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NONLINEAR EQUATIONS FOR FRACTIONAL LAPLACIANS II: EXISTENCE, UNIQUENESS, AND QUALITATIVE PROPERTIES OF SOLUTIONS

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Abstract. This paper, which is the follow-up to part I, concerns the equation $(-\Delta)^s v + G'(v) = 0$ in $\mathbb{R}^n$, with $s \in (0,1)$, where $(-\Delta)^s$ stands for the fractional Laplacian—the infinitesimal generator of a Lévy process.

When $n = 1$, we prove that there exists a layer solution of the equation (i.e., an increasing solution with limits ± 1 at $\pm\infty$) if and only if the potential G has only two absolute minima in $[-1,1]$, located at ± 1 and satisfying $G'(-1) = G'(1) = 0$. Under the additional hypotheses $G''(-1) > 0$ and $G''(1) > 0$, we also establish its uniqueness and asymptotic behavior at infinity. Furthermore, we provide with a concrete, almost explicit, example of layer solution.

For $n \geq 1$, we prove some results related to the one-dimensional symmetry of certain solutions—in the spirit of a well-known conjecture of De Giorgi for the standard Laplacian.

1. Introduction

This paper, which is a follow-up to our work [9], is devoted to the study of the nonlinear problem

$$(-\Delta)^s v = f(v) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $s \in (0,1)$ and

$$(-\Delta)^s v(x) = C_{n,s} \text{ P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(\overline{x})}{|x - \overline{x}|^{n+2s}}\, d\overline{x} \tag{1.2}$$

is the fractional Laplacian. In the previous integral, P.V. stands for the Cauchy principal value and $C_{n,s}$ is a normalizing constant to guarantee that the symbol of the resulting operator is $|\xi|^{2s}$; see [9] for more details. As shown by Caffarelli and Silvestre [11] (see also section 3 of [9]), this problem is equivalent to the nonlinear boundary value problem

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ (1+a)\dfrac{\partial u}{\partial \nu^a} = f(u) & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{1.3}$$

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where $n \geq 1$, $\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y > 0\}$ is a halfspace, $\partial\mathbb{R}^{n+1}_+ = \{y = 0\}$, $u = u(x,y)$ is real valued, and

$$\frac{\partial u}{\partial \nu^a} = -\lim_{y \to 0} y^a \partial_y u$$

is the generalized exterior normal derivative of u. Points in $\mathbb{R}^n$ are denoted by $x = (x_1, \ldots, x_n)$. The parameter a belongs to $(-1,1)$ and is related to the power of the fractional Laplacian $(-\Delta)^s$ by

$$a = 1 - 2s.$$

Indeed, Caffarelli and Silvestre [11] proved the following formula relating the fractional Laplacian $(-\Delta)^s$ to the Dirichlet-to-Neumann operator:

$$(-\Delta)^s \{u(\cdot,0)\} = d_s \frac{\partial u}{\partial \nu^a} \quad \text{in } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+, \tag{1.4}$$

where d_s is a positive constant depending only on s.

The aim of the present paper is to study some special bounded solutions of (1.1). The solutions we consider are the so-called *layer solutions*, i.e., those solutions which are monotone increasing, connecting -1 to 1 at $\mp\infty$, in one of the x-variables. We focus on their existence, uniqueness, symmetry and variational properties, as well as their asymptotic behavior.

In our previous paper [9], we proved a Modica-type estimate which allowed us to derive a necessary condition on the nonlinearity f for the existence of a layer solution in $\mathbb{R}$. More precisely, we proved the following result. Here and throughout the paper, G denotes the potential associated to the nonlinearity, i.e.,

$$G' = -f,$$

which is defined up to an additive constant.

Theorem 1.1 ([9]). *Let $a \in (-1,1)$ and f any $C^{1,\gamma}(\mathbb{R})$ function, for some $\gamma > \max(0,a)$. Let $n = 1$ and u be a layer solution of (1.3), that is, a bounded solution of (1.3) with $n = 1$ such that $u_x(\cdot,0) > 0$ in $\mathbb{R}$ and $u(x,0)$ has limits ± 1 as $x \to \pm\infty$.*

Then, for every $x \in \mathbb{R}$ we have $\int_0^{+\infty} t^a |\nabla u(x,t)|^2 dt < \infty$ and the Hamiltonian equality

$$(1+a)\int_0^{+\infty} \frac{1}{2} t^a \left\{u_x^2(x,t) - u_y^2(x,t)\right\} dt = G(u(x,0)) - G(1).$$

Furthermore, for all $y \geq 0$ and $x \in \mathbb{R}$ we have

$$(1+a)\int_0^{y} \frac{1}{2} t^a \left\{u_x^2(x,t) - u_y^2(x,t)\right\} dt < G(u(x,0)) - G(1).$$

In the previous theorem, the last estimate is uniform as s tends to 1, i.e., as $1+a$ tends to 0. This led in [9] to the convergence of layers, as $s \uparrow 1$, to a layer of $-v'' = f(v)$ in $\mathbb{R}$. In addition, using the Hamiltonian estimates of Theorem 1.1, we established the following necessary conditions for the existence of a layer in $\mathbb{R}$.

Theorem 1.2 ([9]). *Let $s \in (0,1)$ and f any $C^{1,\gamma}(\mathbb{R})$ function, for some $\gamma > \max(0, 1-2s)$. Assume that there exists a layer solution v of*

$$(-\partial_{xx})^s v = f(v) \quad \text{in } \mathbb{R}, \tag{1.5}$$

that is, v is a solution of (1.5) *satisfying*

$$v' > 0 \quad \text{in } \mathbb{R} \qquad \text{and} \qquad \lim_{x\to\pm\infty} v(x) = \pm 1.$$

Then, we have

$$G'(-1) = G'(1) = 0 \tag{1.6}$$

and

$$G > G(-1) = G(1) \quad \text{in } (-1,1). \tag{1.7}$$

In the present paper, we prove that the two necessary conditions in Theorem 1.2 are actually sufficient to ensure the existence of a layer solution in $\mathbb{R}$. Under the additional hypotheses $G''(-1) > 0$ and $G''(1) > 0$, we also prove the uniqueness (up to translations) of a layer solution in $\mathbb{R}$ and we establish its asymptotic behavior at infinity.

To study the asymptotic behavior of the layer solution for a given nonlinearity f, it will be very useful to have the following almost explicit example of layer solution for a particular nonlinearity. For every $t > 0$, a layer solution for some odd nonlinearity $f_s^t \in C^1([-1,1])$ (see Theorem 3.1 below for more details) is provided by the following function:

$$v_s^t(x) = -1 + 2\int_{-\infty}^{x} p_s(t,\overline{x})\,d\overline{x} = \frac{2}{\pi}\int_0^\infty \frac{\sin(xr)}{r} e^{-tr^{2s}}\,dr, \tag{1.8}$$

where p_s is the fundamental solution of the linear fractional heat equation

$$\partial_t w + (-\partial_{xx})^s w = 0, \qquad t > 0,\ x \in \mathbb{R}.$$

When $s = 1/2$, the particular layer solution above agrees with the explicit one used in [10], namely

$$v_{1/2}^t(x) = \frac{2}{\pi}\arctan\frac{x}{t}, \quad \text{with } f_{1/2}^t(v) = \frac{1}{\pi t}\sin(\pi v).$$

In [10], J. Solà-Morales and one of the authors studied layer solutions of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial u}{\partial \nu} = f(u) & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases}$$

which corresponds to the case $a = 0$ (that is, $s = 1/2$) in (1.3). The goal of our paper is to generalize this study to any fractional power of the Laplacian between 0 and 1. We will make a great use of the tools developed in [10].

The study of elliptic equations involving fractional powers of the Laplacian appears to be important in many physical situations in which one has to consider long-range or anomalous diffusions. From a probabilistic point of view, the fractional Laplacian appears as the infinitesimal generator of a Lévy process (see the book of Bertoin [6]). In our case, as in [10], we will concentrate on the problem (1.3) and we will not consider probabilistic aspects.

Problem (1.3) is clearly a degenerate elliptic problem concerning the weight y^a. However, since $a \in (-1,1)$, the weight y^a belongs to the Muckenhoupt class of A_2 functions, i.e., it satisfies

$$\sup_B \Big(\frac{1}{|B|}\int_B w\Big)\Big(\frac{1}{|B|}\int_B w^{-1}\Big) \le C,$$

where $w(x,y) = |y|^a$ and B denotes any ball in $\mathbb{R}^{n+1}$. This fact allows us to develop a regularity theory for weak solutions of (1.3); see [9].

Another important property of the weight y^a is that it just depends on the extension variable y and not on the tangential variable x. The equation is therefore invariant under translations in x, which allows the use of the sliding method to get uniqueness of the layer solution in $\mathbb{R}$, as well as monotonicity of solutions with limits ± 1 at $\pm\infty$.

Remark 1.3. Another interesting problem is to consider the existence of monotone solutions of equation (1.1) connecting $\underline{v}(x_2,\dots,x_n)$ at $-\infty$ to $\overline{v}(x_2,\dots,x_n)$ at $+\infty$, where both $\underline{v}$ and $\overline{v}$ are solutions of $(-\Delta)^s w = f(w)$ in $\mathbb{R}^{n-1}$. We will not address this problem here, but we believe that the methods developed in the present paper (and in [9,10]) allow us to deal with this type of problem.

2. Results

Throughout the paper we will assume that the nonlinearity f is of class $C^{1,\gamma}(\mathbb{R})$ for some $\gamma > \max(0, 1-2s)$. We will denote by G the associated potential, i.e.,

$$G' = -f.$$

The potential G is uniquely defined up to an additive constant.

Let $P_s = P_s(x,y)$ be the Poisson kernel associated to the operator $L_a = \operatorname{div}(y^a \nabla)$, with $a = 1-2s$. We then have (see section 3 of [9]): for v a bounded $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ function, v is a solution of (1.1) if and only if

$$u(\cdot, y) = P_s(\cdot, y) * v,$$

a function having v as trace on $\partial\mathbb{R}^{n+1}_+$, is a solution of (1.3) with f replaced by $(1+a)d_s^{-1} f = 2(1-s)d_s^{-1} f$. Recall that d_s is the constant from (1.4). It turns out that $2(1-s)d_s^{-1}$ has a positive limit as $s \uparrow 1$. This is the reason why we wrote problem (1.3) in [9] with the multiplicative constant $1+a = 2(1-s)$ in it; we wanted uniform estimates as $s \uparrow 1$.

Let us recall some regularity results from [9]. The first one is Lemma 4.4 of [9].

Lemma 2.1 ([9])**.** *Let f be a $C^{1,\gamma}(\mathbb{R})$ function with $\gamma > \max(0, 1-2s)$. Then, any bounded solution of*

$$(-\Delta)^s v = f(v) \ \text{ in } \mathbb{R}^n$$

is $C^{2,\beta}(\mathbb{R}^n)$ for some $0 < \beta < 1$ depending only on s and γ.

Furthermore, given $s_0 > 1/2$ there exists $0 < \beta < 1$ depending only on n, s_0, and γ—and hence independent of s—such that for every $s > s_0$,

$$\|v\|_{C^{2,\beta}(\mathbb{R}^n)} \le C$$

for some constant C depending only on n, s_0, $\|f\|_{C^{1,\gamma}}$, and $\|v\|_{L^\infty(\mathbb{R}^n)}$—and hence independent of $s \in (s_0, 1)$.

In addition, the function defined by $u(\cdot, y) = P_s(\cdot, y) * v$ (where P_s is the Poisson kernel associated to the operator L_a) satisfies for every $s > s_0$,

$$\|u\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} + \|\nabla_x u\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} + \|D^2_x u\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} \le C$$

for some constant C independent of $s \in (s_0, 1)$, indeed depending only on the same quantities as the previous one.

Following [10], we introduce

$$B_R^+ = \{(x,y) \in \mathbb{R}^{n+1} : y > 0, |(x,y)| < R\},$$
$$\Gamma_R^0 = \{(x,0) \in \partial\mathbb{R}_+^{n+1} : |x| < R\},$$
$$\Gamma_R^+ = \{(x,y) \in \mathbb{R}^{n+1} : y \geq 0, |(x,y)| = R\}.$$

We consider the problem in a half-ball

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{in } B_R^+, \\ (1+a)\dfrac{\partial u}{\partial \nu^a} = f(u) & \text{on } \Gamma_R^0. \end{cases} \tag{2.1}$$

In the sequel we will denote by

$$L_a = \operatorname{div}(y^a \nabla)$$

the differential operator in (2.1). Obviously, there is a natural notion of a weak solution of (2.1); see Definition 4.1 of [9].

We have the following regularity result (Lemma 4.5 of [9]).

Lemma 2.2 ([9])**.** *Let $a \in (-1,1)$ and $R > 0$. Let $\varphi \in C^\sigma(\Gamma_{2R}^0)$ for some $\sigma \in (0,1)$ and $u \in L^\infty(B_{2R}^+) \cap H^1(B_{2R}^+, y^a)$ be a weak solution of*

$$\begin{cases} L_a u = 0 & \text{in } B_{2R}^+ \subset \mathbb{R}_+^{n+1}, \\ \dfrac{\partial u}{\partial \nu^a} = \varphi & \text{on } \Gamma_{2R}^0. \end{cases}$$

Then, there exists $\beta \in (0,1)$ depending only on n, a, and σ, such that $u \in C^\beta(\overline{B_R^+})$ and $y^a u_y \in C^\beta(\overline{B_R^+})$.

Furthermore, there exist constants C_R^1 and C_R^2 depending only on n, a, R, $\|u\|_{L^\infty(B_{2R}^+)}$ and also on $\|\varphi\|_{L^\infty(\Gamma_{2R}^0)}$ (for C_R^1) and $\|\varphi\|_{C^\sigma(\Gamma_{2R}^0)}$ (for C_R^2), such that

$$\|u\|_{C^\beta(\overline{B_R^+})} \leq C_R^1 \tag{2.2}$$

and

$$\|y^a u_y\|_{C^\beta(\overline{B_R^+})} \leq C_R^2. \tag{2.3}$$

Problem (2.1) has variational structure, with corresponding energy functional

$$E_{B_R^+}(w) = \int_{B_R^+} \frac{1}{2} y^a |\nabla w|^2 + \int_{\Gamma_R^0} \frac{1}{1+a} G(w), \tag{2.4}$$

where $G' = -f$. This allows us to introduce some of the following notions.

Definition 2.3. a) We say that u is a *layer solution* of (1.3) if it is a bounded weak solution of (1.3),

$$u_{x_1} > 0 \quad \text{on } \partial\mathbb{R}_+^{n+1}, \quad \text{and} \tag{2.5}$$

$$\lim_{x_1 \to \pm\infty} u(x,0) = \pm 1 \quad \text{for every } (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}. \tag{2.6}$$

Note that we will indifferently call a layer solution a solution as above for problem (1.3) or a solution v of equation (1.1) satisfying the same properties.

b) Assume that u is a C^β function in $\overline{\mathbb{R}_+^{n+1}}$ for some $\beta \in (0,1)$, satisfying $-1 < u < 1$ in $\overline{\mathbb{R}_+^{n+1}}$ and such that for all $R > 0$,

$$y^a |\nabla u|^2 \in L^1(B_R^+).$$

We say that u is a *local minimizer* of problem (1.3) if

$$E_{B_R^+}(u) \le E_{B_R^+}(u+\psi)$$

for every $R>0$ and every C^1 function ψ in $\overline{\mathbb{R}^{n+1}_+}$ with compact support in $B_R^+ \cup \Gamma_R^0$ and such that $-1 \le u+\psi \le 1$ in B_R^+. To emphasize this last condition, in some occasions we will say that u is a local minimizer relative to perturbations in $[-1,1]$.

c) We say that u is a *stable solution* of (1.3) if u is a bounded solution of (1.3) and if

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla \xi|^2 - \int_{\partial \mathbb{R}^{n+1}_+} \frac{1}{1+a} f'(u)\, \xi^2 \ge 0 \tag{2.7}$$

for every function $\xi \in C^1(\overline{\mathbb{R}^{n+1}_+})$ with compact support in $\overline{\mathbb{R}^{n+1}_+}$.

It is clear that every local minimizer is a stable solution. At the same time, it is not difficult to prove that every layer solution u is also a stable solution—for this, one uses Lemma 6.1 below and the fact that u_{x_1} is a positive solution of the linearized problem to (1.3).

2.1. **Layer solutions in $\mathbb{R}$.** The following result characterizes the nonlinearities f for which problem (1.1) admits a layer solution in $\mathbb{R}$. In addition, it contains a result on uniqueness of layer solutions.

Theorem 2.4. *Let f be any $C^{1,\gamma}(\mathbb{R})$ function with $\gamma > \max(0, 1-2s)$, where $s \in (0,1)$. Let $G' = -f$. Then, there exists a solution v of*

$$(-\partial_{xx})^s v = f(v) \quad \text{in } \mathbb{R}$$

such that $v' > 0$ in $\mathbb{R}$ and $\lim_{x\to\pm\infty} v(x) = \pm 1$ if and only if

$$G'(-1) = G'(1) = 0 \quad \text{and} \quad G > G(-1) = G(1) \text{ in } (-1,1). \tag{2.8}$$

If in addition $f'(-1) < 0$ and $f'(1) < 0$, then this solution is unique up to translations.

As a consequence, if f is odd and $f'(\pm 1) < 0$, then the solution is odd with respect to some point. That is, $v(x+b) = -v(-x+b)$ for some $b \in \mathbb{R}$.

Remark 2.5. The statement on uniqueness of a layer solution also holds for any nonlinearity f of class $C^1([-1,1])$ satisfying $f'(-1) < 0$ and $f'(1) < 0$. There is no need for f' to be $C^\gamma([-1,1])$. Indeed, we will see that the proof follows that of [5] and thus only requires f to be Lipschitz in $[-1,1]$ and nonincreasing in a neighborhood of -1 and of 1. See also Lemma 5.2 of [10], where this more general assumption is presented.

Note that a layer solution $v = v(x)$, $x \in \mathbb{R}$, as in Theorem 2.4 provides us with a family of layer solutions of the same equation in $\mathbb{R}^n$. More precisely, for each direction $e \in \mathbb{R}^n$, with $|e| = 1$ and $e_1 > 0$, let

$$v^e(x_1, \dots, x_n) := v(\langle e, (x_1, \dots, x_n)\rangle).$$

Then, v^e is a layer solution of

$$(-\Delta)^s v^e = f(v^e) \quad \text{in } \mathbb{R}^n. \tag{2.9}$$

This fact is not immediate from the definition of the fractional Laplacian (1.2) through principal values in $\mathbb{R}$ and in $\mathbb{R}^n$—indeed, the integrals in $\mathbb{R}$ and in $\mathbb{R}^n$

differ, but the normalizing constants $C_{n,s}$ in front make them agree. This fact—that v^e solves (2.9)—follows directly from the equivalence of problem (1.1) with the extension problem (1.3) and the fact that the constant d_s in (1.4) is independent of the dimension n.

The equality $G(-1) = G(1)$ is equivalent to

$$\int_{-1}^{1} f(s)ds = 0.$$

Remark 2.6. Note that G may have one or several local minima in $(-1, 1)$ with higher energy than -1 and 1, and still satisfy condition (2.8). Such a G will therefore admit a layer solution, hence a solution with limits -1 and 1 at infinity. Instead, such a layer solution will not exist if G has a minimum at some point in $(-1, 1)$ with the same height as -1 and 1. In particular, when G is periodic (as in the Peierls-Nabarro problem $f(u) = \sin(\pi u)$; see [23]), the previous theorem proves that there exists no increasing solution connecting two nonconsecutive absolute minima of G.

In [18], with different techniques than ours, Palatucci, Savin, and Valdinoci prove that for potentials G with $G'(-1) = G'(1) = 0$, $G > G(-1) = G(1)$ in $(-1, 1)$, and $G''(\pm 1) > 0$, there exists a layer solution to equation (1.1). They also establish its main properties. As a main difference with our work, they do not use the extension problem (1.3). We also refer to the interesting paper [14] where properties of ground state solutions are investigated.

Our next result gives the asymptotic behavior of layer solutions.

Theorem 2.7. *Let f be any $C^{1,\gamma}(\mathbb{R})$ function with $\gamma > \max(0, 1 - 2s)$, where $s \in (0, 1)$. Assume that $f'(-1) < 0$, $f'(1) < 0$, and that v is a layer solution of*

$$(-\partial_{xx})^s v = f(v) \ \text{ in } \mathbb{R}.$$

Then, there exist constants $0 < c \le C$ such that

$$c|x|^{-1-2s} \le v'(x) \le C|x|^{-1-2s} \ \text{ for } |x| \ge 1. \tag{2.10}$$

As a consequence, for other constants $0 < c \le C$,

$$cx^{-2s} \le 1 - v(x) \le Cx^{-2s} \ \text{ for } x > 1 \tag{2.11}$$

and

$$c|x|^{-2s} \le 1 + v(x) \le C|x|^{-2s} \ \text{ for } x < -1. \tag{2.12}$$

To prove the above theorem for a given nonlinearity f, the following almost explicit layer solution (we emphasize that it is a layer solution for another nonlinearity) will be very useful. More properties and remarks on these concrete layers will be given in section 3.

Theorem 2.8. *Let $s \in (0, 1)$. For every $t > 0$, the $C^\infty(\mathbb{R})$ function*

$$v_s^t(x) := -1 + 2\int_{-\infty}^{x} p_s(t, \overline{x})\, d\overline{x} = \frac{2}{\pi}\int_0^\infty \frac{\sin(xr)}{r} e^{-tr^{2s}}\, dr$$

is the layer solution in $\mathbb{R}$ of (1.1) for a nonlinearity $f_s^t \in C^1([-1, 1])$ which is odd and satisfies $f_s^t(0) = f_s^t(1) = 0$, $f_s^t > 0$ in $(0, 1)$, and $(f_s^t)'(\pm 1) = -1/t$.

In the theorem, since $f_s^t \in C^1([-1,1])$ and $(f_s^t)'(\pm 1) < 0$, Theorem 2.4 and Remark 2.5 guarantee that its corresponding layer v_s^t is unique up to translations.

As we will see in Theorem 2.11 below, every layer solution is a local minimizer and, in particular, a stable solution. This holds in any dimension and for any nonlinearity. Our next result states that the converse is also true in dimension one and under a certain hypothesis on the nonlinearity. That is, under various assumptions on G, we prove that, for $n = 1$, local minimizers, solutions with limits (not monotone a priori), or stable solutions are indeed layer solutions.

Theorem 2.9. *Let f be any $C^{1,\gamma}(\mathbb{R})$ function, with $\gamma > \max(0, 1-2s)$. Let $n = 1$ and u be a function such that*

$$|u| < 1 \quad \text{in } \overline{\mathbb{R}^2_+}.$$

a) *Assume that $G > G(-1) = G(1)$ in $(-1,1)$, and that u is a local minimizer of problem* (1.3) *relative to perturbations in $[-1,1]$. Then, either $u = u(x,y)$ or $u^* = u^*(x,y) := u(-x,y)$ is a layer solution of* (1.3).

b) *Assume $G''(-1) > 0$, $G''(1) > 0$, and that u is a solution of* (1.3) *with*

$$\lim_{x\to\pm\infty} u(x,0) = \pm 1.$$

Then, u is a layer solution of (1.3).

c) *Assume that G satisfies*:

$$\text{if } -1 \le L^- < L^+ \le 1,\ G'(L^-) = G'(L^+) = 0, \tag{2.13}$$

$$\text{and } G > G(L^-) = G(L^+) \text{ in } (L^-, L^+), \tag{2.14}$$

$$\text{then } L^- = -1 \text{ and } L^+ = 1. \tag{2.15}$$

Let u be a nonconstant stable solution of (1.3). *Then, either $u = u(x,y)$ or $u^* = u^*(x,y) := u(-x,y)$ is a layer solution of* (1.3).

Remark 2.10. Notice that the hypothesis (2.13)-(2.15) on G in part c) of the theorem is necessary to guarantee that u connects ± 1. Indeed, assume that $-1 < L^- < L^+ < 1$ are four critical points of G with $G > G(-1) = G(1)$ in $(-1,1)$ and with $G > G(L^-) = G(L^+)$ in (L^-, L^+). Assume also that

$$G(\pm 1) < G(L^\pm).$$

Then, by our existence result (Theorem 2.4) applied twice—in $(-1,1)$ and also in (L^-, L^+) after rescaling it—we have that $(-\partial_{xx})^s v = f(v)$ in $\mathbb{R}$ admits two different increasing solutions: one connecting $L^\pm$ at infinity, and another connecting ± 1.

Instead, as pointed out in Remark 2.6, if $G \ge G(\pm 1) = G(L^\pm)$ in $(-1,1)$, then there is no increasing solution connecting ± 1, as a consequence of our Modica estimate (Theorem 1.1), which gives (1.7).

Note that an identically constant function $u \equiv s$ is a stable solution of (1.1) if and only if $G'(s) = 0$ and $G''(s) \ge 0$. This follows easily from the definition (2.7) of stability. Therefore, regarding part c) of the previous theorem, a way to guarantee that a stable solution u is nonconstant is that $u = s \in (-1,1)$ at some point and that either $G'(s) \ne 0$ or $G''(s) < 0$.

2.2. **Stability, local minimality, and symmetry of solutions.** The following result states that every layer solution in $\mathbb{R}^{n+1}_+$ is a local minimizer. This result is true in every dimension n. See also [18] for a related result which does not use the extension problem.

Theorem 2.11. *Let f be any $C^{1,\gamma}(\mathbb{R})$ function and $\gamma > \max(0, 1-2s)$, where $s \in (0,1)$. Assume that problem* (1.3) *admits a layer solution u in $\mathbb{R}^{n+1}_+$ with $n \geq 1$. Then*:

a) *u is a local minimizer of problem* (1.3).

b) *The potential G satisfies*

$$G'(-1) = G'(1) = 0 \quad \textit{and} \quad G \geq G(-1) = G(1) \ \textit{in} \ (-1,1). \tag{2.16}$$

The strict inequality $G > G(-1) = G(1)$ in (2.16) is known to hold when $n = 1$ or, as a consequence, when $u(\cdot, 0)$ is a one-dimensional solution in $\mathbb{R}^n$. We established this in [9] (it is one of the implications in Theorem 2.4 above). The strict inequality $G > G(\pm 1)$ also holds when $n = 2$ (as a consequence of Theorem 2.12 below) and when $n = 3$ and $s \geq 1/2$ (as a consequence of a result from [8]). It remains an open question in the rest of the cases.

For $n = 2$, we prove that bounded stable solutions u (and hence also local minimizers and layer solutions) are functions of only two variables: y and a linear combination of x_1 and x_2. This statement on the 1D symmetry of $u(\cdot, 0)$ is closely related to a conjecture of De Giorgi on 1D symmetry for interior reactions, proved in [2, 3, 15] in low dimensions and partially settled by Savin [20] up to dimension 8. We also refer the reader to [21, 22] where some rigidity properties of boundary reactions have been established through a more geometric approach. Particularly, in [21], the following symmetry result in dimension $n = 2$ is proved by using a completely different approach than the one used in the present paper, relying on a weighted Poincaré inequality (see also [13]).

Theorem 2.12. *Let f be any $C^{1,\gamma}(\mathbb{R})$ function and $\gamma > \max(0, 1-2s)$, where $s \in (0,1)$. Let v be a bounded solution of*

$$(-\Delta)^s v = f(v) \ \textit{in} \ \mathbb{R}^2.$$

Assume furthermore that its extension u is stable.

Then, v is a function of one variable. More precisely,

$$v(x_1, x_2) = v_0\left(\cos(\theta)x_1 + \sin(\theta)x_2\right) \quad \textit{in } \mathbb{R}^2$$

for some angle θ and some solution v_0 of the one-dimensional problem with the same nonlinearity f, and with either $v_0' > 0$ everywhere or v_0 identically constant.

For $n = 3$ and $s \geq 1/2$, this 1D symmetry result has been proved by E. Cinti and one of the authors in [7, 8]. It remains open for $n = 3$ and $s < 1/2$, and also for $n \geq 4$.

A simpler task than the study of all stable solutions consists of studying solutions u of (1.3) with $|u| \leq 1$ and satisfying the limits (2.6) *uniformly* in $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. Under hypotheses $f'(-1) < 0$ and $f'(-1) < 0$, it is possible to establish in every dimension n that these solutions depend only on the y- and x_1-variables, and are monotone in x_1. Here, by the uniform limits hypothesis, the x-variable in which the solution finally depends on is known a priori—in contrast with the variable of dependence in Theorem 2.12. For the standard Laplacian this result was first established in [4, 5, 12]. We will not provide the proof of the result because it is completely analogous to the one in [10] (which uses the sliding method, as in [5]). Since our operator L_a is invariant under translations in x, one can perform the sliding method together with the maximum principles proved in [9].

Theorem 2.12 is a partial converse in dimension two of Theorem 2.11 a), in the sense that it establishes the monotonicity of stable solutions and, in particular, of local minimizers. The remaining property for being a layer solution (i.e., having limits ± 1 at infinity) requires additional hypotheses on G, as in Theorem 2.9.

2.3. **Outline of the paper.** In section 3 we construct an almost explicit layer solution (Theorem 2.8) and we use it to establish the asymptotic behavior of any layer solution in $\mathbb{R}$ as stated in Theorem 2.7. In section 4 we prove the existence of minimizers to mixed Dirichlet-Neumann problems in bounded domains of $\mathbb{R}^{n+1}_+$—a result needed in subsequent sections. In section 5 we prove the local minimality of layer solutions in any dimension and the necessary conditions on G for such a layer in $\mathbb{R}^n$ to exist, Theorem 2.11. The 1D symmetry result for stable solutions in $\mathbb{R}^2$, Theorem 2.12, is established in section 6. Finally, section 7 concerns layers in $\mathbb{R}$ and establishes the existence result, Theorem 2.4, and the classification result, Theorem 2.9.

3. An example of a layer solution. Asymptotic properties of layer solutions

In this section we provide with an example of a layer solution based on the fractional heat equation. From it, we get the asymptotic behavior of layers for all other nonlinearities. Let us first explain how the concrete layer is found.

The starting point is the fractional heat equation,

$$\partial_t w + (-\partial_{xx})^s w = 0, \qquad t > 0, \ \ x \in \mathbb{R}, \tag{3.1}$$

which is known to have a fundamental solution of the form

$$p_s(t,x) = t^{-\frac{1}{2s}} q_s(t^{-\frac{1}{2s}} x) > 0 \tag{3.2}$$

for $x \in \mathbb{R}, t > 0$. Being the fundamental solution, p_s has total integral in x equal to 1, i.e.,

$$\int_{\mathbb{R}} p_s(t,x)\, dx = 1 \qquad \text{for all } t > 0. \tag{3.3}$$

To compute p_s, one takes the Fourier transform of (3.1) to obtain

$$\partial_t \widehat{p_s} + |\xi|^{2s} \widehat{p_s} = 0,$$

where $\widehat{p_s} = \widehat{p_s}(t,\xi)$ is the Fourier transform in x of $p_s(t,x)$. Thus, since $p_s(0,\cdot)$ is the Delta at zero and hence $\widehat{p_s}(0,\cdot) \equiv 1$, we deduce that

$$\widehat{p_s}(t,\xi) = \exp\{-t|\xi|^{2s}\}.$$

From this, by the inversion formula for the Fourier transform, we find

$$p_s(t,x) = \frac{1}{\pi} \int_0^\infty \cos(xr) e^{-tr^{2s}}\, dr. \tag{3.4}$$

It follows that the function

$$v_s^t(x) := -1 + 2\int_{-\infty}^x p_s(t,\overline{x})\, d\overline{x} = 2\int_0^x p_s(t,\overline{x})\, d\overline{x} \tag{3.5}$$

is increasing and has limits ± 1 at $\pm\infty$. The concrete expression (3.6) below for v_s^t is obtained by interchanging the order of the two integrals when using (3.4) to compute the primitive of p_s.

That v_s^t is a layer solution is stated in the next theorem, which contains all statements in Theorem 2.8 and also the asymptotic behavior of v_s^t, among other facts. The proof of the theorem is given at the end of this section.

Theorem 3.1. *Let $s \in (0,1)$. For every $t > 0$, the $C^\infty(\mathbb{R})$ function*

$$\begin{aligned} v_s^t(x) &:= -1 + 2\int_{-\infty}^{x} p_s(t,\overline{x})\,d\overline{x} = 2\int_0^x p_s(t,\overline{x})\,d\overline{x} \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sin(xr)}{r} e^{-tr^{2s}}\,dr \\ &= sign(x)\,\frac{2}{\pi}\int_0^\infty \frac{\sin(z)}{z} e^{-t(z/|x|)^{2s}}\,dz \end{aligned} \tag{3.6}$$

is the layer solution in $\mathbb{R}$ of (1.1) for a nonlinearity $f_s^t \in C^1([-1,1])$ which is odd and twice differentiable in $[-1,1]$ and which satisfies

$$f_s^t(0) = f_s^t(1) = 0, \quad f_s^t > 0 \text{ in } (0,1), \quad (f_s^t)'(\pm 1) = -\frac{1}{t}$$

and

$$(f_s^t)''(1) = -\frac{\pi}{t}\frac{\cos(\pi s)}{\sin(\pi s)}\frac{\Gamma(4s)}{(\Gamma(2s))^2} \begin{cases} < 0 & \text{if } 0 < s < 1/2, \\ = 0 & \text{if } s = 1/2, \\ > 0 & \text{if } 1/2 < s < 1. \end{cases} \tag{3.7}$$

In addition, the following limits exist:

$$\lim_{|x|\to\infty} |x|^{1+2s}(\partial_x v_s^t)(x) = t\frac{4s}{\pi}\sin(\pi s)\Gamma(2s) > 0 \tag{3.8}$$

and, as a consequence,

$$\lim_{x\to\pm\infty} |x|^{2s}|v_s^t(x) \mp 1| = t\frac{2}{\pi}\sin(\pi s)\Gamma(2s) > 0.$$

Remark 3.2. As stated in the theorem, we have $f_s^t \in C^1([-1,1])$ and $(f_s^t)'(\pm 1) < 0$ for every $s \in (0,1)$. In particular, by Theorem 2.4 and Remark 2.5, its corresponding layer v_s^t is unique up to translations.

When $s = 1/2$, the particular layer above agrees with the explicit one used in [10], namely

$$v_{1/2}^t(x) = \frac{2}{\pi}\arctan\frac{x}{t}, \quad \text{with } f_{1/2}^t(v) = \frac{1}{\pi t}\sin(\pi v).$$

This can be easily seen computing (3.4) explicitly when $s = 1/2$, using integration by parts, to obtain

$$\partial_x v_{1/2}^t(x) = 2p_{1/2}(t,x) = \frac{2}{\pi}\frac{1}{t}\frac{1}{1 + x^2/t^2}.$$

We may try to see which function we obtain in the above formulas setting $s = 1$. In this case, (3.4) can be checked to be equal to a Gaussian, and thus v_1^t, two times its primitive, is the error function $\mathrm{erf}(x)$—up to a scaling constant. Its derivative is therefore e^{-cx^2}, which does not have the correct decay e^{-cx} at $+\infty$ for the derivative v' of a layer solution to $-v'' = f(v)$. This is due to the fact that the limit as $s \to 1$ of f_s^t will not be a $C^1([-1,1])$ nonlinearity at the value 1—even if they all satisfy $(f_s^t)'(1) = -1/t$. The reason is that their second derivatives at 1, $(f_s^t)''(1)$, blow-up as $s \to 1$ as shown by (3.7).

Note also that (3.7) shows that, when $1/2 < s < 1$, the nonlinearity f_s^t is positive but not concave in $(0,1)$.

The following immediate consequence of Theorem 3.1 will give the asymptotic behavior of layer solutions for any nonlinearity f.

Corollary 3.3. *Let $s \in (0,1)$ and $t > 0$ be a constant. Then, the function*

$$\varphi^t = \partial_x v_s^t > 0,$$

where v_s^t is the explicit layer of Theorem 3.1, satisfies

$$(-\partial_{xx})^s \varphi^t(x) + 2t^{-1}\varphi^t(x) \geq 0 \qquad \text{for } |x| \text{ large enough,} \tag{3.9}$$

$$(-\partial_{xx})^s \varphi^t(x) + 2^{-1}t^{-1}\varphi^t(x) \leq 0 \qquad \text{for } |x| \text{ large enough,} \tag{3.10}$$

and that the following limit exists and is positive:

$$\lim_{|x|\to\infty} |x|^{1+2s}\varphi^t(x) \in (0,+\infty). \tag{3.11}$$

Proof. Clearly $\varphi^t = \partial_x v_s^t > 0$ satisfies the linearized equation

$$(-\partial_{xx})^s \varphi^t - (f_s^t)'(v_s^t(x))\,\varphi^t = 0 \quad \text{in } \mathbb{R}.$$

Using that $\varphi^t > 0$, v_s^t has limits ± 1 at $\pm\infty$, f_s^t is $C^1([-1,1])$, and that $(f_s^t)'(\pm 1) = -1/t$, both (3.9) and (3.10) follow. The statement (3.11) follows from (3.8). □

With this corollary at hand, we can now prove the asymptotics of any layer.

Proof of Theorem 2.7. The proof uses Corollary 3.3 above and a very easy maximum principle, Lemma 4.13 and Remark 4.14 of [9]. Its statement in dimension one is the following.

Let $w \in C^2_{\text{loc}}(\mathbb{R})$ be a continuous function in $\mathbb{R}$ such that $w(x) \to 0$ as $|x| \to \infty$ and

$$(-\partial_{xx})^s w + d(x)w \geq 0 \qquad \text{in } \mathbb{R} \tag{3.12}$$

for some bounded function d. Assume also that, for some nonempty closed set $H \subset \mathbb{R}$, one has $w > 0$ in H and that d is continuous and nonnegative in $\mathbb{R} \setminus H$. Then, $w > 0$ in $\mathbb{R}$.

Now let f and v be a nonlinearity and a layer as in Theorem 2.7. We then have

$$(-\partial_{xx})^s v' - f'(v)v' = 0 \quad \text{in } \mathbb{R}. \tag{3.13}$$

To prove the upper bound for v' in (2.10), we take t large enough such that $2t^{-1} < \min\{-f'(-1), -f'(1)\}$. Then, for any positive constant $C > 0$,

$$w := C\varphi^t - v'$$

satisfies, by (3.9) and (3.13), $(-\partial_{xx})^s w + 2t^{-1}w \geq 0$ for $|x|$ large enough, say for x in the complement of a compact interval H. Next, take the constant $C > 0$ so that $w \geq 1$ in the compact set H, and now define d in H so that $(-\partial_{xx})^s w + dw = 0$ in H—recall that $w \geq 1$ in H and hence d is well defined and bounded in H. We take $d = 2t^{-1}$ in $\mathbb{R} \setminus H$. Thus, (3.12) is satisfied and, since $w \to 0$ at infinity, the maximum principle above leads to $w > 0$ in $\mathbb{R}$. This is the desired upper bound for v' in (2.10), since φ^t satisfies (3.11).

To prove the lower bound for v' in (2.10), we proceed in the same way but replacing the roles of v' and φ^t. For this, we now take $t > 0$ small enough such

that $\max\{-f'(-1), -f'(1)\} < 2^{-1}t^{-1}$. Thus, $\tilde{w} := Cv' - \varphi^t$ satisfies $(-\partial_{xx})^s\tilde{w} + 2^{-1}t^{-1}\tilde{w} \geq 0$ for $|x|$ large enough. One proceeds exactly as before to obtain $\tilde{w} > 0$ in $\mathbb{R}$ for C large enough, which is the desired lower bound for v' in (2.10). □

It remains to establish Theorem 3.1. For this, we use the following well-known technical lemma due to G. Pólya [19], 1923. We prove it here for completeness; in fact, the proof as explained in [19] only works for $s \leq 1/2$. For $s > 1/2$, we follow the proof given in [16].

Lemma 3.4. *For $\kappa > 0$ and $s \in (0,1)$, we have*

$$\lim_{x\to+\infty} \int_0^\infty \sin(z) z^{\kappa s-1} e^{-(z/x)^{2s}}\, dz = \sin(\kappa s\pi/2)\Gamma(\kappa s).$$

Proof. For every $x > 0$, we have

$$\begin{aligned}\int_0^\infty \sin(z) z^{\kappa s-1} e^{-(z/x)^{2s}}\, dz &= \operatorname{Im} \int_0^\infty e^{iz} z^{\kappa s-1} e^{-(z/x)^{2s}}\, dz \\ &= \operatorname{Im} \int_0^\infty h_x(z)\, dz,\end{aligned}$$

where

$$h_x(z) := z^{\kappa s-1} e^{iz-(z/x)^{2s}}.$$

Let us also denote

$$h_\infty(z) := z^{\kappa s-1} e^{iz}.$$

For $0 \leq \theta \leq \pi/2$, let γ_θ be the half-line from the origin making an angle θ with the positive x-axis. We will next see that, for certain angles θ, $\operatorname{Im}\int_{\gamma_\theta} h_x(z)\, dz$ are all equal and independent of those θ. For this, given two angles $0 \leq \theta_1 < \theta_2 \leq \pi/2$ and $R > 0$, we integrate counterclockwise on the contour given by the segments of length R starting from 0 on γ_{θ_1} and on γ_{θ_2}, and by the arc $\Gamma^R_{\theta_1,\theta_2}$ of radius R with center at the origin and joining the two end points of the previous segments. We also need to remove a neighborhood of zero and add a small arc with center at the origin connecting the two half-lines. The integrals of h_x and of h_∞ in this small arc will tend to zero as the radius tends to zero, since $|h_x(z)| + |h_\infty(z)| \leq C|z|^{\kappa s-1}$ near the origin.

The key point is to make sure that the integral of h_x, and later of h_∞, on the arc $\Gamma^R_{\theta_1,\theta_2}$ of radius R tends to zero as $R \to \infty$ if we choose the angles $0 \leq \theta_1 < \theta_2 \leq \pi/2$ correctly. Note that if $z \in \mathbb{C}$ belongs to such an arc, then z belongs to the sector

$$S_{\theta_1,\theta_2} := \{z \in \mathbb{C} : \theta_1 \leq \operatorname{Arg}(z) \leq \theta_2\}.$$

To guarantee the convergence to zero of the integral on the arc, note that

$$|h_x(z)| = |z|^{\kappa s-1} \exp\{-\operatorname{Im}(z) - x^{-2s}\operatorname{Re}(z^{2s})\} \tag{3.14}$$

and

$$|h_\infty(z)| = |z|^{\kappa s-1} \exp\{-\operatorname{Im}(z)\} \tag{3.15}$$

for all $z \in \mathbb{C}$ in the first quadrant.

We need to distinguish two cases.

Case 1. Suppose that $s \leq 1/2$. In this case we take $\theta_1 = 0$ and $\theta_2 = \pi/2$. Then, if z lies in the sector $S_{0,\pi/2}$ (the first quadrant), z^{2s} is also in the first quadrant, since $2s \leq 1$. Thus, the real and imaginary parts appearing in (3.14) are both nonnegative, and at least one of them is positive up to the boundary of the quadrant.

Thus, by (3.14), $|h_x| \to 0$ exponentially fast—as $\exp\{-c(x)|z|^{2s}\}$—uniformly in all of the quadrant. Hence, the integral on the arc $\Gamma^R_{0,\pi/2}$ tends to zero as $R \to \infty$. We deduce that

$$\int_0^\infty \sin(z) z^{\kappa s-1} e^{-(z/x)^{2s}}\, dz = \operatorname{Im} \int_{\gamma_0} h_x(z)\, dz = \operatorname{Im} \int_{\gamma_{\pi/2}} h_x(z)\, dz$$
$$= \operatorname{Im} \left\{ e^{i\kappa s\pi/2} \int_0^\infty y^{\kappa s-1} e^{-y-i^{2s}(y/x)^{2s}}\, dy \right\}.$$

Note that the function in the last integral is integrable since

$$|e^{-y-i^{2s}(y/x)^{2s}}| = |e^{-y-(\cos(s\pi)+i\sin(s\pi))(y/x)^{2s}}| = e^{-y-\cos(s\pi)(y/x)^{2s}} \le e^{-y}$$

due to $s \le 1/2$. Thus, the limit as $x \to +\infty$ exists and is equal to

$$\begin{aligned} \lim_{x\to+\infty} \int_0^\infty \sin(z) z^{\kappa s-1} e^{-(z/x)^{2s}}\, dz &= \operatorname{Im} \left\{ e^{i\kappa s\pi/2} \int_0^\infty y^{\kappa s-1} e^{-y}\, dy \right\} \\ &= \sin(\kappa s\pi/2) \int_0^\infty y^{\kappa s-1} e^{-y}\, dy \\ &= \sin(\kappa s\pi/2)\Gamma(\kappa s), \end{aligned} \tag{3.16}$$

as claimed.

Case 2. Suppose now that $1/2 < s < 1$. In this case (3.14) does not tend to zero at infinity in all of the first quadrant, since $2s > 1$, and thus $\operatorname{Re}(z^{2s})$ becomes negative somewhere in the quadrant. Here, we need to take

$$\theta_1 = 0 \quad \text{and} \quad \theta_2 = \frac{\pi}{4s}.$$

Now, in the sector $S_{0,\pi/(4s)}$, the real and imaginary parts appearing in (3.14) are both nonnegative, and at least one of them positive up to the boundary of the sector. Thus, as before, we now deduce

$$\int_0^\infty \sin(z) z^{\kappa s-1} e^{-(z/x)^{2s}}\, dz = \operatorname{Im} \int_{\gamma_0} h_x(z)\, dz = \operatorname{Im} \int_{\gamma_{\pi/(4s)}} h_x(z)\, dz.$$

Note that in the last integral on $\gamma_{\pi/(4s)}$, we have

$$\begin{aligned} |h_x(z)| &= |z|^{\kappa s-1} \exp\{-\operatorname{Im}(z) - x^{-2s}\operatorname{Re}(z^{2s})\} \\ &= |z|^{\kappa s-1} \exp\{-\operatorname{Im}(z)\} = |h_\infty(z)| \end{aligned}$$

for $z \in \gamma_{\pi/(4s)}$. Besides, by the last expression, h_∞ is integrable on $\gamma_{\pi/(4s)}$. Thus, by dominated convergence, we have

$$\lim_{x\to+\infty} \int_0^\infty \sin(z) z^{\kappa s-1} e^{-(z/x)^{2s}}\, dz = \operatorname{Im} \int_{\gamma_{\pi/(4s)}} h_\infty(z)\, dz. \tag{3.17}$$

Finally, for this last integral we work on the sector $S_{\pi/(4s),\pi/2}$. By (3.15), $h_\infty(z)$ tends to zero exponentially fast and uniformly as $|z| \to \infty$ on the sector. Thus,

$$\begin{aligned} \operatorname{Im} \int_{\gamma_{\pi/(4s)}} h_\infty(z)\, dz &= \operatorname{Im} \int_{\gamma_{\pi/2}} h_\infty(z)\, dz \\ &= \operatorname{Im} \left\{ e^{i\kappa s\pi/2} \int_0^\infty y^{\kappa s-1} e^{-y}\, dy \right\}. \end{aligned}$$

Recalling (3.17), one concludes as in (3.16). □

Finally, we can prove our results on the explicit layer.

Proof of Theorem 3.1. Let v_s^t be defined by (3.6). It is clear that

$$v_s^t(x) = v_s^1(t^{-1/(2s)}x). \tag{3.18}$$

Hence, by the definition (1.2) of the fractional Laplacian, we have $(-\partial_{xx})^s v_s^t(x) = t^{-1}(-\partial_{xx})^s v_s^1(t^{-1/(2s)}x)$. Thus, having proved all the statements for v_s^1, they will also hold for v_s^t with nonlinearity $f_s^t(v) = t^{-1} f_s^1(v)$.

Hence, we may take $t = 1$. To simplify notation, we denote

$$v := v_s^1 \qquad \text{and} \qquad f := f_s^1.$$

From $v'(x) = 2p_s(1, x)$ and expression (3.4), it is clear that $v \in C^\infty(\mathbb{R})$. By expression (3.5), we have $v(-\infty) = -1$. Since $v' = 2q_s = 2p_s(1, \cdot) > 0$, v is increasing.

The fact that $v(+\infty) = 1$ is a consequence of (3.3), $\int_{\mathbb{R}} p_s(1, y)\, dy = 1$. It also follows from expression (3.6) and the well-known fact that $\int_0^\infty \sin(z) z^{-1}\, dz = \pi/2$. This can also be proved by adding a factor $z^{\kappa s}$ in the integral (3.6), and then using Lemma 3.4 and that $\sin(\kappa s\pi/2)\Gamma(\kappa s) = \sin(\kappa s\pi/2)(\kappa s)^{-1}\Gamma(\kappa s+1) \to \pi/2$ as $\kappa \downarrow 0$.

We now prove that there exists a function f such that

$$(-\partial_{xx})^s v = f(v) \ \text{ in } \mathbb{R}.$$

For this, we use the expression (3.2) and that p_s solves the fractional heat equation (3.1). Because of the commutation of the derivative with the fractional Laplacian (recall that $v \in C^\infty(\mathbb{R})$ and that v and v' are bounded), we deduce that

$$\begin{aligned} \{(-\partial_{xx})^s v\}'(x) &= (-\partial_{xx})^s v'(x) = 2(-\partial_{xx})^s q_s(x) = -2\partial_t p_s(1, x) \\ &= \frac{1}{s}\left\{q_s(x) + x q_s'(x)\right\}. \end{aligned}$$

Therefore, integrating by parts,

$$(-\partial_{xx})^s v(x) = \frac{1}{s}\int_{-\infty}^x \{q_s(z) + z q_s'(z)\}\, dz = \frac{1}{s} x q_s(x) = \frac{1}{2s} x v'(x).$$

Since $v' > 0$, the C^∞ function $v = v(x)$ is invertible on $\mathbb{R}$, with inverse $x = x(v)$, a C^∞ function on the open interval $(-1, 1)$. We now set

$$f(v) := \frac{1}{2s} x(v) v'(x(v)), \tag{3.19}$$

so that our semilinear fractional equation is satisfied. We know that $f \in C^\infty(-1, 1)$. Also, since v is an odd function, its inverse x is also odd and therefore f is odd, by (3.19). This expression also gives that $f > 0$ in $(0, 1)$.

It remains to verify that $f \in C^1([-1, 1])$ once we set $f(\pm 1) = 0$ and $f'(\pm 1) = -1$, and that f is twice differentiable in $[-1, 1]$ and having values for $f''(\pm 1)$ given by (3.7) with $t = 1$. It also remains to establish the asymptotic behavior of v'.

For all this, using (3.4) we compute

$$\begin{aligned} (\pi/2) v'(x) &= \pi q_s(x) = \frac{1}{x}\int_0^\infty \cos(z) e^{-(z/x)^{2s}}\, dz \\ &= \frac{1}{x}\int_0^\infty \{\sin(z)\}' e^{-(z/x)^{2s}}\, dz \\ &= 2s x^{-1-2s}\int_0^\infty \sin(z) z^{2s-1} e^{-(z/x)^{2s}}\, dz, \end{aligned} \tag{3.20}$$

by integration by parts. Hence using Lemma 3.4 with $\kappa = 2$, we deduce

$$\lim_{x\to+\infty} x^{1+2s} v'(x) = \lim_{x\to+\infty} 2x^{1+2s} q_s(x) = \frac{4s}{\pi}\sin(\pi s)\Gamma(2s), \tag{3.21}$$

as claimed in (3.8)—for other values of t, simply use (3.18). In particular,

$$\lim_{x\to+\infty} xv'(x) = 0$$

and thus, by (3.19), f is continuous on $[-1,1]$ defining $f(\pm 1) = 0$. In addition, we also deduce

$$1 - v(x) = \int_x^\infty v'(y)\,dy = \frac{2}{\pi}\sin(\pi s)\Gamma(2s)x^{-2s} + \mathrm{o}(x^{-2s}) \tag{3.22}$$

as $x \to +\infty$.

Next, we differentiate (3.19), that is, $f(v(x)) = (2s)^{-1}xv'(x) = (2s)^{-1}x2q_s(x)$, to obtain

$$f'(v)v' = \frac{1}{2s}\{v' + x(v)2q_s'(x(v))\}$$

and hence

$$f'(v) = \frac{1}{2s}\left\{1 + x(v)\frac{q_s'(x(v))}{q_s(x(v))}\right\}. \tag{3.23}$$

Thus, using (3.4) we compute

$$\begin{aligned}
\pi x q_s'(x) &= -\int_0^\infty xr\sin(xr)e^{-r^{2s}}\,dr \\
&= -x^{-1}\int_0^\infty z\sin(z)e^{-(z/x)^{2s}}\,dz \\
&= -x^{-1}\int_0^\infty \{\sin(z) - z\cos(z)\}' e^{-(z/x)^{2s}}\,dz \\
&= -2sx^{-1-2s}\int_0^\infty \{\sin(z) - z\cos(z)\}z^{2s-1}e^{-(z/x)^{2s}}\,dz.
\end{aligned}$$

We also compute $\pi\{(1+2s)q_s + xq_s'\}$ by adding (3.20) (multiplied by $1+2s$) to the previous expression. Integrating by parts, and at the end invoking Lemma 3.4 with $\kappa = 4s$, we obtain

$$\begin{aligned}
&\pi\{(1+2s)q_s + xq_s'\} \\
&\quad= 2sx^{-1-2s}\int_0^\infty \{2s\sin(z) + z\cos(z)\}z^{2s-1}e^{-(z/x)^{2s}}\,dz \\
&\quad= 2sx^{-1-2s}\int_0^\infty \{\sin(z)z^{2s}\}' e^{-(z/x)^{2s}}\,dz \\
&\quad= (2s)^2 x^{-1-4s}\int_0^\infty \sin(z)z^{4s-1}e^{-(z/x)^{2s}}\,dz \\
&\quad= x^{-1-4s}\{4s^2\sin(2\pi s)\Gamma(4s) + \mathrm{o}(1)\} \\
&\quad= x^{-1-4s}\{8s^2\sin(\pi s)\cos(\pi s)\Gamma(4s) + \mathrm{o}(1)\} \qquad (3.24)
\end{aligned}$$

as $x \to +\infty$.

Therefore, from (3.23), (3.24), and (3.21), one has

$$f'(v(x)) = -1 + \frac{1}{2s}\,\frac{(1+2s)q_s + xq_s'}{q_s} = -1 + \mathrm{O}(x^{-2s}). \tag{3.25}$$

Thus, setting $f'(\pm 1) = -1$ and using that f' is even, we have that f is differentiable at ± 1.

Finally, using (3.25), (3.24), (3.22), and (3.21), we have

$$\begin{aligned}\frac{f'(v(x)) - f'(1)}{v(x) - 1} &= \frac{f'(v(x)) + 1}{v(x) - 1} \\ &= \frac{1}{2s}\,\frac{(1+2s)q_s + xq_s'}{(v-1)q_s} \to -\pi\frac{\cos(\pi s)\Gamma(4s)}{\sin(\pi s)(\Gamma(2s))^2}\end{aligned}$$

as $x \to +\infty$. This establishes that $f \in C^1([-1,1])$ and also that f is twice differentiable in all of $[-1,1]$ with

$$f''(\pm 1) = \mp\pi\frac{\cos(\pi s)}{\sin(\pi s)}\,\frac{\Gamma(4s)}{(\Gamma(2s))^2}. \tag{3.26}$$

The proof is now complete. □

4. Minimizers of the Dirichlet-Neumann problem in bounded domains

In this section, we concentrate on the existence of absolute minimizers of the functional $E_\Omega(u)$ on bounded domains Ω. This is an important step since, as in [10], the existence theory of layer solutions goes through a localization argument in half-balls of $\mathbb{R}^{n+1}_+$.

Let $\Omega \subset \mathbb{R}^{n+1}_+$ be a bounded Lipschitz domain. We define the following subsets of $\partial\Omega$:

$$\partial^0\Omega = \{(x,0) \in \partial\mathbb{R}^{n+1}_+ \,:\, B^+_\varepsilon(x,0) \subset \Omega \text{ for some } \varepsilon > 0\} \quad \text{and} \tag{4.1}$$

$$\partial^+\Omega = \overline{\partial\Omega \cap \mathbb{R}^{n+1}_+}. \tag{4.2}$$

Let $H^1(\Omega, y^a)$ denote the weighted Sobolev space

$$H^1(\Omega, y^a) = \left\{u : \Omega \to \mathbb{R} \,:\, y^a(u^2 + |\nabla u|^2) \in L^1(\Omega)\right\}$$

endowed with its usual norm.

Let $u \in C^\beta(\overline{\Omega}) \cap H^1(\Omega, y^a)$ be a given function with $|u| \le 1$, where $\beta \in (0,1)$. We consider the energy functional

$$E_\Omega(v) = \int_\Omega \frac{y^a}{2}|\nabla v|^2 + \int_{\partial^0\Omega} \frac{1}{1+a}G(v) \tag{4.3}$$

in the class

$$\mathcal{C}_{u,a}(\Omega) = \{v \in H^1(\Omega, y^a) : -1 \le v \le 1 \text{ a.e. in } \Omega \text{ and } v \equiv u \text{ on } \partial^+\Omega\},$$

which contains u and thus is nonempty.

The set $\mathcal{C}_{u,a}(\Omega)$ is a closed convex subset of the affine space

$$H_{u,a}(\Omega) = \{v \in H^1(\Omega, y^a) : v \equiv u \text{ on } \partial^+\Omega\}, \tag{4.4}$$

where the last condition should be understood in terms of the fact that $v - u$ vanishes on $\partial^+\Omega$ in the weak sense.

Lemma 4.1. *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^{n+1}_+$ be a bounded Lipschitz domain. Let $u \in C^\beta(\overline{\Omega}) \cap H^1(\Omega, y^a)$ be a given function with $|u| \le 1$, where $\beta \in (0,1)$. Assume that*

$$f(1) \le 0 \le f(-1). \tag{4.5}$$

Then, the functional E_Ω admits an absolute minimizer w in $\mathcal{C}_{u,a}(\Omega)$. In particular, w is a weak solution of

$$\begin{cases} L_a w = 0 & \text{in } \Omega, \\ (1+a)\dfrac{\partial w}{\partial \nu^a} = f(w) & \text{on } \partial^0\Omega, \\ w = u & \text{on } \partial^+\Omega. \end{cases} \tag{4.6}$$

Moreover, w is a stable solution of (4.6)*, in the sense that*

$$\int_\Omega y^a |\nabla \xi|^2 - \int_{\partial^0\Omega} \frac{1}{1+a} f'(w)\xi^2 \geq 0 \tag{4.7}$$

for every $\xi \in H^1(\Omega, y^a)$ such that $\xi \equiv 0$ on $\partial^+\Omega$ in the weak sense.

Hypothesis (4.5) states simply that -1 and 1 are a subsolution and a supersolution, respectively, of (4.6).

Proof of Lemma 4.1. As in [10], it is useful to consider the following continuous extension $\tilde{f}$ of f outside $[-1,1]$:

$$\tilde{f}(t) = \begin{cases} f(-1) & \text{if } s \leq -1, \\ f(s) & \text{if } -1 \leq s \leq 1, \\ f(1) & \text{if } 1 \leq s. \end{cases}$$

Let

$$\tilde{G}(s) = -\int_0^s \tilde{f},$$

and consider the new functional

$$\tilde{E}_\Omega(v) = \int_\Omega \frac{y^a}{2}|\nabla v|^2 + \int_{\partial^0\Omega} \frac{1}{1+a}\tilde{G}(v),$$

in the affine space $H_{u,a}(\Omega)$ defined by (4.4).

Note that $\tilde{G} = G$ in $[-1,1]$, up to an additive constant. Therefore, any minimizer w of $\tilde{E}_\Omega$ in $H_{u,a}(\Omega)$ such that $-1 \leq w \leq 1$ is also a minimizer of E_Ω in $\mathcal{C}_{u,a}(\Omega)$.

To show that $\tilde{E}_\Omega$ admits a minimizer in $H_{u,a}(\Omega)$, we use a standard compactness argument. Indeed, let $v \in H_{u,a}(\Omega)$. Since $v-u \equiv 0$ on $\partial^+\Omega$, we can extend $v-u$ to be identically 0 in $\mathbb{R}^{n+1}_+ \backslash \Omega$, and we have $v-u \in H^1(\mathbb{R}^{n+1}_+, y^a)$. By Nekvinda's result [17], the trace space of $H^1(\mathbb{R}^{n+1}_+, y^a)$ is the Gagliardo space $W^{\frac{1-a}{2},2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$. The Sobolev embedding (see [1])

$$H^s(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$$

(or into any $L^p(\mathbb{R}^n)$ if $n = 1 \leq 2s$) and the classical Rellich compactness theorem immediately give the compactness of the inclusion

$$H_{u,a}(\Omega) \Subset L^2(\partial^0\Omega).$$

Now, since $H_{u,a}(\Omega) \subset L^2(\partial^0\Omega)$ and $\tilde{G}$ has linear growth at infinity, it follows that $\tilde{E}_\Omega$ is well defined, bounded below, and coercive in $H_{u,a}(\Omega)$. Hence, using the compactness of the inclusion $H_{u,a}(\Omega) \Subset L^2(\partial^0\Omega)$, taking a minimizing sequence in $H_{u,a}(\Omega)$ and a subsequence convergent in $L^2(\partial^0\Omega)$, we conclude that $\tilde{E}_\Omega$ admits an absolute minimizer w in $H_{u,a}(\Omega)$.

Since $\tilde{f}$ is a continuous function, $\tilde{E}$ is a C^1 functional in $H_{u,a}(\Omega)$. Making first and second order variations of $\tilde{E}$ at the minimum w, we obtain that w is a

weak solution of (4.6) which satisfies (4.7), with f and f' replaced by $\tilde{f}$ and $\tilde{f}'$, respectively, in both (4.6) and (4.7).

Therefore, it only remains to show that the minimizer w satisfies

$$-1 \leq w \leq 1 \quad \text{a.e. in } \Omega.$$

We use that -1 and 1 are, respectively, a subsolution and a supersolution of (4.6), due to hypothesis (4.5). We proceed as follows. We use that the first variation of $\tilde{E}_\Omega$ at w in the direction $(w-1)^+$ (the positive part of $w-1$), is zero. Since $|w| = |u| \leq 1$ on $\partial^+\Omega$ and hence $(w-1)^+$ vanishes on $\partial^+\Omega$, we have that $w+\varepsilon(w-1)^+ \in H_{u,a}(\Omega)$ for every ε. We deduce that

$$\begin{aligned} 0 &= \int_\Omega y^a \nabla w \, \nabla (w-1)^+ - \int_{\partial^0\Omega} \tilde{f}(w)(w-1)^+ \\ &= \int_{\Omega\cap\{w\geq 1\}} y^a |\nabla (w-1)^+|^2 - \int_{\partial^0\Omega\cap\{w\geq 1\}} f(1)(w-1)^+ \\ &\geq \int_\Omega y^a |\nabla (w-1)^+|^2, \end{aligned}$$

where we have used that $\tilde{f}(s) = f(1)$ for $s \geq 1$, and that $f(1) \leq 0$ by assumption. We conclude that $(w-1)^+$ is constant, and hence identically zero. Therefore, $w \leq 1$ a.e. The inequality $w \geq -1$ is proved in the same way, now using $f(-1) \geq 0$. □

5. Local minimality of layers and consequences. Proof of Theorem 2.11

The fact that for reactions in the interior (that is, $s = 1$ in our equation), layer solutions in $\mathbb{R}^n$ are necessarily local minimizers was found by Alberti, Ambrosio, and one of the authors in [2]. For the fractional case, this is the statement in Theorem 2.11 a) above. The proof in [2] also works in the fractional case, working with the extension problem. It uses two ingredients: the existence result from the previous section (Lemma 4.1) and the following uniqueness result in the presence of a layer.

Lemma 5.1. *Assume that problem* (1.3) *admits a layer solution u. Then, for every $R > 0$, u is the unique weak solution of the problem*

$$\begin{cases} L_a w = 0 & \text{in } B_R^+ \subset \mathbb{R}^{n+1}_+, \\ -1 \leq w \leq 1 & \text{in } B_R^+, \\ (1+a)\dfrac{\partial w}{\partial \nu^a} = f(w) & \text{on } \Gamma_R^0, \\ w = u & \text{on } \Gamma_R^+. \end{cases} \tag{5.1}$$

Proof. We refer the reader to the proof of Lemma 3.1 in [10] since the proof is identical in our case. Indeed, since the operator L_a is invariant under translations in x, this allows us to use the sliding method as in Lemma 3.1 of [10] to get the uniqueness. The only other important ingredient in the proof is the Hopf boundary lemma; in our present context it can be found in Proposition 4.11 and Corollary 4.12 of [9]. □

Part b) of Theorem 2.11 will follow from the following proposition. It will be useful also in other future arguments. Notice that the result for $n = 1$ follows from our Modica estimate, Theorem 2.3 of [9] (rewritten in Theorem 1.1 of the present

paper). Instead, the following proof also works in higher dimensions but only gives $G \geq G(L^-) = G(L^+)$ in $[-1,1]$—in contrast with the strict inequality $G > G(-1) = G(1)$ obtained in dimension one from the Modica estimate (Theorem 1.1) when $L^\pm = \pm 1$.

Proposition 5.2. *Let u be a solution of* (1.3) *such that $|u| < 1$, and*

$$\lim_{x_1 \to \pm\infty} u(x,0) = L^\pm \quad \textit{for every } (x_2, \dots, x_n) \in \mathbb{R}^{n-1},$$

for some constants L^- and L^+ (that could be equal). Assume that u is a local minimizer relative to perturbations in $[-1,1]$. Then,

$$G \geq G(L^-) = G(L^+) \quad \textit{in } [-1,1].$$

Proof. It suffices to show that $G \geq G(L^-)$ and $G \geq G(L^+)$ in $[-1,1]$. It then follows that $G(L^-) = G(L^+)$. By symmetry, it is enough to establish that $G \geq G(L^+)$ in $[-1,1]$. Note that this inequality, as well as the notion of local minimizer, is independent of adding a constant to G. Hence, we may assume that

$$G(s) = 0 < G(L^+) \quad \text{for some } s \in [-1,1],$$

and we need to obtain a contradiction. Since $G(L^+) > 0$, we have that

$$\frac{1}{1+a} G(t) \geq \varepsilon > 0 \quad \text{for } t \text{ in a neighborhood in } [-1,1] \text{ of } L^+$$

for some $\varepsilon > 0$.

Consider the points $(b,0,0) = (x_1 = b, x_2 = 0, \dots, x_n = 0, y = 0)$ on $\partial \mathbb{R}^{n+1}_+$. Since for $R > 0$,

$$E_{B_R^+(b,0,0)}(u) \geq \int_{\Gamma_R^0(b,0)} \frac{1}{1+a} G(u(x,0))\, dx$$

and $u(x,0) \underset{x_1 \to +\infty}{\longrightarrow} L^+$, we deduce that

$$\varliminf_{b \to +\infty} E_{B_R^+(b,0,0)}(u) \geq c(n)\, \varepsilon R^n \qquad \text{for all } R > 0. \tag{5.2}$$

The constant $c(n)$ depends only on n.

The lower bound (5.2) will be a contradiction with an upper bound for the energy of u, that we obtain using the local minimality of u.

For $R > 1$, let ξ_R be a smooth function in $\mathbb{R}^{n+1}$ such that $0 \leq \xi_R \leq 1$,

$$\xi_R = \begin{cases} 1 & \text{in } B^+_{(1-\eta)R}, \\ 0 & \text{on } \mathbb{R}^{n+1}_+ \setminus B^+_R, \end{cases}$$

and $|\nabla \xi_R| \leq C(n)(\eta R)^{-1}$, where $\eta \in (0,1)$ is to be chosen later. Let

$$\xi_{R,b}(x,y) := \xi_R(x_1 + b, x_2, \dots, x_n, y).$$

Since

$$(1 - \xi_{R,b})u + \xi_{R,b} s = u + \xi_{R,b}(s - u)$$

takes values in $[-1,1]$ and agrees with u on $\Gamma_R^+(b,0,0)$, we have that

$$E_{B_R^+(b,0,0)}(u) \leq E_{B_R^+(b,0,0)}(u + \xi_{R,b}(s-u)).$$

Next, we bound by above this last energy. Since $G(s) = 0$, the potential energy is only nonzero in $B_R^+ \setminus B_{(1-\eta)R}^+$, which has measure bounded above by $C(n)\eta R^n$. On the other hand, since we proved in Lemma 4.8(i) of [9] that

$$\|\nabla_x u\|_{L^\infty(B_R^+(x,0))} \to 0 \text{ as } x_1 \to \pm\infty,$$

we deduce that

$$\begin{aligned}\overline{\lim_{b\to+\infty}} \int_{B_R^+(b,0,0)} y^a |\nabla\{u + \xi_{R,b}(s-u)\}|^2 &\le 2\int_{B_R^+} y^a |\nabla \xi_R|^2 \\ &\le \frac{C(n)}{\eta^2 R^2} R^n \int_0^R y^a\, dy = C(n)\frac{R^{n+1+a}}{\eta^2 R^2} = C(n)\frac{R^{n-2s}}{\eta^2}.\end{aligned}$$

Putting together the bounds for Dirichlet and potential energies, we conclude that

$$\begin{aligned}\overline{\lim_{b\to+\infty}} E_{B_R^+(b,0,0)}(u) &\le \overline{\lim_{b\to+\infty}} E_{B_R^+(b,0,0)}(u + \xi_{R,b}(s-u)) \\ &\le C\{\eta R^n + \eta^{-2} R^{n-2s}\},\end{aligned}$$

for some constant $C > 0$ depending only on n, a, and G.

Recalling the lower bound (5.2), we now choose η small enough so that $C\eta = (1/2)c(n)\varepsilon$. In this way, (5.2) and the last upper bound lead to $(1/2)c(n)\varepsilon R^n \le C\eta^{-2}R^{n-2s}$. This is a contradiction when R is large enough. □

Proof of Theorem 2.11. We proceed exactly as in the proof of Theorem 1.4 in [10], page 1708.

To prove part a), for $R > 1$ we consider problem (5.1) in a half-ball. Lemma 4.1 gives the existence of a minimizer w with $-1 \le w \le 1$. Note that in the lemma one needs condition (4.5). But in the presence of a layer, we showed in Lemma 4.8(i) of [9] that one has $f(-1) = f(1) = 0$.

On the other hand, Lemma 5.1 states that the layer u is the unique solution of (5.1). Thus, $u \equiv w$ in B_R^+. This shows that u is a local minimizer.

To prove part b), $G'(-1) = G'(1) = 0$ was shown in Lemma 4.8(i) of [9]. We have established the other relation, $G \ge G(-1) = G(1)$ in $[-1, 1]$, in Proposition 5.2 above. □

6. Monotonicity and 1D symmetry of stable solutions in $\mathbb{R}^2$. Proof of Theorem 2.12

To prove Theorem 2.12, we need two lemmas. The following one, applied with $d(x) = -(1+a)^{-1} f'(u(x,0))$, establishes an alternative criterium for a solution u of (1.3) to be stable.

Lemma 6.1. *Let d be a bounded and Hölder continuous function on $\partial\mathbb{R}^{n+1}_+$. Then,*

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla\xi|^2 + \int_{\partial\mathbb{R}^{n+1}_+} d(x)\xi^2 \ge 0 \tag{6.1}$$

for every function $\xi \in C^1(\overline{\mathbb{R}^{n+1}_+})$ with compact support in $\overline{\mathbb{R}^{n+1}_+}$, if and only if there exists a Hölder continuous function φ in $\overline{\mathbb{R}^{n+1}_+}$ such that $\varphi > 0$ in $\overline{\mathbb{R}^{n+1}_+}$,

$\varphi \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}, y^a)$, *and*

$$\begin{cases} L_a\varphi = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial \varphi}{\partial \nu^a} + d(x)\varphi = 0 & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{6.2}$$

Proof. First, assume the existence of a positive solution φ of (6.2), as in the statement of the lemma. Let $\xi \in C^1(\overline{\mathbb{R}^{n+1}_+})$ have compact support in $\overline{\mathbb{R}^{n+1}_+}$. We multiply $L_a\varphi = 0$ by ξ^2/φ, integrate by parts and use the Cauchy-Schwarz inequality to obtain (6.1).

For the other implication, we follow [10]. Assume that (6.1) holds for every $\xi \in C^1(\overline{\mathbb{R}^{n+1}_+})$ with compact support in $\overline{\mathbb{R}^{n+1}_+}$. For every $R > 0$, let λ_R be the infimum of the quadratic form

$$Q_R(\xi) = \int_{B^+_R} y^a |\nabla \xi|^2 + \int_{\Gamma^0_R} d(x)\xi^2 \tag{6.3}$$

among functions in the class S_R, defined by

$$\begin{aligned} S_R &= \left\{ \xi \in H^1(B^+_R, y^a) \,:\, \xi \equiv 0 \text{ on } \Gamma^+_R \text{ and } \textstyle\int_{\Gamma^0_R} \xi^2 = 1 \right\} \\ &\subset H_{0,a}(B^+_R) = \left\{ \xi \in H^1(B^+_R, y^a) \,:\, \xi \equiv 0 \text{ on } \Gamma^+_R \right\}. \end{aligned}$$

We recall that the space $H_{0,a}(B^+_R)$ was already defined in (4.4).

By our assumption, $\lambda_R \geq 0$ for every R. By definition it is clear that λ_R is a nonincreasing function of R. Next, we show that λ_R is indeed a decreasing function of R. As a consequence, we deduce that $\lambda_R > 0$ for every R, and this will be important in the sequel.

To show that λ_R is decreasing in R, note first that since d is assumed to be a bounded function, the functional Q_R is bounded below in the class S_R. For the same reason, any minimizing sequence (ξ_k) has $(\nabla \xi_k)$ uniformly bounded in $L^2(B^+_R, y^a)$. Hence, by the compact inclusion $H_{0,a}(B^+_R) \Subset L^2(\Gamma^0_R)$ (already mentioned in the proof of Lemma 4.1), we conclude that the infimum of Q_R in S_R is achieved by a function $\phi_R \in S_R$.

Moreover, we may take $\phi_R \geq 0$, since $|\phi|$ is a minimizer whenever ϕ is a minimizer. Note that $\phi_R \geq 0$ is a solution, not identically zero, of

$$\begin{cases} L_a\phi_R = 0 & \text{in } B^+_R, \\ \dfrac{\partial \phi_R}{\partial \nu^a} + d(x)\phi_R = \lambda_R \phi_R & \text{on } \Gamma^0_R, \\ \phi_R = 0 & \text{on } \Gamma^+_R. \end{cases}$$

It follows from the strong maximum principle that $\phi_R > 0$ in B^+_R.

We can now easily prove that λ_R is decreasing in R. Indeed, arguing by contradiction, assume that $R_1 < R_2$ and $\lambda_{R_1} = \lambda_{R_2}$. Multiply $L_a\phi_{R_1} = 0$ by ϕ_{R_2}, integrate by parts, use the equalities satisfied by ϕ_{R_1} and ϕ_{R_2}, and also the assumption $\lambda_{R_1} = \lambda_{R_2}$. We obtain

$$\int_{\Gamma^+_{R_1}} \frac{\partial \phi_{R_1}}{\partial \nu^a} \phi_{R_2} = 0,$$

and this is a contradiction since, on $\Gamma^+_{R_1}$, we have $\phi_{R_2} > 0$ and the derivative $\partial\phi_{R_1}/\partial\nu^a < 0$.

Next, using that $\lambda_R > 0$ we obtain

$$\int_{B_R^+} y^a|\nabla\xi|^2 + \int_{\Gamma_R^0} d(x)\xi^2 \geq \lambda_R \int_{\Gamma_R^0} \xi^2 \geq -\delta_R \int_{\Gamma_R^0} d(x)\xi^2,$$

for all $\xi \in H_{0,a}(B_R^+)$, where δ_R is taken such that $0 < \delta_R \leq \lambda_R/\|d\|_{L^\infty}$. From the last inequality, we deduce that

$$\int_{B_R^+} y^a|\nabla\xi|^2 + \int_{\Gamma_R^0} d(x)\xi^2 \geq \varepsilon_R \int_{B_R^+} y^a|\nabla\xi|^2 \tag{6.4}$$

for all $\xi \in H_{0,a}(B_R^+)$, for $\varepsilon_R > 0$ given by $\varepsilon_R = 1 - 1/(1+\delta_R)$.

It is now easy to prove that, for every constant $c_R > 0$, there exists a solution φ_R of

$$\begin{cases} L_a\varphi_R = 0 & \text{in } B_R^+, \\ \dfrac{\partial\varphi_R}{\partial\nu^a} + d(x)\varphi_R = 0 & \text{on } \Gamma_R^0, \\ \varphi_R = c_R & \text{on } \Gamma_R^+. \end{cases} \tag{6.5}$$

Indeed, rewriting this problem for the function $\psi_R = \varphi_R - c_R$, we need to solve

$$\begin{cases} L_a\psi_R = 0 & \text{in } B_R^+, \\ \dfrac{\partial\psi_R}{\partial\nu^a} + d(x)\psi_R + c_R d(x) = 0 & \text{on } \Gamma_R^0, \\ \psi_R = 0 & \text{on } \Gamma_R^+. \end{cases}$$

This problem can be solved by minimizing the functional

$$\int_{B_R^+} \frac{y^a}{2}|\nabla\xi|^2 + \int_{\Gamma_R^0} \left\{\frac{1}{2}d(x)\xi^2 + c_R d(x)\xi\right\}$$

in the space $H_{0,a}(B_R^+)$. Note that the functional is bounded below and coercive, thanks to inequality (6.4). Finally, the compact inclusion $H_{0,a}(B_R^+) \Subset L^2(\Gamma_R^0)$ gives the existence of a minimizer.

Next, we claim that

$$\varphi_R > 0 \quad \text{in } \overline{B_R^+}.$$

Indeed, the negative part φ_R^- of φ_R vanishes on Γ_R^+. Using this, (6.5), and the definition (6.3) of Q_R, it is easy to verify that $Q_R(\varphi_R^-) = 0$. By definition of the first eigenvalue λ_R and the fact that $\lambda_R > 0$, this implies that $\varphi_R^- \equiv 0$, i.e., $\varphi_R \geq 0$. Now, Hopf's maximum principle (Corollary 4.12 of [9]) gives $\varphi_R > 0$ up to the boundary.

Finally, we choose the constant $c_R > 0$ in (6.5) to have $\varphi_R(0,0) = 1$. Then, by the Harnack inequality in Lemma 4.9 of [9] applied to φ_S with $S > 4R$, we deduce that

$$\sup_{B_R^+} \varphi_S \leq C_R \quad \text{for all } S > 4R.$$

Now that (φ_S) is uniformly bounded in B_R^+, we use (2.2) in Lemma 2.2 to get a uniform $C^\beta(\overline{B_{R/2}^+})$ bound for the sequence. Note that the constant C_R^1 in (2.2) depends on the L^∞ (and not on the C^σ) of $d\varphi_S$, which we already controlled. However, to apply Lemma 2.2 we need to know that $d\varphi_S$ is C^σ. This is a consequence of the linear problem solved by φ_S and the fact that $d\varphi_S \in L^\infty$. This leads to $\varphi_S \in C^\sigma$ as shown in the beginning of the proof of Lemma 4.5 of [9].

Now, the uniform $C^\beta(\overline{B^+_{R/2}})$ bound gives that a subsequence of (φ_S) converges locally in $\overline{\mathbb{R}^{n+1}_+}$ to a $C^\beta_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ solution $\varphi > 0$ of (6.2). $\square$

The previous lemma provides a direct proof of the fact that every layer solution u of (1.1) is stable, which was already known by the local minimality property established in section 5. Indeed, we simply note that $\varphi = u_{x_1}$ is strictly positive and solves the linearized problem (6.2), with $d(x) = -(1+a)^{-1} f'(u(x,0))$. Hence, the stability of u follows from Lemma 6.1.

We now use the previous lemma to establish a result that leads easily to the monotonicity and the 1D symmetry of stable solutions in dimensions $n = 1$ and $n = 2$, respectively.

Lemma 6.2. *Assume that $n \le 2$ and that u is a bounded stable solution of* (1.3). *Then, there exists a Hölder continuous function $\varphi > 0$ in $\overline{\mathbb{R}^{n+1}_+}$ such that, for every $i = 1, \dots, n$,*

$$u_{x_i} = c_i \varphi \quad \text{in } \mathbb{R}^{n+1}_+$$

for some constant c_i.

Proof. Since u is assumed to be a stable solution, then (6.1) holds with $d(x) := -(1+a)^{-1} f'(u(x,0))$. Note that $d \in C^\beta$ by Lemma 2.2. Hence, by Lemma 6.1, there exists a Hölder continuous function $\varphi > 0$ in $\overline{\mathbb{R}^{n+1}_+}$ such that

$$\begin{cases} L_a \varphi = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial \varphi}{\partial \nu^a} - (1+a)^{-1} f'(u(x,0))\varphi = 0 & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

For $i = 1, \dots, n$ fixed, consider the function

$$\sigma = \frac{u_{x_i}}{\varphi}.$$

The goal is to prove that σ is constant in $\mathbb{R}^{n+1}_+$.

Note first that

$$\varphi^2 \nabla \sigma = \varphi \nabla u_{x_i} - u_{x_i} \nabla \varphi.$$

Thus, we have that

$$\operatorname{div}(y^a \varphi^2 \nabla \sigma) = 0 \quad \text{in } \mathbb{R}^{n+1}_+.$$

Moreover, we have $\frac{\partial \sigma}{\partial \nu^a} = 0$ on $\partial\mathbb{R}^{n+1}_+$ since

$$\varphi^2 \sigma_y = \varphi u_{yx_i} - u_{x_i} \varphi_y = 0,$$

due to the fact that u_{x_i} and φ both satisfy the same linearized boundary condition.

We can use the Liouville property that we established in [9] (Theorem 4.10 of [9]), and deduce that σ is constant, provided that the growth condition

$$\int_{B^+_R} y^a (\varphi\sigma)^2 \le C R^2 \qquad \text{for all } R > 1 \tag{6.6}$$

holds for some constant C independent of R. But note that $\varphi\sigma = u_{x_i}$, and therefore

$$\int_{B^+_R} y^a (\varphi\sigma)^2 \le \int_{B^+_R} y^a |\nabla u|^2.$$

Thus, we need to estimate this last quantity.

To do this, we perform a simple energy estimate. Multiply the equation

$$\operatorname{div}(y^a \nabla u) = 0$$

by $\xi^2 u$ and integrate in B^+_{2R}, where $0 \le \xi \le 1$ is a C^∞ cutoff function with compact support in B_{2R} such that $\xi \equiv 1$ in B_R and $|\nabla \xi| \le 2/R$. We obtain

$$\int_{B^+_{2R}} y^a \left\{\xi^2 |\nabla u|^2 + 2\xi u \nabla \xi \cdot \nabla u\right\} = \int_{\Gamma^0_{2R}} (1+a)^{-1} f(u) \xi^2 u.$$

Thus, by the Cauchy-Schwarz inequality and since u and ξ are bounded,

$$\int_{B^+_{2R}} y^a\, \xi^2 |\nabla u|^2 \le \frac{1}{2} \int_{B^+_{2R}} y^a\, \xi^2 |\nabla u|^2 + C \int_{B^+_{2R}} y^a |\nabla \xi|^2 + C|\Gamma^0_{2R}|$$

for a constant C independent of R. Absorbing the first term on the left hand side, using that $\xi \equiv 1$ in B_R and $|\nabla \xi| \le 2/R$, and computing $\int_0^{2R} y^a\, dy$, we deduce that

$$\int_{B^+_R} y^a\, |\nabla u|^2 \le C\{R^{-2} R^n R^{1+a} + R^n\} = C\{R^{n-2s} + R^n\} \le CR^2$$

since $n \le 2$. This establishes (6.6) and finishes the proof. □

We can now give the

Proof of Theorem 2.12. Let $n = 2$. The extension u of v is a bounded stable solution of (1.3) with f replaced by $(1+a)d_s^{-1} f$.

Lemma 6.2 establishes that $u_{x_i} \equiv c_i \varphi$ for some constants c_i, for $i = 1, 2$. If $c_1 = c_2 = 0$, then u is a constant. Otherwise we have that $c_2 u_{x_1} - c_1 u_{x_2} \equiv 0$ and we conclude that u depends only on y and on the variable parallel to $(0, c_1, c_2)$. That is,

$$u(x_1, x_2, y) = u_0\left((c_1 x_1 + c_2 x_2)/(c_1^2 + c_2^2)^{1/2}, y\right) = u_0(z, y),$$

where z denotes the variable parallel to $(0, c_1, c_2)$. We have that u_0 is a solution of the same nonlinear problem now for $n = 1$ thanks to the extension characterization; recall that the constant d_s in (1.4) does not depend on the dimension.

In particular $\partial_x u_0 = (c_1^2 + c_2^2)^{1/2} \varphi$, and hence $\partial_x u_0 > 0$ everywhere. This finishes the proof of the theorem. □

7. Layer solutions in $\mathbb{R}$

This section is devoted to the case $n = 1$. The Modica estimate that we proved in [9] (see Theorems 1.1 and 1.2 above) gave that

$$G > G(-1) = G(1) \text{ in } (-1, 1)$$

is a necessary condition for the existence of a layer solution in $\mathbb{R}$. Note the strict inequality in $G > G(\pm 1)$.

The rest of the section is dedicated to proving the existence of a layer solution under the above condition on G, in addition to $G'(-1) = G'(1) = 0$, as stated in Theorem 2.4. The existence part of Theorem 2.4 is entirely contained in the following lemma.

Lemma 7.1. *Assume that $n = 1$, and that*

$$G'(-1) = G'(1) = 0 \quad \text{and} \quad G > G(-1) = G(1) \ \text{ in } (-1,1).$$

Then, for every $R > 0$, there exists a function $u_R \in C^\beta(\overline{B_R^+})$ for some $\beta \in (0,1)$ independent of R, such that

$$-1 < u_R < 1 \quad \text{in } \overline{B_R^+},$$
$$u_R(0,0) = 0,$$
$$\partial_x u_R \geq 0 \quad \text{in } B_R^+,$$

and u_R is a minimizer of the energy in B_R^+, in the sense that

$$E_{B_R^+}(u_R) \leq E_{B_R^+}(u_R + \psi)$$

for every $\psi \in C^1(\overline{B_R^+})$ with compact support in $B_R^+ \cup \Gamma_R^0$ and such that $-1 \leq u_R + \psi \leq 1$ in B_R^+.

Moreover, as a consequence of the previous statements, we will deduce that a subsequence of (u_R) converges in $C^\beta_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ to a layer solution u of (1.3).

Proof. For $R > 1$, let

$$Q_R^+ = (-R, R) \times (0, R^{1/8}).$$

Consider the function

$$v^R(x,y) = v^R(x) = \frac{\arctan x}{\arctan R} \quad \text{for } (x,y) \in \overline{Q_R^+}.$$

Note that $-1 \leq v^R \leq 1$ in Q_R^+.

Let u^R be an absolute minimizer of $E_{Q_R^+}$ in the set of functions $v \in H^1(Q_R^+, y^a)$ such that $|v| \leq 1$ in Q_R^+ and $v \equiv v^R$ in $\partial^+ Q_R^+$ in the weak sense. Since we are assuming $G'(-1) = G'(1) = 0$, the existence of such a minimizer was proved in Lemma 4.1. We have that u^R is a weak solution of

$$\begin{cases} L_a u^R = 0 & \text{in } Q_R^+, \\ (1+a)\dfrac{\partial u^R}{\partial \nu^a} = f(u^R) & \text{on } \partial^0 Q_R^+, \\ u^R = v^R & \text{on } \partial^+ Q_R^+, \end{cases}$$

and, by the strong maximum principle and Hopf's lemma (Corollary 4.12 of [9]),

$$|u^R| < 1 \quad \text{in } \overline{Q_R^+}.$$

The function u^R is Hölder continuous by Lemma 2.2.

We follow the method developed in [10] and proceed in three steps. First we show:

$$\text{Claim 1:} \quad E_{Q_R^+}(u^R) \leq C R^{1/4} \tag{7.1}$$

for some constant C independent of R. Here we take $G - G(-1) = G - G(1)$ as a boundary energy potential. We will use this energy bound to prove in a second step that, for R large enough,

$$\text{Claim 2:} \quad |\{u^R(\cdot,0) > 1/2\}| \geq R^{3/4} \text{ and } |\{u^R(\cdot,0) < -1/2\}| \geq R^{3/4}. \tag{7.2}$$

Finally, in a third step independent of the two previous ones, we prove that

$$\text{Claim 3:} \quad u_x^R = \partial_x u^R \geq 0 \quad \text{in } Q_R^+. \tag{7.3}$$

With the above three claims, we can easily finish the proof of the lemma, as follows. Since $u^R(\cdot,0)$ is nondecreasing (here, this is a key point) and continuous in $(-R,R)$, we deduce from (7.2) that for R large enough,

$$u^R(x_R,0)=0 \quad \text{for some } x_R \text{ such that } |x_R|\le R-R^{3/4}.$$

Since $|x_R|\le R-R^{3/4}<R-R^{1/8}$, we have that

$$\overline{B^+_{R^{1/8}}}(x_R,0)\subset(-R,R)\times[0,R^{1/8}]\subset\overline{Q^+_R}.$$

We slide u^R and define

$$u_{R^{1/8}}(x,y)=u^R(x+x_R,y) \quad \text{for} \quad (x,y)\in\overline{B^+_{R^{1/8}}}(0,0).$$

Then, relabeling the index by setting $S=R^{1/8}$, we have that $u_S\in C^\beta(\overline{B^+_S}(0,0))$, $-1<u_S<1$ in $\overline{B^+_S}(0,0)$, $u_S(0,0)=0$, and $\partial_x u_S\ge 0$ in $B^+_S(0,0)$. Moreover, u_S is a minimizer in $B^+_S(0,0)$ in the sense of Lemma 7.1. This follows from extending a given H^1 function ψ with compact support in $(B^+_S\cup\Gamma^0_S)(x_R,0)$, and with $|u+\psi|\le 1$ in $B^+_S(x_R,0)$, by zero in $Q^+_R\setminus B^+_S(x_R,0)$. Hence ψ is a $H^1(Q^+_R)$ function. Then one uses the minimality of u^R in Q^+_R and the fact that the energies of u^R and $u^R+\psi$ coincide in $Q^+_R\setminus B^+_S(x_R,0)$ to deduce the desired relation between the energies in $B^+_S(x_R,0)$.

Now we prove the last statement of the lemma: a subsequence of (u_R) converges to a layer solution. Note that we use the sequence (u_R) just constructed, and not the sequence (u^R) in the beginning of the proof.

Let $S>0$. Since $|u_R|<1$, Lemma 2.2 gives $C^\beta(\overline{B^+_S})$ estimates for u_R, uniform for $R\ge 2S$. Hence, for a subsequence (that we still denote by u_R), we have that u_R converges locally uniformly as $R\to\infty$ to some function $u\in C^\beta_{\rm loc}(\overline{\mathbb{R}^2_+})$. By the additional bound (2.3) on $y^a u_y$ given by Lemma 2.2, one can pass to the limit in the weak formulation and u weakly solves (1.3).

We also have that $|u|\le 1$,

$$u(0,0)=0 \quad \text{and} \quad u_x\ge 0 \text{ in } \mathbb{R}^2_+.$$

Since $u(0,0)=0$, we have $|u|\not\equiv 1$ and hence $|u|<1$ in $\overline{\mathbb{R}^2_+}$, by the strong maximum principle and Hopf's lemma. Note that ±1 are solutions of the problem since, by hypothesis, $G'(\pm1)=f(\pm1)=0$.

Let us now show that u is a local minimizer relative to perturbations in $[-1,1]$. Indeed, let $S>0$ and ψ be a C^1 function with compact support in $B^+_S\cup\Gamma^0_S$ and such that $|u+\psi|\le 1$ in B^+_S. Extend ψ to be identically zero outside B^+_S, so that $\psi\in H^1_{\rm loc}(\overline{\mathbb{R}^2_+})$. Note that, since $-1<u<1$ and $-1\le u+\psi\le 1$, we have $-1<u+(1-\varepsilon)\psi<1$ in $\overline{B^+_S}$ for every $0<\varepsilon<1$. Hence, by the local convergence of (u_R) towards u, for R large enough we have $B^+_S\subset B^+_R$ and $-1\le u_R+(1-\varepsilon)\psi\le 1$ in B^+_S, and hence also in B^+_R. Then, since u_R is a minimizer in B^+_R, we have $E_{B^+_R}(u_R)\le E_{B^+_R}(u_R+(1-\varepsilon)\psi)$ for R large. Since ψ has support in $B^+_S\cup\Gamma^0_S$, this is equivalent to

$$E_{B^+_S}(u_R)\le E_{B^+_S}(u_R+(1-\varepsilon)\psi) \quad \text{for } R \text{ large}.$$

Letting $R\to\infty$, we deduce that $E_{B^+_S}(u)\le E_{B^+_S}(u+(1-\varepsilon)\psi)$. We conclude now by letting $\varepsilon\to 0$.

Finally, since $u_x \geq 0$, the limits $L^{\pm} = \lim_{x\to\pm\infty} u(x,0)$ exist. To establish that u is a layer solution, it remains only to prove that $L^{\pm} = \pm 1$. For this, note that we can apply Proposition 5.2 to u, a local minimizer relative to perturbations in $[-1,1]$, and deduce that

$$G \geq G(L^-) = G(L^+) \quad \text{in } [-1,1].$$

Since in addition $G > G(-1) = G(1)$ in $(-1,1)$ by hypothesis, we infer that $|L^{\pm}| = 1$. But $u(0,0) = 0$ and thus u cannot be identically 1 or -1. We conclude that $L^- = -1$ and $L^+ = 1$, and therefore u is a layer solution.

We now go back to the functions u^R defined in the beginning of the proof, and proceed to establish the three claims made above.

Step 1. Here we prove (7.1) for some constant C independent of R. We take $G - G(-1) = G - G(1)$ as a boundary energy potential.

Since $E_{Q_R^+}(u^R) \leq E_{Q_R^+}(v^R)$, we simply need to bound the energy of v^R. We have

$$|\nabla v^R| = |\partial_x v^R| = \frac{1}{\arctan R}\,\frac{1}{1+x^2} \leq C\frac{1}{1+x^2},$$

and hence

$$\int_{Q_R^+} y^a |\nabla v^R|^2 \leq C R^{\frac{1+a}{8}} \int_{-R}^{R} \frac{dx}{(1+x^2)^2} \leq C R^{1/4},$$

since $0 < 1 + a < 2$.

Next, since $G \in C^{2,\gamma}$, $G'(-1) = G'(1) = 0$ and $G(-1) = G(1)$, we have that

$$G(s) - G(1) \leq C(1 + \cos(\pi s)) \quad \text{for all } s \in [-1,1],$$

for some constant $C > 0$. Therefore, using that $\pi/\arctan R > 2$, we have

$$\begin{aligned} G(v^R(x,0)) - G(1) &\leq C\left\{1 + \cos\left(\pi\frac{\arctan x}{\arctan R}\right)\right\} \\ &\leq C\big(1 + \cos(2\arctan x)\big) \\ &= C2\cos^2(\arctan x) = \frac{2C}{1+x^2}. \end{aligned}$$

We conclude that

$$\int_{-R}^{R} \{G(v^R(x,0)) - G(1)\}\,dx \leq C\int_{-R}^{R} \frac{dx}{1+x^2} \leq C.$$

This, together with the above bound for the Dirichlet energy, proves (7.1).

Step 2. Here we prove (7.2) for R large enough.

Since $u^R \equiv v^R$ on $\{y = R^{1/8}\}$ and $\int_{-R}^{R} v^R(x)\,dx = 0$, we have

$$\int_{-R}^{R} u^R(x,0)\,dx = \int_{-R}^{R} u^R(x,0)\,dx - \int_{-R}^{R} u^R(x,R^{1/8})\,dx = -\int_{Q_R^+} u_y^R.$$

The energy bound (7.1) and the hypothesis that $G - G(1) \geq 0$ give that the Dirichlet energy alone also satisfies the bound in (7.1). We use this together with the previous

equality and Cauchy-Schwarz inequality (writing $|u_y^R| = y^{-a/2}y^{a/2}|u_y^R|$), to deduce

$$\begin{aligned}\Big|\int_{-R}^{R} u^R(x,0)dx\Big| &\le \int_{Q_R^+} |u_y^R| \le \Big\{\int_{Q_R^+} y^{-a}\cdot\int_{Q_R^+} y^a|\nabla u^R|^2\Big\}^{1/2} \\ &\le C\Big\{RR^{(1-a)/8}R^{1/4}\Big\}^{1/2} \le CR^{3/4},\end{aligned} \tag{7.4}$$

since $0 < 1-a < 2$.

Next, by (7.1) we know that $\int_{-R}^{R}\{G(u^R(x,0))-G(1)\}\,dx \le CR^{1/4} \le CR^{3/4}$. On the other hand, $G(s)-G(1) \ge \varepsilon > 0$ if $s \in [-1/2, 1/2]$, for some $\varepsilon > 0$ independent of R. Moreover, $G - G(1) \ge 0$ in $(-1,1)$. We deduce

$$\varepsilon\big|\{|u^R(\cdot,0)| \le 1/2\}\big| \le \int_{-R}^{R}\{G(u^R(x,0)) - G(1)\}\,dx \le CR^{3/4},$$

and therefore $\big|\{|u^R(\cdot,0)| \le 1/2\}\big| \le CR^{3/4}$. This combined with (7.4) leads to

$$\Big|\int_{(-R,R)\cap\{|u^R(\cdot,0)|>1/2\}} u^R(x,0)\,dx\Big| \le CR^{3/4}. \tag{7.5}$$

We claim that

$$|\{u^R(\cdot,0) > 1/2\}| \ge R^{3/4} \quad \text{for } R \text{ large enough.}$$

Suppose not. Then, using (7.5) and $|\{u^R(\cdot,0) > 1/2\}| \le R^{3/4}$, we obtain

$$\frac{1}{2}|\{u^R(\cdot,0) < -1/2\}| \le \Big|\int_{(-R,R)\cap\{u^R(\cdot,0)<-1/2\}} u^R(x,0)\,dx\Big| \le CR^{3/4}.$$

Hence, all the three sets $\{|u^R(\cdot,0)| \le 1/2\}$, $\{u^R(\cdot,0) > 1/2\}$, and $\{u^R(\cdot,0) < -1/2\}$ would have length smaller than $CR^{3/4}$. This is a contradiction for R large, since these sets fill $(-R,R)$.

Step 3. Here we establish the monotonicity result (7.3). This is done exactly as in Step 3 in the proof in [10], to which we refer. One simply uses the sliding method with the aid of the Hopf boundary lemma of [9]. □

Proof of Theorem 2.4. The necessary conditions on G follow from our previous paper [9]; see Theorem 1.2 above.

That the conditions are sufficient for the existence of a layer $v = v(x)$ follows from Lemma 7.1, which gives a layer solution $u = u(x,y)$ of the corresponding nonlinear extension problem (1.3), and then by taking $v := u(\cdot,0)$. Note that we consider the extension problem with f replaced by $(1+a)d_s^{-1}f$ due to the relation (1.4) between the fractional Laplacian and the Neumann derivative.

Finally, the proof of the uniqueness result follows exactly that of Lemma 5.2 in [10] for the half-Laplacian. It uses the sliding method combined with the maximum principle Lemma 4.13 and Remark 4.14 in our previous paper [9]. □

Proof of Theorem 2.9. The proof is identical to that of Proposition 6.1 in [10], page 1727. □

References

[1] Robert A. Adams, *Sobolev spaces*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65. MR0450957 (56 #9247)

[2] Giovanni Alberti, Luigi Ambrosio, and Xavier Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math. **65** (2001), no. 1-3, 9–33, DOI 10.1023/A:1010602715526. Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday. MR1843784 (2002f:35080)

[3] Luigi Ambrosio and Xavier Cabré, *Entire solutions of semilinear elliptic equations in* $\mathbf{R}^3$ *and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13** (2000), no. 4, 725–739 (electronic), DOI 10.1090/S0894-0347-00-00345-3. MR1775735 (2001g:35064)

[4] Martin T. Barlow, Richard F. Bass, and Changfeng Gui, *The Liouville property and a conjecture of De Giorgi*, Comm. Pure Appl. Math. **53** (2000), no. 8, 1007–1038, DOI 10.1002/1097-0312(200008)53:8<1007::AID-CPA3>3.3.CO;2-L. MR1755949 (2001m:35095)

[5] H. Berestycki, F. Hamel, and R. Monneau, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. **103** (2000), no. 3, 375–396, DOI 10.1215/S0012-7094-00-10331-6. MR1763653 (2001j:35069)

[6] Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR1406564 (98e:60117)

[7] Xavier Cabré and Eleonora Cinti, *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*, Discrete Contin. Dyn. Syst. **28** (2010), no. 3, 1179–1206, DOI 10.3934/dcds.2010.28.1179. MR2644786 (2011c:35139)

[8] Xavier Cabré and Eleonora Cinti, *Sharp energy estimates for nonlinear fractional diffusion equations*, Calc. Var. Partial Differential Equations **49** (2012), no. 1–2, 233–269. MR3148114

[9] Xavier Cabré and Yannick Sire, *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), no. 1, 23–53. MR3165278

[10] Xavier Cabré and Joan Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. **58** (2005), no. 12, 1678–1732, DOI 10.1002/cpa.20093. MR2177165 (2006i:35116)

[11] Luis Caffarelli and Luis Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260, DOI 10.1080/03605300600987306. MR2354493 (2009k:35096)

[12] Alberto Farina, *Symmetry for solutions of semilinear elliptic equations in* $\mathbf{R}^N$ *and related conjectures*, Ricerche Mat. **48** (1999), no. suppl., 129–154. Papers in memory of Ennio De Giorgi (Italian). MR1765681 (2001h:35056)

[13] Alberto Farina, *Propriétés qualitatives de solutions d'équations et systèmes d'équations non-linéaires*, 2002. Habilitation à diriger des recherches, Paris VI.

[14] Rupert L. Frank and Enno Lenzmann, *Uniqueness of non-linear ground states for fractional Laplacians in* $\mathbb{R}$, Acta Math. **210** (2013), no. 2, 261–318, DOI 10.1007/s11511-013-0095-9. MR3070568

[15] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311** (1998), no. 3, 481–491, DOI 10.1007/s002080050196. MR1637919 (99j:35049)

[16] Vassili Kolokoltsov, *Symmetric stable laws and stable-like jump-diffusions*, Proc. London Math. Soc. (3) **80** (2000), no. 3, 725–768, DOI 10.1112/S0024611500012314. MR1744782 (2001f:60016)

[17] Aleš Nekvinda, *Characterization of traces of the weighted Sobolev space* $W^{1,p}(\Omega, d_M^\varepsilon)$ *on* M, Czechoslovak Math. J. **43(118)** (1993), no. 4, 695–711. MR1258430 (95i:46041)

[18] Giampiero Palatucci, Ovidiu Savin, and Enrico Valdinoci, *Local and global minimizers for a variational energy involving a fractional norm*, Ann. Mat. Pura Appl. (4) **192** (2013), no. 4, 673–718. MR3081641

[19] George Pólya, *On the zeros of an integral function represented by Fourier's integral*, Messenger of Math. **52** (1923), 185–188.

[20] Ovidiu Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math. (2) **169** (2009), no. 1, 41–78, DOI 10.4007/annals.2009.169.41. MR2480601 (2009m:58025)

[21] Yannick Sire and Enrico Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (2009), no. 6, 1842–1864, DOI 10.1016/j.jfa.2009.01.020. MR2498561 (2010c:35201)

[22] Yannick Sire and Enrico Valdinoci, *Rigidity results for some boundary quasilinear phase transitions*, Comm. Partial Differential Equations **34** (2009), no. 7-9, 765–784, DOI 10.1080/03605300902892402. MR2560300 (2010k:35161)

[23] J. F. Toland, *The Peierls-Nabarro and Benjamin-Ono equations*, J. Funct. Anal. **145** (1997), no. 1, 136–150, DOI 10.1006/jfan.1996.3016. MR1442163 (97m:35235)

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