MULTI-SCALING LIMITS FOR RELATIVISTIC DIFFUSION EQUATIONS WITH RANDOM INITIAL DATA

GI-REN LIU AND NARN-RUEIH SHIEH

Abstract. Let \( u(t, x) \), \( t > 0, \ x \in \mathbb{R}^n \), be the spatial-temporal random field arising from the solution of a relativistic diffusion equation with the spatial-fractional parameter \( \alpha \in (0, 2) \) and the mass parameter \( m > 0 \), subject to a random initial condition \( u(0, x) \) which is characterized as a subordinated Gaussian field. In this article, we study the large-scale and the small-scale limits for the suitable space-time rescalings of the solution field \( u(t, x) \). Both the Gaussian and the non-Gaussian limit theorems are discussed. The small-scale scaling involves not only scaling on \( u(t, x) \) but also re-scaling the initial data; this is a new type result for the literature. Moreover, in the two scalings the parameter \( \alpha \in (0, 2) \) and the parameter \( m > 0 \) play distinct roles for the scaling and the limiting procedures.

1. Introduction

The relativistic operator

\[
m - (m^{\frac{\alpha}{2}} - \Delta)^{\frac{\alpha}{2}},
\]

with \( \alpha = 1 \) and \( m > 0 \), appeared in the 1970’s (to our knowledge) in an article of Lieb [23] studying the stability theory of matters. Its connection with Lévy processes is investigated by Carmona et al. in [8]. In Ryznar [29], the version for general \( 0 < \alpha < 2 \) is studied; see also Baeumer et al. [4], Kumara et al. [18], and Chen et al. [9] for more recent studies. The operator should be understood as a pseudo-differential operator, as that in [8] and [34]. The Fourier transform of the heat kernel associated with the operator shows an interesting multi-scaling property, as that observed in [30,31].

The purpose of this article is to present this multi-scaling property from the associated PDE. Namely, we consider the following Cauchy problem for the relativistic diffusion equation (RDE for brevity), subject to some random initial data, and aim to discuss the multi-scaling limits for the spatial-temporal random field arising from the solution of this random initial value problem:

\[
(1.1) \quad \frac{\partial}{\partial t} u(t, x) = (m - (m^{\frac{\alpha}{2}} - \Delta)^{\frac{\alpha}{2}}) u(t, x), \quad u(0, x) = u_0(x), \quad t \geq 0, \ x \in \mathbb{R}^n.
\]

Received by the editors July 7, 2012 and, in revised form, October 1, 2012 and May 7, 2013.

2010 Mathematics Subject Classification. Primary 60G60, 60H05, 62M15, 35K15.

Keywords and phrases. Large-scale limits, small-scale limits, relativistic diffusion equations, random initial data, multiple Itô-Wiener integrals, subordinated Gaussian fields, Hermite ranks.

The first author was partially supported by a Taiwan NSC grant for graduate students.
RDEs (1.1) also appear to be an essential role in the theory of computer vision; see a special volume edited by Kimmel et al. [16], in which PDE and scale-space methods are the focus and RDEs are particularly employed.

We describe the structure and the goal of this article as follows. We consider the random initial data \( u_0 \) to be subordinated Gaussian random fields and study the large-scale and the small-scale limits for the properly re-scaled solution field. We prove that the two parameters \( \alpha \) and \( m > 0 \) play distinct roles in the two scaling behaviors. For the large-scale limit (Theorem 1 and Theorem 3), it is the mass \( m > 0 \) that dominates the space-time scaling and also the limiting field, which brings the \( m > 0 \) into its structure. While for the small-scale limit (Theorem 2 and Theorem 4), it is the spatial index \( \alpha \) that dominates both the scaling factor and the limiting field, and it appears to be irrelevant for \( m \) if it is positive or zero.

In our discussions, the large-scale Theorem 1 and Theorem 3 are respectively comparable to the Central Limit Theorem for local functionals of random fields with weak dependence in [7] and to a certain non-Gaussian Central Limit Theorem in [11,33]. For the small-scale Theorem 2 and Theorem 4, they involve not only the space-time scaling \( u(t, x), \ t > 0, \ x \in \mathbb{R}^n \), but also need to re-scale the initial data; to our knowledge, these are new type results for the literature; see [24] for the authors’ very recent study.

As for the methodology for proofs, for the Gaussian limits we employ the moments and the Feynman-type diagrams used notably in [7], and for the non-Gaussian limits we employ the truncation of Hermite expansions used notably in [12].

We remark that in the non-relativistic case, i.e. \( m = 0 \), the large-scale limits for the random initial value problem with multiple Itô-Wiener integrals as input have been discussed in Anh and Leonenko [11,2]; subsequent works, together with Burgers’ equation, in this direction by the authors and collaborators can be seen in [3,5,14,20,22,28] and the references therein. However, the multi-scaling limits due to the different roles of the mass and the fractional-index, the target of this article, are not all in the cited papers. Moreover, in this article we are able to drop-off the usually imposed isotropic assumption of the initial datum.

We should also mention that in an article discussing tempered stable Lévy processes by Rosiński [27], the author proves rigorously, among others, the statement that such a process in a short time looks like a stable process, while in a large time scale it looks like a Brownian motion. This article has shown nicely how the multi-scaling limits appear in the context of stochastic processes. (We are indebted to the referee for indicating to us the article [27] and the relevant concepts.)

In Section 2, we present some preliminaries; we state our main results in Section 3, and all the proofs of our results are given in Section 4.

Finally, we mention that the study of the PDEs with random initial conditions can be traced back to [15] and [26]. Besides the above-mentioned literature, there also has been very significant progress on Burgers’ equation with different types of random input; see the monograph of Woyczyński [36] and Chapter 6 of Bertoin [6].

2. Preliminaries

2.1. Green function for RDEs. As understood, we regard the spatial operator in the RDE (1.1) as a pseudo-differential operator; see for example the book and the paper by Wong [34,35]. The Green function, denoted by \( G_{\alpha,m}(t, x), \ t > 0, \ x \in \mathbb{R}^n \), for the Cauchy problem (1.1) is thus determined by the (spatial) Fourier transform
\( \hat{G}_{\alpha,m}(t, \lambda), \alpha \in (0, 2), \ m > 0, \) which is given by
\[
\int_{\mathbb{R}^n} e^{i(\lambda, x)} G_{\alpha,m}(t, x) dx = e^{-t((m^2 + |\lambda|^2) \frac{\alpha}{2} - m)}, \ \lambda \in \mathbb{R}^n.
\]

See Carmona et al. [8] for general \( \alpha \in (0, 2) \) [29] also considers the boundary problem). These papers also study \( G_{\alpha,m}(t, x), \ m > 0, \) as the transition probability density of a Lévy process \( X_{\alpha,m}(t) \) which is the subordination of the Brownian motion by a certain subordinator. The explicit expression for the Green function is known only in the case \( \alpha = 1; \) see for example the recent works [11,29], which give explicit calculations to show that the subordinator is normal inverse Gaussian.

The solution of (1.1) is given in the form
\[
u(t, x; u_0(\cdot)) = \int_{\mathbb{R}^n} G_{\alpha,m}(t, x - y)u_0(y)dy.
\]

In this work, our initial data is a second-order homogeneous random field on \( \mathbb{R}^n, \) and thus the solution of (1.1) should be understood as a mean-square solution, resulting in a spatial-temporal random solution field \( u(t, x), \) See [28] Proposition 1] for some discussions on the mean-square solutions of parabolic PDEs with mean-square random initial data.

### 2.2. Subordinated Gaussian fields as initial data

Let \((\Omega, \mathcal{F}, \mathcal{P})\) be an underlying probability space such that all random elements appearing in this article are measurable with respect to it.

We specify the initial data \( u_0(x) \) be a subordinated Gaussian field, introduced by Dobrushin [10, as follows; see also [12] for more recent discussions.

**Condition A.** The initial data of (1.1) is assumed to be a random field on \( \mathbb{R}^n \) given by
\[
u_0(x) = h(\zeta(x)), \ x \in \mathbb{R}^n,
\]
where \( \zeta(x) \) is a mean-square continuous and homogeneous Gaussian random field with mean zero and variance 1, and its spectral measure \( F(d\lambda) \) has the (spectral) density \( f(\lambda), \ \lambda \in \mathbb{R}^n; \) moreover, \( h : \mathbb{R} \to \mathbb{R} \) is a (non-random) function such that
\[
\mathbb{E} h^2(\zeta(0)) = \int_{\mathbb{R}} h^2(r)p(r)dr < \infty; \quad p(r) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2}, \ r \in \mathbb{R}.
\]

Under Condition A, by the Bochner-Khintchine theorem, we have the following spectral representation for the covariance function of the Gaussian field \( \zeta(x): \)
\[
R(x) = \text{Cov}(\zeta(0), \zeta(x)) = \int_{\mathbb{R}^n} e^{i(\lambda, x)} f(\lambda)d\lambda.
\]

Moreover, by the Karhunen Theorem, \( \zeta(x) \) has the representation
\[
\zeta(x) = \int_{\mathbb{R}^n} e^{i(\lambda, x)} \sqrt{f(\lambda)}W(d\lambda), \ x \in \mathbb{R}^n,
\]
where \( W(d\lambda) \) is the standard complex-valued Gaussian white noise on the Fourier domain \( \mathbb{R}^n, \) that is, a centered orthogonal-scattered Gaussian random measure on \( \mathbb{R}^n \) such that \( W(\Delta_1) = W(-\Delta_1) \) and \( \mathbb{E}W(\Delta_1)W(\Delta_2) = \text{Leb}(\Delta_1 \cap \Delta_2) \) for any \( \Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R}^n). \) See, for example, the book of Leonenko [19, Theorem 1.1.3] for
the above facts. We need the following expansion of $h(r)$ in the Hilbert space $L^2(\mathbb{R}, p(r)dr)$:

$$h(r) = C_0 + \sum_{l=1}^{\infty} C_l \frac{H_l(r)}{\sqrt{l!}},$$

where

$$C_l = \int_{\mathbb{R}} h(r) \frac{H_l(r)}{\sqrt{l!}} p(r) dr$$

and $\{H_l(r), l = 0, 1, 2, \ldots \}$ are the Hermite polynomials, that is,

$$H_l(r) = (-1)^l e^{\frac{r^2}{2}} \frac{d^l}{dr^l} e^{-\frac{r^2}{2}}, \text{ for } l \in \{0, 1, 2, \ldots \}.$$  

Accordingly, the Hermite rank of the function $h(\cdot)$ is defined by

$$m := \inf \{l \geq 1 : C_l \neq 0 \}.$$  

It is well-known that (see, for example, Major [25, Corollary 5.5 and p. 30])

$$E[H_{l_1}(\zeta(y))H_{l_2}(\zeta(z))] = \delta_{l_1 l_2} l_1! R^{l_1}(y-z), \quad y, z \in \mathbb{R}^n$$  

($\delta_{l_1 l_2}$ is the Kronecker symbol), and

$$H_l(\zeta(x)) = \int_{\mathbb{R}^n}^l e^{i(x, \lambda_1 + \cdots + \lambda_l)} \prod_{k=1}^{l} \sqrt{f(\lambda_k)} W(d\lambda_k).$$

In the above, (2.10) means the multiple Itô-Wiener integral representation, and the integration $\int^l$ means that it excludes the diagonal hyperplanes $z_i = \pm z_j$, $i, j = 1, \ldots, l, i \neq j$.

We impose two different conditions on the singularity of the spectral density $f(\lambda)$ at $0$ which yield, respectively, the Gaussian and the non-Gaussian scaling limits.

**Condition B.** The spectral density function $f(\lambda)$ of the Gaussian random field $\zeta(x)$ in Condition A can be written as

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}} \text{ for some } \kappa > \frac{n}{m},$$

where $m$ is the Hermite rank of the function $h$, and the $B(\cdot) \in C(\mathbb{R}^n)$ is of suitable decay at infinity to ensure $f \in L^1(\mathbb{R}^n)$.

**Condition C.** The spectral density function $f(\lambda)$ of the Gaussian random field $\zeta(x)$ in Condition A can be written as

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}}, \quad 0 < \kappa < \frac{n}{m},$$

where $m$ is the Hermite rank of the function $h$, and the $B(\cdot) \in C(\mathbb{R}^n)$ is of suitable decay at infinity to ensure $f \in L^1(\mathbb{R}^n)$, and moreover $B(0) > 0$.

Note that in the two conditions, we do not assume that the $B(\cdot)$ is radial in $\cdot$, so that the field $u_0(x)$ is not necessary to be isotropic. We also mention that Condition B means that the density $f$ either is regular at $0$ or has a singularity for which the order is less than $n(1 - 1/m)$, while Condition C means that $f$ has a singularity at $0$ for which the order is higher than $n(1 - 1/m)$. 


By (2.5) and the convolutions, we have, for each \( l \geq 1 \),
\[
R^l(x) = \int_{\mathbb{R}^n} e^{i(\lambda, x)} f^{*l}(\lambda) d\lambda, \quad l \in \mathbb{N},
\]
where \( f^{*l}(\lambda) \) is the \( l \)-fold convolution of \( f \) defined recursively as \( f^{*1} = f \) and
\[
f^{*l}(\lambda) = \int_{\mathbb{R}^n} f(\lambda - \eta) f^{*(l-1)}(\eta) d\eta, \quad l \geq 2.
\]
The following analytic lemma asserts the behavior of \( f^{*l}, \, l \in \mathbb{N} \); for completeness, we give its proof in Appendix A.

**Lemma 1.** Suppose that the spectral density function \( f \) has the form
\[
f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}}, \quad \kappa > 0,
\]
for some non-negative bounded and continuous function \( B(\lambda) \) so that \( f \in L^1(\mathbb{R}^n) \).

Then for any \( k \geq 2 \) there exists a bounded function \( B_k \in C(\mathbb{R}^n \setminus \{0\}) \) such that the
\( k \)-fold convolution \( f^{*k} \) of \( f \) can be written as
\[
f^{*k}(\lambda) = \begin{cases} 
B_k(\lambda)|\lambda|^{k\kappa-n}, & \text{for } k\kappa < n, \\
B_k(\lambda)\ln(2 + \frac{1}{|\lambda|}), & \text{for } k\kappa = n, \\
B_k(\lambda) \in C(\mathbb{R}^n), & \text{for } k\kappa > n.
\end{cases}
\]
Moreover, for any \( k_1 > k_2 > n/\kappa \) the inequality \( \sup_{\lambda \in \mathbb{R}^n} B_{k_1}(\lambda) \leq \sup_{\lambda \in \mathbb{R}^n} B_{k_2}(\lambda) \) holds.

To understand the difference of Conditions B and C, in view of Lemma 1, Condition B implies that the \( k \)-fold convolution \( f^{*k}, \, k \geq m \), has no singularity at the origin \( \lambda = 0 \), which in turn asserts that the spectral density of the random initial data \( u_0 \) has no singularity at \( \lambda = 0 \), while Condition C asserts that the initial data \( u_0 \) has a spectral density which is singularity at \( \lambda = 0 \). The situation can be described as, respectively, the long-range and the short-range dependence of the initial field \( u_0 \), a central notion in vast applications; one may refer to the special volume by Doukhan, Oppenheim, and Taqqu [12].

3. Main results

The significant difference between Condition B and Condition C, as remarked at the end of the last section, is employed to obtain the Gaussian and respectively the non-Gaussian scaling limits. We will present them in two subsections.

In the context henceforth, the notation \( \Rightarrow \) denotes the convergence of random variables (respectively, random families) in the sense of distribution (respectively, finite-dimensional distributions).

3.1. Gaussian limits with initial data in \((A, B)\). As mentioned in Section 1, we will present the large-scale and the small-scale limit theorems. We remark that our Theorems 1 and 2 in this subsection are comparable to the Central Limit Theorem for local functionals of random fields with weak dependence in Breuer and Major [7]. The novel feature is that the mass \( m > 0 \) and the fractional-index \( \alpha \) play different roles in the two-scales.
Theorem 1. Let \( u(t, x; u_0(\cdot)), t > 0, x \in \mathbb{R}^n \), be the mean-square solution of (1.1) with \( m > 0 \) and let the initial data \( u_0(x) = h(\zeta(x)) \) satisfy Conditions A and B with the Hermite rank \( m \geq 1 \). Then when \( T \to \infty \),
\[
T^{\frac{3}{2}} \left\{ u(Tt, \sqrt{T}x; u_0(\cdot)) - C_0 \right\} \Rightarrow U(t, x),
\]
where \( U(t, x), t > 0, x \in \mathbb{R}^n \), is a Gaussian field with the following spectral representation:
\[
(3.1) \quad U(t, x) = \int_{\mathbb{R}^n} e^{i\lambda \cdot x} \sigma_m e^{-t \frac{3}{2} m^{-1} - \frac{3}{2} |\lambda|^2} W(d\lambda), \quad \sigma_m = \left( \sum_{r=m}^{\infty} f^{r^2}(0)C_r^2 \right)^{\frac{1}{2}},
\]
where \( W(d\lambda) \) is a complex-valued standard Gaussian noise measure on \( \mathbb{R}^n \) (cf. (2.6)).

For the small-scale limit, we need to re-scale the initial data too; thus the notation \( u_0(\varepsilon^{-\frac{1}{n}} - x) \) imposed on \( u_0 \) means that the variable of \( u_0 \) is under the indicated dilation factor \( \varepsilon^{-\frac{1}{n}} - x \).

Theorem 2. Let \( u(t, x; u_0(\cdot)), t > 0, x \in \mathbb{R}^n \), be the mean-square solution of (1.1) with \( m > 0 \) and let the initial data \( u_0(x) = h(\zeta(x)) \) satisfy Conditions A and B with the Hermite rank \( m \geq 1 \). For any \( \varepsilon > 0 \), when \( \varepsilon \to 0 \),
\[
(3.2) \quad \varepsilon^{-\frac{3}{2}} \left\{ u(\varepsilon t, \varepsilon^{-\frac{1}{n}} x; u_0(\varepsilon^{-\frac{1}{n}} - x)) - C_0 \right\} \Rightarrow V(t, x),
\]
where \( V(t, x), t > 0, x \in \mathbb{R}^n \), is a Gaussian field with the following spectral representation:
\[
(3.3) \quad V(t, x) = \int_{\mathbb{R}^n} e^{i\lambda \cdot x} \sigma_m e^{-t |\lambda|^2} W(d\lambda), \quad \sigma_m = \left( \sum_{r=m}^{\infty} f^{r^2}(0)C_r^2 \right)^{\frac{1}{2}},
\]
where \( W(d\lambda) \) is a complex-valued standard Gaussian noise measure on \( \mathbb{R}^n \).

Remark. The typical case for Theorem 2 is \( \alpha = 1, \chi = 1/2 \). In this critical case, the scaling order for Theorems 1 and 2 is the same, namely \( n/4 \). However, the spatial scaling is square-root in Theorem 1, while it is linear in Theorem 2. Moreover, the integral kernel for the limiting field in the two theorems is Gauss vs. Poisson. The latter situation can be comparable with an analytic discussion in Wong [34].

3.2. Non-Gaussian limits with initial data in (A,C). As in the above subsection, we have the large-scale and the small-scale limits; however the high singularity order in Condition C asserts that our limiting fields are now non-Gaussian. The non-Gaussian limits are of the convolution type, which can be seen in the pioneering papers of Taqqu [33] and Dobrushin and Major [11], and more recently in Anh and Leonenko [12].

Theorem 3. Let \( u(t, x; u_0(\cdot)), t > 0, x \in \mathbb{R}^n \), be the mean-square solution of (1.1) whose initial data \( \{ u_0(x) = h(\zeta(x)), x \in \mathbb{R}^n \} \) satisfy Conditions A and C with \( \kappa \in (0, \frac{n}{m}) \) and \( 1 < m \), where \( m \) is the Hermite rank of the non-random function \( h \) on \( \mathbb{R} \), which has the Hermite coefficients \( C_j, j = 0, 1, \ldots \). Then when \( T \to \infty \),
\[
(3.4) \quad T^{\frac{m}{n}} \left\{ u(Tt, \sqrt{T}x; h(\zeta(\cdot))) - C_0 \right\} \Rightarrow U_m(t, x),
\]
where $U_m(t, x)$ is represented by the following multiple Wiener integrals:

$$U_m(t, x) = B^m(0) \frac{C_m}{\sqrt{m!}} \int_{\mathbb{R}^n \times m} e^{i(x, \lambda_1 + \cdots + \lambda_m)} \frac{\exp(-t \frac{m}{2} m^{1-\frac{\alpha}{2}} |\lambda_1 + \cdots + \lambda_m|^2)}{(|\lambda_1| \cdots |\lambda_m|)^{\frac{n-\alpha}{2}}} \prod_{l=1}^m W(d\lambda_l),$$

where $\int_{\mathbb{R}^n \times m} \cdots$ denotes an $m$-fold Wiener integral with respect to the complex Gaussian white noise $W(\cdot)$ on $\mathbb{R}^n$.

**Theorem 4.** Let $u(t, x; u_0(\cdot))$ be the mean-square solution to (1.1) whose initial data $\{u_0(x) = h(\zeta(x)), x \in \mathbb{R}^n\}$ satisfy Conditions $A$ and $C$ with $\kappa \in (0, \frac{n}{m})$ and $1 < m$, where $m$ is the Hermite rank of the function $h$. Then, for any fixed parameter $\chi > 0$, when $\varepsilon \to 0$,

$$\varepsilon^{-\frac{m\alpha}{2}} \{u(\varepsilon t, \varepsilon^{\frac{1}{\chi}} x; h(\varepsilon^{-(\frac{1}{\chi} - \chi)})) - C_0\} \Rightarrow V_m(t, x),$$

where $V_m(t, x)$ is represented by the multiple Wiener integrals

$$V_m(t, x) = B^m(0) \frac{C_m}{\sqrt{m!}} \int_{\mathbb{R}^n \times m} e^{i(x, \lambda_1 + \cdots + \lambda_m)} \frac{\exp(-t |\lambda_1 + \cdots + \lambda_m|^\alpha)}{(|\lambda_1| \cdots |\lambda_m|)^{\frac{n-\alpha}{2}}} \prod_{l=1}^m W(d\lambda_l).$$

**Remark.** In [2] the authors considered a hybrid differential operator in the spatial variable (the Riesz-Bessel operator) as follows:

$$-(\Delta)^{\alpha/2} (I - \Delta)^{\gamma/2}, \quad \alpha \in (0, 2), \quad \gamma \geq 0.$$

However, in their main Theorem 2.3, a large-scale limit in our context, only the Riesz parameter $\alpha$ plays the role and the Bessel parameter $\gamma$ is invisible. This intriguing situation is now justified by the RFD (1.1), in which we could say that it is “physically correct” to consider the relativistic operator $(m - (m^{\frac{\alpha}{2}} - \Delta)^{\frac{\alpha}{2}})$ rather than the Bessel operator in the form presented in [2].

### 4. Proofs of Theorems

The following two-scale property of the relativistic Green function $G_{\alpha, m}$ is the key to our results; when one deals with the Laplacian or the fractional Laplacian operator, it is instead only the mono-scaling. We describe this two-scale property in terms of Fourier transforms:

$$\widehat{G}_{\alpha, m}(Tt, T^{-\frac{\alpha}{2}} \lambda) = \exp \left\{ Tt (m - (m^{\frac{\alpha}{2}} + T^{-1} |\lambda|^2)^{\frac{\alpha}{2}}) \right\} \to \exp \left\{ -t \frac{\alpha}{2} m^{1-\frac{\alpha}{2}} |\lambda|^2 \right\},$$

as $T \to \infty$; (4.1) is a consequence of Taylor’s expansion,

$$m - (m^{\frac{\alpha}{2}} + T^{-1} |\lambda|^2)^{\frac{\alpha}{2}} = m - \left( m + \frac{\alpha}{2} (m^{\frac{\alpha}{2}})^{\frac{\alpha}{2}-1} T^{-1} |\lambda|^2 + \frac{\alpha}{4} (\frac{\alpha}{2} - 1) c_T^{\frac{\alpha}{2}-2} T^{-2} |\lambda|^4 \right)$$

$$= - \frac{\alpha}{2} (m^{\frac{\alpha}{2}})^{\frac{\alpha}{2}-1} T^{-1} |\lambda|^2 + \frac{\alpha}{4} (1 - \frac{\alpha}{2}) c_T^{\frac{\alpha}{2}-2} T^{-2} |\lambda|^4$$

for some $c_T \in (m^{\frac{\alpha}{2}}, m^{\frac{\alpha}{2}} + T^{-1} |\lambda|^2)$. In contrast to the large-scale (4.1), we have the following small-scale, as $\varepsilon \to 0$:

$$\widehat{G}_{\alpha, m}(\varepsilon t, \varepsilon^{-\frac{1}{\chi}} \lambda) = e^{\varepsilon t m} e^{-\varepsilon \varepsilon (m^{\frac{\alpha}{2}} + \varepsilon^{-\frac{\alpha}{2}} |\lambda|^2)^{\frac{\alpha}{2}}} \to e^{-t |\lambda|^\alpha}.$$

We observe that (4.2) indeed holds no matter whether $m$ is $> 0$ or $= 0$. 

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Proofs of Theorems 1 and 2. We apply the Hermite expansion \((2.7)\) to \(u(t, x)\). For the large-scale, we set

\[
X_T(t, x) = T^{-\frac{n}{2}} u(Tt, \sqrt{T}x; u_0(\cdot)) - C_0
\]

\[
= T^{-\frac{n}{2}} \sum_{k=m}^{\infty} \frac{C_k}{\sqrt{k!}} \int_{\mathbb{R}^n} G_{\alpha,m}(Tt, \sqrt{T}x - y) H_k(\zeta(y)) dy,
\]

and for the small-scale, we set

\[
Y_\varepsilon(t, x) := \varepsilon^{-\frac{n}{2}} u(\varepsilon t, \varepsilon^{\frac{1}{2}} x; u_0(\varepsilon^{-\frac{1}{2}} x)) - C_0
\]

\[
= \varepsilon^{-\frac{n}{2}} \sum_{l=m}^{\infty} \frac{C_l}{\sqrt{l!}} \int_{\mathbb{R}^n} G_{\alpha,m}(\varepsilon t, \varepsilon^{\frac{1}{2}} x - y) H_l(\zeta(\varepsilon^{-\frac{1}{2}} x y)) dy.
\]

Below, we proceed only with the proof of Theorem 2, the small-scale limit, and see how the re-scaling of the initial data is needed to obtain the desired limit. The proof of Theorem 1 is parallel and does not require the re-scaling of the initial data. Since the proof in the following does not require the \(m\) to be strictly positive, our Theorem 2 also provides a small-scale version of the large-scale, i.e. the usual, limit result in [2]. The methodology of the proof can be traced back to [7].

For any \(M \in \mathbb{N}\) and any set of real numbers \(\{a_1, a_2, \cdots, a_M\}\), denote

\[
(4.3)
\]

\[
\xi_\varepsilon := \sum_{j=1}^{M} a_j Y_\varepsilon(t_j, x_j),
\]

where \(\{t_1, \cdots, t_M\} \subset \mathbb{R}_+\) and \(\{x_1, \cdots, x_M\} \subset \mathbb{R}^n\) are arbitrary. In order to apply the Method of Moments to prove the statement of Theorem 2, we need to verify the following:

\[
\lim_{\varepsilon \to 0} \mathbb{E} \xi_\varepsilon^p = \begin{cases} 
0, & p = 2\nu + 1, \\
(p - 1)!! \left\{ \mathbb{E} \left[ \left( \sum_{j=1}^{M} a_j V(t_j, x_j) \right)^2 \right] \right\}^{\nu}, & p = 2\nu,
\end{cases}
\]

where \(V(t, x)\) is defined in (3.3). We remark that the high (i.e. \(p > 2\)) moments are needed, since \(\xi_\varepsilon\) is not Gaussian, though the desired limit is Gaussian. Firstly, we split \(\xi_\varepsilon\) into two parts:

\[
(4.5)
\]

\[
\xi_\varepsilon = \xi_\varepsilon,_{\leq N} + \xi_\varepsilon,_{> N},
\]

where (henceforth, we will suppress the indices \(\alpha\) and \(m\) for \(G_{\alpha,m}\) and \(\tilde{G}_{\alpha,m}\))

\[
(4.6)
\]

\[
\xi_\varepsilon,_{> N} = \sum_{j=1}^{M} a_j \varepsilon^{-\frac{n}{2}} \sum_{l=N+1}^{\infty} \frac{C_l}{\sqrt{l!}} \int_{\mathbb{R}^n} G(\varepsilon t_j, \varepsilon^{\frac{1}{2}} x_j - y) H_l(\zeta(\varepsilon^{-\frac{1}{2}} x y)) dy,
\]
and we prove that \( E[\ell^2_{\epsilon, > N}] \to 0 \), whenever \( N \) is chosen large enough. Observe that for any \( N \geq m - 1 \), by \((2.9)\),

\[
(4.7) \quad \mathbb{E}(\ell^2_{\epsilon, > N})^2 = \mathbb{E}\left[ \left( \sum_{j_1,j_2=1}^{M} a_{j_1} a_{j_2} e^{-n\chi} \sum_{l=N+1}^{\infty} C_l \int_{\mathbb{R}^n} G(\varepsilon t_{j_1}, \varepsilon \frac{1}{\alpha} x_{j_1} - y)H_l(\xi(\varepsilon \frac{1}{\alpha} x_{j_1} - y))dy \right)^2 \right]
\]

\[
= \sum_{j_1,j_2=1}^{M} a_{j_1} a_{j_2} e^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} G(\varepsilon t_{j_1} + t_{j_2}, \varepsilon \frac{1}{\alpha} (x_{j_1} - x_{j_2}) - z)R_l^l(\varepsilon \frac{1}{\alpha} - \chi (y_1 - y_2)) \int_{\mathbb{R}^n} G(\varepsilon t_{j_1}, \varepsilon \frac{1}{\alpha} x_{j_1} - (z - z'))G(\varepsilon t_{j_2}, \varepsilon \frac{1}{\alpha} x_{j_2} - z')d\lambda d\zeta,
\]

where the last equality is followed by a change of variables, the symmetry property \( G(t, z) = G(t, -z) \) of the transition probability density function \( G \), and its semigroup property

\[
\int_{\mathbb{R}^n} G(\varepsilon t_{j_1}, \varepsilon \frac{1}{\alpha} x_{j_1} - (z - z'))G(\varepsilon t_{j_2}, \varepsilon \frac{1}{\alpha} x_{j_2} - z) \int_{\mathbb{R}^n} G(\varepsilon t_{j_1} + t_{j_2}, \varepsilon \frac{1}{\alpha} (x_{j_1} - x_{j_2}) - z)R_l^l(\varepsilon \frac{1}{\alpha} - \chi (y_1 - y_2)) d\lambda d\zeta.
\]

Continuing to \((4.7)\), by the spectral representation \((2.7)\) for the \( k \)-th power of the covariance function \( R(\cdot) \), we see that \((4.7)\) is equal to

\[
(4.8) \quad \sum_{j_1,j_2=1}^{M} a_{j_1} a_{j_2} e^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} G(e^{i(\varepsilon \frac{1}{\alpha} - \chi) \lambda}) f^{*l}(\lambda) d\lambda dz
\]

\[
= \sum_{j_1,j_2=1}^{M} a_{j_1} a_{j_2} e^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} e^{i(\varepsilon \frac{1}{\alpha} - \chi) \lambda} G(e^{i\lambda (x_{j_1} - x_{j_2})}) f^{*l}(\lambda) d\lambda
\]

\[
= \sum_{j_1,j_2=1}^{M} a_{j_1} a_{j_2} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} e^{i\lambda (x_{j_1} - x_{j_2})} \int_{\mathbb{R}^n} \exp\{e^{i\lambda (x_{j_1} - x_{j_2})} \exp\{-(t_{j_1} + t_{j_2})|\lambda|^\alpha\} d\lambda < \infty
\]

when \( \varepsilon \to 0 \), where \( f^{*l}(\cdot) \), \( l \geq m \), are continuous and uniformly bounded on \( \mathbb{R}^n \) since Condition B and Lemma 1 imply:

\[
(4.9) \quad f^{*l}(\lambda) = \int_{\mathbb{R}^n} f^{*m}(\lambda - \eta) f^{*(l-m)}(\eta) d\eta \leq \|B_m\|_\infty \int_{\mathbb{R}^n} f^{*(l-m)}(\eta) d\eta = \|B_m\|_\infty \forall l > m.
\]
From (4.8), for any $\delta > 0$ there exists $N_0 \in \mathbb{N}$, $\varepsilon_0 > 0$ such that
\begin{equation}
E(\xi_{\varepsilon, > N})^2 < \delta, \text{ for any } N \geq N_0, \varepsilon < \varepsilon_0,
\end{equation}
which implies that we need to prove a truncated version of (4.4) as follows:
\begin{equation}
\lim_{\varepsilon \to 0} E_{\varepsilon, \leq N_0} = \left\{ 0, \begin{array}{ll}
(p - 1)!! \left\{ E \left[ \left( \sum_{j=1}^{M} a_j V_{m,N_0}(t_j, x_j) \right)^2 \right] \right\}^{\nu}, & p = 2\nu + 1, \\
& p = 2\nu,
\end{array} \right.
\end{equation}
where
\begin{equation}
V_{m,N_0}(t, x) = \int_{\mathbb{R}^n} e^{i(\lambda, x)} \sigma_m, N_0 e^{-t|\lambda|^n} W(d\lambda) \text{ with } \sigma_m, N_0 = \left( \sum_{r=m}^{N_0} f^r(0) C_r^2 \right)^{\frac{1}{2}}.
\end{equation}
By (1.5) for the definition of $\xi_{\varepsilon, \leq N_0} (= \xi_{\varepsilon} - \xi_{\varepsilon, > N_0})$ and our rescaling of the initial data, we have
\begin{equation}
E(\xi_{\varepsilon, \leq N_0})^p = \varepsilon^{\frac{p}{a}} \sum_{j_1, \cdots, j_p = 1}^{M} \sum_{l_1, \cdots, l_p = m}^{N_0} \left[ \prod_{i=1}^{p} a_j \frac{C_i}{\sqrt{l_i!}} \right]
\times \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^{p} G(\varepsilon^{\frac{i}{\lambda}} x_j, \varepsilon^{\frac{i}{\lambda}} y_i) \right\} \left[ \prod_{i=1}^{p} H_i(\varepsilon^{-\frac{1}{\lambda}} y_i) \right] dy_1 \cdots dy_p
\end{equation}
\begin{equation}
= \varepsilon^{-\frac{p}{a}} \sum_{j_1, \cdots, j_p = 1}^{M} \sum_{l_1, \cdots, l_p = m}^{N_0} \left[ \prod_{i=1}^{p} a_j \frac{C_i}{\sqrt{l_i!}} \right]
\times \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^{p} \varepsilon^{\frac{i}{\lambda}} G(\varepsilon^{\frac{i}{\lambda}} x_j, \varepsilon^{\frac{i}{\lambda}} y_i) \right\} \left[ \prod_{i=1}^{p} H_i(\varepsilon^{-\frac{1}{\lambda}} y_i) \right] dy_1 \cdots dy_p.
\end{equation}
To analyze $E(\xi_{\varepsilon, \leq N_0})^p$, $p = 2\nu$ (the odd $p = 2\nu + 1$ is unnecessary, since all the involved random variables are centered), we employ the diagram method (see [7] or [3, p. 72]). A graph $\Gamma$ with $l_1 + \cdots + l_p$ vertices is called a (complete) diagram of order $(l_1, \cdots, l_p)$ if:
(a) the set of vertices $V$ of the graph $\Gamma$ is of the form $V = \bigcup_{j=1}^{p} W_j$, where $W_j = \{(j, l) : 1 < l < l_j \}$ is the $j$-th level of the graph $\Gamma$, $1 \leq j \leq p$;
(b) each vertex is of degree 1; that is, each vertex is just an endpoint of an edge;
(c) if $((j_1, l_1), (j_2, l_2)) \in \Gamma$, then $j_1 \neq j_2$; that is, the edges of the graph $\Gamma$ may connect only different levels.
Let $T = T(l_1, \cdots, l_p)$ be a set of (complete) diagrams of $\Gamma$ of order $(l_1, \cdots, l_p)$. Denote by $E(\Gamma)$ the set of edges of the graph $\Gamma \in T$. For the edge $e = ((j_1, l'_1), (j_2, l'_2)) \in E(\Gamma)$, $j_1 < j_2$, $1 < l'_1 \leq l_1$, $1 \leq l'_2 \leq l_2$, we set $d_1(e) = j_1$, $d_2(e) = j_2$, to denote the location of the edge $e$ in $\Gamma$. We call a diagram $\Gamma$ regular if its levels can be split into pairs in such a manner that no edge connects the levels belonging to different pairs. Denote by $T^* = T^*(l_1, \cdots, l_p)$ the set of all regular diagrams in $T$. Therefore, if $\Gamma \in T^*$ is a regular diagram, then it can be divided into $p/2$ sub-diagrams (denoted by $\Gamma_1, \cdots, \Gamma_{p/2}$), which cannot be separated again; in this case, we naturally define $d_1(\Gamma_i) \equiv d_1(e)$ and $d_2(\Gamma_i) \equiv d_2(e)$ for any $e \in E(\Gamma_i)$, $i = 1, \cdots, \nu = p/2$. We denote by $\sharp E(\Gamma)$ (resp. $\sharp E(\Gamma_j)$) the number of edges belonging to the specific diagram $\Gamma$ (resp. the sub-diagram $\Gamma_j$).
Based on the notation above let
\[ D_p = \{(J, L) : J = (j_1, \cdots, j_p), 1 \leq j_i \leq M, \]
\[ L = (l_1, \cdots, l_p), m \leq l_i \leq N_0, i = 1, \cdots, p\}, \]
we see that (4.13) can be rewritten as
\[ \mathbb{E}(\xi_{\epsilon, \leq N_0})^p = \sum_{(J, L) \in D_p} K(J, L) \sum_{\Gamma \in T^*} F_\Gamma(J, L, \epsilon) + \sum_{(J, L) \in D_p} K(J, L) \sum_{\Gamma \in T \setminus T^*} F_\Gamma(J, L, \epsilon), \]
where
\[ F_\Gamma(J, L, \epsilon) = \varepsilon^{-\frac{\nu}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^{p} \varepsilon^{\frac{\alpha}{n}} G(\varepsilon t_{j_i}, \varepsilon^{\frac{1}{\nu}}(x_{j_i} - y_i)) \right\}
\times \left[ \prod_{\epsilon \in E(\Gamma)} R(\varepsilon^{-\chi}(y_{d_1(\epsilon)} - y_{d_2(\epsilon)})) \right] dy_1 \cdots dy_p. \]

Next, we want to verify two things:
\[ \begin{cases} 
(1) \lim_{\varepsilon \to 0} \sum_{(J, L) \in D_p} K(J, L) \sum_{\Gamma \in T^*} F_\Gamma(J, L, \epsilon) = (p - 1)!! \left\{ \mathbb{E} \left[ \left( \sum_{j=1}^{M} a_j V_{m, N_0} (t_j, x_j) \right)^2 \right] \right\}^{p/2}, \\
(2) \lim_{\varepsilon \to 0} \sum_{(J, L) \in D_p} K(J, L) \sum_{\Gamma \in T \setminus T^*} F_\Gamma(J, L, \epsilon) = 0.
\end{cases} \]

Proof of (1). As argued above, for each \( \Gamma \in T^* \), the case \( p = 2\nu, \nu \in \mathbb{N} \), has a unique decomposition into sub-diagrams \( \Gamma = (\Gamma_1, \cdots, \Gamma_\nu) \), for which each one cannot be further decomposed. Accordingly, we can rewrite \( F_\Gamma(J, L, \epsilon) \) as the following \( \nu = p/2 \) products:
\[ \begin{align*}
F_\Gamma(J, L, \epsilon) &= \varepsilon^{-\frac{\nu}{2}} \prod_{i=1}^{\nu} \int_{\mathbb{R}^{2n}} \varepsilon^{\frac{\alpha}{n}} G(\varepsilon t_{d_1(\Gamma_i)}, \varepsilon^{\frac{1}{\nu}}(x_{d_1(\Gamma_i)} - y)) \varepsilon^{\frac{\alpha}{n}} G(\varepsilon t_{d_2(\Gamma_i)}, \varepsilon^{\frac{1}{\nu}}(x_{d_2(\Gamma_i)} - y')) \\
&\quad \times R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi}(y - y')) \\
&= \varepsilon^{-\frac{\nu}{2}} \prod_{i=1}^{\nu} \varepsilon^{\frac{\alpha}{n}} G(\varepsilon (t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)}), \varepsilon^{\frac{1}{\nu}}(x_{d_1(\Gamma_i)} - x_{d_2(\Gamma_i)} - z)) R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi} z) dz.
\end{align*} \]

We note that
\[ R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi} z) = \varepsilon^{\eta \chi} \int_{\mathbb{R}^{n}} e^{i(\lambda, z)} f^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi} \lambda) d\lambda, \quad i = 1, \cdots, \nu, \]
and \( \sharp E(\Gamma_i) > n/\kappa \) (since \( \kappa > n/m \) in Condition B). By the Fourier transform of \( G, \)
\[ \int_{\mathbb{R}^{n}} e^{i(\lambda, z)} \varepsilon^{\frac{\alpha}{n}} G(\varepsilon (t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)}), \varepsilon^{\frac{1}{\nu}}(x_{d_1(\Gamma_i)} - x_{d_2(\Gamma_i)} - z)) dz
\]

\[ = e^{i(\lambda, x_{d_1(\Gamma_i)} - x_{d_2(\Gamma_i)})} \exp \{ \varepsilon (t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)}) [m - (m^2 + |\varepsilon^{-\frac{1}{\nu}}\lambda|^2)^{\frac{1}{2}}] \}, \]
\[ (4.20) \]
\[ K(J, L) = \prod_{i=1}^{\nu} a_{d_1(\Gamma_i)}a_{d_2(\Gamma_i)} \frac{C_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}. \]

Therefore, by (4.19) and (4.20),
\[ (4.21) \]
\[ \lim_{\varepsilon \to 0} \sum_{(J, L) \in D_{2\nu}} K(J, L) \sum_{\Gamma \in T^*} F_{\Gamma}(J, L, \varepsilon) \]
\[ = \sum_{(J, L) \in D_{2\nu}} \sum_{\Gamma \in T^*} \left[ \prod_{i=1}^{\nu} a_{d_1(\Gamma_i)}a_{d_2(\Gamma_i)} \int e^{i(\lambda, x_{d_1(\Gamma_i)} - x_{d_2(\Gamma_i)})} \exp\{- (t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)}) |\lambda|^\alpha \} d\lambda \right] \left[ \prod_{i=1}^{\nu} f^{\sharp E(\Gamma_i)}(0) \frac{C_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!} \right]. \]

We note that all components in the first bracket in (4.21) are independent of the index set \( L \) and the summation \( \sum_{\Gamma \in T^*} \) depends only on \( \sum_{L} \); therefore
\[ (4.22) \]
\[ \lim_{\varepsilon \to 0} \sum_{(J, L) \in D_{2\nu}} K(J, L) \sum_{\Gamma \in T^*} F_{\Gamma}(J, L, \varepsilon) \]
\[ = \sum_{L} \sum_{\Gamma \in T^*} \sum_{J} \left[ \prod_{i=1}^{\nu} a_{d_1(\Gamma_i)}a_{d_2(\Gamma_i)} \int e^{i(\lambda, x_{d_1(\Gamma_i)} - x_{d_2(\Gamma_i)})} \exp\{- (t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)}) |\lambda|^\alpha \} d\lambda \right] \left[ \prod_{i=1}^{\nu} f^{\sharp E(\Gamma_i)}(0) \frac{C_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!} \right] \]
\[ \times \left[ \prod_{j, j' = 1}^{M} a_{j}a_{j'} \int e^{i(\lambda, x_j - x_{j'})} \exp\{- (t_j + t_{j'}) |\lambda|^\alpha \} d\lambda \right]^\nu \]
\[ \times \left[ \prod_{L} \sum_{\Gamma \in T^*} \sum_{i=1}^{\nu} f^{\sharp E(\Gamma_i)}(0) \frac{C_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!} \right]. \]

To handle the summation in the above, we note that \( \prod_{i=1}^{\nu} f^{\sharp E(\Gamma_i)}(0) \frac{C_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!} \) only depends on \( \{\sharp E(\Gamma_i), i = 1, \ldots, \nu\} \), not on the structures of sub-diagrams \( \Gamma_i, i = 1, \ldots, \nu \); thus we may rewrite the above summation based on the following observation. Let \( s \) be the number of different integers \( r_1, \ldots, r_s \) in \( \{l_1, \ldots, l_{2\nu}\} \) with \( m \leq r_1 < \ldots < r_s \leq N_0 \). A regular diagram requires \( 1 \leq s \leq \nu \), which also implies that the set \( \{l_1, \ldots, l_{2\nu}\} \) can be split into \( s \) subsets \( Q_1, \ldots, Q_s \) and all elements within \( Q_i \) have the common value \( r_i, i = 1, \ldots, s \). For the number of \( \text{pairs} \).
within each subset \( Q_i \), we denote it by \( q_i \), which satisfies \( q_i \geq 1 \), \( i = 1, \ldots, s \), and \( q_1 + \cdots + q_s = \nu \). Thus, the above summation is

\[
\sum_{1 \leq s \leq \nu} \left( \frac{2\nu}{(2q_1)! \cdots (2q_s)!} \right) [\cdots].
\]

However, for any \((s; r_1, \ldots, r_s; q_1, \ldots, q_s)\) in the above sum, there correspond \((2q_1)! \cdots (2q_s)! (r_1)! \cdots (r_s)! q_s\) different regular diagrams. Therefore,

(4.23)

\[
\sum_{L} \sum_{\Gamma \in T^*} \left[ \prod_{i=1}^{\nu} f^{(2E(\Gamma_i))(0)} C_{2E(\Gamma_i)}^{2} \right]^{s}
\]

\[
= (2\nu - 1)! \sum_{1 \leq s \leq \nu} \sum_{m \leq r_1 < \cdots < r_s = N_0} \sum_{q_1 + \cdots + q_s = \nu} \frac{(2\nu)!}{2^q_1! \cdots q_s!} (r_1)! \cdots (r_s)! q_s \left[ \prod_{i=1}^{s} (f^{(r)}(0)C_{r_i}^{2}) q_i \right]^{\nu}
\]

\[
= (2\nu - 1)! \sum_{r = m}^{N_0} \left[ \sum_{r} f^{(r)}(0)C_{r}^{2} \right]^{\nu}.
\]

Substituting (4.23) into (4.22) and recalling \( \sigma_{m, N_0} = \left( \sum_{r = m}^{N_0} f^{(r)}(0)C_{r}^{2} \right)^{\frac{1}{2}} \), we get

(4.24)

\[
\lim_{\varepsilon \to 0} \sum_{(J, L) \in D_{2\nu}} K(J, L) \sum_{\Gamma \in T^*} F_{\Gamma}(J, L, \varepsilon)
\]

\[
= (2\nu - 1)! \left[ \sum_{j, j'}^{M} a_j a_{j'} \int e^{i(\lambda, x_j - x_{j'})} \exp\left\{ - (t_j + t_{j'}) |\lambda|^\alpha \right\} d\lambda \right]^{\nu} \left[ \sum_{r = m}^{N_0} f^{(r)}(0)C_{r}^{2} \right]^{\nu}
\]

\[
= (2\nu - 1)! \left[ \mathbb{E} \left( \sum_{j = 1}^{M} a_j \int e^{i(\lambda, x_j)} \sigma_{m, N_0} e^{-t_j |\lambda|^{\alpha}} W(d\lambda) \right) \right]^{\nu}.
\]

\[\square\]

Proof of (2). \( \lim_{\varepsilon \to 0} \sum_{(J, L) \in D_p} K(J, L) \sum_{\Gamma \in T \setminus T^*} F_{\Gamma}(J, L, \varepsilon) = 0 \).

By (4.11), the number of elements in the summation of \( \sum_{(J, L) \in D_p} \) is finite; thus it suffices to show that \( \lim_{\varepsilon \to 0} F_{\Gamma}(J, L, \varepsilon) = 0 \), i.e.

(4.25)

\[
\varepsilon^{-\frac{\min}{\mathbb{E}^{\nu}} \int_{\mathbb{R}_{np}} \left( \prod_{i=1}^{p} \mathbb{P} G(\varepsilon t_j, \varepsilon \frac{1}{c} (x_j - y_j)) \right) \prod_{e \in E(\Gamma)} R(\varepsilon^{-\chi}(y_{d_1(e)} - y_{d_2(e)})) \right] dy_1 \cdots dy_p \to 0
\]

for each \( \Gamma \in T(l_1, \cdots, l_p) \setminus T^* \). Without loss of generality, we may just prove (4.25) for \( t_j = 1 \) and \( x_j = 0 \), \( i = 1, \ldots, p \), and also just consider the case \( l_1 \leq l_2 \leq \cdots \leq l_p \).
Let
\[ A_{j,j'} := \{ e \in E(\Gamma) \mid d_1(e) = j, d_2(e) = j' \}, \quad B(i) := \bigcup_{j' > i} A_{i,j'}, \quad 1 \leq i, j < j' \leq p. \]

We observe that the number \( \sharp B(i) \) of \( B(i) \) must be \( \leq l_i \), and a non-regular diagram \( \Gamma \) must contain a non-empty \( B(i) \) with \( \sharp B(i) < l_i \); moreover, it has (\( \sharp B(i) \))
\[ (4.26) \quad \sum_{i=1}^{p} \frac{\sharp B(i)}{l_i} \geq \frac{p}{2}. \]
\[ (4.27) \quad F_{\Gamma}(J,L,\varepsilon) \]
\[ = \varepsilon^{-\frac{p\lambda}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^{p} \varepsilon^{\frac{1}{n}} G(\varepsilon, \varepsilon^{\frac{1}{n}} y_i) \right\} \left[ \prod_{i: B(i) \neq \emptyset} \prod_{e \in B(i)} R(\varepsilon^{-\lambda}(y_i - y_{d_2(e)})) \right] dy_1 \cdots dy_p \]
\[ \leq \varepsilon^{-\frac{p\lambda}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^{p} \varepsilon^{\frac{1}{n}} G(\varepsilon, \varepsilon^{\frac{1}{n}} y_i) \right\} \]
\[ \times \left[ \prod_{i: B(i) \neq \emptyset} \sum_{j: A_{i,j} \neq \emptyset} \frac{1}{\sharp B(i)} R_{\sharp B(i)}(\varepsilon^{-\lambda}(y_i - y_{d_2(e)})) \right] dy_1 \cdots dy_p \]
\[ \leq c \varepsilon^{-\frac{p\lambda}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^{p} \varepsilon^{\frac{1}{n}} G(\varepsilon, \varepsilon^{\frac{1}{n}} y_i) \right\} \]
\[ \times \left[ \prod_{i: B(i) \neq \emptyset} \sum_{j: A_{i,j} \neq \emptyset} \frac{1}{\sharp B(i)} R_{\sharp B(i)}(\varepsilon^{-\lambda}(y_i - y_j)) \right] dy_1 \cdots dy_p, \]
where \( c = \prod_{i: B(i) \neq \emptyset} \sum_{j: A_{i,j} \neq \emptyset} \sharp A_{i,j}/\sharp B(i). \)

For any \( i \in \{1, \ldots, p - 1\} \) with \( B(i) \neq \emptyset \), let \( j(i) \) be any term in \( \{j' : A_{ij'} \neq \emptyset\} \).

To prove (4.27) \( \to 0 \), by the spectral representation, it suffices to show that
\[ (4.28) \quad \varepsilon^{-\frac{p\lambda}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^{p} \varepsilon^{\frac{1}{n}} G(\varepsilon, \varepsilon^{\frac{1}{n}} y_i) \right\} \]
\[ \times \left[ \prod_{i: B(i) \neq \emptyset} \int e^{i(y_i - y_{j(i)}, \lambda_{i,j(i)})} f^{*\sharp B(i)}(\varepsilon^\lambda \lambda_{i,j(i)}) \varepsilon^{m} d\lambda_{i,j(i)} \right] dy_1 \cdots dy_p \]
converges to zero when \( \varepsilon \to 0 \).

Applying Lemma 1 to \( k = \sharp B(i) \), the number of \( B(i) \), we see that
\[ (4.29) \quad f^{*\sharp B(i)}(\lambda) \leq \begin{cases} o(1) & \text{if } \sharp B(i) = l_i, \\ o(|\lambda|^{(\frac{\sharp B(i)}{l_i} - 1)}) & \text{if } 1 < \sharp B(i) < l_i, \end{cases} \]
when \( |\lambda| \to 0 \).

For example, using the first case in (2.14) of Lemma 1, we can write (4.29) as follows:
\[ (4.30) \quad f^{*\sharp B(i)}(\lambda) = C_{\sharp B(i)}(\lambda)|\lambda|^{\sharp B(i)\frac{\kappa}{n}} - n, \quad C_{\sharp B(i)}(\lambda) = B_{\sharp B(i)}(\lambda)|\lambda|^{\sharp B(i)(\kappa - \frac{n}{m})}, \]
where \( \lim_{|\lambda| \to 0} C_{\sharp B(i)}(\lambda) = 0 \) because \( \kappa > n/m \geq n/l_i \).
Thus,

\begin{equation}
\label{4.31}
\varepsilon^{-\frac{\nu_n}{2}} o(\varepsilon^\lambda \left( \sum_i \frac{t_B(i)}{\varepsilon} \right)) Q_\varepsilon,
\end{equation}

where

\[
Q_\varepsilon = \int_{\mathbb{R}^{n_p}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{\nu_n}{2}} G(\varepsilon, \varepsilon^{\frac{1}{2}} y_i) \right\} \times \left[ \prod_{i; B(i) \neq \emptyset} \int e^{i(y_i - y_j(i), \lambda_{i,j(i)})} |\lambda_{i,j(i)}|^{n(t_B(i)/\varepsilon - 1)} d\lambda_{i,j(i)} \right] dy_1 \cdots dy_p,
\]

which is bounded in $0 < \varepsilon \ll 1$, because, firstly, for each $\lambda_{i,j(i)}$, by (4.2) the following is bounded in $0 < \varepsilon \ll 1$:

\begin{equation}
\label{4.32}
\int_{\mathbb{R}^{n_p}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{\nu_n}{2}} G(\varepsilon, \varepsilon^{\frac{1}{2}} y_i) \right\} \left[ \prod_{i; B(i) \neq \emptyset} e^{i(y_i - y_j(i), \lambda_{i,j(i)})} \right] dy_1 \cdots dy_p,
\end{equation}

and moreover

\[
\prod_{i; B(i) \neq \emptyset} |\lambda_{i,j(i)}|^{n(t_B(i)/\varepsilon - 1)}
\]

is integrable with respect to $\prod_{i; B(i) \neq \emptyset} d\lambda_{i,j(i)}$ near the origin. Finally, the convergence of (4.31) to zero is followed by the inequality cited above, i.e. $\sum_{i=1}^p \frac{t_B(i)}{\varepsilon} \geq \frac{p}{2}$. \hfill \square

**Proof of Theorem 3.** By the solution form (2.2) and $\int_{\mathbb{R}^n} G_{\alpha,m}(t, x) dx = 1$,

\[
T^{\frac{m_p}{2}} \left\{ u(Tt, \sqrt{T} x; h(\zeta(\cdot))) - C_0 \right\}
= T^{\frac{m_p}{2}} \left\{ \int_{\mathbb{R}^n} G_{\alpha,m}(Tt, \sqrt{T} x - y) \left[ C_0 + \sum_{k=m}^{\infty} C_k H_k(\zeta(y)) \right] dy - C_0 \right\}
\]

\begin{equation}
\label{4.33}
= \sum_{k=m}^{\infty} T^{\frac{m_p}{2}} \frac{C_k}{\sqrt{k!}} \int_{\mathbb{R}^n} G_{\alpha,m}(Tt, \sqrt{T} x - y) H_k(\zeta(y)) dy =: \sum_{k=m}^{\infty} u_{k,T}(t, x).
\end{equation}

By the Slutsky argument (see, for example, [19, p. 6]), Theorem 3 will be proved if we can show that

\begin{equation}
\label{4.34}
(i) \ u_{m,T}(t, x) \Rightarrow U_m(t, x), \quad (ii) \ \sum_{k=m+1}^{\infty} u_{k,T}(t, x) \to 0 \text{ in probability, as } T \to \infty.
\end{equation}

**Proof of (i).** Replacing the component $H_m(\zeta(y))$ in the expression of $u_{m,T}(t, x)$ with its Itô-Wiener expansion (2.10) and using the Fourier transform $\hat{G}_{\alpha,m}(t, \lambda)$ of
\( G_{\alpha, m}(t, x) \) in (2.1), we have

\[
(4.35) \quad u_{m,T}(t, x) = T^{\frac{m}{2}} \frac{C_m}{\sqrt{m!}} \int_{\mathbb{R}^n} G_{\alpha, m}(Tt, \sqrt{T}x - y) \left\{ \int_{\mathbb{R}^{n \times m}} e^{i(y, y_1 + \cdots + y_m)} \prod_{\sigma=1}^m \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma) \right\} dy
\]

By the definition of \( \int_{\mathbb{R}^{n \times m}} \) in (2.10) and the self-similarity property \( W(T^{-\frac{1}{2}} d\lambda) \overset{\text{d}}{=} T^{-\frac{n}{2}} W(d\lambda) \), \( u_{m,T} \) has the same finite-dimensional distributions (\( \overset{\text{d}}{=} \)) as \( \tilde{u}_{m,T} \), where

\[
(4.36) \quad \tilde{u}_{m,T}(t, x) = \frac{C_m}{\sqrt{m!}} T^{\frac{m(n-m)}{4}} \int_{\mathbb{R}^{n \times m}} e^{i(x, y_1 + \cdots + y_m)} \hat{G}_{\alpha, m}(Tt, T^{-\frac{1}{2}} (\lambda_1 + \cdots + \lambda_m))
\]

From the isometry property of the multiple Wiener integrals and the integral representation of the limiting field \( U_m(t, x) \) in (3.5),

\[
(4.37) \quad \mathbb{E} |\tilde{u}_{m,T}(t, x) - U_m(t, x)|^2 = C^2 \int_{\mathbb{R}^n} \left| T^{\frac{m(n-m)}{4}} \hat{G}_{\alpha, m}(Tt, T^{-\frac{1}{2}} (\lambda_1 + \cdots + \lambda_m)) \prod_{\sigma=1}^m \sqrt{f(T^{-\frac{1}{2}} \lambda_\sigma)} \right|^2 d\lambda_\sigma
\]

Condition C and (4.1) allow us to apply the dominated convergence theorem to show that (4.37) will converge to zero when \( T \to \infty \). We note that the convergence in (4.1) can be shown to be monotone decreasing when \( T \uparrow \infty \) for each \( t > 0 \) and \( \lambda \in \mathbb{R}^n \).

Thus, we get

\[
(4.38) \quad \lim_{T \to \infty} \mathbb{E} |\tilde{u}_{m,T}(t, x) - U_m(t, x)|^2 = 0,
\]

and claim (i) is concluded by \( u_{m,T} \overset{\text{d}}{=} \tilde{u}_{m,T} \) and the Cramer-Wold theorem.
Proof of (ii). By the orthogonal property (2.39), the semigroup property of $G_{\alpha,m}(t,x)$, and (2.13),

\[ (4.39) \]

\[ \mathbb{E}\left[\left(\sum_{k=m+1}^{\infty} u_{k,T}(t,x)\right)^2\right] \]

\[= T^{\frac{m-n}{2}} \sum_{k=m+1}^{\infty} C_k^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_{\alpha,m}(Tt, \sqrt{T}x - y)G_{\alpha,m}(Tt, \sqrt{T}x - y')R^k(y - y')dy \, dy' \]

\[= T^{\frac{m-n}{2}} \sum_{k=m+1}^{\infty} C_k^2 \int_{\mathbb{R}^n} G_{\alpha,m}(2Tt, z)R^k(z)dz \]

\[= T^{\frac{m-n}{2}} \sum_{k=m+1}^{\infty} C_k^2 \int_{\mathbb{R}^n} \tilde{G}_{\alpha,m}(2Tt, \lambda) f^{*k}(\lambda)d\lambda \quad \text{(by (2.43))} \]

\[= T^{\frac{m-n}{2}} \left( \sum_{k=m+1}^{k^*} + \sum_{k=k^*+1}^{\infty} \right) C_k^2 \int_{\mathbb{R}^n} \tilde{G}_{\alpha,m}(2Tt, T^{-\frac{1}{2}}\lambda)f^{*k}(T^{-\frac{1}{2}}\lambda)d\lambda =: (I) + (II), \]

where $k^* = \max\{k \in \mathbb{N} \mid k \geq m + 1, \; k\kappa \leq n\}$. 

For the case $k^*\kappa < n$, by Lemma 1 and (4.1),

\[ \lim_{T \to \infty} (I) = \lim_{T \to \infty} T^{\frac{m-n}{2}} \sum_{k=m+1}^{k^*} C_k^2 \int_{\mathbb{R}^n} \tilde{G}_{\alpha,m}(2Tt, T^{-\frac{1}{2}}\lambda)B_k(T^{-\frac{1}{2}}\lambda)|T^{-\frac{1}{2}}\lambda|^{k\kappa-n}d\lambda \]

\[\leq \lim_{T \to \infty} \sum_{k=m+1}^{k^*} T^{\frac{m-n-k\kappa}{2}} C_k^2 \|B_k\|_{\infty} \int_{\mathbb{R}^n} e^{-t^{\frac{3}{2}}m^{1-\frac{2}{n}}|\lambda|^2} |\lambda|^{|k\kappa-n|}d\lambda \]

\[\leq \lim_{T \to \infty} T^{-\frac{\gamma}{2}} \sum_{k=m+1}^{k^*} C_k^2 \|B_k\|_{\infty} \int_{\mathbb{R}^n} e^{-t^{\frac{3}{2}}m^{1-\frac{2}{n}}|\lambda|^2} |\lambda|^{|k\kappa-n|}d\lambda = 0. \]

For the case $k^*\kappa = n$, we still have $\lim_{T \to \infty} (I) = 0$ because

\[ \lim_{T \to \infty} T^{\frac{m-n}{2}} C_{k^*}^2 \int_{\mathbb{R}^n} \tilde{G}_{\alpha,m}(2Tt, T^{-\frac{1}{2}}\lambda)B_{k^*}(T^{-\frac{1}{2}}\lambda)\ln(2 + T^{\frac{1}{2}}|\lambda|^{-1})d\lambda = 0. \]

On the other hand, by the assumption $m\kappa < n$ in Condition C and Lemma 1, for any $k > k^* + 1$ we have $\|f^{*k}\|_{\infty} \leq \|f^{*k+1}\|_{\infty}$, so

\[ \lim_{T \to \infty} (II) \leq \lim_{T \to \infty} T^{\frac{m-n}{2}} \sum_{k=k^*+1}^{\infty} C_k^2 \|f^{*(k+1)}\|_{\infty} \int_{\mathbb{R}^n} \tilde{G}_{\alpha,m}(2Tt, T^{-\frac{1}{2}}\lambda)d\lambda = 0. \]

Hence $\lim_{T \to \infty} \mathbb{E}\left[\left(\sum_{k=m+1}^{\infty} u_{k,T}(t,x)\right)^2\right] = 0$, and the claim (ii) is proved by the Markov inequality. \qed

Proof of Theorem 4. The following proof is a hybrid of the proofs of Theorems 2 and 3; we give a full presentation mainly to see how the re-scaling of the initial data

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proceeds. By the Hermite expansion and the solution form (2.2) we can rewrite
\begin{equation}
\tag{4.42}
\end{equation}
\begin{equation}
\tag{4.41}
\end{equation}

Proof of (i). By substituting the Itô-Wiener expansion (2.10) for the random field $H_m(\zeta(\cdot))$ into $I^\varepsilon_m(t, x)$ and exchanging the order of integration:
\begin{equation}
\tag{4.42}
\end{equation}
we have used the self-similarity property $W(\varepsilon^\alpha d\lambda) \overset{d}{=} \varepsilon^{\frac{\alpha}{\alpha}} W(d\lambda)$ in the last equality.

Now, applying the isometry property of the multiple Wiener integrals to the difference of $I^\varepsilon_m(t, x)$ and the random field $V_m(t, x)$ in (3.7), we have
\begin{equation}
\tag{4.43}
\end{equation}
when $\varepsilon \to 0$, by Condition C and (4.2).
By the Markov inequality, (4.43) implies $\bar{I}^\varepsilon_m(t,x) \to V_m(t,x)$ in probability. However, because $I^\varepsilon_m(t,x) \overset{d}{=} \bar{I}^\varepsilon_m(t,x)$, the claim (i) is concluded by the Cramer-Wold argument.

**Proof of (ii).** From (4.40), by the orthogonal property (2.9) and the semigroup property of $G_{\alpha,m}(t,x)$,

$$
\mathbb{E}\left( \sum_{k=m+1}^{\infty} I^\varepsilon_k(t,x) \right)^2 = \sum_{k=m+1}^{\infty} \mathbb{E}(I^\varepsilon_k(t,x))^2
$$

$$
= \sum_{k=m+1}^{\infty} \varepsilon^{-\chi nk} C^2_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_{\alpha,m}(\varepsilon t, y) G_{\alpha,m}(\varepsilon t, y') R^k(\varepsilon^{-\frac{1}{\alpha}} - \chi(y - y')) dy dy'
$$

$$
= \sum_{k=m+1}^{\infty} \varepsilon^{-\chi nk} C^2_k \int_{\mathbb{R}^n} G_{\alpha,m}(2\varepsilon t, z) R^k(\varepsilon^{-\frac{1}{\alpha}} - \chi z) dz
$$

$$
= \sum_{k=m+1}^{\infty} \varepsilon^{-\chi nk} C^2_k \int_{\mathbb{R}^n} \tilde{G}_{\alpha,m}(2\varepsilon t, \varepsilon^{-\frac{1}{\alpha}} - \chi z) f^*k(\chi) d\lambda
$$

$$
= \left( \sum_{k=m+1}^{k^*} + \sum_{k=k^*+1}^{\infty} \right) \varepsilon^{\chi(n-m\kappa)} C^2_k \int_{\mathbb{R}^n} \tilde{G}_{\alpha,m}(2\varepsilon t, \varepsilon^{-\frac{1}{\alpha}} - \chi) f^*k(\chi) d\lambda =: (I) + (II),
$$

where $k^* = \max\{k \in \mathbb{N} | k \geq m + 1, k\kappa \leq n\}$.

For the case $k^\ast \kappa < n$, by Lemma 1,

$$
\lim_{\varepsilon \to 0} (I) = \lim_{\varepsilon \to 0} \sum_{k=m+1}^{k^*} \varepsilon^{\chi(n-m\kappa)} C^2_k \int_{\mathbb{R}^n} \tilde{G}_{\alpha,m}(2\varepsilon t, \varepsilon^{-\frac{1}{\alpha}} - \chi) B_k(\varepsilon^\chi) |\varepsilon^\chi\lambda|^{k\kappa-n} d\lambda
$$

$$
\leq \lim_{\varepsilon \to 0} \sum_{k=m+1}^{k^*} \varepsilon^{\chi(n-m\kappa)} C^2_k \|B_k\|_\infty \int_{\mathbb{R}^n} e^{-2t|\lambda|^n} |\lambda|^{k\kappa-n} d\lambda = 0.
$$

For the case $k^\ast \kappa = n$, we still have $\lim_{\varepsilon \to 0} (I) = 0$ because

$$
\lim_{\varepsilon \to 0} \varepsilon^{\chi(n-m\kappa)} C^2_k \int_{\mathbb{R}^n} \tilde{G}_{\alpha,m}(2\varepsilon t, \varepsilon^{-\frac{1}{\alpha}} - \chi) B_{k^*}(\varepsilon^\chi) \ln(2 + |\varepsilon^\chi\lambda|^{-1}) d\lambda = 0.
$$

On the other hand, by the assumption $\kappa < n/m$ in Condition C and Lemma 1, for any $k > k^* + 1$ we have $\|f^k\|_\infty \leq \|f^*(k^*+1)\|_\infty$, so

$$
\lim_{\varepsilon \to 0} (II) \leq \lim_{\varepsilon \to 0} \sum_{k=k^*+1}^{\infty} \varepsilon^{\chi(n-m\kappa)} C^2_k \|f^*(k^*+1)\|_\infty \int_{\mathbb{R}^n} e^{-2t|\lambda|^n} d\lambda = 0.
$$

Hence $\lim_{\varepsilon \to 0} \mathbb{E}\left( \sum_{k=m+1}^{\infty} I^\varepsilon_k(t,x) \right)^2 = 0$, and the claim (ii) is proved by the Markov inequality.

**Appendix: Proof of Lemma 1**

The idea of the following proofs comes from [32, p. 115, Theorem 3] and [17, p. 160, Theorem 8.8]. We only consider their results for the density functions on
the whole space. Suppose that two spectral density functions \( f_1 \) and \( f_2 \) are in the form

\[
(A.1) \quad 0 \leq f_j(\lambda) = \frac{K_j(\lambda)}{|\lambda|^{n-\kappa_j}}, \quad \kappa_j > 0, \quad j = 1, 2,
\]

where \( K_1(\lambda) \) and \( K_2(\lambda) \) are non-negative functions belonging to \( C(\mathbb{R}^n \setminus \{0\}) \).

Let \( g(\lambda) = \int_{\mathbb{R}^n} f_1(\lambda - \eta) f_2(\eta) d\eta, \quad \lambda \in \mathbb{R}^n \). To prove Lemma 1, we show that \( g \) can be written as

\[
g(\lambda) = \begin{cases} 
B(\lambda)|\lambda|^{\kappa_1 + \kappa_2 - n}, & \text{for } \kappa_1 + \kappa_2 < n, \\
B(\lambda) \ln(2 + \frac{1}{|\lambda|}), & \text{for } \kappa_1 + \kappa_2 = n, \\
B(\lambda) \in C(\mathbb{R}^n), & \text{for } \kappa_1 + \kappa_2 > n,
\end{cases}
\]

for some bounded function \( B(\lambda) \in C(\mathbb{R}^n \setminus \{0\}) \).

**Case 1** \((\kappa_1 + \kappa_2 < n)\). For any \( \lambda_0 \neq 0 \), we divide \( \mathbb{R}^n \) into four parts: \( \mathbb{R}^n = D_1 \cup D_2 \cup D_3 \cup D_4 \), where

\[
D_1 = \{ \eta \in \mathbb{R}^n | |\eta - \lambda_0| < |\lambda_0|/2 \}, \\
D_2 = \{ \eta \in \mathbb{R}^n | |\eta| < |\lambda_0|/2 \}, \\
D_3 = \{ \eta \in (D_1 \cup D_2)^c | |\eta - \lambda_0| < |\eta| \}, \\
D_4 = \{ \eta \in (D_1 \cup D_2)^c | |\eta - \lambda_0| > |\eta| \}.
\]

Therefore,

\[
g(\lambda_0) = \sum_{j=1}^{4} \int_{D_j} f_1(\lambda_0 - \eta) f_2(\eta) d\eta =: I_1 + I_2 + I_3 + I_4,
\]

\[
I_1(\lambda_0) = \left( \sup_{\eta \in D_1} f_2(\eta') \right) \int_{D_1} f_1(\lambda_0 - \eta') d\eta \\
\leq \left( \sup_{\eta \in D_1} K_2(\eta) \right) \left( \frac{|\lambda_0|}{2} \right)^{\kappa_2 - n} \left( \sup_{\eta \in D_2} K_1(\eta) \right) c_0 \int_0^{\frac{|\lambda_0|}{2}} r^{\kappa_1 - 1} dr = C|\lambda_0|^{\kappa_1 + \kappa_2 - n},
\]

where \( c_0 \) is the surface area of the unit sphere on \( \mathbb{R}^n \) and \( C \) is a constant independent of \( \lambda_0 \). Similarly, \( I_2(\lambda_0) \leq C|\lambda_0|^{\kappa_1 + \kappa_2 - n} \).

By the fact that \( I_3 \in C(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n) \), we know that \( \sup_{|\lambda_0| \geq 1} I_3(\lambda_0) < \infty \). So we suffice to study the behavior of \( I_3(\cdot) \) on the domain \( \{ \lambda_0 | |\lambda_0| < 1 \} \).

By the requirement \( [A.1] \), \( \lim_{|\eta| \to \infty} K_j(\eta)|\eta|^{\kappa_j} = 0 \); that is, for any \( \varepsilon > 0 \), there exists a constant \( M = M(\varepsilon) > 0 \) such that

\[
(A.2) \quad K_j(\eta) \leq \varepsilon |\eta|^{-\kappa_j} \quad \text{for all } |\eta| > M.
\]

Because \( |\eta - \lambda_0| < |\eta| \) for \( \eta \in D_3 \),

\[
I_3(\lambda_0) \leq \left( \int_{D_3 \cap \{|\eta-\lambda_0| > M+1\}} + \int_{D_3 \cap \{|\eta-\lambda_0| < M+1\}} \right) \frac{K_1(\lambda_0 - \eta)K_2(\eta)}{|\lambda_0 - \eta|^{2n-\kappa_1-\kappa_2}} d\eta \\
=: I_{3,1}(\lambda_0) + I_{3,2}(\lambda_0).
\]
By using (A.2) and $|\eta - \lambda_0| < |\eta|$ again,

\begin{align}
I_{3,1}(\lambda_0) &\leq \varepsilon \int_{D_3 \cap \{|\eta - \lambda_0| > M + 1\}} \frac{K_1(\lambda_0 - \eta)|\eta|^{-\kappa_2}}{|\lambda_0 - \eta|^{2n - \kappa_1 - \kappa_2}} d\eta \\
&\leq \varepsilon \int_{\{|\eta - \lambda_0| > M + 1\}} \frac{K_1(\lambda_0 - \eta)}{|\lambda_0 - \eta|^{2n - \kappa_1}} d\eta \\
&\leq \varepsilon (M + 1)^{-n} \int_{\mathbb{R}^n} \frac{K_1(\eta)}{|\eta|^{n - \kappa_1}} d\eta = \varepsilon (M + 1)^{-n},
\end{align}

(A.3)

\begin{align}
I_{3,2}(\lambda_0) &\leq \|K_1\|_\infty \|K_2\|_\infty c_n \int_{\frac{\lambda_0}{2}}^{\frac{\lambda_0}{M + 1}} \frac{\eta^{n - 1}}{\eta^{2n - \kappa_1 - \kappa_2}} d\eta \\
&< \|K_1\|_\infty \|K_2\|_\infty c_n \left( \frac{|\lambda_0|}{2} \right)^{\kappa_1 + \kappa_2 - n}.
\end{align}

(A.4)

Combining (A.3) and (A.4), we get $I_3(\lambda) = B(\lambda)|\lambda|^{|\kappa_1 + \kappa_2 - n|}$ for some bounded function $B$. This observation still holds for $I_4$. Therefore, the proof for the case $\kappa_1 + \kappa_2 < n$ is finished.

\[\square\]

**Case 2** ($\kappa_1 + \kappa_2 = n$, $\kappa_1, \kappa_2 > 0$). Let $\hat{\lambda}_0 = \lambda_0/|\lambda_0|$,

\[
g(\lambda_0) = \int_{\mathbb{R}^n} \frac{K_1(\lambda_0 - \eta)K_2(\eta)}{|\lambda_0 - \eta|^{n - \kappa_1}|\eta|^{n - \kappa_2}} d\eta \\
= (\int_{\{|\eta| < 2\}} + \int_{\{|\eta| > 2\}}) \frac{K_1(|\lambda_0|)(\hat{\lambda}_0 - \eta))K_2(|\lambda_0|\eta)}{|\lambda_0 - \eta|^{n - \kappa_1}|\eta|^{n - \kappa_2}} d\eta \\
=: J_1(\lambda_0) + J_2(\lambda_0),
\]

(A.5)

\[
J_1(\lambda_0) \leq \int_{\{|\eta| < 2\}} \frac{\|K_1\|_\infty \|K_2\|_\infty}{|\lambda_0 - \eta|^{n - \kappa_1}|\eta|^{n - \kappa_2}} d\eta = \int_{\{|\eta| < 2\}} \frac{\|K_1\|_\infty \|K_2\|_\infty}{|\lambda_0 - \eta|^{n - \kappa_1}|\eta|^{n - \kappa_2}} d\eta < \infty,
\]

where the last equality holds for any unit vector $\hat{x}$. Moreover, we also have that

\[
J_2(\lambda_0) = (\int_{\{2 < |\eta| < 2 + \frac{1}{|\lambda_0|}\}} + \int_{\{|\eta| > 2 + \frac{1}{|\lambda_0|}\}}) \frac{K_1(|\lambda_0|)(\hat{\lambda}_0 - \eta))K_2(|\lambda_0|\eta)}{|\lambda_0 - \eta|^{n - \kappa_1}|\eta|^{n - \kappa_2}} d\eta \\
=: J_{2,1}(\lambda_0) + J_{2,2}(\lambda_0).
\]

Because $|\hat{\lambda}_0 - \eta| \geq |\eta| - 1$,

\[
J_{2,1}(\lambda_0) \leq \int_{\{2 < |\eta| < 2 + \frac{1}{|\lambda_0|}\}} \frac{\|K_1\|_\infty \|K_2\|_\infty}{(|\eta| - 1)^{n - \kappa_1}|\eta|^{n - \kappa_2}} d\eta = c_n \int_{\frac{1}{2}}^{2(2 + \frac{1}{|\lambda_0|})} \frac{\|K_1\|_\infty \|K_2\|_\infty}{(r - 1)^{n - \kappa_1}|\eta|^{n - \kappa_2}} dr
\]

\[
= c_n \|K_1\|_\infty \|K_2\|_\infty \int_{\frac{1}{2}}^{2(2 + \frac{1}{|\lambda_0|})} \frac{dr}{(1 - \frac{1}{r})^{n - \kappa_1}} \leq 2^{n - \kappa_1} c_n \|K_1\|_\infty \|K_2\|_\infty \ln\left(2 + \frac{1}{|\lambda_0|}\right).
\]

(A.6)

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Changing from variable \( \eta \) to \( \frac{\tau}{|\lambda_0|} \) and using the inequality \(|\lambda_0 - \tau| \geq |\tau| - |\lambda_0| \geq 2(1 + 2|\lambda_0|) - |\lambda_0| \geq 2(1 + |\lambda_0|)\),

\[
J_{2, 2}(\lambda_0) = \int_{\{|\tau| > 2(1 + 2|\lambda_0|)\}} \frac{K_1(\lambda_0 - \tau)K_2(\eta)}{|\lambda_0 - \tau|^{n-\kappa_1}|\tau|^{n-\kappa_2}} d\tau
\leq \frac{\|K_1\|_{\infty}}{2^{n-\kappa_1}} \int_{\{|\tau| > 2(1 + 2|\lambda_0|)\}} \frac{K_2(\tau)}{|\tau|^{n-\kappa_2}} d\eta \leq \frac{\|K_1\|_{\infty}}{2^{n-\kappa_1}}.
\]

The last estimation, together with \((A.5)\) and \((A.6)\), implies that there exist bounded and positive functions \(B(\lambda)\) and \(C(\lambda)\) such that \(g(\lambda) = B(\lambda)\ln(2 + \frac{1}{|\lambda|}) + C(\lambda) = B(\lambda)\ln(2 + \frac{C(\lambda)}{\ln(2 + |\lambda|)})\) is also a bounded function.

**Case 3** \( (\kappa_1 + \kappa_2 > n, \kappa_1, \kappa_2 > 0) \). Because \( \kappa_1 + \kappa_2 > n \) implies that there exist \( p, p' \in (1, \infty) \) such that \( p(n - \kappa_1), p'(n - \kappa_2) < n \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). For any \( \lambda \in \mathbb{R}^n \), by Hölder’s inequality, \( g(\lambda) \leq ||f_1||_p||f_2||_{p'} \). Meanwhile, it also implies the continuity of \( g \) as follows:

\[
|g(\lambda) - g(\lambda_0)| = \left| \int_{\mathbb{R}^n} (f_1(\lambda - \eta) - f_1(\lambda_0 - \eta))f_2(\eta)d\eta \right| \\
\leq ||f_1(\lambda - \cdot) - f_1(\lambda_0 - \cdot)||_p||f_2||_{p'} \rightarrow 0
\]

when \( \lambda \rightarrow \lambda_0 \) for any \( \lambda_0 \in \mathbb{R}^n \).

Finally, by taking successive convolutions and using the result of Case 1, for any \( k_1 > k_2 > n/\kappa_1, f^{*k_1}(\lambda) \) and \( f^{*k_2}(\lambda) \) are bounded functions, which implies

\[
f^{*k_1}(\lambda) = \int_{\mathbb{R}^n} f^{*k_2}(\lambda - \eta)f^{*(k_1-k_2)}(\eta)d\eta \leq ||f^{*k_2}||_{\infty} \int_{\mathbb{R}^n} f^{*(k_1-k_2)}(\eta)d\eta = ||f^{*k_2}||_{\infty}.
\]

**ACKNOWLEDGEMENTS**

This article was mainly worked on while the second author visited York University (Canada) in Fall 2011 and Chinese University of Hong Kong in Springs 2012 and 2013; their hospitality and financial support are appreciated. The contents of this article were reported by the second author at the probability scientific session of the Canadian Mathematical Society 2011 Winter Meeting.

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