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A NOTE ON THE COHOMOLOGY OF THE LANGLANDS GROUP

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Abstract. We begin with a comparison of various cohomology theories for topological groups. Using the continuity result for Moore cohomology, we establish a Hochschild-Serre spectral sequence for a slightly larger class of groups. We use these properties to compute the cohomology of the Langlands group of a totally imaginary field. The appendix answers a question raised by Flach concerning the cohomological dimension of the group $\mathbb{R}$.

1. Introduction

In the first half of this article, we try to collect the recent continuity result of Moore cohomology of compact groups defined via measurable cochains announced in [2] and compare it with the parallel result on the topological group cohomology defined via topos theory [5] with the aid of a series of comparison results [2, 5, 10, 18] between various cohomology theories. We will recall the definitions of different cohomology theories of topological groups here and briefly discuss the relations between them. In the second half, we will give an application to compute the topological group cohomology of the Langlands group L_F. As an extension of the result in [5], we show that $H^i(L_F, \mathbb{Z})$ has infinite rank for all even $i \geq 4$ in the case where F is a totally imaginary number field. This result gives a negative answer to the conjectural use of the Langlands group to modify the definition of a Weil-etale topos $\overline{Y}_W$ for the spectrum of the ring of integers of a number field F, whose cohomologies $H^i(Y_W, \mathbb{Z})$ are expected to vanish for $i \geq 4$.

The appendix answers a question raised in [5, 9.3]. Since the present article is in some sense a natural continuation of [5], we decided to include this answer here.

2. Basic definitions

In this section, we will briefly recall the definition of various cohomology theories for topological groups with coefficients on their corresponding topological modules.

2.1. **Topos theoretic cohomology of topological groups.** Let Top be the (small) category of topological spaces with continuous maps, and J_{open} the open covering topology on Top. Then $(\text{Top}, J_{\text{open}})$ is a site and set $\mathcal{T} = Sh(\text{Top}, J_{\text{open}})$ to be the associated category of sheaves of sets on it. Given a topological group G and a topological G-module A (i.e. a topological abelian group with a linear continuous G-action), let $y : \text{Top} \to \mathcal{T}$ be the Yoneda embedding, where $y(X) = \text{Hom}_{\text{Top}}(\_\_, X)$. Then yG is a group object in $\mathcal{T}$ and set BG to be the classifying topos of G, i.e. the

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category of yG-objects in $\mathcal{T}$. In particular, yA is an abelian group object in BG. We define the topos theoretic cohomology of a topological group G with coefficients in a G-module A to be

$$H^i(G, A) = \left(R^i\Gamma\right)(yA)\,, \tag{2.1}$$

where $\Gamma(\mathcal{A}) := \mathrm{Hom}_{\mathrm{BG}}(*, \mathcal{A})$ is the global section functor. (Here $*$ denotes the trivial yG-object, i.e. the constant sheaf of a point with trivial G-action.)

2.2. **Continuous cochain cohomology of topological groups.** As above, suppose G is a topological group and A is a topological G-module. Let $C^i(G, A)$ be the set of continuous maps $G^i \to A$ for $i \geq 0$ and 0 otherwise. Then we define the coboundary maps $d : C^i(G, A) \to C^{i+1}(G, A)$ by

$$\begin{aligned} df(g_1, \ldots, g_{i+1}) :=& g_1 f(g_2, \ldots, g_{i+1}) \\ &+ \sum_{j=1}^{i} (-1)^j f(g_1, g_2, \ldots, g_j g_{j+1}, \ldots, g_{i+1}) \\ &+ (-1)^{i+1} f(g_1, \ldots, g_i). \end{aligned}$$

Simple computation shows that $d^2 = 0$. Thus $\left(C^i(G, A), d\right)$ form a cochain complex and the corresponding cohomology is defined to be the continuous cochain cohomology of G with coefficients in A, denoted

$$H^i_{\mathrm{cts}}(G, A) := \frac{\mathrm{Ker}\left(d : C^i(G, A) \to C^{i+1}(G, A)\right)}{\mathrm{Im}\left(d : C^{i-1}(G, A) \to C^i(G, A)\right)}. \tag{2.2}$$

2.3. **Moore measurable cochain cohomology for locally compact groups.** Let G be a locally compact group and A a complete metric G-module. It is well known that there is a Haar measure on G so that both G and A are equipped with a Borel structure (i.e. a σ-field of subsets of G or A which contains all singletons and where the group operations are Borel functions) generated by open subsets [12]. Let $C^i_M(G, A)$ be the set of Haar-a.e. equivalence classes of Borel maps $G^i \to A$ for $i \geq 0$ and 0 otherwise. Define the coboundary maps $\delta : C^i_M(G, A) \to C^i_M(G, A)$ by

$$\begin{aligned} \delta f(g_1, \ldots, g_{i+1}) :=& g_1 f(g_2, \ldots, g_{i+1}) \\ &+ \sum_{j=1}^{i} (-1)^j f(g_1, g_2, \ldots, g_j g_{j+1}, \ldots, g_{i+1}) \\ &+ (-1)^{i+1} f(g_1, \ldots, g_i), \end{aligned}$$

where all equalities hold almost everywhere in corresponding Haar measures. It is easy to see that $\delta^2 = 0$, henceforth $(C^i_M(G, A), \delta)$ form a cochain complex and its cohomology defines the Moore measurable cochain cohomology of G with coefficients in A, denoted

$$H^i_M(G, A) := \frac{\mathrm{Ker}\left(\delta : C^i_M(G, A) \to C^{i+1}(G, A)\right)}{\mathrm{Im}\left(\delta : C^{i-1}_M(G, A) \to C^i(G, A)\right)}. \tag{2.3}$$

2.4. **Algebraic cohomology of topological groups.** Let G be a topological group, $\mathcal{M}_G$ the category of complete metric G-modules, and A a complete metric G-module. Set $EXT^n_{\mathcal{M}_G}(\mathbb{Z}, A)$ to be the set of n-term exact sequences $0 \to A \to B_1 \to \cdots \to B_n \to \mathbb{Z} \to 0$ in $\mathcal{M}_G$, where $\mathbb{Z}$ is the group of integers with the discrete topology and trivial G-action and $Ext^n_{\mathcal{M}_G}(\mathbb{Z}, A)$ is the quotient of $EXT^n_{\mathcal{M}_G}(\mathbb{Z}, A)$ by the equivalence relation generated by chain maps with identity on the first and last terms. Then the algebraic cohomology of G with coefficients in A is defined to be

$$H^i_a(G, A) := Ext^i_{\mathcal{M}_G}(\mathbb{Z}, A). \tag{2.4}$$

2.5. **Ordinary cohomology of classifying spaces.** Let G be a topological group and A be an abelian group with trivial G-action. The cohomology of G with coefficients in A defined via classifying space is given by

$$H^i_{sp}(G, A) = H^i_{sing}(\mathbf{B}G, A), \tag{2.5}$$

where $\mathbf{B}G$ denotes the topological classifying space of G and H^i_{sing} is the ordinary ith singular cohomology.

3. Relations between various cohomology theories of topological groups

This entire section is devoted to gathering up comparison results of various cohomology theories of topological groups appearing in the literature. From this hull of information, we conclude a few useful consequences which facilitate the calculation of the cohomology of the Langlands group in the next section.

3.1. **Recollections of comparison results.** We will summarize the comparison theorems between the five cohomology theories for topological groups mentioned above that appeared in [2, 5, 10, 18]. All the results here are not new, but by putting them together, we get a relatively complete picture of the relations among different cohomology theories. It turns out that the comparison theorems extend the availability of the Hochschild-Serre spectral sequences for infinite dimensional compact groups under some mild conditions.

Theorem 3.1 ([5], Proposition 5.1). *Let G be a topological group and A be a topological G-module. If $H^q_{\mathcal{T}}(G^n, yA) = 0$ for all $n, q > 0$, then $H^i(G, A) = H^i_{cts}(G, A)$. Here $H^q_{\mathcal{T}}(G^n, yA)$ denotes the sheaf cohomology of the topological space G^n with coefficients in the sheaf represented by A.*

Theorem 3.2 ([5], Proposition 9.5). *Let G be a paracompact, locally compact group and let A be a vector group with continuous G-action. Then $H^i(G, A) = H^i_{cts}(G, A)$.*

Theorem 3.3 ([5], Proposition 5.2). *Let G be a topological group and A be a discrete G-module with trivial G-action. Suppose that G^n is locally contractible for all $n > 0$; then $H^i(G, A) = H^i_{sp}(G, A)$.*

Theorem 3.4 ([18]). *Let G be a locally compact group and A be a complete metric G-module. If G is Hausdorff, then $H^i_M(G, A) = H^i_a(G, A)$.*

Theorem 3.5 ([18], Theorems 1 and 3)**.** *Let G be a finite dimensional paracompact, locally compact group. Then*

(1) If G is zero dimensional, then

$$H^i_M(G, A) = H^i_{cts}(G, A)$$

for any locally compact G-module A.

(2) If A is a vector group with continuous G-action, then

$$H^i_M(G, A) = H^i_{cts}(G, A).$$

Theorem 3.6 ([5], Corollary 2, [10], Propositions 1.4, [18], Theorem 2)**.** *Let G be a finite dimensional paracompact, locally compact group. Suppose A is a complete metric G-module with the following property: For any short exact sequence of complete metric abelian topological groups*

$$0 \to A \to B \xrightarrow{\tau} C \to 0,$$

τ has the homotopy lifting property for every finite dimensional paracompact space. Then $H^i_a(G, A) = H^i(G, A)$.

Theorem 3.7 ([18], Theorem 4)**.** *Let G be a finite dimensional paracompact, locally compact group and A be a discrete abelian group with trivial G-action. Then $H^i_M(G, A) = H^i_{sp}(G, A)$.*

Theorem 3.8 ([2], Theorem C)**.** *Let G be a compact group and A be a discrete abelian group with trivial G-action. Then $H^i_M(G, A) = H^i_{sp}(G, A)$.*

Corollary 3.9. *Let G be a compact group and A be a discrete abelian group with trivial G-action. Then $H^i(G, A), H^i_M(G, A), H^i_a(G, A), H^i_{sp}(G, A)$ all coincide.*

Proof. From Theorems 3.1, 3.4, 3.8, we see that $H^i_M(G, A), H^i_a(G, A), H^i_{sp}(G, A)$ coincide.

Note that $G^n, n > 0$, is locally contractible for every compact Lie group since they are locally Euclidean, and $H^i(G, A) = H^i_{sp}(G, A)$ for every compact Lie group G. Now since (3.1) holds for $\mathcal{H}^i$ equals H^i, H^i_{sp}([5], Corollary 7; [9], Proposition III-1.11), the equality $H^i(G, A) = H^i_{sp}(G, A)$ extends through all compact groups G by virtue of the fact that every compact group is a projective limit of compact Lie groups [16]. □

3.2. **Continuity theorems and spectral sequences.** From the comparison results listed above, we now further establish some useful machinery for calculating the cohomologies, namely the continuity of the cohomology bifunctors under limits and the spectral sequences associated to certain short exact sequences of topological groups.

Theorem 3.10. *Let $\{G_\alpha\}$ be a filtered inverse system of compact groups, and for each α let A_α be a discrete G_α-module so that $\{A_\alpha\}$ forms a direct system of G-modules. Then*

$$\mathcal{H}^i(\varprojlim_\alpha G_\alpha, \varinjlim_\alpha A_\alpha) = \varinjlim_\alpha \mathcal{H}^i(G_\alpha, A_\alpha) \tag{3.1}$$

for all $i \geq 0$, where $\mathcal{H}^i$ can be any of $H^i, H^i_M, H^i_a, H^i_{sp}$.

Proof. It follows immediately from Corollary 3.9 and that (3.1) holds for H^i ([5], Corollary 7). □

Theorem 3.11 ([5], Corollary 6). *Let $1 \to H \xrightarrow{i} G \xrightarrow{p} Q \to 1$ be an exact sequence of topological groups so that p is a local section cover and let A be a topological G-module. Then there is a Hochschild-Serre spectral sequence*

$$H^p(Q, \underline{H}^q(H, A)) \Rightarrow H^{p+q}(G, A),$$

where $\underline{H}^q(H, A) = R^q e_{H*}(Bi)^*(yA) = \underline{Ext}^q(H, Res_H^G(A))$.

Proposition 3.12. *Let $1 \to H \xrightarrow{i} G \xrightarrow{p} Q \to 1$ be an exact sequence of topological groups. Then $p : G \to Q$ is a local section cover if one of the following conditions is satisfied:*

(1) *H is a Lie group and G is Hausdorff.*
(2) *G is compact with countable basis and Q is profinite.*
(3) *G is locally compact and finite dimensional.*

Proof. (1) and (2) follows from Proposition 2.1 of [5] and (3) follows from Theorem 3 of [14]. □

Theorem 3.13 ([13], Theorem 9). *Let $1 \to H \xrightarrow{i} G \xrightarrow{p} Q \to 1$ be an exact sequence of locally compact groups and let A be a locally compact G-module. Then if $H_M^q(H, A)$ is Hausdorff for all $q \geq 0$, there is a Hochschild-Serre spectral sequence*

$$H_M^p(Q, H_M^q(H, A)) \Rightarrow H_M^{p+q}(G, A).$$

Corollary 3.14. *Let $1 \to H \xrightarrow{i} G \xrightarrow{p} Q \to 1$ be an exact sequence of compact groups and let A be a discrete abelian group with trivial G-action. If $H_M^q(H, A)$ is a discrete Q-module with trivial action, then there is a Hochschild-Serre spectral sequence*

$$H^p(Q, H^q(H, A)) \Rightarrow H^{p+q}(G, A).$$

Proof. It follows from Corollary 3.9 that $H^i(G, A), H^q(H, A)$ and $H^p(Q, H^q(H, A))$ are the same as $H_M^i(G, A), H_M^q(H, A)$ and $H_M^p(Q, H_M^q(H, A))$ respectively. Since $H_M^q(H, A)$ is discrete, in particular, it is Hausdorff. Thus the result follows immediately from Theorem 3.13. □

Theorem 3.15 ([7], V, Corollary 3.3). *Let G be a topological group and $\mathcal{A}$ an abelian sheaf on Top with yG-action. Then there is a Cartan-Leray spectral sequence*

$$H^p\left(H_{\mathcal{T}}^q(G^\bullet, \mathcal{A})\right) \Rightarrow H^{p+q}(G, \mathcal{A}).$$

Corollary 3.16. *Let G be a compact group and A a torsion topological G-module. Then $H^p(G, A)$ is a torsion topological G-module for all $p \geq 0$.*

Proof. It is an immediate consequence of Theorem 3.15 since all the terms $H^p\left(H_{\mathcal{T}}^q(G^\bullet, yA)\right)$ of the above spectral sequence are torsion G-modules. □

4. Application to the Langlands group

In [1], Arthur has constructed a conjectural candidate for a Langlands group in order to classify automorphic representations. However for a more general purpose, we give an axiomatic definition of the Langlands groups which includes Arthur's example.

Definition 4.1. Let F be a number field, G_F and W_F be the absolute Galois group and the Weil group of F respectively. A locally compact group L_F is called a Langlands group if it is equipped with a surjective homomorphism $L_F \to W_F$ which satisfies the following three properties:

(1) $\ker(L_F \to W_F \to G_F)$ is connected.
(2) $L_F^{ab} \cong W_F^{ab} \cong C_F$, where C_F is the idele class group of F.
(3) $\ker(L_F \to L_F^{ab} \cong C_F \xrightarrow{\log|\cdot|} \mathbb{R})$ is compact.

Our goal here is to compute $H^i(L_F, \mathbb{Z})$ in the case where F is a totally imaginary number field. The main idea is to make use of Flach's result [5] on the cohomology of the Weil group W_F and extend the result to the Langlands group through a series of spectral sequence calculations. The main result we obtained is that $H^i(L_F, \mathbb{Z})$ is an infinite rank abelian group for every even $i \geq 4$, and as a consequence this shows that the attempt to modify the definition of Weil-etale topology by using the Langlands group instead of the Weil group in order to exhibit the full relationship with the Dedekind zeta-function of F at $s = 0$ fails.

Let $W_F^1 = \ker(W_F \to \mathbb{R})$ and $L_F^1 = \ker(L_F \to \mathbb{R})$. From property (3) of L_F, we see that L_F^1 is a compact group, and the splitting $W_F \cong W_F^1 \times \mathbb{R}$ of W_F lifts to a splitting $L_F \cong L_F^1 \times \mathbb{R}$ of L_F. Let $K_F = \ker(L_F \to W_F) = \ker(L_F^1 \to W_F^1)$. Since K_F is a closed subgroup of L_F^1, it is also compact. Note that we have a commutative diagram of groups with rows and columns being exact:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & L_F^{1,0} \arrow[r] \arrow[d] & W_F^{1,0} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_F \arrow[r] \arrow[d] & L_F^1 \arrow[r] \arrow[d] & W_F^1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1 \arrow[r] \arrow[d] & G_F \arrow[r, equal] \arrow[d] & G_F \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

where $L_F^{1,0} = \ker(L_F^1 \to G_F), W_F^{1,0} = \ker(W_F^1 \to G_F)$. Since both $L_F^{1,0}$ and $W_F^{1,0}$ are connected and G_F is totally disconnected, they are the identity components of L_F^1 and W_F^1 respectively. Then by the Snake lemma for groups, we can complete it into a short exact sequence of short exact sequences of rows:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & K_F \arrow[r] \arrow[d, equal] & L_F^{1,0} \arrow[r, "\pi_{1,0}"] \arrow[d] & W_F^{1,0} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_F \arrow[r] \arrow[d] & L_F^1 \arrow[r, "\pi_1"] \arrow[d] & W_F^1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1 \arrow[r] \arrow[d] & G_F \arrow[r, equal] \arrow[d] & G_F \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{4.1}$$

By definition, all the exact sequences except possibly the first row are exact as topological groups. To see that it is also the case for the first row, first note that K_F is endowed with the subspace topology from L_F^1 which is the same as that from $L_F^{1,0}$ and that K_F is a compact subspace of the Hausdorff group $L_F^{1,0}$, so $K_F \to L_F^{1,0}$ is a closed embedding. Second, clearly $\pi_{1,0}$ is continuous since it is the restriction of the continuous map π_1 onto $L_F^{1,0}$, and then replace W_F^1 by its image $W_F^{1,0}$. It remains to show that it is a strict homomorphism. But it is guaranteed by the fact that $L_F^{1,0}$ is a compact group and that $W_F^{1,0}$ is a Hausdorff group ([4], Remark 1 of Proposition 25, Ch. III, §2.9). Consider the short exact sequence of topological groups $1 \to K_F \to L_F^{1,0} \to W_F^{1,0} \to 1$. We would like to study the cohomology of $L_F^{1,0}$ through a Hochschild-Serre spectral sequence associated to this short exact sequence. Unfortunately, $L_F^{1,0} \to W_F^{1,0}$ fails to be a local section cover, so we cannot apply Theorem 3.11 to such a spectral sequence as in [5]. To overcome this technical difficulty, we invoke the Moore cohomology and apply Theorem 3.13. Note that K_F is compact, by Corollary 3.9, $H_M^q(K_F, \mathbb{Z}) = H^q(K_F, \mathbb{Z})$ is discrete and it is endowed with the trivial $W_F^{1,0}$-action since $W_F^{1,0}$ is connected. Thus according to Corollary 3.14 we get a Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(W_F^{1,0}, H^q(K_F, \mathbb{Z})) \Longrightarrow H^{p+q}(L_F^{1,0}, \mathbb{Z}). \tag{4.2}$$

On the other hand, $L_F^{1,0}$ fits into the exact sequence $1 \to L_F^{1,0} \to L_F^1 \to G_F \to 1$, so Theorem 3.11 together with Proposition 3.12(2) give the second Hochschild-Serre spectral sequence

$$\mathcal{E}_2^{p,q} = H^p(G_F, H^q(L_F^{1,0}, \mathbb{Z})) \Longrightarrow H^{p+q}(L_F^1, \mathbb{Z}), \tag{4.3}$$

where $H^q(L_F^{1,0}, \mathbb{Z})$ is a discrete G_F-module. Then the Hochschild-Serre spectral sequence

$$H^p(L_F^1, \underline{H}^q(\mathbb{R}, \mathbb{Z})) \Longrightarrow H^{p+q}(L_F, \mathbb{Z})$$

associated to the group extension $0 \to \mathbb{R} \to L_F \to L_F^1 \to 1$ degenerates trivially and we obtain isomorphisms $H^i(L_F, \mathbb{Z}) \cong H^i(L_F^1, \mathbb{Z})$.

Lemma 4.2. *Denote by $A^D = Hom_{cont}(A, \mathbb{R}/\mathbb{Z})$ the Pontryagin dual of a locally compact abelian group A. Then*

(1) $E_2^{p,0} = H^p(W_F^{1,0}, \mathbb{Z})$ are torsion free for all p;
(2) $E_2^{p,q}$ are torsion abelian groups for all p and all odd q;
(3) $E_2^{p,q}$ are torsion abelian groups for all q and all odd p;
(4) $E_2^{p,1} = 0$ for all p;
(5) $E_2^{p,2} = H^p(W_F^{1,0}, (K_F^{ab})^D)$ for all p;
(6) $E_2^{0,2} = (K_F^{ab})^D$.

Proof. (1) $E_2^{p,0} = H^p(W_F^{1,0}, \mathbb{Z})$ follows immediately from $H^0(K_F, \mathbb{Z}) = \mathbb{Z}$. Following the notation used in [5, p. 18], they are torsion free since

$$H^p(W_F^{1,0}, \mathbb{Z}) = \varprojlim_K H^p(C_K^{1,0}, \mathbb{Z}) \subset \prod_K H^p(C_K^{1,0}, \mathbb{Z})$$

and

$$H^p(C_K^{1,0}, \mathbb{Z}) = \bigoplus_{\sum_\nu 2p_\nu + q = i} \bigotimes_{\nu \in S_\infty(F)} N^{p_\nu}(h_\nu) \otimes H^q(\mathbb{V}(K), \mathbb{Z})$$

are torsion free for all finite Galois extensions K/F and all p.

(2) Since K_F is compact, it can be written as a projective limit of compact Lie groups [16], namely $K_F = \varprojlim_i K_i$, where each K_i is a compact Lie group. By Corollary 3.9 and Theorem 3.10,

$$H^{2k+1}(K_F, \mathbb{Q}) = \varinjlim_i H^{2k+1}(K_i, \mathbb{Q}) = \varinjlim_i H_{sp}^{2k+1}(K_i, \mathbb{Q}) = 0$$

since $H_{sp}^*(K_i, \mathbb{Q}) = H_{sp}^*(K_i^0, \mathbb{Q})^{\Gamma_i}$ and $H_{sp}^*(K_i^0, \mathbb{Q}) = H_{sp}^*(T_i, \mathbb{Q})^{W_i}$, where K_i^0 is the connected component of K_i, T_i is the maximal torus of K_i^0, W_i is the Weyl group of K_i^0, and $H_{sp}^*(T_i, \mathbb{Q})$ is a polynomial algebra with generators in $H_{sp}^2(T_i, \mathbb{Q})$ [6]. Then from the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of trivial discrete K_F-modules, we have the derived long exact sequence on the cohomology,

$$H^{2k}(K_F, \mathbb{Q}/\mathbb{Z}) \to H^{2k+1}(K_F, \mathbb{Z}) \to H^{2k+1}(K_F, \mathbb{Q}) = 0.$$

Since $\mathbb{Q}/\mathbb{Z}$ is a discrete torsion abelian group, by Corollary 3.16, $H^{2k}(K_F, \mathbb{Q}/\mathbb{Z})$ is a torsion abelian group. In particular, $H^{2k+1}(K_F, \mathbb{Z})$ being a quotient of $H^{2k}(K_F, \mathbb{Q}/\mathbb{Z})$ is a torsion abelian group. By Corollary 3.16 again, we have that $E_2^{p,2k+1} = H^p(W_F^{1,0}, H^{2k+1}(K_F, \mathbb{Z}))$ are torsion abelian groups for all p and all k.

(3) Note that $H^p(W_F^{1,0}, \mathbb{Z}) = 0$ for all odd p. It follows that $H^p(W_F^{1,0}, A) = 0$ for any discrete finitely generated free abelian group A with trivial $W_F^{1,0}$-action. Using the same notation as above, by Theorem 3.10, $K_F = \varprojlim_i K_i$ gives

$$E_2^{p,q} = H^p(W_F^{1,0}, H^q(K_F, \mathbb{Z})) = \varinjlim_i H^p(W_F^{1,0}, H^q(K_i, \mathbb{Z})).$$

Let $T^q(K_i, \mathbb{Z})$ be the torsion subgroup of $H^q(K_i, \mathbb{Z})$. Since $H_{sp}^*(K_i, \mathbb{Q}) = H_{sp}^*(T_i, \mathbb{Q})^{W_i}$, $H^q(K_i, \mathbb{Z})$ is finitely generated. It follows that $T^q(K_i, \mathbb{Z})$

is finite and that $H^q(K_i,\mathbb{Z})/T^q(K_i,\mathbb{Z})$ is a finitely generated torsion free abelian group, which must be a free abelian group of finite rank. Hence,

$$\begin{aligned}&H^p(W_F^{1,0},H^q(K_i,\mathbb{Z}))\\=&H^p(W_F^{1,0},T^q(K_i,\mathbb{Z})\oplus H^p(W_F^{1,0},H^q(K_i,\mathbb{Z}))/T^p(W_F^{1,0},H^q(K_i,\mathbb{Z})))\\=&H^p(W_F^{1,0},T^q(K_i,\mathbb{Z}))\oplus H^p(W_F^{1,0},H^q(K_i,\mathbb{Z})/T^q(K_i,\mathbb{Z}))\\=&H^p(W_F^{1,0},T^q(K_i,\mathbb{Z})).\end{aligned}$$

Now since $W_F^{1,0}$ is compact and $T^q(K_i,\mathbb{Z})$ is a discrete torsion abelian group, by considering the continuous cochain theory as above, we see that $H^p(W_F^{1,0},H^q(K_i,\mathbb{Z}))=H^p(W_F^{1,0},T^q(K_i,\mathbb{Z}))$ are torsion abelian groups for all q and all odd p. As an inductive limit of torsion abelian groups, $E_2^{p,q}$ are torsion abelian groups for all q and all odd p.

(4) Note that $H^1(K_F,\mathbb{Z})=\mathrm{Hom}_{cont}(K_F,\mathbb{Z})=0$ since every continuous homomorphism from K_F to $\mathbb{Z}$ has its image being a compact subgroup of $\mathbb{Z}$ which must be trivial. Therefore $E_2^{p,1}=H^p(W_F^{1,0},H^1(K_F,\mathbb{Z}))=0$.

(5) Applying the derived long exact sequence to the short exact sequence $0\to\mathbb{Z}\to\mathbb{R}\to\mathbb{R}/\mathbb{Z}\to 0$, we obtain

$$\begin{aligned}H^2(K_F,\mathbb{Z})&=H^1(K_F,\mathbb{R}/\mathbb{Z})=\mathrm{Hom}_{cont}(K_F,\mathbb{R}/\mathbb{Z})\\&=\mathrm{Hom}_{cont}(K_F^{ab},\mathbb{R}/\mathbb{Z})=(K_F^{ab})^D.\end{aligned}$$

Thus for all p,

$$E_2^{p,2}=H^p(W_F^{1,0},H^2(K_F,\mathbb{Z}))=H^p(W_F^{1,0},(K_F^{ab})^D).$$

(6) Since K_F^{ab} is a compact abelian group, its Pontryagin dual $(K_F^{ab})^D$ must be discrete. As $W_F^{1,0}$ is connected, it must have trivial action on $(K_F^{ab})^D$. Therefore by (5), $E_2^{0,2}=H^0(G_F,(K_F^{ab})^D)=(K_F^{ab})^D$. □

Theorem 4.3. *Denote by $A^D=Hom_{cont}(A,\mathbb{R}/\mathbb{Z})$ the Pontryagin dual of a locally compact abelian group A. Let F be a totally imaginary number field and $\mathbb{Z}$ the discrete $L_F^{1,0}$-module with trivial action. Then*

(1) $H^0(L_F^{1,0},\mathbb{Z})=\mathbb{Z}$;
(2) $H^1(L_F^{1,0},\mathbb{Z})=0$;
(3) $H^2(L_F^{1,0},\mathbb{Z})=H^2(W_F^{1,0},\mathbb{Z})=(C_F^{1,0})^{ab}$, in particular $(K_F^{ab})^D=0$;
(4) $H^{2k}(L_F^{1,0},\mathbb{Z})$ contains $H^{2k}(W_F^{1,0},\mathbb{Z})$ as a subgroup for all k;
(5) $H^{2k+1}(L_F^{1,0},\mathbb{Z})$ are torsion abelian groups for all k.

Proof. From Lemma 4.2(4), $E_2^{0,0}$,$E_2^{0,1}$,$E_2^{1,0}$,$E_2^{2,0}$,$E_2^{1,1}$,$E_2^{0,2}$ survive in the limit, so by Lemma 4.2(1),(6),

$$\begin{aligned}H^0(L_F^{1,0},\mathbb{Z})=E_2^{0,0}=H^0(W_F^{1,0},\mathbb{Z})=\mathbb{Z},\\H^1(L_F^{1,0},\mathbb{Z})=E_2^{1,0}=H^1(W_F^{1,0},\mathbb{Z})=0,\end{aligned}$$

and $H^2(L_F^{1,0}, \mathbb{Z})$ is an extension of $E_2^{0,2} = (K_F^{ab})^D$ by $E_2^{2,0} = H^2(W_F^{1,0}, \mathbb{Z})$. Note that

$$H^2(L_F^{1,0}, \mathbb{Z}) = H^1(L_F^{1,0}, \mathbb{R}/\mathbb{Z}) = \left(\left(L_F^{1,0}\right)^{ab}\right)^D = (C_F^{1,0})^D.$$

Thus $H^2(L_F^{1,0}, \mathbb{Z}) = H^2(W_F^{1,0}, \mathbb{Z}) = (C_F^{1,0})^{ab}$ and $(K_F^{ab})^D = 0$.

Next we need to show that $E_2^{p,0} = H^p(W_F^{1,0}, \mathbb{Z})$ survive in the limit for all p so that statement (4) follows immediately. To see this, note that from Lemma 4.2(1)-(3), the terms in the bottom row of E_2 are torsion free, and all rows and columns of odd indices consist of torsion abelian groups. Clearly, it follows that at any stage of the spectral sequence, the terms in rows and columns of odd indices are torsion abelian groups since they are successive subquotients of the corresponding E_2 terms. Thus at every stage, the differentials mapping to the bottom rows are mapping from torsion abelian groups to torsion free abelian groups which must be trivial. The last statement follows readily from the above discussion that all terms in the limit filtrations of $H^{2k+1}(L_F^{1,0}, \mathbb{Z})$ are torsion abelian groups. □

Lemma 4.4. *Denote by $A^D = Hom_{cont}(A, \mathbb{R}/\mathbb{Z})$ the Pontryagin dual of a locally compact abelian group A. Then*

(1) $\mathcal{E}_2^{p,q}$ are torsion abelian groups for all p and all odd q;
(2) $\mathcal{E}_2^{p,q} = 0$ for all $p \geq 3$ and all q;
(3) $\mathcal{E}_2^{0,2} = (C_F^{1,0})^D$ and $\mathcal{E}_2^{p,2} = 0$ for all $p \geq 1$;
(4) $\mathcal{E}_2^{p,1} = 0$ for all p;
(5) for all p,

$$\begin{aligned}\mathcal{E}_2^{p,0} &= H^p(G_F, \mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}, & p = 0, \\ (G_F^{ab})^D, & p = 2, \\ 0, & \textit{otherwise}. \end{cases}\end{aligned}$$

Proof. (1) By Theorem 4.3(5), $H^q(L_F^{1,0}, \mathbb{Z})$ are torsion abelian groups for all odd q. Since G_F is compact, by considering the continuous cochain theory, it is easy to see that $\mathcal{E}_2^{p,q} = H^p(G_F, H^q(L_F^{1,0}, \mathbb{Z}))$ are also torsion abelian groups for all p and all odd q.

(2) It follows from the fact that the strict cohomological dimension of the Galois group G_F of the totally imaginary field F is 2.

(3) By Theorem 4.3(3), we have an isomorphism $H^2(L_F^{1,0}, \mathbb{Z}) \simeq H^2(W_F^{1,0}, \mathbb{Z})$, thus $H^0(G_F, H^2(L_F^{1,0}, \mathbb{Z})) = H^0(G_F, H^2(W_F^{1,0}, \mathbb{Z})) = (C_F^{1,0})^D$ and $H^p(G_F, H^2(L_F^{1,0}, \mathbb{Z})) = H^p(G_F, H^2(W_F^{1,0}, \mathbb{Z})) = 0$.

(4) It follows immediately from Theorem 4.3(2).

(5) The first equality follows directly from Theorem 4.3(1). For the second equality, the case $p \geq 3$ follows from (2). Since G_F acts on $\mathbb{Z}$ trivially, $H^0(G_F, \mathbb{Z}) = \mathbb{Z}$. $H^1(G_F, \mathbb{Z}) = \mathrm{Hom}_{cont}(G_F, \mathbb{Z}) = 0$ since all continuous homomorphic images of G_F in $\mathbb{Z}$ are compact subgroups, which are essentially trivial. $H^2(G_F, \mathbb{Z}) = H^1(G_F, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}_{cont}(G_F, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}_{cont}(G_F^{ab}, \mathbb{R}/\mathbb{Z}) = (G_F^{ab})^D$. □

Theorem 4.5. *Denote by $A^D = Hom_{cont}(A, \mathbb{R}/\mathbb{Z})$ the Pontryagin dual of a locally compact abelian group A. Let F be a totally imaginary number field and $\mathbb{Z}$ the discrete L_F-module with trivial action. Then*

(1) $H^0(L_F, \mathbb{Z}) = \mathbb{Z}$;
(2) $H^1(L_F, \mathbb{Z}) = 0$;
(3) $H^2(L_F, \mathbb{Z}) = (C_F^1)^D$;
(4) $H^{2k}(L_F, \mathbb{Z})$ are abelian groups of infinite rank for all $k \geq 2$.

Proof. The computation is based on Lemma 4.4 and the isomorphisms

$$H^i(L_F, \mathbb{Z}) \cong H^i(L_F^1, \mathbb{Z}), \qquad H^i(W_F, \mathbb{Z}) \cong H^i(W_F^1, \mathbb{Z}).$$

(1) $H^0(L_F^1, \mathbb{Z}) = \mathcal{E}_2^{0,0} = H^0(G_F, \mathbb{Z}) = \mathbb{Z}$.
(2) It follows from Lemma 4.4(4),(5) that $\mathcal{E}_2^{0,1} = \mathcal{E}_2^{1,0} = 0$.
(3) Since $\mathcal{E}_2^{p,1} = 0$ for all p. It is easy to see that $H^2(L_F, \mathbb{Z})$ is an extension of $\mathcal{E}_2^{0,2}$ by $(G_F^{ab})^D$. We then obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (G_F^{ab})^D \arrow[r] \arrow[d, equal] & H^2(W_F, \mathbb{Z}) \arrow[r] \arrow[d] & (C_F^{1,0})^D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (G_F^{ab})^D \arrow[r] & H^2(L_F, \mathbb{Z}) \arrow[r] & \mathcal{E}_2^{0,2} \arrow[r] & 0
\end{tikzcd}$$

with rows exact. Thus the five lemma gives $H^2(W_F, \mathbb{Z}) \to H^2(L_F, \mathbb{Z})$ being injective. Now by the Snake lemma, we can complete the commutative diagram into

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (G_F^{ab})^D \arrow[r] \arrow[d, equal] & H^2(W_F, \mathbb{Z}) \arrow[r] \arrow[d] & (C_F^{1,0})^D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (G_F^{ab})^D \arrow[r] \arrow[d] & H^2(L_F, \mathbb{Z}) \arrow[r] \arrow[d] & \mathcal{E}_2^{0,2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(G_F, (K_F^{ab})^D) \arrow[r, equal] \arrow[d] & H^0(G_F, (K_F^{ab})^D) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

It follows that $H^2(L_F, \mathbb{Z})$ is an extension of $H^2(W_F, \mathbb{Z}) = (C_F^1)^D$ by $H^0(G_F, (K_F^{ab})^D) = 0$ by Theorem 4.3(3) and therefore $H^2(L_F, \mathbb{Z}) = (C_F^1)^D$.
(4) Note that for $k \geq 2$, we have that $\mathcal{E}_3^{0,2k} = \ker(d_2 : \mathcal{E}_2^{0,2k} \to \mathcal{E}_2^{2,2k-1})$ survive in the limit of the spectral sequence. Thus we always have a surjective homomorphism $H^{2k}(L_F, \mathbb{Z}) \to \mathcal{E}_3^{0,2k}$. Hence to show that $H^{2k}(L_F, \mathbb{Z})$ are abelian groups of infinite rank, it suffices to show this for $\mathcal{E}_3^{0,2k}$. From Lemma 4.4(1), $\mathcal{E}_2^{2,2k-1}$ is a torsion abelian group, so if $\mathcal{E}_2^{0,2k}$ is an abelian group of infinite rank, then $\mathcal{E}_3^{0,2k}$ being its kernel to $\mathcal{E}_2^{2,2k-1}$ must also be of

infinite rank. Now from Theorem 4.3(4), the inclusion $H^{2k}(W_F^{1,0},\mathbb{Z})$ into $H^{2k}(L_F^{1,0},\mathbb{Z})$ induces an inclusion

$$H^0(G_F, H^{2k}(W_F^{1,0},\mathbb{Z})) \to H^0(G_F, H^{2k}(L_F^{1,0},\mathbb{Z})) = \mathcal{E}_2^{0,2k}.$$

Note that $H^0(G_F, H^{2k}(W_F^{1,0},\mathbb{Z}))$ is of countable infinite rank (cf. [5], in the proof of Theorem 10.1, p. 652), so $H^0(G_F, H^{2k}(L_F^{1,0},\mathbb{Z})) = \mathcal{E}_2^{0,2k}$ is also of infinite rank. This completes the proof. $\square$

5. Appendix: On the cohomological dimension of topological groups by M. Flach

In this appendix we answer a question raised in [5, 9.3] as to the cohomological dimension of the group $G = \mathbb{R}$. By comparison with Lie(G)-cohomology one shows that $H^i(\mathbb{R},\mathcal{A}) = 0$ for $i > 1$ if $\mathcal{A} = y(V)$ is represented by a locally convex, Hausdorff, quasi-complete topological $\mathbb{R}$-vector space V and the G-action on V is differentiable, and it was asked if this vanishing holds for more general sheaves.

This turns out to be false. We shall in fact show that the cohomological dimension of BG is infinite for any non-discrete locally compact group G, even restricting to coefficient sheaves $y(V)$ represented by a topological $\mathbb{R}$-vector space V if G has non-torsion elements in every neighborhood of the identity. This result follows from the simple observation that $H^i(G, s_*M) = H^i(G^\delta, M)$, where G^δ is the underlying discrete group of a topological group G, M is a G^δ-module and $s : \mathrm{Set} \to \mathcal{T}$ is a certain point of the topos $\mathcal{T}$. This observation is explained in the first section and the application to locally compact groups is given in the second.

5.1. **Comparison to the underlying discrete group.** We continue the notation introduced in the main body of this article. As in [5] we shall abbreviate $H^i(G,A) := H^i(G, y(A))$, i.e. the notation $H^i(G,A)$ will always denote the topos cohomology rather than the continuous cochain groups. We set $\tilde{\mathbb{R}} = y(\mathbb{R})$, where $\mathbb{R}$ carries its Euclidean topology, whereas we write $\mathbb{R}$ for $y(\mathbb{R}^\delta)$, where X^δ always denotes the underlying set of a topological space X with the discrete topology.

For any morphism $f : \mathcal{E}' \to \mathcal{E}$ of topoi and group object $\mathcal{G}$ of $\mathcal{E}$ there is a commutative diagram of morphisms of topoi

$$\begin{tikzcd} Bf^*\mathcal{G} \arrow[r, "f"] \arrow[d, "p'"'] & B\mathcal{G} \arrow[d, "p"] \\ \mathcal{E}' \arrow[r, "f"] & \mathcal{E} \end{tikzcd}$$

where p and p' are the canonical projections and the top row we also denote by f since it coincides with f on objects and morphisms. The $\mathcal{G}$-action on $f_*(X')$ is given by

$$\mathcal{G} \times f_*(X') \to f_*f^*(\mathcal{G}) \times f_*(X') \cong f_*(f^*(\mathcal{G}) \times X') \to f_*(X'),$$

where the first arrow is induced by the unit of the adjunction and the last by the $f^*(\mathcal{G})$-action on X'. The $f^*(\mathcal{G})$-action on $f^*(X)$ arises from the (right) exactness of f^*.

Lemma 5.1. *If $f_* : \mathcal{E}' \to \mathcal{E}$ is exact, then so is $f_* : Bf^*\mathcal{G} \to B\mathcal{G}$, and we have*

$$(R^ip_*)f_*\mathcal{A} \cong f_*(R^ip'_*\mathcal{A})$$

and

$$H^i(\mathcal{G}, f_*\mathcal{A}) \cong H^i(f^*(\mathcal{G}), \mathcal{A})$$

for all abelian group objects $\mathcal{A}$ of $Bf^*(\mathcal{G})$. If $f_* : \mathcal{E}' \to \mathcal{E}$ is fully faithful, then so is $f_* : Bf^*\mathcal{G} \to B\mathcal{G}$.

Proof. The exactness of $f_* : Bf^*\mathcal{G} \to B\mathcal{G}$ follows since it is identical to the exact functor $f_* : \mathcal{E}' \to \mathcal{E}$ on objects and morphisms. Choosing an injective resolution $\mathcal{B} \to I^\bullet$ in $Bf^*\mathcal{G}$ and noting that f_* maps injectives to injectives gives the statements about derived functors. Since the vertical maps in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{B\mathcal{G}}(f_*X', f_*Y') \arrow[r] & \mathrm{Hom}_{\mathcal{E}}(f_*X', f_*Y') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathrm{Hom}_{\mathcal{E}}(\mathcal{G} \times f_*X', f_*Y') \arrow[d] \\
 & \mathrm{Hom}_{\mathcal{E}'}(f^*f_*X', Y') \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}_{\mathcal{E}'}(f^*\mathcal{G} \times f^*f_*X', Y') \\
\mathrm{Hom}_{Bf^*\mathcal{G}}(X', Y') \arrow[r] & \mathrm{Hom}_{\mathcal{E}'}(X', Y') \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & \mathrm{Hom}_{\mathcal{E}'}(f^*\mathcal{G} \times X', Y') \arrow[u]
\end{tikzcd}$$

are isomorphisms by full faithfulness of f_* and the horizontal sequences are equalizers as explained in [11, VII.3, eq (5)], we get full faithfulness of f_* on $Bf^*\mathcal{G}$. □

Lemma 5.2. *There is a morphism of topoi,*

$$s : \mathrm{Set} \to \mathcal{T},$$

with $s^*(\mathcal{F}) = \mathcal{F}(\{*\})$ and $s_*(S)(U) = \mathrm{Hom}_{\mathrm{Set}}(U, S)$ for any sheaf $\mathcal{F}$ on Top *for the open covering topology, set S and topological space U. Moreover, the functor s_* is fully faithful and exact.*

Proof. One has a pair of adjoint functors

$$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left, "d"] & \mathrm{Top} \arrow[l, shift left, "u"] \end{tikzcd}$$

where $u(X)$ is the underlying set of a topological space X and $d(S)$ is the set S with the discrete topology. The functor d is the left adjoint, and the functor u is continuous for the open covering topology J_{open} on Top and the canonical topology J_{can} on Set (which has surjective families as covering morphisms). Therefore, by [11, VII. 10 Th 4], there exists a morphism of topoi

$$s : \mathrm{Set} = \mathrm{Sh}(\mathrm{Set}, J_{can}) \to \mathrm{Sh}(\mathrm{Top}, J_{open}) = \mathcal{T},$$

with

$$s^*(\mathcal{F}) = a(\mathcal{F} \circ d); \quad s_*(S) = \mathrm{Hom}_{\mathrm{Set}}(-, S) \circ u$$

where a is the associated sheaf. But since $\mathcal{F}$ is a sheaf for J_{open} the functor

$$S \mapsto \mathcal{F}(d(S)) = \prod_{s \in S} \mathcal{F}(\{*\}) = \mathrm{Hom}_{\mathrm{Set}}(S, \mathcal{F}(\{*\}))$$

on Set is represented by the set $\mathcal{F}(\{*\})$, and is in particular already a sheaf. This gives the description of s^*. The adjunction

$$s^*s_*(S) = \mathrm{Hom}_{\mathrm{Set}}(\{*\}, S) = S$$

is an isomorphism, i.e. s_* is fully faithful. Finally, the functor s_* is exact since it is already exact as a functor with values in presheaves on Top where on each object

U it coincides with the product functor in Set indexed by the underlying set of U. In Set product functors over arbitrary index sets are exact. $\square$

Remark 1. We note that $s_*(S) = y(S^c)$, where S^c is the set S with the chaotic topology which has S and $\emptyset$ as its only open sets. So all objects of $\mathcal{T}$ which lie in the essential image of s_* are representable.

For any topological space X the set

$$s^*y(X) = y(X)(\{*\}) = \mathrm{Hom}_{\mathrm{Set}}(\{*\}, X) = X$$

is simply the underlying set of X.

Corollary 5.3. *For any topological group G with underlying discrete group G^δ, and any G^δ-module M we have*

$$H^i(G, s_*M) = H^i(s^*G, M) = H^i(G^\delta, M).$$

Proof. Combine Lemmas 5.1 and 5.2. $\square$

5.2. **Locally compact groups.** We first consider the case where $G \cong \mathbb{R}, S^1$ or $\prod_{p\in S} \mathbb{Z}_p$ for some set of primes S and then generalize to locally compact groups.

Lemma 5.4. *If $G \cong \mathbb{R}, S^1$ or $\prod_{p\in S} \mathbb{Z}_p$, then $H^i(G^\delta, \mathbb{R})$ is a $\mathbb{R}$-vector space of infinite dimension for any $i > 0$.*

Proof. The group $\mathbb{R}^\delta$ is a $\mathbb{Q}$-vector space of uncountable dimension, and by [3, Ch. V, Thm. 6.4(ii)] we have an isomorphism $H_i(\mathbb{R}^\delta, \mathbb{Z}) \cong \bigwedge^i_{\mathbb{Z}}(\mathbb{R}^\delta) \cong \bigwedge^i_{\mathbb{Q}}(\mathbb{R}^\delta)$ which is likewise a $\mathbb{Q}$-space of uncountable dimension for $i > 0$. The universal coefficient exact sequence [3, Ch. III, §1, Exer. 3] gives a surjection (actually an isomorphism)

$$H^i(\mathbb{R}^\delta, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(H_1(\mathbb{R}^\delta, \mathbb{Z}), \mathbb{R})$$

which shows the lemma for $G = \mathbb{R}$. In the case $G = S^1$ one has an exact sequence

$$1 \to F \to G^\delta \to E \to 1 \tag{5.1}$$

where $F = G_{tor} \cong \mathbb{Q}/\mathbb{Z}$ and E is uniquely divisible, i.e. a $\mathbb{Q}$-vector space (again of uncountable dimension). The Hochschild-Serre spectral sequence

$$H^i(E, H^j(F, \mathbb{R})) \Rightarrow H^{i+j}(G^\delta, \mathbb{R})$$

gives isomorphisms

$$H^i(E, \mathbb{R}) \to H^i(G^\delta, \mathbb{R}) \tag{5.2}$$

since F is torsion. Indeed, again by [3, Ch. V, Thm. 6.4(ii)] we have $H_i(F, \mathbb{R}) \cong \bigwedge^i_{\mathbb{R}}(F \otimes_{\mathbb{Z}} \mathbb{R}) = 0$ for $i > 0$, and the universal coefficient sequences for homology and cohomology [3, Ch. III, §1, Exer. 3] show that $H^i(F, \mathbb{R}) = 0$ for $i > 0$.

Finally, for $G = \prod_{p\in S} \mathbb{Z}_p$ there is an exact sequence (5.1) where E is a $\mathbb{Q}$-vector space of uncountable dimension and F lies in an exact sequence

$$0 \to \mathbb{Z} \to F \to \bigoplus_{p\notin S} \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

One easily checks that $H^j(F, \mathbb{R})$ is finite dimensional so that the same is true for the kernel of the edge homomorphism (5.2). Therefore $H^i(G^\delta, \mathbb{R})$ is again of uncountable dimension for any $i > 0$. $\square$

Proposition 5.5. *Let G be a locally compact group that has non-torsion elements in every neighborhood of the identity. Then there is topological $\mathbb{R}$-vector space V (with continuous G-action) so that $H^i(G, V) \neq 0$ for all i.*

Proof. If the connected component G^0 of G is non-trivial it is known that there are non-trivial continuous homomorphisms $\phi : \mathbb{R} \to G^0 \subseteq G$ (see e.g. [8, Prop. 3.30]). The kernel of ϕ is a closed subgroup of $\mathbb{R}$, and hence either 0 or infinite cyclic. So G^δ contains a subgroup H isomorphic to $\mathbb{R}^\delta$ or $(S^1)^\delta$. The coinduced module $V = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], \mathbb{R})$ (with the chaotic topology) is a topological $\mathbb{R}$-vector space and satisfies

$$H^i(G, y(V)) = H^i(G, s_*(V)) = H^i(G^\delta, V) = H^i(H, \mathbb{R})$$

by Shapiro's Lemma. We conclude by Lemma 5.4.

If $G^0 = \{1\}$, then G is totally disconnected and therefore has an open neighborhood basis at the identity consisting of open compact subgroups [17, Prop. 4.13]. By assumption there is a non-torsion element in each such group, therefore generating a procyclic subgroup H',

$$H' \cong \prod_{p \in T} \mathbb{Z}_p / p^{n_p} \mathbb{Z}_p \times \prod_{p \in S} \mathbb{Z}_p,$$

with $S \neq \emptyset$ and T some set of primes disjoint from S (see e.g. [15, p. 78]). This group in turn contains a subgroup $H \cong \prod_{p \in S} \mathbb{Z}_p$ from which we can induce as before. □

Corollary 5.6. *If G is a non-discrete locally compact group, then there exists a topological G-module A with $H^i(G, A) \neq 0$ for arbitrary large i.*

Proof. In the above proof we might have $S = \emptyset$, i.e. any element in a given neighborhood of the identity might be torsion. Since it is well known that $H^{2i}(H, \mathbb{Z}) \neq 0$ for a *non-trivial* finite cyclic group H [3, Ch. III, §1, Example 2], we can continue the above argument with $V = A = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], \mathbb{Z})$ if G is not discrete. □

References

[1] James Arthur, *A note on the automorphic Langlands group*, Canad. Math. Bull. **45** (2002), no. 4, 466–482, DOI 10.4153/CMB-2002-049-1. Dedicated to Robert V. Moody. MR1941222 (2004a:11120)

[2] T. Austin, *Continuity properties of Moore cohomology*, Preprint, available online at, arXiv.org:1004.4937, 2010.

[3] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1982. MR672956 (83k:20002)

[4] N. Bourbaki, *General Topology*, Springer-Verlag, Berlin (1989).

[5] M. Flach, *Cohomology of topological groups with applications to the Weil group*, Compos. Math. **144** (2008), no. 3, 633–656, DOI 10.1112/S0010437X07003338. MR2422342 (2009f:14033)

[6] R. Gonzales, *Localization in equivariant cohomology and GKM theory*, http://www.math.uwo.ca/ rgonzal3/qfy.pdf.

[7] A. Grothendieck, M. Artin, J. L. Verdier, *Theorie des topos et cohomologie etale des schemas (SGA4)*, Lecture Notes in Math. Soc. **269, 270, 271**, Springer, Berlin (1972).

[8] Karl H. Hofmann and Sidney A. Morris, *The Lie theory of connected pro-Lie groups*, EMS Tracts in Mathematics, vol. 2, European Mathematical Society (EMS), Zürich, 2007. A structure theory for pro-Lie algebras, pro-Lie groups, and connected locally compact groups. MR2337107 (2008h:22001)

[9] Karl H. Hofmann and Paul S. Mostert, *Cohomology theories for compact abelian groups*, Springer-Verlag, New York, 1973. With an appendix by Eric C. Nummela. MR0372113 (51 #8330)

[10] Stephen Lichtenbaum, *The Weil-étale topology for number rings*, Ann. of Math. (2) **170** (2009), no. 2, 657–683, DOI 10.4007/annals.2009.170.657. MR2552104 (2011a:14035)

[11] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1994. A first introduction to topos theory; corrected reprint of the 1992 edition. MR1300636 (96c:03119)

[12] Calvin C. Moore, *Extensions and low dimensional cohomology theory of locally compact groups. I, II*, Trans. Amer. Math. Soc. **113** (1964), 40–63. MR0171880 (30 #2106)

[13] Calvin C. Moore, *Group extensions and cohomology for locally compact groups. III*, Trans. Amer. Math. Soc. **221** (1976), no. 1, 1–33. MR0414775 (54 #2867)

[14] Paul S. Mostert, *Local cross sections in locally compact groups*, Proc. Amer. Math. Soc. **4** (1953), 645–649. MR0056614 (15,101d)

[15] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2000. MR1737196 (2000j:11168)

[16] John F. Price, *Lie groups and compact groups*, Cambridge University Press, Cambridge, 1977. London Mathematical Society Lecture Note Series, No. 25. MR0450449 (56 #8743)

[17] Markus Stroppel, *Locally compact groups*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006. MR2226087 (2007d:22001)

[18] David Wigner, *Algebraic cohomology of topological groups*, Trans. Amer. Math. Soc. **178** (1973), 83–93. MR0338132 (49 #2898)

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