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LINEAR INDEPENDENCE OF POINCARÉ SERIES OF EXPONENTIAL TYPE VIA NON-ANALYTIC METHODS

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Abstract. Given a finite set $\{T\}$ of symmetric, positive definite, half-integral n by n matrices over $\mathbf{Z}$ which are inequivalent under the action of $\mathrm{GL}(n, \mathbf{Z})$, we show that the corresponding set of Poincaré series $\{P_k^n(T)\}$ attached to them are linearly independent for weights k in infinitely many arithmetic progressions. We also give a quite explicit description of those arithmetic progressions for all even degrees, when the matrices T have no improper automorphisms and their level is an odd prime. Our main tools are theta series with simple harmonic polynomials as coefficients and techniques familiar from the theory of modular forms mod p.

1. Introduction

The space of Siegel cusp forms S_k^n of weight k and degree n is spanned by the collection of Poincaré series $\{P_k^n(T)\}$ indexed by symmetric, positive definite, half-integral n by n matrices T over $\mathbf{Z}$ (denoted as Λ_n^+). They are characterized as the unique cusp forms (up to scalars) which represent the linear functional (with respect to the Petersson inner product on S_k^n) obtained by mapping a cusp form f to its T-th Fourier coefficient $a(f, T)$. It is a pertinent question to decide which of these Poincaré series might be identically zero and to determine linear relations among them. In fact, one of our early motivations for these questions were the three open problems on Poincaré series mentioned in [18, §3].

The question of non-vanishing is typically investigated by analytic methods, e.g. by analyzing the Fourier expansion of such Poincaré series (see [12, 13, 27]). For example, in [12, Satz 6], U. Christian proved that there is a positive constant $\kappa_n(T)$ such that $P_k^n(T)$ does not vanish identically for all $k \geq \kappa_n(T)$. However, the dependence on T in $\kappa_n(T)$ was *unspecified.* On the other hand, in [13] it was proved that uniformly for all large weights k and T with $\det T \ll_\varepsilon k^{2-\varepsilon}$ and the "content"of $T \ll k^{2/3-\varepsilon}$, the $P_k^2(T)$ does not vanish identically. The question on linear relations is more delicate, and only a few results are available in the literature; see [11, 14] for the case of degree 1. The only instance (to our knowledge) in the case of higher degrees where this problem has been treated is in [5], by looking at the asymptotic properties of certain generating series involving Poincaré series. However, by the methods of [5], one can only prove the existence of infinitely many *unspecified* weights for which a family $\{P_k^n(T)\}$ is linearly independent with inequivalent indices.

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In this paper we look at the merits and limits of using theta series (with simple harmonic polynomials as coefficients) to construct Siegel cusp forms with non-vanishing properties for certain Fourier coefficients. We are concerned with Siegel cusp forms for the full modular group for arbitrary degree n. Our main tools will be theta series (with level) introduced by Maaß [25] and techniques of level changing, familiar from the theory of modular forms modulo p. The information about the Fourier coefficients of the cusp forms constructed then translates into linear independence of Poincaré series. We mention in this context that non-analytic methods were also used in [15] (degree 2) on the non-vanishing problem via Maaß lifts.

After recalling some prerequisites in section 2, we consider in section 3 cusp forms with prescribed "minimal" non-vanishing Fourier coefficients. The level of these cusp forms will be square-free. To go down to level one, we use congruence properties and the trace map from higher to lower levels; see section 4. Given a set $\{T_1, \dots, T_h\}$ of inequivalent matrices in Λ_n^+, by this method we can construct cusp forms f_i of level one ($1 \leq i \leq h$) such that the matrix $(a(F_i, T_j))_{i,j}$ has full rank, with the weight a certain *explicit* function of levels of the T_i. The main result, which we formulate in terms of Poincaré series, says that the Poincaré series $P_k^n(T_1), \dots, P_k^n(T_h)$ are linearly independent under the same condition on the weight. It is presented in section 5; see Theorem 5.1. Although this theorem is stated for even degree n, a similar statement is true for odd degrees as well; see section 5.2.3. The merit of this theorem is that under suitable conditions, one can give quite explicit bounds on the several parameters involved therein, which in general should be "large enough". We feel that this result might be useful in other situations.

In section 6, we show that the linear independence of Poincaré series of a weight k for a given set $\{T_1, \dots, T_h\}$ of inequivalent matrices in Λ_n^+ directly implies the linear independence of the same set, for weights in infinitely many arithmetic progressions. We again rely on simple properties of congruences between modular forms, invoking a result of [7]. See Proposition 6.1.

In principle, our results are very explicit as long as the level of the T_i in question is a power of a fixed prime. In the final section we explain this in detail in two cases: degree 1 and arbitrary even degree, thus making Theorem 5.1 effective for these cases. In the degree 1 case, we also consider a variant of our method: we argue via the difference of two appropriate binary theta series; see Proposition 7.3. In the even degree case (see Theorem 7.5) we make the assumption that the T_i are of level an odd prime p, and that they do not have improper automorphisms. The weights for which Theorem 7.5 holds are completely explicit. It is expected that Theorem 5.1, Theorem 7.5 applied for $h = 1$ (i.e., considering the non-vanishing problem) and Proposition 7.3 should hold for all weights $k > 2n + 1$ and $k > 3$ respectively.

2. Notation and preliminaries

2.1. **Siegel modular forms.** For basic facts about Siegel modular forms we refer to [2] or [22]. The symplectic group $\mathrm{Sp}(n, \mathbf{R})$ acts on Siegel's half-space in the usual way by $g\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$ and on functions $F : \mathbb{H}_n \longrightarrow \mathbb{C}$ by the "stroke" operator:

$$(F \mid_k g)(Z) = \det(CZ + D)^{-k} F(g\langle Z \rangle) \qquad (g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbf{R})).$$

We will mainly consider modular forms for the congruence subgroups

$$\Gamma_0^n(N) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbf{Z}) \mid C \equiv 0 \bmod N\}.$$

We set $\Gamma_n := \mathrm{Sp}(n, \mathbf{Z})$. For a Dirichlet character χ mod N a holomorphic function f on $\mathbf{H}_n$ is called a modular form for $\Gamma_0^n(N)$ of weight k and nebentypus χ if it satsfies the transformation law

$$f\,|_k\, \gamma = \chi(\gamma) \cdot f \qquad (\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^n(N))$$

with $\chi(\gamma) := \chi(\det(D))$ and with the additional condition of being holomorphic at the cusps when $n = 1$. We denote the space of all such functions by $M_k^n(N, \chi)$, and $S_k^n(N, \chi)$ will be the subspace of cusp forms. An element $f \in M_k^n(N, \chi)$ has a Fourier expansion

$$f(Z) = \sum_{T \in \Lambda_n} a(f, T) \exp(\mathrm{tr}\ TZ),$$

where tr A denotes the trace of the matrix $2\pi i A$. Further, Λ_n is the set of n by n symmetric semi-integral positive semi-definite matrices, and Λ_n^+ denotes the subset of positive definite elements. Sometimes we abbreviate the function $\exp(\mathrm{tr}\ TZ)$ simply by q^T. The space of modular forms with all the Fourier coefficients in a ring $R \subset \mathbf{C}$ is denoted as $M_k^n(N, \chi)(R)$, the corresponding space of cusp forms by $S_k^n(N, \chi)(R)$. For the full modular group, we use the notation $M_k^n(R)$ (resp. $S_k^n(R)$) for the space of modular (resp. cusp) forms with Fourier coefficients in R.

The unimodular group $\mathrm{GL}(n, \mathbf{Z})$ acts on Λ_n^+ by the rule $(U, T) \mapsto T[U] := U^t T U$. We say that S is equivalent (resp. inequivalent) to T if the orbit of S is equal (resp. disjoint) from that of T under this action. We sometimes identify $T \in \Lambda_n^+$ with the quadratic form represented by it.

2.2. **Hecke operators at bad primes.** As usual, we define the Hecke operator $U(L)$ on $M_k^n(N, \chi)$ for $L \mid N$ by

$$f = \sum_T a(f, T) \exp(\mathrm{tr}\ TZ) \longmapsto f \mid U(L) = \sum_T a(f, L \cdot T) \exp(\mathrm{tr}\ TZ).$$

We also need a twisted version of this operator; for a prime p with $p \mid\mid N$ (i.e., p exactly dividing N) we choose a matrix $g \in \mathrm{Sp}(n, \mathbf{Z})$ such that

$$g \equiv 1_{2n} \bmod N/p \quad \text{and} \quad g \equiv \left(\begin{smallmatrix} 0_n & -1_n \\ 1_n & 0_n \end{smallmatrix}\right) \bmod p.$$

Then $f\,|_k\, g$ has a Fourier expansion $f(Z) = \sum_{T \in \frac{1}{p} \cdot \Lambda_n} b(f, T) \exp(\mathrm{tr}\ TZ)$; we put

$$f \mid W_N(p) := \sum_{T \in \Lambda_n} b(f, T) \exp(\mathrm{tr}\ TZ).$$

Then $W_N(p)$ defines a map from $M_k^n(N, \chi)$ to $M_k^n(N, \overline{\chi})$. By the main result of [6], this map is an isomorphism if $p \mid\mid N$.

2.3. **Congruences.** For a prime p we denote by ν_p the usual p-adic valuation on $\mathbf{Q}$, normalized by $\nu_p(p) = 1$; we will tacitly extend it to field extensions of $\mathbf{Q}$ if necessary. A modular form f on an arbitrary congruence subgroup of $\mathrm{Sp}(n, \mathbf{Z})$ has a Fourier expansion

$$f(Z) = \sum_{T \in \Lambda_n} a(f, T) \exp(\mathrm{tr}\ TZ/M)$$

for an appropriate natural number M. We put

$$\nu_p(f) := \inf\{\nu_p(a(f,T)) \mid T \in \Lambda_n\}.$$

Then two modular forms f and g will be called congruent mod p if

$$\nu_p(f-g) > \nu_p(f).$$

2.4. **Characters.** For an imaginary quadratic field K with discriminant D, we denote the quadratic character associated to it by χ_D (as in [29]); it satisfies $\chi_D(p) = \left(\frac{D}{p}\right)$ for a prime $p \nmid D$ (where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol), $\chi_D(-1) = -1$ and is a primitive character $\bmod |D|$.

We recall that the level of $T \in \Lambda_n^+$ is the smallest positive integer satisfying $q^{-1}T \in \Lambda_n^+$. For $T \in \Lambda_n^+$ we define the discriminant disc T by disc $T = (-1)^{n/2}\det T$ if n is even and by $(-1)^{\frac{n-1}{2}}\frac{1}{2}\det T$ if n is odd. Later we would also need the real Dirichlet character χ_T associated with $T \in \Lambda_n^+$ (for n even) defined by

$$\chi_T\left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\right) = \left(\frac{\text{disc } T}{\det D}\right),$$

where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^n(q)$. The conductor of χ_T is equal to the square–free part $\mathrm{sf}(|\text{disc } T|)$ of $|\text{disc}T|$ if $(-1)^{[n/2]}\mathrm{sf}(|\text{disc } T|) \equiv 1 \bmod 4$ and equal to $4\mathrm{sf}(|\text{disc } T|)$ otherwise. See [3, p. 37], for example. Finally, we denote by χ_T^o the primitive character associated with χ_T.

2.5. **Poincaré series.** We define the Poincaré series of exponential type (see, e.g., [21, 22]) only for the full modular group $\Gamma_n := \mathrm{Sp}(n, \mathbf{Z})$ by

$$P_k^n(T)(Z) := \sum_{\gamma \in \Gamma_{n,\infty}\backslash\Gamma_n} e_T \mid_k \gamma \quad (Z \in \mathbf{H}_n),$$

where $e_T(Z) := \exp(\text{tr } T \cdot Z)$ and $\Gamma_{n,\infty} := \{\pm\left(\begin{smallmatrix} 1_n & B \\ 0_n & 1_n \end{smallmatrix}\right) \mid B = B^t \in \mathbf{Z}^{(n,n)}\}$. These series have nice convergence properties for $k > 2n+1$ and they generate the full space of cusp forms; see [22, p. 90] for example. We remark that these series are equal to zero if both k and n are odd. Using the Petersson inner product $\langle , \rangle$ on S_k^n we have the important formula

$$\langle F, P_k^n(T)\rangle \sim_{n,k} \det(T)^{k-\frac{n+1}{2}} \cdot a(F,T) \tag{2.1}$$

(where $\sim_{n,k}$ means up to a non-zero constant depending only on n and k), valid for all cusp forms F with Fourier coefficients $a(F,T)$, $(T \in \Lambda_n^+)$. By (2.1), one can thus translate questions about non-vanishing of Poincaré series into questions about existence of certain cusp forms with non-vanishing Fourier coefficients and vice versa.

3. T-minimal cusp forms of square–free level

We begin with the notion of the "order" and T-minimality of a cusp form. Let $F = \sum a(F,T)q^T \in S_k^n(M,\chi)$.

Definition 3.1. (i) We define the order of F by

$$\mathrm{ord}(F) := \min\{\det T \mid a(F,T) \neq 0\}.$$

(ii) For $T \in \Lambda_n^+$ with $\det T = \mathrm{ord}(F)$, we call F a T-minimal cusp form if

$$\{S \in \Lambda_n^+ \mid \det(S) = \mathrm{ord}(F), a(F,S) \neq 0\} = \{T[U] \mid U \in \mathrm{GL}(n,\mathbf{Z})\}.$$

We remark here that in general we cannot expect that a cusp form be T-minimal for some T (unless $n = 1$). In fact to our knowledge, the only statement regarding this is supplied by Proposition 3.4.

The next lemma is about the "concavity" of the determinant of positive definite matrices. This notion will be quite useful in dealing with the T-minimality of cusp forms.

Lemma 3.2 (Concavity of determinant)**.** *Let $T_1, \ldots, T_r$ be n by n positive definite matrices and $\lambda_1, \ldots, \lambda_r$ be in the interval $[0, 1]$ such that $\sum_{i=1}^r \lambda_i = 1$. Then*

$$\det(\lambda_1 T_1 + \cdots + \lambda_r T_r) \geq (\det T_1)^{\lambda_1} \cdots (\det T_r)^{\lambda_r},$$

with an equality if and only if all the T_i are equal. In particular,

$$\det(T_1 + \cdots + T_r) > r^n \cdot \min\{\det T_i,\ 1 \leq i \leq r\},$$

unless all the T_i are equal.

Proof. The statement for $r = 2$ follows from [17, Theorem 7.8.8]. The rest follows by a straightforward induction on r. The second part follows from the first by taking all the $\lambda_i = 1/r$. □

We next state a result which shows that the notion of T-minimality is preserved under the U-operator. This allows us to pass to square–free levels.

Proposition 3.3. *Assume that $F \in S_k^n(M, \chi)$ is T-minimal and $r \in \mathbf{N}$. Then all powers $F^r \in S_{kr}^n(M, \chi^r)$ are $r \cdot T$-minimal. In particular, $F^r \mid U(r)$ is again T-minimal; moreover, if $r^2 \mid M$ and $\mathrm{cond}(\chi^r) \mid \frac{M}{r}$, then $F^r \mid U(r)$ is of level $\frac{M}{r}$.*

Proof. Let $d = \mathrm{ord}(F) = \det T$, where F is T-minimal. Consider the Fourier expansion of F^r:

$$F^r(Z) = \sum_{T_1, \ldots, T_r :\ \sum_{i=1}^r T_i = S} a(F, T_1) \cdots a(F, T_r) q^S. \tag{3.1}$$

Suppose that $a(F, S) \neq 0$ for some S. From (3.1) we find an r-tuple $\{T_1, \ldots, T_r\}$ with $\sum_{i=1}^r T_i = S$ such that $a(F, T_i) \cdots a(F, T_r) \neq 0$. Also, $\det T_i \geq d$ for all i. Using Lemma 3.2, we get $\det S > r^n d$ unless all the T_i are equal. Thus if $\det S = \det rT$, we must have $T_i = T_1$ for all i and $S = rT_1$. So, the T-minimality of F implies that of F^r. This proves the first part of the lemma; the assertion about the $U(r)$ operator and minimality is easy to see.

The statement about the level follows from general properties of the U-operator described in [23, Lemma 1], which carry over to Siegel modular forms easily. This completes the proof of the proposition. □

The existence of T-minimal cusp forms of square-free level is asserted by the following result:

Proposition 3.4. *For n even, $T \in \Lambda_n^+$, let M be the level of T. Let D be the discriminant of an imaginary quadratic number field. We decompose $MD^2 = M_0 \cdot N$ with $N \mid M_0$. Then there exists a cusp form $F \in S_k^n(M_0, (\chi_T \chi_D)^N)$ which is T-minimal, where*

$$k = (n/2 + 1) \cdot N \qquad (r \in \mathbf{N}).$$

Moreover, F can be chosen such that all its Fourier coefficients are integers, the coefficient at T being 1.

Proof. The starting point is the theta series

$$\theta^n_{\det}(T,\chi_D)(Z) := \frac{1}{\epsilon(T)} \sum_{X\in \mathbf{Z}^{(n,n)}} \det(X)\chi_D(\det X)\exp(\mathrm{tr}\ X^tTXZ),$$

where

$$\epsilon(T) = \#\{G \in \mathrm{GL}(n,\mathbf{Z}) \mid G^tTG = T\}.$$

Following Maaß [25, Theorem 1] (see also [4, Theorem 3]) this is an element of $S^n_{n/2+1}(MD^2, \chi_T\chi_D)$ with integral Fourier coefficients. The automorphisms of T act on the non-degenerate X without fixed points. Thus we easily see that $a(\theta^n_{\det}(T,\chi_D),T) = 1$. Clearly this shows that $\theta^n_{\det}(T,\chi_D)$ is T-minimal. Then from Proposition 3.3 we see that

$$F := \theta^n_{\det}(T,\chi_D)^N \mid U(N)$$

has all the requested properties, if we observe, that for N even, the character $(\chi_T\chi_D)^N$ is trivial. If N is odd, the condition $N \mid \frac{MD^2}{\mathrm{cond}((\chi_T\chi_D)^N)}$ is satisfied, because the odd part of this conductor is square-free. □

Remark 3.5. The theta series

$$\theta^n_{\det}(T)(Z) = \frac{1}{\epsilon(T)}\sum_X \det(X)\exp(\mathrm{tr}\ X^tTXZ)$$

without the character χ_D is identically zero if and only if T has an automorphism $G \in \mathrm{GL}(n,\mathbf{Z})$ with $\det G = -1$. This is the reason for incorporating the characters χ_D. If T had no improper integral automorphisms, we could have as well worked with $\theta^n_{\det}(T)$.

In the sequel, we will not really need the strong property of being T-minimal; anyway, it will get lost when we go down to level one. The only thing that will be necessary is the following.

Corollary 3.6. *Let $T_1,\dots,T_h$ be pairwise inequivalent matrices of level M in Λ^+_n, n even, with M, D, N as in Proposition* 3.4. *We assume that* $\det T_1 \le \det T_2 \le \cdots \le \det T_h$. *Then there exist cusp forms $F_i \in S^n_k(M_0,(\chi_{T_i}\chi_D)^N)$ having integral Fourier coefficients with weight $k = (\frac{n}{2}+1)N$ such that*

$$\det(a(F_i,T_j)) \neq 0.$$

Proof. We choose the F_i as in Proposition 3.4 above. Then, by the T_i-minimality of the F_i, the matrix $A := (a(F_i,T_j))$ is a unipotent upper triangular integral matrix (note that we have ordered the determinants of the T_i). □

Remark 3.7. In Proposition 3.4 and hence in Corollary 3.6 we can assure that all the nebentypus characters are trivial by enlarging the level M by a factor 4 (by taking the square of the theta series and applying the $U(2)$ operator). This will not really be necessary, but is sometimes convenient. In any case, the conductor of $(\chi_{T_i}\chi_D)^N$ is automatically odd if M/N is square–free.

4. Removing primes from the level

In this section, we start from pairwise inequivalent matrices $T_1, \dots, T_h$ and $F_i \in S_k^n(M,\chi)(\mathbf{Q})$ $(j = 1, \dots, h)$ such that

$$\det(a(F_i, T_j)) \neq 0.$$

Here M will be square-free and χ is a quadratic character (of odd conductor, but not necessarily primitive).

We call this the *Situation* (M, k, χ, T_i). We want to "remove" a prime p from the level M by increasing the weight. We may assume (without loss of generality) that the F_i have integral Fourier coefficients with $\nu_p(F_i) = 0$ and

$$\nu_p(\det(a(F_i), T_j)) = \lambda \text{ with } \lambda \geq 0. \tag{4.1}$$

We decompose the quadratic character as

$$\chi = \chi' \cdot \chi_p,$$

where χ' is a character $\bmod M/p$ and χ_p is either trivial or equal to the quadratic character given by $\left(\frac{(-1)^{\frac{p-1}{2}}p}{*}\right)$.

Let us recall from [7] that a $\mathbf{Z}$-lattice L is called p-special if there exists an automorphism σ of L such that σ is of order p and acts freely on $L \backslash \{\mathbf{0}\}$.

Next, we recall from [9, p. 6] the existence of a modular form $\mathcal{K}_{p-1}$ on $\Gamma_0^n(p)$ with character χ_p of weight

$$\alpha(p) = \begin{cases} p-1 & \text{for} \quad \chi_p = 1,\, p \neq 2, \\ \frac{p-1}{2} & \text{for} \quad \chi_p \neq 1, \\ 2 & \text{for} \quad p = 2. \end{cases}$$

This modular form is congruent to zero modulo p not only for the usual Fourier expansion but also for the expansions in all cusps, except in the cusp $\left(\begin{smallmatrix} 0_n & -1_n \\ 1_n & 0_n \end{smallmatrix}\right)$, where it is congruent 1 mod p.

In fact it is shown in [9] that the modular form

$$\mathcal{K}_{p-1} := (\det S)^{n/2} \theta^n(S)$$

has all the requested properties if S is a p-special lattice. Here $\theta^n(S)$ is the usual theta series associated with the lattice S of size $2p-2$, level p and determinant p^2 (in the case of trivial character) or size $p-1$ (in the case of non-trivial character):

$$\theta^n(S)(Z) := \sum_{X \in \mathbf{Z}^{(n,n)}} \exp\left(\pi i \operatorname{tr} X^t S X \cdot Z\right).$$

The existence of such p-special lattices S as described above is assured by [8, Corollary 1,2] for odd p. For $p = 2$ we may choose for S any quaternary quadratic form of level 2 with determinant 2.

Furthermore, we recall that for a modular form h in $M_l^n(\Gamma_0(M), \chi')$ the trace from $\Gamma_0^n(M)$ to $\Gamma_0^n(M/p)$ is defined by

$$\operatorname{tr}_{M/p}^M(h) = \sum_{\gamma \in \Gamma_0^n(M) \backslash \Gamma_0^n(M/p)} \chi'(\gamma) \cdot h \mid_l \gamma;$$

it defines a modular form of weight l for $\Gamma_0^n(M/p)$ with nebentypus χ'.

We define, for any $\tilde{F}_i \in M_k^n(\Gamma_0(M), \chi)$ $(i = 1, \dots, h)$ and $\beta = \kappa \cdot p^\gamma$,

$$G_i := p^{-\frac{n(n+1)}{2}} \cdot \operatorname{tr}_{M/p}^M(\tilde{F}_i \cdot \mathcal{K}_{p-1}^\beta) \in M_{k'}^n(M/p, \chi')$$

with $k' = k + \alpha(p)\kappa p^\gamma$; here κ has to be odd if χ_p is non-trivial. We can achieve the congruence

$$G_i \equiv F_i \bmod p^{\lambda+1}, \quad \text{for all } i \tag{4.2}$$

if we choose $\tilde{F}_i \in M_k^n(M,\chi)(\mathbf{Q})$ with $\tilde{F}_i \mid W_M(p) = F_i$ and $\beta = \kappa \cdot p^\gamma$ with γ sufficiently large. We refer the reader to [9, p. 7], where a detailed proof of the above fact can be found. The existence of such $\tilde{F}_i$ is assured by the injectivity of the $U(p)$-operator; see [6, 9].

From (4.2) we see that $\det(a(G_i,T_j)) \equiv \det(a(F_i,T_j) \bmod p^{\lambda+1}$ holds, and thus we get in particular (by our choice of λ in (4.1)) that $\det(a(G_i,T_j)) \neq 0$.

We summarize the above discussion into the following. Recall the notion of *Situation* (M,k,χ,T_i) from the beginning of this section and keep the rest of the notation of this section.

Proposition 4.1. *It is possible to pass from Situation (M,k,χ,T_i) to Situation $(M/p,k',\chi',T_i)$ with weight*

$$k' = k + \alpha(p)\kappa p^\gamma$$

when γ is sufficiently large. We may choose κ arbitrarily, but it has to be odd if χ_p is non-trivial.

Remark 4.2. When we iterate this process to remove all primes, there remains the delicate point that we know nothing about the factorization of the new determinant $\det(a(G_i,T_j))$ at the other primes. This is the main obstacle against making this procedure explicit. In the case of prime level, this problem does not occur and our method is quite explicit; see section 7.

5. Linear independence of Poincaré series for level one

5.1. **The main result for level one.** When we sum up the two procedures described in the previous sections, we get the following existence theorem for linearly independent Poincaré series (taking into account (2.1) to pass from Fourier coefficients to Poincaré series).

Theorem 5.1. *Let $T_1,\ldots,T_h$ be mutually inequivalent elements of Λ_n^+ of level M with n even. Let D be the discriminant of an imaginary quadratic number field; if the T_i do not have improper integral automorphisms, we may just put $D=1$. Let $M_0 = \prod_{j=1}^t p_j$ be the square-free part of MD^2 and define N by $MD^2 = M_0 \cdot N$. We assume that N is even (after possibly enlarging the level M, if necessary). Then the Poincaré series $P_k^n(T_i)$ of weight k are linearly independent with*

$$k = (\frac{n}{2}+1)\cdot N + \sum_{j=1}^{t} \alpha(p_i)\kappa_i p_i^{\gamma_i} \tag{5.1}$$

provided that the γ_i are chosen "sufficiently" large.

We have to explain what "sufficiently large" means: We start from the theta series

$$F_i := \frac{1}{\epsilon(T_i)}\theta^n_{\det}(T_i,\chi_D) \in S^n_{\frac{n}{2}+1}(MD^2,\chi_{T_i}\chi_D)$$

and we first go to $G_i := F_i^N \mid U(N)$ as in section 4 . Then we apply the trace procedure to remove the primes $p_1,\ldots,p_t$. We first choose γ_1 sufficiently large according to section 4. Then the condition for γ_2 will depend on κ_1,γ_1, and so on.

Finally, we may choose κ_t, γ_t arbitrarily with a lower bound for γ_t depending on all the previous choices of $\kappa_1, \gamma_1, \ldots, \kappa_{t-1}, \gamma_{t-1}$. We add here that in (5.1), the κ_i all should be equal to $p-1$ (for odd p) or equal to 2 (for $p = 2$).

5.2. **Some variants and remarks.**

5.2.1. *Variants concerning characters.* The method of proof of the theorem above allows several choices for the F_i; they all lead to variants of the theorem with slight modifications (concerning the parity of the κ_i and the explicit shape of the $\alpha(p_i)$); the basic structure of formula (5.1) remains the same in all cases. Under the assumption that all the ${\chi_{T_i}}^o$ are the same, denoted by χ, the following choices are more appropriate:

Choice 1. Assume that $n \equiv 2 \bmod 4$; then $\chi (= \chi_{T_i})$ is odd and we may start from

$$F_i := \frac{1}{\epsilon(T_i)} \theta^n_{\det}(T_i, \chi) \in S^n_{\frac{n}{2}+1}(Mc^2).$$

Here c is the conductor of χ and we have to decompose $M \cdot c^2 = M_0 \cdot N$ with M_0 being the square–free part. For the odd primes dividing N we should then have $\alpha(p) = p - 1$.

Choice 2. Assume that $4 \mid n$. Then $\chi \cdot \chi_D$ has non-trivial conductor c and we may choose

$$F_i := \frac{1}{\epsilon(T_i)} \theta^n_{\det}(T_i, (\chi\chi_D)^o) \in S^n_{\frac{n}{2}+1}(Mc^2, \chi_D).$$

Here we have to decompose $M \cdot c^2 = M_0 \cdot N$, and we have for odd primes p

$$\alpha(p) = \begin{cases} \frac{p-1}{2} & \text{if} \quad p \mid c, \\ p-1 & \text{if} \quad p \nmid c. \end{cases}$$

Furthermore, κ must be odd if $\alpha(p)$ is $\frac{p-1}{2}$.

5.2.2. *Levels.* Our method can be applied to any level chosen independently of the levels of the matrices T_i. Of course the results for a higher level follow from the corresponding ones for a smaller level.

5.2.3. *Odd degrees.* When n is odd, the theta series $\theta^n_{\det}(T, \chi_T\chi_D)$ becomes a modular form of half-integral weight. To get integral weights, we can then take

$$\theta^n_{\det}(T, \chi_T\chi_D)^2 \mid U(2)$$

as a starting point and proceed in the same way as in the case of even degree.

5.2.4. *Non-cusp forms.* If one is only interested in the existence of modular forms F_i such that the matrix $(a(F_i, T_j))$ has full rank, one may start with theta series $\theta^n(T_i)$ without the harmonic polynomial det. However, in section 2, one then gets weaker results; in particular, the notion of T-minimal can no longer be used. Instead, one has to use congruences and arguments similar to those of section 3, using the key property that for modular forms F we have congruences of type

$$F^p \mid U(p) \equiv F \bmod p.$$

6. The linear independence of Poincaré series for infinitely many arithmetic progressions

The main point of Theorem 5.1 is that under suitable conditions, it allows us to effectively determine an upper bound for the lowest weight k for which the linear independence of the Poincaré series holds. The theorem gives such a bound as a quite explicit function of the known parameters (e.g., degree n, the T_i). Once the existence of such a k is assured, one can argue differently to get infinitely many such weights, as can be seen from the following elementary remark, which is based on the boundedness of denominators of Fourier coefficients of modular forms:

Proposition 6.1. *Suppose we have $T_1, \ldots, T_h$ pairwise inequivalent and $F_i \in S_k^n(\mathbf{Q})$, $i = 1, \ldots, h$, such that $d := \det(a(F_i, T_j)) \neq 0$. Then for almost all odd primes p (provided that $p \equiv 1 \bmod 4$ or $p > n+3$ or $p \geq \frac{n}{2} + 3$ and p regular) and for all weights*

$$k' = k + (p-1)l \qquad (l \in \mathbb{N}),$$

the Poincaré series $P_{k'}^n(T_i)$ are linearly independent.

Proof. By the bounded denominator property, for almost all primes p, the Fourier coefficients of the F_i are p-integral (see [30, Theorem 1]) and d is a p-adic unit. For all such odd primes p satisfying the conditions imposed in the proposition, there exists a level one modular form E_{p-1} of weight $p-1$ (see [7]) such that

$$E_{p-1} \equiv 1 \bmod p.$$

Then we consider the cusp forms $G_i := F_i \cdot E_{p-1}^l$. They satisfy

$$\det(a(G_i, T_j)) \equiv d \bmod p,$$

in particular, this determinant is non-zero. This implies the linear independence of the Poincaré series. □

7. Explicit results

Our methods can be made more explicit, as long as we only handle one prime; to write the congruences in question, we need some information about denominators of Fourier expansions of theta series (in all cusps) and some more details about the inverse of $U(p)$. We exhibit below some cases, where explicit results are possible. Note that no unknown constants appear. We consider the case $n = 1$ (with some variants of the methods employed so far) and arbitrary even degrees. With more effort, better results might be possible. In this section we use the notation $e(z) := \exp(2\pi i z)$.

7.1. **A remark on the integrality properties of theta series.** For an even integral m-dimensional quadratic form S of level q, we consider the theta series with characteristic (of degree n):

$$\theta^n(S, P, \mathfrak{a}) = \sum_{X \equiv \mathfrak{a} \bmod 1} P(X) \exp(\mathrm{tr}\ X^t S X Z), \qquad \text{with } \mathfrak{a} \in \mathbf{Q}^{m,n},\ S \cdot \mathfrak{a} \in \mathbf{Z}^{m,n}.$$

Moreover, P should be a harmonic polynomial, homogeneous of degree ν, with coefficients in $\mathbb{Z}^{m,n}$.

Then for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n$ with $\det C \neq 0$,

$$q^\nu \det S^{\frac{n}{2}} \det C^k\ \theta^n(S, P, \mathfrak{a}) \mid_k M \tag{7.1}$$

is a Fourier series whose Fourier coefficients are integers of a cyclotomic number field. This statement is probably well known (at least implicitly). For $P = 1$ it follows directly from [20, formula (23)]. The case of arbitrary P can be treated similarly (or can be obtained from loc. cit. by applying appropriate differential operators to the more general theta series treated there).

7.2. **Degree 1.** The degree one case has some special features: first of all our theta series will be of half-integral weight; secondly, we do not need the procedure employed in [9], i.e. we do not need the inverse of $U(p)$; the somewhat simpler method of Serre [28] is sufficient.

7.2.1. *Theta series of weight* $\frac{3}{2}$. In order to stick to just one (odd) prime $p > 3$, we fix a positive integer j and we let j' run between 0 and j. We put $t = p^j$, $s := p^{j-1}$ and more generally $t' := p^{j'}$. Then the theta series (see [29, p. 457])

$$f_{t'} := \sum_m \chi_{-4}(m) m\, e(m^2 t' \tau)$$

are weight 3/2 cusp forms for $\Gamma_0(64t)$, where $\chi_{-4}(m) = \left(\frac{-4}{m}\right)$.

Further, $g_{t'}(\tau) := f_{t'}^{32s} \mid U(32s)$ is a modular form of weight $48s$ for $\Gamma_0(2p)$ with rational integral Fourier coefficients. Moreover, it is a t'-minimal modular form by Proposition 3.3. To remove the prime p from the level by congruences, one can proceed directly as in Serre [28, p. 226] and does not need to invert the operator $U(p)$. We have to consider (recall $t' = p^{j'}$)

$$\begin{aligned} h_{j'} &= \mathrm{tr}_2^{2p}(g_{t'} \cdot E_{p-1}^{\beta}) \\ &= g_{t'} \cdot E_{p-1}^{\beta} + p \cdot \left(g_{t'} \cdot E_{p-1}^{\beta}\right) |_{48s+(p-1)\beta}\, M \mid \tilde{U}(p). \end{aligned} \tag{7.2}$$

Here M is any element of $\mathrm{SL}(2, \mathbf{Z})$ satisfying

$$M \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod 2, \quad M \equiv \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \bmod p,$$

and E_{p-1} is an element of $M_{p-1}(p)$ which satisfies the congruences

$$E_{p-1} \equiv 1 \bmod p, \qquad E_{p-1} \mid_{p-1} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \equiv 0 \bmod p.$$

For the existence of E_{p-1} we refer to [28, `LEMME 8`] (for $p > 3$). The operator $\tilde{U}(p)$ is defined on any holomorphic function on $\mathbf{H}$ with period p by

$$f(\tau) = \sum_n a(n)\, e(n\tau/p) \longmapsto f \mid \tilde{U}(p)(\tau) = \sum_n a(n)\, e(n\tau).$$

To estimate the possible p-denominator of $g_{t'} \mid_{2s} M$, we keep in mind that $f_{t'}$ is a theta series; therefore, $g_{t'}$ is a theta series with congruence conditions. More precisely, it can be expressed as an integral linear combination of series of type (see [4, p. 678] for example):

$$\sum_{\mathfrak{m} \equiv \mathfrak{m}^o \bmod 32s} (m_1 \dots m_{32s})\, e\left(t' \left(\sum m_i^2\right) \cdot \tau\right)$$

with a fixed element $\mathfrak{m}^o \in \mathbf{Z}^{32s}$. This is of type $\theta^1(S, P, \mathfrak{a})$ with $S := t' \cdot (32s)^2 \cdot 1_{32s}$ and some P of degree $\nu = 32s$. The level q is $1024 \cdot p^{j'+2j-2}$.

Then (considering the "worst" case $t' = t$) at most

$$\left(p^{3j-2}\right)^{\nu} \cdot \left(t \cdot s^2\right)^{\frac{32s}{2}} = p^{(144j-96)p^{j-1}}$$

should appear as a power of p in the denominator of $g_{t'} \mid_{2s} M$ (see (7.1)).

To assure that the second term in (7.2) is congruent to zero modulo p for all j', we must assure that

$$1+\beta-p^{j-1}(144j-96)>0. \tag{7.3}$$

We should keep in mind that these cusp forms $h_{j'}$ are no longer t'-minimal, but t'-minimal $\pmod p$ in an obvious sense. This is evident from (7.2) and (7.3). As before, the t'-minimality implies the following proposition.

Proposition 7.1. *Let p be a prime with $p>3$. For any $j\geq 1$ there are cusp forms*

$$h_{j'}=\sum_{n=1}^{\infty} a(j',n)\ e(n\cdot\tau)\in S_k(\Gamma_0(2))$$

for $0\leq j'\leq j$ such that

$$\det(a(u,p^v)_{0\leq u,v\leq j})\neq 0$$

for all weights k satisfying

$$k=2p^{j-1}+(p-1)\beta,$$

with $\beta\geq p^{j-1}(144j-96)$.

Remark 7.2. Along the same lines as above, one can treat the case of $p=2$ to get $t+1$ cusp forms of level one with the matrix of their Fourier coefficents at indices $1,2,4,\ldots,2^t$ having maximal rank. The effective bound, which one can obtain here, may be of interest in the context of Ahlgren's work [1] on Maeda's conjecture.

7.2.2. *Binary theta series.* This is a slight digression from our basic strategy. We fix a prime $q\equiv 3 \bmod 4$; for a prime p with $\left(\frac{-q}{p}\right)=1$ there exists a binary quadratic form T of discriminant $-q$ (exactly one up to equivalence), which represents p integrally. This fact can be viewed as a special case of a remarkable result of Kitaoka [19, p. 155] on representations of elements of Λ_{m-1} by elements of Λ_m or can be read from the arithmetic of quadratic number fields. The number of integral representations of p by S is then 4. Assume further that the class number of integral binary quadratic forms of discriminant $-q$ is not one. We choose a second such quadratic form S which does not represent p. We note that S and T are in the same genus. Then we consider the cusp form (see, e.g., [4])

$$f:=\theta(T)-\theta(S)\in S_1(\Gamma_0(q),\chi_{-q}),$$

where

$$\theta(T)(z):=\sum_{X\in\mathbf{Z}^{2,1}} e(X^tTX\cdot z)$$

is the theta series attached to T and χ_{-q} is the character attached to T (see section 2.4). The Fourier coefficient of f at p is then equal to 4. We now look at the level one form

$$g:=\mathrm{tr}_1^q(f\cdot\mathcal{E})\in S_{\frac{q+1}{2}},$$

where $\mathcal{E}\in M_{\frac{q-1}{2}}(q,\chi_{-q})$ with

$$\mathcal{E}\equiv 1 \bmod q,\qquad \nu_p(\mathcal{E}\mid_{\frac{q-1}{2}}\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right))=-\frac{1}{2}.$$

The existence of such $\mathcal{E}$ for $q\geq 7$ is discussed in [8] via theta series; more precisely, see [8, Corollary 2] and first few lines of section 6 there. It can also be constructed as a linear combination of appropriate Eisenstein series.

Then $\nu_q(f-g) > 0$ follows from the formula of the trace map (see (7.2)) and we have $f \equiv g \bmod q$. The same procedure also works with any odd power of $\mathcal{E}$.

Proposition 7.3. *Let q be an arbitrary prime with $-q \equiv 1 \bmod 4$, and assume that the class number of positive definite integral binary quadratic forms of discriminant $-q$ is not one. Then for all primes p with $\left(\frac{-q}{p}\right) = +1$, there exists a cusp form g of level 1 and weight $k = 1 + t \cdot \frac{q-1}{2}$ with $a(g,p) \neq 0$ for all odd $t \in \mathbf{N}$.*

Example 7.4. For $q = 23$ and $k = 12$ the proposition above shows that for all primes p with $\left(\frac{-23}{p}\right) = +1$ we have by the above proposition, for the Fourier coefficients of Ramanujan's Δ-function, that $\tau(p) \neq 0$. This is well known (see, e.g., [31]). With a little more care we can rediscover Wilton's congruences [32] for Δ by considering

$$f = \theta(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 6 \end{pmatrix}) - \theta(\begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 3 \end{pmatrix}).$$

7.2.3. *Binary theta series with harmonic polynomials.* A similar procedure also works with binary theta series $\theta(S,P)$ with harmonic polynomials P of degree 2ν; here no assumption on class numbers is necessary. We omit the details.

7.3. **Arbitrary even degree n, prime level.** We treat only the simplest possible case: we choose an odd prime p and we consider finitely many pairwise inequivalent quadratic forms $T_i \in \Lambda_n^+$ with $\det T_i = p$. The level of the T_i is then also equal to p. We further assume that the T_i do not have improper integral automorphisms, so we do not need to deal with quadratic characters χ_D (as in Proposition 3.4).

We have to make the procedure of section 3 more explicit; for that purpose we have to recall some of the considerations of [9] in our special situation. For $h \in M_l^n(p)$ we look at

$$\mathrm{tr}_1^p(h) := p^{-\frac{n(n+1)}{2}} \sum_{g \in \Gamma_0^n(p)\backslash\Gamma_n} h \mid_l g.$$

The Bruhat decomposition over the finite field $\mathbf{F}_p$ implies that (see also [8, 9])

$$\begin{aligned} \mathrm{Sp}(n,\mathbf{Z}) &= \bigcup_{i=0}^{n} \Gamma_0^n(p) \cdot \omega_i \cdot \mathrm{Sp}(n,\mathbf{Z})_\infty \\ &= \bigcup_{i=0}^{n} \bigcup_{r,s} \Gamma_0^n(p) \cdot \omega_i \cdot \begin{pmatrix} 1_n & B_{ir} \\ 0_n & 1_n \end{pmatrix} \cdot \begin{pmatrix} A_{is}^t & 0_n \\ 0_n & A_{is}^{-1} \end{pmatrix} \end{aligned} \tag{7.4}$$

with

$$\omega_i = \begin{pmatrix} 1_{n-i} & 0 & 0_{n-i} & 0 \\ 0 & 0_i & 0 & -1_i \\ 0_{n-i} & 0 & 1_{n-i} & 0 \\ 0 & 1_i & 0 & 0_i \end{pmatrix} \qquad (0 \le i \le n)$$

and certain (finitely many) elements $B_{ir} \in \mathbf{Z}^{(n,n)}$ and $A_{is} \in \mathrm{GL}(n,\mathbf{Z})$.

We decompose the trace operator into $n+1$ pieces Y_i collecting the contributions, which belong to a fixed ω_i (see, e.g., [8, 9]). We note that the contribution of the B_{ir} gives rise to a scalar factor $p^{\frac{i(i+1)}{2}}$; then Y_i is of the form

$$Y_i = p^{-\frac{n(n+1)}{2}} \cdot p^{\frac{i(i+1)}{2}} \sum_s h \mid_l \omega_i \begin{pmatrix} A_{is}^t & 0_n \\ 0_n & A_{is}^{-1} \end{pmatrix}.$$

The unimodular matrices A_{is} do not affect the integrality of the Fourier coefficients; the Y_i with $i < n$ are then congruent to zero mod p as long as

$$\nu_p(h\,|_l\,\omega_i) - \frac{n(n+1)}{2} + \frac{i(i+1)}{2} > 0 \qquad (0 \le i < n). \tag{7.5}$$

Furthermore, recalling $W_p(p)$ from section 2.2, we have

$$Y_n = h \mid W_p(p).$$

We recall from [8] the existence of a special quadratic form $S \in \Lambda^+_{p-1}$ with level p and $\det S = p$ such that

$$\mathcal{K}_{p-1} := p^{\frac{n}{2}} \cdot \theta^n(S) \in M^n_{\frac{p-1}{2}}(p, \chi_p)$$

satisfies the congruences

$$\begin{aligned} &\nu_p(\mathcal{K}_{p-1}\,|_{\frac{p-1}{2}}\,\omega_i) \ge \frac{n-i}{2} \qquad (0 \le i < n),\\ &\mathcal{K}\,|_{\frac{p-1}{2}}\,\omega_n \equiv 1 \bmod p. \end{aligned}$$

Let T be any of the T_i. We apply the trace operator from above to

$$h := \tilde{F} \cdot \mathcal{K}^{\beta}_{p-1}$$

with

$$\tilde{F} := W_p(p)^{-1}(\theta^n_{det}(T))$$

(we use the notation $\theta^n_{\mathrm{det}}(T) := \theta^n_{\mathrm{det}}(T,1)$ as in section 4) and some odd $\beta = \kappa \cdot p^{\gamma}$ to be specified below.

The calculation in [9, §3] shows that

$$Y_n \equiv \theta^n_{\mathrm{det}}(T) \bmod p$$

if

$$\nu_p(\tilde{F}\,|_{\frac{n}{2}+1}\,\omega_n) + \gamma + 1 > 0. \tag{7.6}$$

To get the desired congruence

$$\mathrm{tr}^p_1(\tilde{F} \cdot \mathcal{K}^{\beta}_{p-1}) \equiv \theta^n_{\mathrm{det}}(T) \bmod p$$

we need more information on the inverse of $W_p(p)$ as follows.

In [6, p. 816] a $(n+1) \times (n+1)$ matrix is given which describes how the multiplication by the special double-coset $\Gamma^n_0(p) \cdot \omega_n \cdot \Gamma^n_0(p)$ acts on the Hecke algebra associated to the Hecke pair $(\Gamma^n_0(p), \Gamma_n)$ (more precisely, a variant of it attached to a character χ). That matrix has integer coefficients (explicitly computable, but complicated); its determinant is however easily computed to be p^{δ} with

$$\delta := \frac{n}{6}(2n^2 + 3n + 1).$$

Note that this determinant is independent of the character χ. The first column of the inverse of that matrix gives us the requested inverse; we do not need the explicit shape here. If we write that inverse abstractly as a linear combination of left cosets we get as a weak consequence an expression

$$W_p(p)^{-1} = p^{-\delta - \frac{n(n+1)}{2}} \sum_{\gamma \in \Gamma^n_0(p)\backslash\Gamma_n} a(\gamma) \cdot \Gamma^n_0(p) \cdot \gamma \tag{7.7}$$

with certain integers $a(\gamma)$. We note that the double coset $\Gamma_n \cdot \left(\begin{smallmatrix} 0_n & -1_n \\ 1_n & 0_n \end{smallmatrix}\right) \cdot \Gamma_n$ treated in [6] differs from $W_p(p)$ by a factor factor $p^{-\frac{n(n+1)}{2}}$ when viewed as an operator; this is the reason why this factor appears in (7.7).

We need an estimate for $\nu_p(\tilde{F} \mid_{\frac{n}{2}+1} g)$ for an arbitrary $g \in \Gamma_n$. In view of (7.4) and (7.7) we have (using the automorphy property of $\tilde{F}$)

$$\nu_p(\tilde{F} \mid_{\frac{n}{2}+1} g) \geq -\delta - \frac{n(n+1)}{2} + \min\{\nu_p(\theta^n_{\det}(T) \mid_{\frac{n}{2}+1} \omega_i) \mid 0 \leq i \leq n\}.$$

The action of the ω_i on theta series is well understood (see, e.g., [10, Lemma 4.8]):

$$\theta^n_{\det} \mid_{\frac{n}{2}+1} \omega_i = \pm \det(T)^{-\frac{i}{2}} \sum_{X=X_1,X_2} \det(X) \exp(\mathrm{tr} X^t T X Z)$$

with $X_1 \in \mathbf{Z}^{n,n-i},\ X_2 \in T^{-1} \cdot \mathbf{Z}^{n,i}$. From this we get

$$\nu_p(\theta^n_{\det}(T) \mid_{\frac{n}{2}+1} \omega_i) \geq -i - \frac{i}{2},$$

and then

$$\nu_p(\tilde{F} \mid_{\frac{n}{2}+1} g) \geq -\delta - \frac{n(n+1)}{2} - \frac{3n}{2}.$$

Then (7.6) is satisfied if

$$\gamma \geq \delta + \frac{n(n+1)}{2} + \frac{3n}{2}. \tag{7.8}$$

On the other hand, the congruences (7.5) hold if the inequalities

$$-\delta - \frac{n(n+1)}{2} - \frac{3n}{2} + \beta \cdot \frac{n-i}{2} - \frac{n(n+1)}{2} + \frac{i(i+1)}{2} > 0 \tag{7.9}$$

are satisfied for all $i < n$.

An elementary calculation shows that (recall $\beta = \kappa p^\gamma$) with the choice (7.8) of γ the inequalities (7.9) are automatically satisfied. Summarizing:

Theorem 7.5. *For even n and an odd prime p, let $T_1, \ldots, T_h$ be pairwise inequivalent elements of Λ_n^+ with $\det(T_i) = p$; we assume that the T_i have no improper integral automorphisms. Then the Poincaré series $P^n_k(T_i)$, $i = 1, \ldots, h$, are linearly independent for all weights*

$$k = \frac{n}{2} + 1 + \kappa \cdot p^\gamma \cdot \frac{p-1}{2}$$

with κ odd and

$$\gamma \geq \frac{n^3}{3} + n^2 + \frac{13n}{6}.$$

Remark 7.6. In any particular situation (with n fixed) one may obtain a better result by using an explicit version of (7.7); there may be a considerable amount of cancellations of powers of p in the denominators. For example, in degree 4, nebentypus χ_p, the inversion of the matrix in example 5.4 of [6] yields much better denominators:

$$\Delta := \frac{(p^3-1)(p-1)}{p^8} \cdot \Gamma_0(p) + \frac{p-1}{p^{13}} \cdot \Gamma^4_0(p)\omega_2\Gamma^4_0(p) + \frac{1}{p^{10}} \cdot \Gamma^4_0(p)\omega_4\Gamma^4_0(p).$$

Remark 7.7. Along the same lines one can deal with $T_i \in \Lambda_n^+$ with the level being an arbitrary power of a prime p. The necessary formulas for applying the $U(p)$-operator to theta series are available from [10, 16]. To remove the condition on the automorphisms (no improper automorphisms) seems to be more delicate.

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