CORRECTION TO
“COMBINATORICS AND GEOMETRY OF POWER IDEALS”: TWO COUNTEREXAMPLES FOR POWER IDEALS OF HYPERPLANE ARRANGEMENTS

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Abstract. We disprove Holtz and Ron’s conjecture that the power ideal \( C_{A,-2} \) of a hyperplane arrangement \( A \) (also called the internal zonotopal space) is generated by \( A \)-monomials. We also show that, in contrast with the case \( k \geq -2 \), the Hilbert series of \( C_{A,k} \) is not determined by the matroid of \( A \) for \( k \leq -6 \).

Remark. This note is a corrigendum to our article [1], and we follow the notation of that paper.

1. Introduction

Let \( A = \{H_1, \ldots, H_n\} \) be a hyperplane arrangement in a vector space \( V \); say \( H_i = \{x \mid l_i(x) = 0\} \) for some linear functions \( l_i \in V^* \). Call a product of (possibly repeated) \( l_i \)'s an \( A \)-monomial in the symmetric algebra \( \mathbb{C}[V^*] \). Let Lines(\( A \)) be the set of lines of intersection of the hyperplanes in \( A \). For each \( h \in V \) with \( h \neq 0 \), let \( \rho_A(h) \) be the number of hyperplanes in \( A \) not containing \( h \). Let \( \rho = \rho(A) = \min_{h \in V} (\rho_A(h)) \). For all integers \( k \geq - (\rho + 1) \), consider the power ideals

\[
I_{A,k} := \left\langle h^{\rho_A(h)+k+1} \mid h \in V, h \neq 0 \right\rangle, \quad I'_{A,k} := \left\langle h^{\rho_A(h)+k+1} \mid h \in \text{Lines}(A) \right\rangle
\]

in the symmetric algebra \( \mathbb{C}[V] \). It is convenient to regard the polynomials in \( I_{A,k} \) as differential operators, and to consider the space of solutions to the resulting system of differential equations:

\[
C_{A,k} = I'_{A,k} := \left\{ f(x) \in \mathbb{C}[V^*] \mid \frac{\partial}{\partial x} h^{\rho_A(h)+k+1} f(x) = 0 \text{ for all } h \neq 0 \right\}
\]

which is known as the inverse system of \( I_{A,k} \). Define \( C'_{A,k} \) similarly. These objects arise naturally in numerical analysis, algebra, geometry, and combinatorics. For references, see [1,3].

One important question is to compute the Hilbert series of these spaces of polynomials, graded by degree, as a function of combinatorial invariants of \( A \). Frequently, the answer is expressed in terms of the Tutte polynomial of \( A \). This has been done
successfully in many cases. One strategy used independently by different authors has been to prove the following:

(i) There is a spanning set of $A$-monomials for $C_{A,k}$.
(ii) There is an exact sequence $0 \to C_{A \setminus H,k}(-1) \to C_{A,k} \to C_{A/H,k} \to 0$ of graded vector spaces.
(iii) Therefore, the Hilbert series of $C_{A,k}$ is an evaluation of the Tutte polynomial of $A$.

Here $A \setminus H$ and $A/H$ are the deletion and contraction of $H$, respectively.

For $k \geq -1$, this method works very nicely. Dahmen and Michelli [2] were the first ones to do this for $C_{A,-1}$. Postnikov, Shapiro, and Shapiro [3] did it for $C_{A,0}$, while Holtz and Ron [4] did it for $C_{A,0}$. In [1] we did it for $C_{A,k}$ for all $k \geq -1$, and showed that $C'_{A,0} = C_{A,0}$ and $C'_{A,-1} = C_{A,-1}$.

For $k \leq -3$ this approach does not work in full generality. In [1] we showed that (i) is false in general for $C_{A,k}$, and left (ii) and (iii) open, suggesting the problem of measuring $C_{A,k}$. For $k \leq -6$, (ii) and (iii) are false, as we will show in Propositions 4 and 5, respectively. In fact, we will see that the Hilbert series of $C_{A,k}$ is not even determined by the matroid of $A$.

The intermediate cases are interesting and subtle, and deserve further study; notably the case $k = -2$, which Holtz and Ron call the internal zonotopal space. In [3] they proved (ii) and (iii) and conjectured (i) for $C'_{A,-2}$. In [1, Proposition 4.5.3] – a restatement of Holtz and Ron’s Conjecture 6.1 in [3] – we put forward an incorrect proof of this conjecture; the last sentence of our argument is false. In fact their conjecture is false, as we will see in Proposition 2.

2. THE CASE $k = -2$: INTERNAL ZONOTOPAL SPACES

Before showing why Holtz and Ron’s conjecture is false, let us point out that the remaining statements about $C_{A,-2}$ that we made in [1] are true. The easiest way to derive them is to prove that $C_{A,-2} = C'_{A,-2}$, and simply note that Holtz and Ron already proved those statements for $C'_{A,-2}$:

Lemma 1. We have $C_{A,k} = C'_{A,k}$ for any $k$ with $-(\rho + 1) \leq k \leq 0$.

Proof. By [1, Theorem 4.17] we have $I_{A,0} = I'_{A,0}$, so it suffices to show that $I_{A,j} = I'_{A,j}$ implies that $I_{A,j-1} = I'_{A,j-1}$ as long as these ideals are defined. If $I_{A,j} = I'_{A,j}$, then for any $h \in V \setminus \{0\}$ we have $h^{\rho_A(h) + j + 1} = \sum f_i h_i^{\rho_A(h_i) + j + 1}$ for some polynomials $f_i$, where the $h_i$’s are the lines of the arrangement. As long as the exponents are positive, taking partial derivatives in the direction of $h$ gives $h^{\rho_A(h) + j} = \sum g_i h_i^{\rho_A(h_i) + j}$ for some polynomials $g_i$. \hfill \square

The following result shows that (i) does not hold for $C_{A,-2}$.

Proposition 2. [4, Conjecture 6.1] is false: The “internal zonotopal space” $C_{A,-2}$ is not necessarily spanned by $A$-monomials.

Proof. Let $H$ be the hyperplane arrangement in $\mathbb{C}^4$ determined by the linear forms $y_1, y_2, y_3, y_1 - y_4, y_2 - y_4, y_3 - y_4$. We have

$$I'_{H,-2} = \langle x_1^1, x_2^1, x_3^1, (\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + x_4)^2 \rangle = \langle x_1, x_2, x_3, x_4 \rangle$$

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as \( \epsilon_1, \epsilon_2, \epsilon_3 \) range over \( \{0,1\} \). The other generators of \( I_{\mathcal{H},-2} \) are of degree at least 3, and are therefore in \( I_{\mathcal{H},-2}^I \) already, so

\[
I_{\mathcal{H},-2} = \langle x_1, x_2, x_3, x_4^3 \rangle, \quad C_{\mathcal{H},-2} = \text{span}(1, y_4).
\]

Therefore \( C_{\mathcal{H},-2} \) is not spanned by \( \mathcal{H} \)-monomials.

As Holtz and Ron pointed out, if [3, Conjecture 6.1] had been true, it would have implied [3, Conjecture 1.8], an interesting spline-theoretic interpretation of \( C_{\mathcal{A},-2} \) when \( \mathcal{A} \) is unimodular. The arrangement above is unimodular, but it does not provide a counterexample to [3, Conjecture 1.8]. In fact, Matthias Lenz [4] has recently put forward a proof of this weaker conjecture.

3. THE CASE \( k \leq -6 \)

In this section we show that when \( k \leq -6 \), the Hilbert series of \( C_{\mathcal{A},k} \) is not a function of the Tutte polynomial of \( \mathcal{A} \). In fact, it is not even determined by the matroid of \( \mathcal{A} \). Recall that \( \rho = \rho(\mathcal{A}) := \min_{h \in \mathcal{V}}(\rho_{\mathcal{A}}(h)) \). Say \( h \in \mathcal{V} \) is large if it is on the maximum number of hyperplanes, so \( \rho_{\mathcal{A}}(h) = \rho \).

**Lemma 3.** The degree 1 component of \( C_{\mathcal{A},-\rho} \) is

\[
(C_{\mathcal{A},-\rho})_1 = \langle \text{span}\{h \in \mathcal{V} : h \text{ is large} \} \rangle^\perp
\]

in \( \mathcal{V}^* \).

**Proof.** An element \( f \) of \( C_{\mathcal{A},-\rho} \) needs to satisfy the differential equations

\[
h(\partial/\partial x)^{\rho_{\mathcal{A}}(h)-\rho+1} f(x) = 0
\]

for all non-zero vectors \( h \in \mathcal{V} \). If \( f \) is linear, then this condition is trivial unless \( h \) is large; and in that case it says that \( f \perp h \). \( \square \)

**Proposition 4.** For \( k \leq -6 \), the Hilbert series of \( C_{\mathcal{A},k} \) is not determined by the matroid of \( \mathcal{A} \).

**Proof.** First assume \( k = -2m \). Let \( L_1, L_2, L_3 \) be three different lines through 0 in \( \mathbb{C}^3 \) and consider an arrangement \( \mathcal{A} \) of 3m (hyper)planes consisting of \( m \) generically chosen planes \( H_{i1}, \ldots, H_{im} \) passing through \( L_i \) for \( i = 1, 2, 3 \). Then \( \rho = 2m \) and the only large lines are \( L_1, L_2, \) and \( L_3 \). Therefore \( \dim(C_{\mathcal{A},-2m})_1 \) equals 1 if \( L_1, L_2, L_3 \) are coplanar, and 0 otherwise. However, the matroid of \( \mathcal{A} \) does not know whether \( L_1, L_2, L_3 \) are coplanar.

More precisely, consider two versions \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) of the above construction; in \( \mathcal{A}_1 \) the lines \( L_1, L_2, L_3 \) are coplanar, and in \( \mathcal{A}_2 \) they are not. Notice that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) have the same matroid: the rank 3 matroid whose non-bases are the triples \( \{H_{ia}, H_{ib}, H_{ic}\} \) for \( 1 \leq i \leq 3 \) and \( 1 \leq a < b < c \leq m \). However, \( \dim(C_{\mathcal{A}_1,-2m})_1 \neq \dim(C_{\mathcal{A}_2,-2m})_1 \).

The case \( k = -2m-1 \) is similar. It suffices to add a generic plane to the previous arrangements. \( \square \)

**Proposition 5.** For \( k \leq -6 \), the sequence of graded vector spaces

\[
0 \to C_{\mathcal{A}\setminus H,k}(-1) \to C_{\mathcal{A},k} \to C_{\mathcal{A}/H,k} \to 0
\]

of [11, Proposition 4.4.1] is not necessarily exact, even if \( H \) is neither a loop nor a coloop.
Proof. We will not need to recall the maps that define this sequence; we will simply show an example where right exactness is impossible because \( \dim(C_{A,k})_1 = 0 \) and \( \dim(C_{A/H,k})_1 = 1 \). We do this in the case \( k = -2m \); the other one is similar.

Consider the arrangement \( A = A_2 \) of the proof of Proposition 4 and the plane \( H = H_{11} \). We have \( \dim(C_{A,-2m})_1 = 0 \). In the contraction \( A/H \), the planes \( H_{12}, \ldots, H_{1m} \) become the same line \( L_1 \) in \( H \), while the other \( 2m \) planes of \( A \) become generic lines in \( H \). Therefore \( \rho(A \setminus H) = 2m \) and \( (C_{A/H,-2m})_1 = L_1^\perp \) in \( H^* \), which is one-dimensional. \( \Box \)

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