned} \tag{9.7}$$

For the denominator we have

$$\begin{aligned}
F_2(N(z_1)/D(z_1), N(z_2)/D(z_2))D(z_1)D(z_2) &= 3N(z_1)N(z_2) \\
&\quad - (N(z_1)D(z_2) + N(z_2)D(z_1))(\alpha_1 + \alpha_2 + \alpha_3) \\
&\quad + D(z_1)D(z_2)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) \\
&= (N(z_2) - \alpha_1 D(z_2))(N(z_1) - \alpha_3 D(z_1)) \\
&\quad + (N(z_2) - \alpha_2 D(z_2))(N(z_1) - \alpha_1 D(z_1)) \\
&\quad + (N(z_2) - \alpha_3 D(z_2))(N(z_1) - \alpha_2 D(z_1)) \\
&= (z_2e_1 + e_2)(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(z_1e_3 + e_3)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) \\
&\quad + (z_2e_2 + e_1)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_1)(z_1e_1 + e_2)(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \\
&\quad + (z_2e_3 + e_3)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)(z_1e_2 + e_1)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_1) \\
&= (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)\big((z_2e_1 + e_2)(z_1e_3 + e_3)(\alpha_1 - \alpha_3) \\
&\quad + (z_2e_2 + e_1)(z_1e_1 + e_2)(\alpha_2 - \alpha_1) + (z_2e_3 + e_3)(z_1e_2 + e_1)(\alpha_3 - \alpha_2)\big). \qquad (9.8)
\end{aligned}$$

Again using that $e_1 - e_2 = i\sqrt{3}e_3$, $e_1 - e_3 = -i\sqrt{3}e_2$, and $e_2 - e_3 = i\sqrt{3}e_1$, we have the identities

$$\begin{aligned}
(z_2e_1 + e_2)(z_1e_3 + e_3) - (z_2e_2 + e_1)(z_1e_1 + e_2) &= z_1z_2(e_1e_3 - e_1e_2) + (e_2e_3 - e_1e_2) \\
&= i\sqrt{3}(z_1z_2e_2 - e_1), \\
(z_2e_2 + e_1)(z_1e_1 + e_2) - (z_2e_3 + e_3)(z_1e_2 + e_1) &= z_1z_2(e_1e_2 - e_2e_3) + (e_1e_2 - e_1e_3) \\
&= i\sqrt{3}(z_1z_2e_1 - e_2), \\
(z_2e_3 + e_3)(z_1e_2 + e_1) - (z_2e_1 + e_2)(z_1e_3 + e_3) &= z_1z_2(e_2e_3 - e_1e_3) + (e_1e_3 - e_2e_3) \\
&= i\sqrt{3}(z_1z_2e_3 - e_3).
\end{aligned}$$

Substituting into (9.8) we continue:

$$\begin{aligned}
F_2(N(z_1)/D(z_1), N(z_2)/D(z_2))D(z_1)D(z_2) &= i\sqrt{3}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) \\
&\quad \times (\alpha_1(z_1z_2e_2 - e_1) + \alpha_2(z_1z_2e_1 - e_2) + \alpha_3(z_1z_2e_3 - e_3)) \\
&= -i\sqrt{3}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)(z_1z_2)D(-1/z_1z_2). \qquad (9.9)
\end{aligned}$$

The proof concludes after dividing (9.7) by (9.9). □

Proof of Theorem 5.1. Suppose $\alpha_3 \in \operatorname{int} S[\alpha_1, \alpha_2, \alpha_3]$ and let Γ be the semi-circle of $\partial S[\alpha_1, \alpha_2, \alpha_3]$ with endpoints α_1, α_2 that is in $D[\alpha_1, \alpha_2, \alpha_3]$. Without loss of generality, assume that α_3 is on the side of the line through α_1, α_2 such that the positive angle between $S[\alpha_1, \alpha_2, \alpha_3]$ and $D[\alpha_1, \alpha_2, \alpha_3]$, at the point α_2, is in the interval $(0, \pi/2]$. (Otherwise, exchange the points α_1 and α_2. Note that the angle 0 is excluded since $\alpha_3 \in \operatorname{int} S[\alpha_1, \alpha_2, \alpha_3]$.) Recall that $W^{-1}(\alpha_k) = e_k$, $k = 1, 2, 3$; hence $W^{-1}(\partial D[\alpha_1, \alpha_2, \alpha_3]) = C(e_1, e_2, e_3)$. Let $s \in \mathbb{R}$ be such that $W^{-1}(\partial S[\alpha_1, \alpha_2, \alpha_3]) = C(e_1, s, e_2)$. The non-degenerate Möbius transformation W^{-1} preserves oriented angles. Hence, the positive angle between $C(e_1, s, e_2)$ and $C(e_1, e_2, e_3)$, at the point e_2, is in the interval $(0, \pi/2]$. This implies that s is in $[2 - \sqrt{3}, 1)$. Since $W^{-1}(\alpha_3) = e_3$, W^{-1} maps $S[\alpha_1, \alpha_2, \alpha_3]$ onto the closure of the connected component of $\mathbb{C} \setminus C(e_1, s, e_2)$ that contains the point $e_3 = -1$. In other words, W^{-1} maps $S[\alpha_1, \alpha_2, \alpha_3]$ onto $\mathbb{C} \setminus B(e_2, s, e_1)$. Since the non-degenerate Möbius transformation W^{-1} preserves oriented circles (oriented by three points on

them), the disk $D[\alpha_1, \alpha_2, \alpha_3]$ is mapped onto the the disk $D[e_1, e_2, e_3]$. Hence, the arc Γ (being in $D[\alpha_1, \alpha_2, \alpha_3]$) is mapped onto $\operatorname{arc}[e_1, s, e_2]$.

Our goal is to find a locus contained in $S[\alpha_1, \alpha_2, \alpha_3]$. By the previous paragraph and Proposition 5.1, it is sufficient to find a locus of $z^3 + 1$ in $\mathbb{C} \setminus B(e_2, s, e_1)$. By Theorem 7.1, the closed domain Ω_3 with boundary $\operatorname{arc}(e_1, s, e_2) \cup g(\operatorname{arc}[e_1, s, e_2])$, where $g(z) = -1/z^2$, is such a locus of $z^3 + 1$. Hence, by Proposition 5.1, we have that $\Omega := W(\Omega_3)$ is a locus of $p(z)$ in $S[\alpha_1, \alpha_2, \alpha_3]$. At the end, we clarify the boundary structure of $W(\Omega_3)$. On the one hand, $W(\operatorname{arc}[e_1, s, e_2]) = \Gamma$. On the other hand, if $u \in g(\operatorname{arc}[e_1, s, e_2])$, then $u = -1/z^2$ for some $z \in \operatorname{arc}[e_1, s, e_2]$ and by Lemma 9.1 we have $W(u) = W(-1/z^2) = Q(W(z))$. Thus, the boundary of $W(\Omega_3)$ is $\Gamma \cup Q(\Gamma)$. □

Proof of Proposition 5.2. Without loss of generality assume that $\alpha_1 = -1$, $\alpha_2 = 1$ and α_3 is in the closed unit disk, $B[0, 1]$, and in the closed upper half plane. (Otherwise, apply the appropriate affine transformation.) The quadratic map (5.4) becomes

$$Q_{\alpha_3}(z) := \frac{\alpha_3 z^2 + 2z - 3\alpha_3}{3z^2 - 2\alpha_3 z - 1}. \tag{9.10}$$

The points where the denominator is zero are

$$w_1 := \frac{\alpha_3 + \sqrt{\alpha_3^2 + 3}}{3} \quad \text{and} \quad w_2 := \frac{\alpha_3 - \sqrt{\alpha_3^2 + 3}}{3}.$$

One can see that for any $\alpha_3 \in B[0, 1]$ we have $|w_k| \leq 1$ for $k = 1, 2$, and equality occurs if and only if $\alpha_3 = \pm 1$. Since $\Gamma = \operatorname{arc}[-1, -i, 1] \subset \partial B[0, 1]$ it should be evident that $Q_{\alpha_3}(\Gamma)$ changes continuously when $\alpha_3 \in \operatorname{int} B[0, 1]$. For $\epsilon \in \{-1, 1\}$, we have

$$\lim_{\alpha_3 \to \epsilon} Q_{\alpha_3}(z) = \frac{\epsilon z + 3}{3z + \epsilon}.$$

Now, it is evident that for any $\epsilon \in \partial B[0, 1] \cap \{z : \operatorname{Im}(z) \geq 0\}$ we have

$$\lim_{\alpha_3 \to \alpha} Q_{\alpha_3}(\Gamma) = Q_\epsilon(\Gamma).$$

In both cases, $\epsilon \in \{-1, 1\}$, the Möbius transformation $z \mapsto \frac{\epsilon z + 3\epsilon}{3z + \epsilon}$ maps the arc Γ into $\operatorname{arc}[-1, \epsilon, 1]$, hence $Q_\epsilon(\Gamma) = \operatorname{arc}[-1, i, 1]$. What is left to show is that for any $\epsilon \in \partial B[0, 1] \cap \{z : \operatorname{Im}(z) > 0\}$ we also have $Q_\epsilon(\Gamma) = \operatorname{arc}[-1, \epsilon, 1]$. Fix such an ϵ and consider the Möbius transformation

$$W(z) = -\frac{z(\epsilon i\sqrt{3} - 1) - (\epsilon i\sqrt{3} + 1)}{z(\epsilon - i\sqrt{3}) + (\epsilon + i\sqrt{3})},$$

defined by $W(e_k) = \alpha_k$ for $k = 1, 2$ and $W(e_3) = \epsilon$. That is, $W(z)$ is given by (5.3) when α_k, $k = 1, 2, 3$, is replaced by $-1, 1$, and ϵ, respectively. Since $W(1) = 1/\epsilon \in \Gamma$, we have $W^{-1}(\Gamma) = \operatorname{arc}[e_1, 1, e_2]$, and since $W(-1) = \epsilon$, we have $W^{-1}(\operatorname{arc}[-1, \epsilon, 1]) = \operatorname{arc}[e_1, -1, e_2]$. It is trivial to see that the function $g(z) = -1/z^2$ maps $\operatorname{arc}[e_1, 1, e_2]$ onto $\operatorname{arc}[e_1, -1, e_2]$. By Lemma 9.1, for every $z \in \mathbb{C}$, we have $Q_\epsilon(z) = (W \circ g \circ W^{-1})(z)$, and by the above, we finally obtain

$$\begin{aligned} Q_\epsilon(\Gamma) &= (W \circ g \circ W^{-1})(\Gamma) = (W \circ g)(\operatorname{arc}[e_1, 1, e_2]) = W(\operatorname{arc}[e_1, -1, e_2]) \\ &= \operatorname{arc}[-1, \epsilon, 1]. \end{aligned}$$

The proof is complete. □

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References

[1] Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press Oxford University Press, Oxford, 2002. MR1954841 (2004b:30015)

[2] J.H. Grace, *The zeros of a polynomial*, Proc. Cambridge Philos. Soc. **11** (1902), 352–357.

[3] G. Szegö, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen* (German), Math. Z. **13** (1922), no. 1, 28–55, DOI 10.1007/BF01485280. MR1544526

[4] David G. Wagner, *Multivariate stable polynomials: theory and applications*, Bull. Amer. Math. Soc. (N.S.) **48** (2011), no. 1, 53–84, DOI 10.1090/S0273-0979-2010-01321-5. MR2738906 (2012d:32006)

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