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A SOLUTION TO THE FOCUSING 3D NLS THAT BLOWS UP ON A CONTRACTING SPHERE

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ABSTRACT. We rigorously construct radial H^1 solutions to the 3d cubic focusing NLS equation $i\partial_t\psi + \Delta\psi + 2|\psi|^2\psi = 0$ that blow-up along a contracting sphere. With blow-up time set to $t = 0$, the solutions concentrate on a sphere at radius $\sim t^{1/3}$ but focus towards this sphere at the faster rate $\sim t^{2/3}$. Such dynamics were originally proposed heuristically by Degtyarev-Zakharov-Rudakov in 1975 and independently later by Holmer-Roudenko in 2007, where it was demonstrated to be consistent with all conservation laws of this equation. In the latter paper, it was proposed as a solution that would yield divergence of the L^3_x norm within the "wide" radius $\sim \|\nabla u(t)\|_{L^2_x}^{-1/2}$ but not within the "tight" radius $\sim \|\nabla u(t)\|_{L^2_x}^{-2}$, the second being the rate of contraction of self-similar blow-up solutions observed numerically and described in detail by Sulem-Sulem in 1999.

1. INTRODUCTION

Consider the 3d cubic focusing NLS equation on $\mathbb{R}^3$:

$$i\partial_t\psi + \Delta\psi + 2|\psi|^2\psi = 0, \tag{1.1}$$

where $\psi = \psi(x,t) \in \mathbb{C}$ and $x \in \mathbb{R}^3$, $t \in \mathbb{R}$. The initial-value problem posed with initial-data in H^1_x is locally well-posed and there is a unique solution in $C([0,T); H^1_x)$ on a maximal forward lifespan $[0,T)$. If $T < \infty$, then $\lim_{t\nearrow T} \|\nabla\psi(t)\|_{L^2_x} = \infty$.

During their lifespan, the solutions ψ to (1.1) satisfy mass, energy and momentum conservation laws.

$$M(\psi(t)) \equiv \int |\psi(x,t)|^2 dx = \int |\psi_0(x)|^2 dx = M(\psi_0), \tag{1.2}$$

$$E(\psi(t)) \equiv \int \left(|\nabla\psi(x,t)|^2 - |\psi(x,t)|^4\right) dx = E(\psi_0), \tag{1.3}$$

$$P(\psi(t)) \equiv \int \left(\bar\psi\nabla\psi - \psi\nabla\bar\psi\right) dx = P(\psi_0). \tag{1.4}$$

The case $T < \infty$ occurs for a large class of initial data (see the discussion in Holmer-Platte-Roudenko [6]) and in this case we say that the solution *blows-up in finite time*. Numerical results (see Sulem-Sulem [13, Chapter 7]) describe the existence of self-similar radial blow-up solutions of the form

$$u(x,t) \approx \frac{1}{\lambda(t)} U\left(\frac{x}{\lambda(t)}\right) e^{i\log(T-t)} \text{ with } \lambda(t) = \sqrt{2b(T-t)}, \tag{1.5}$$

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where $U = U(x)$ is a stationary profile satisfying the nonlinear elliptic equation

$$\Delta U - U + ib(U + y \cdot \nabla U) + 2|U|^2 U = 0. \tag{1.6}$$

Numerics suggest that $b > 0$ can be selected so that a nontrivial *zero-energy* solution U to (1.6) exists. Asymptotics as $|x| \to \infty$ show that any solution of (1.6) fails to belong to $\dot{H}_x^{1/2}$ (and hence H_x^1). Nevertheless, it is expected that H_x^1 blow-up solutions to (1.1) well-modeled by (1.5) *near the origin* exist, and the proof remains an important open problem.

In this paper we consider only radial solutions, thus, $P(\psi) = 0$. We construct a family of finite-time blow-up solutions to (1.1) with different dynamics – they focus toward a sphere, while at the same time the radius of the sphere is shrinking. We study the problem by converting it (via the time-reversal, $u(t) \mapsto \bar{u}(t)$, and time-translation symmetry of (1.1)) to one with solutions defined on $(0, t_0]$ for $t_0 > 0$ that "start" at $t_0 > 0$ and blow-up at $t = 0$ as they evolve backward in time.

Our main result is the following:

Theorem 1.1. *For any $e \in \mathbb{R}$ there exists a radial solution $\psi \in C((0, t_0], H^1(\mathbb{R}^3))$ of (1.1) with $E(\psi) = e$, which starts at time $t_0 > 0$ and, while evolving backwards in time, blows-up at $t = 0$, and has the following form:*

$$\psi(r,t) = e^{i\theta(t)} e^{iv(t)r/2} \lambda(t)\, \varphi\big(\lambda(t)(r - q(t))\big) + h(r,t), \quad r = |x|, x \in \mathbb{R}^3, \tag{1.7}$$

where $\varphi(r) = \operatorname{sech}(r)$ and the time-dependent parameters (q, v, λ, θ) satisfy

$$\begin{aligned} q(t) &= q_0 t^{1/3}, & q_0 &= 2^{1/3} 3^{1/6}, \\ \lambda(t) &= \lambda_0 t^{-2/3}, & \lambda_0 &= 2^{-2/3} 3^{-1/3}, \\ v(t) &= v_0 t^{-2/3}, & v_0 &= 2^{1/3} 3^{-5/6}, \\ \theta(t) &= \theta_0 t^{-1/3}, & \theta_0 &= 2^{-1/3} 3^{-2/3} \end{aligned} \tag{1.8}$$

and

$$\|h(t)\|_{L^2(\mathbb{R}^3)} + t^{2/3} \|h(t)\|_{H^1(\mathbb{R}^3)} + t^{-1/3} \||x|\, h(t)\|_{L^2(\mathbb{R}^3)} \le C\, t^{1/3}. \tag{1.9}$$

Note that it follows from (1.8) and (1.9) that

$$\|\nabla \psi(t)\|_{L^2(\mathbb{R}^3)} \sim t^{-2/3} \quad \text{and} \quad \||x|\, \psi(t)\|_{L^2(\mathbb{R}^3)} \sim t^{1/3}.$$

Remarks. 1. Let φ be a solution of the nonlinear elliptic equation

$$-\varphi + \Delta\varphi + 2|\varphi|^2 \varphi = 0. \tag{1.10}$$

The unique (up to translation) minimal mass H^1 solution of (1.10) is called the *ground state*. It is smooth, radial, real-valued and positive, and exponentially decaying (see, for example, Tao [11, Apx. B]). Note that $\psi(x,t) = e^{it}\varphi(x)$ is a solution of (1.1). In one dimension, i.e., $d = 1$, the ground state of (1.10) is explicit, namely, $\varphi(x) = \operatorname{sech}(x)$. It is this function we use for the profile of the blow-up in (1.7) (and not the 3d ground state solution of (1.10)).

2. It follows from (1.8) that the solutions given by Theorem 1.1 satisfy $\|\psi\|_{L^2(\mathbb{R}^3)} = \|\varphi\|_{L^2(\mathbb{R})} = \sqrt{2}$. Applying scaling, one then obtains contracting sphere blow-up solutions with an arbitrary mass.

3. We do not address in this note the questions of stability of these solutions (the numerical and heuristical results of [3] indicate that the behavior (1.7), (1.8) should be stable with respect to radial perturbations). Still, let us mention that the

arguments developed in the proof of Theorem 1.1 can be easily adjusted in order to produce blow-up solutions of the form

$$\psi(t) = S(t) + f + o_{H^1}(1), \quad t \to 0,$$

where $S(t)$ is a solution of (1.1) given by Theorem 1.1, and f is an arbitrarily smooth radial decaying function vanishing to a sufficiently high order at the origin, very much in the spirit of [1].

4. A numerical visualization of the blow-up described by Theorem 1.1 is given in Figure 1, courtesy of R. Platte, Arizona State University.

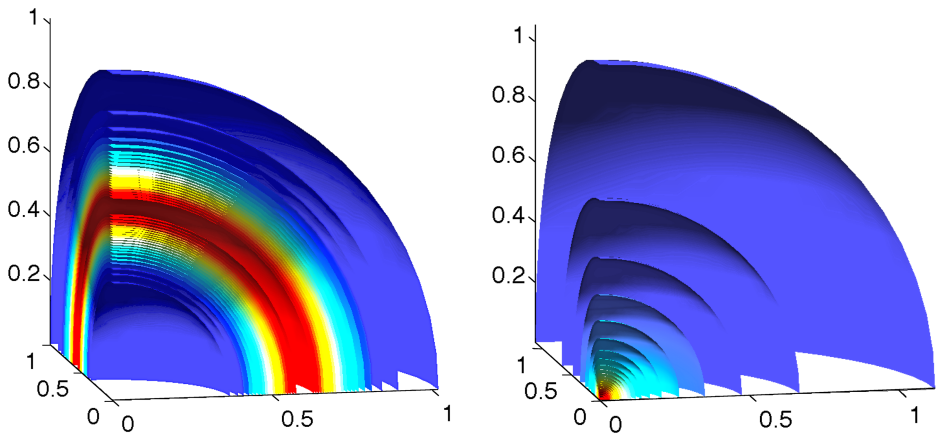

FIGURE 1. A contracting sphere collapse in the 3d cubic NLS: (a) a concentrating layer of the collapsing solution is around $r \sim 0.6$, (b) the eventual collapse at the origin. Courtesy of R. Platte, Arizona State University.

The paper is organized as follows: In §2, we construct (Proposition 2.1) for arbitrary $N \in \mathbb{N}$, an approximate solution $\psi^{(N)}$ satisfying

$$i\partial_t \psi^{(N)} + \Delta \psi^{(N)} + 2|\psi^{(N)}|^2 \psi^{(N)} = iH^{(N)}, \tag{1.11}$$

where

$$t^{2k/3} \|\nabla^k H^{(N)}(t)\|_{L^2(\mathbb{R}^3)} \lesssim_k t^{(2N-1)/3}, \qquad 0 < t \le t_0(N),$$

with implicit constant independent of N. To accomplish this, we set

$$\psi^{(N)}(r,t) = e^{i\theta} e^{ivr/2} \lambda U^{(N)}(\rho, t), \tag{1.12}$$

where $\rho = \lambda(r - q)$ is the recentered and rescaled spatial variable and q, v, θ, λ are t-dependent parameters. We then seek to suitably define the parameters and profile $U^{(N)}$. We impose the *a priori* conditions

$$\begin{cases} v = q', \\ \theta' = -\frac{1}{4}v^2 - \frac{1}{2}v'q + \lambda^2, \end{cases}$$

which achieve convenient simplifications of the equation that arises when (1.12) is substituted into (1.11), and enable us to solve for v and θ, once q and λ have been

specified. For additional convenience, we set $\omega = \lambda^{-1}q^{-2}$ (effectively replacing λ by ω) and[1] $\chi = U^{(N)} - \varphi$ (recall $\varphi(\rho) = \operatorname{sech}\rho$). Then (1.11) will follow from

$$|\partial_\rho^\ell(\mathcal{H}\vec{\chi}+\ell(q,\omega)\vec{\chi}+\mathcal{F}(q,\omega)+F^{\geq 2}(\vec{\chi}))| \leq C_{\ell,\mu}\, t^{(2N+3)/3}e^{-\mu|\rho|}\,, \quad 0<\mu<1, \tag{1.13}$$

where $\mathcal{H}$ is the 1D cubic linearized Hamiltonian – $\mathcal{H}$, ℓ, $\mathcal{F}$, and $F^{\geq 2}$ are defined precisely in §2. We then seek a solution to (1.13) by expanding

$$\omega(t) = \sum_{k=0}^{N} \omega_k t^{2k/3}\,, \qquad q(t) = \sum_{k=0}^{N} q_k t^{(2k+1)/3}\,, \qquad \chi(\rho,t) = \sum_{k=1}^{2N+2} t^{k/3}\chi_k(\rho)\,. \tag{1.14}$$

Substituting the expansions (1.14) into (1.13), we can organize terms by their order in t and carry out a recursive construction. The parameter coefficients q_k and ω_k are selected at each stage to satisfy solvability conditions for χ_{k+1}.

In §3, we consider the exact solution ψ with data $\psi|_{t=\epsilon} = \psi^{(N)}(\epsilon)$ for any $0 < \epsilon \leq t_0(N)$, and prove (Proposition 3.5) that for N sufficiently large, the comparison estimate

$$\sum_{k=0}^{2} t^{2k/3}\|\nabla^k(\psi(t)-\psi^{(N)}(t))\|_{L^2(\mathbb{R}^3)} \lesssim t^{2N/3}\,, \qquad \epsilon \leq t \leq t_0(N), \tag{1.15}$$

and localization bound (Proposition 3.6)

$$\|x\psi(t)\|_{L^2(\mathbb{R}^3)} \lesssim t^{1/3} \tag{1.16}$$

hold, independently of $0 < \epsilon \leq t_0(N)$. The estimates (1.15) and (1.16) are proved using a Lyapunov functional $G(h)$ (see (3.8)), built from energy, mass, and radial momentum.

Finally, in §4, we complete the proof of Theorem 1.1 using estimates (1.15), (1.16) and a compactness argument.

This result was announced and described in detail [9] by G. Perelman at the Analysis seminar at Université de Cergy-Pontoise in December 2011. In February 2012, Merle, Raphaël, and Szeftel (MRS) posted a paper [8] on arxiv.org proving a closely related result. We stress that our result was obtained independently of [8]. Let us now provide a brief comparison between the two results. MRS provide a construction of radial ring-type blow-up solutions for dimension $n \geq 2$ and (1.1) with nonlinearity $|u|^{p-1}u$, where $1 + \frac{4}{n} < p < 5$. The specialization of their result to the case $p = 3$ and $n = 3$ (the physical case) can then be compared to our result. As we do, MRS construct an approximate solution to arbitrarily high order using an ansatz similar to our (1.7) except that a factor e^{-ibr^2} is included, adding a fifth parameter b. They then reduce to two free parameters[2] b, β by imposing three *a priori* conditions (see the beginning of step 2 on p. 7 in [8]). On the other hand, we start with four parameters and reduce to two free parameters q, ω (see (2.5)) by imposing two *a priori* conditions (2.3). We expand q, ω in series in powers of $t^{1/3}$. Instead, MRS reduce to a profile equation (see [8, (3.11)]) with *a priori* unknown polynomials $\mathcal{P}_1$, $\mathcal{P}_2$ depending on b, β but independent of t (see [8, (3.12)]). In both papers, the coefficients are chosen to meet solvability conditions at each stage of the iteration. MRS use the numerical and heuristic work of [3] to suggest a form of the expansion, in particular, the introduction of the parameter

[1] In fact, once χ is constructed, we take $U^{(N)}$ to be a *cutoff* of $\varphi + \chi$.

[2] Their β is similar to our v.

b. We feel that our presentation, however, requires less *a priori* knowledge of the structure of the expansion and results in a simpler exposition. The two papers proceed with different compactness arguments to construct an exact solution close to the high order approximate solution. MRS prove localized bounds and obtain convergence only in L^2. We prove localized and H^2 bounds to obtain convergence in H^1, allowing us to control the energy of the exact solution. Thus, we construct solutions with arbitrarily prespecified energy. Finally, we are able to report more precise values (1.8) for the parameters (compare to [8, (1.14)]).

2. Construction of an approximate solution

We start with writing a solution to (1.1) as

$$\psi(r,t) = e^{i\theta(t)} e^{iv(t)r/2} \lambda(t)\, U\big(\lambda(t)(r-q(t)), t\big). \tag{2.1}$$

Substituting it into (1.1) and introducing a rescaled coordinate

$$\rho = \lambda(r-q),$$

we obtain that $U = U(\rho,t)$ solves

$$\begin{aligned} i\,\frac{1}{\lambda^2}\,U_t + U_{\rho\rho} - \frac{1}{\lambda^2}\left(\theta' + \frac{v^2}{4} + \frac{v'\,q}{2} + \frac{v'\rho}{2\lambda}\right) U + 2\,|U|^2 U + i\left(\frac{v}{\lambda} - \frac{q'}{\lambda}\right) U_\rho \\ = -\frac{2}{\rho+\lambda\,q}\,U_\rho - i\frac{\lambda'}{\lambda^3}\big(1+\rho\,\partial_\rho\big)\,U - i\frac{v}{\lambda(\rho+\lambda\,q)}\,U. \end{aligned} \tag{2.2}$$

Imposing the following two conditions:

$$\begin{aligned} &v(t) = q'(t), \\ &\theta'(t) + \frac{v(t)^2}{4} + \frac{v'(t)\,q(t)}{2} = \lambda(t)^2, \end{aligned} \tag{2.3}$$

we obtain

$$\begin{aligned} i\,\frac{1}{\lambda^2}\,U_t + U_{\rho\rho} - U + 2\,|U|^2 U = -\frac{2}{\rho+\lambda\,q}\,U_\rho \\ + \frac{q''\rho}{2\lambda^3}\,U - i\frac{\lambda'}{\lambda^3}\big(1+\rho\,\partial_\rho\big)\,U - i\frac{q'}{\lambda(\rho+\lambda\,q)}\,U. \end{aligned} \tag{2.4}$$

We next denote

$$\omega = \frac{1}{\lambda\,q^2} \tag{2.5}$$

and rewrite (2.4) as

$$\begin{aligned} i\,q^4\omega^2\,U_t + U_{\rho\rho} - U + 2\,|U|^2 U = -\frac{2\,q\omega}{\rho\omega q + 1}\,U_\rho \\ + \frac{1}{2}q''q^6\omega^3\rho\,U + i(2q'q^3\omega^2 + \omega'\omega q^4)\big(1+\rho\,\partial_\rho\big)\,U - i\frac{q'q^3\omega^2}{\rho\omega q+1}\,U. \end{aligned} \tag{2.6}$$

Define the Pauli matrices by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Split U into the localized part $\varphi(\rho) = \operatorname{sech}\rho$ and the external part $\chi(\rho,t)$:

$$U(\rho,t) = \varphi(\rho) + \chi(\rho,t). \tag{2.7}$$

Substituting (2.7) into (2.6) and using the vector notation $\vec{\chi} = \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix}$, we obtain the following equation for $\vec{\chi}$:

$$\mathcal{H}\vec{\chi} + l(q,\omega)\vec{\chi} + \mathcal{F}(q,\omega) + F^{\geq 2}(\vec{\chi}) = 0, \tag{2.8}$$

where the components of the left-hand side are defined as follows:

• the Hamiltonian corresponding to the linearization of the 1D cubic NLS $i\partial_t u + \partial_\rho^2 u + 2|u|^2 u = 0$ around the soliton $u(t,\rho) = e^{it}\operatorname{sech}\rho$:

$$\mathcal{H} = (-\partial_\rho^2 + 1)\,\sigma_3 - 4\varphi^2(\rho)\sigma_3 - 2\varphi^2(\rho)\sigma_3\sigma_1\,,$$

• the linear operator

$$l(q,\omega) = -iq^4\omega^2\partial_t - \frac{2q\omega}{\rho\omega q + 1}\partial_\rho\sigma_3 + i(2q'q^3\omega^2 + \omega'\omega q^4)(1+\rho\partial_\rho) - i\frac{q'q^3\omega^2}{\rho\omega q+1} + \frac{1}{2}q''q^6\omega^3\rho\sigma_3,$$

• the terms independent of χ:

$$\mathcal{F}(q,\omega) = l(q,\omega)\varphi\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and lastly,

• the higher order terms grouped into

$$F^{\geq 2}(\vec{\chi}) = -4\varphi|\chi|^2\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 2\varphi\begin{pmatrix} \chi^2 \\ -\bar{\chi}^2 \end{pmatrix} - 2|\chi|^2\begin{pmatrix} \chi \\ -\bar{\chi} \end{pmatrix}.$$

Proposition 2.1. *There exist coefficients $(\omega_k)_{k\geq 0}$, $(q_k)_{k\geq 0}$, with q_2 arbitrary and $\omega_0 = 1$, and smooth exponentially decaying functions $(\chi_k(\rho))_{k\geq 1}$,*

$$|e^{\mu|\rho|}\partial_\rho^l\chi_k(\rho)| \leq C_{l,\mu,k}, \quad \rho\in\mathbb{R}, \tag{2.9}$$

for any $k\geq 1$, $l\geq 0$, and $0\leq\mu<1$, verifying the orthogonality conditions

$$(\operatorname{Re}\chi_k, \varphi_\rho)_{L^2(\mathbb{R})} = (\operatorname{Im}\chi_k, \varphi)_{L^2(\mathbb{R})} = 0, \quad \forall k\geq 1, \tag{2.10}$$

such that the following holds. For any $N\geq 2$, if we define

$$\omega(t) \equiv \omega^{(N)}(t) \stackrel{\text{def}}{=} 1 + \sum_{k=1}^{N}\omega_k t^{2k/3}, \qquad q(t) \equiv q^{(N)}(t) \stackrel{\text{def}}{=} \sum_{k=0}^{N} q_k t^{(2k+1)/3},$$

$$\text{and} \quad \chi(\rho,t) \equiv \chi^{(N)}(\rho,t) \stackrel{\text{def}}{=} \sum_{k=1}^{2N+2} t^{k/3}\chi_k(\rho),$$

then (2.8) *approximately holds:*

$$\left|\partial_\rho^l\big(\mathcal{H}\vec{\chi} + l(q,\omega)\vec{\chi} + \mathcal{F}(q,\omega) + F^{\geq 2}(\vec{\chi})\big)\right| \leq C_{l,\mu}t^{(2N+3)/3}e^{-\mu|\rho|}, \tag{2.11}$$

for $0\leq\mu<1$, $\rho > -\frac{1}{2}q_0^{-1}t^{-1/3}$, $0<t\leq t_0(N)$, with some $t_0(N)>0$ and with $C_{\ell,\mu}$ independent of N.

Proof. Writing $\omega(t) = \sum_{k=0}^{N} \omega_k t^{2k/3}$, $q(t) = \sum_{k=0}^{N} q_k t^{(2k+1)/3}$, $\chi(\rho,t) = \sum_{k=1}^{2N+2} t^{k/3}\chi_k(\rho)$, $\omega_0 = 1$ and substituting this ansatz into the expression $\mathcal{H}\vec{\chi} + l(q,\omega)\vec{\chi} + \mathcal{F}(q,\omega) + F^{\geq 2}(\vec{\chi})$, we get

$$\begin{aligned} &\mathcal{H}\vec{\chi} + l(q,\omega)\vec{\chi} + \mathcal{F}(q,\omega) + F^{\geq 2}(\vec{\chi}) = \sum_{k=1}^{2N+2} t^{k/3}(\mathcal{H}\vec{\chi}_k - \vec{D}_k) + S(t), \\ &S(t) = -\sum_{k\geq 2N+3} t^{k/3}\vec{D}_k + l_1(q,\omega)(\varphi \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \vec{\chi}), \end{aligned} \tag{2.12}$$

where

$$l_1(q,\omega) = \frac{2\,q\omega(q\omega\rho)^{2N+3}}{\rho\omega q + 1}\partial_\rho\,\sigma_3 + i\frac{q' q^3\omega^2(q\omega\rho)^{2N+3}}{\rho\omega q + 1},$$

$$\vec{D}_k = \begin{pmatrix} D_k \\ -\bar{D}_k \end{pmatrix}, \quad D_k = D_k^{(0)} + D_k^{(1)} + D_k^{(2)},$$

$D_k^{(0)}$, $D_k^{(1)}$, $D_k^{(2)}$ being contributions of $\mathcal{F}(q,\omega) - l_1(q,\omega)\varphi \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $(l(q,\omega) - l_1(q,\omega))\vec{\chi}$ and $F^{\geq 2}(\vec{\chi})$ respectively:

$$\begin{aligned} &\mathcal{F}(q,\omega) - l_1(q,\omega)\varphi \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\sum_{k\geq 1} t^{k/3}\vec{D}_k^{(0)}(\rho), \\ &(l(q,\omega) - l_1(q,\omega))\vec{\chi} = -\sum_{k\geq 2} t^{k/3}\vec{D}_k^{(1)}(\rho), \\ &F^{\geq 2}(\vec{\chi}) = -\sum_{k\geq 2} t^{k/3}\vec{D}_k^{(2)}(\rho), \end{aligned}$$

$\vec{D}_k^{(i)} = \binom{D_k^{(i)}}{-\bar{D}_k^{(i)}}$. Note that all these sums contain only finite number of terms.

The following structural properties of $D_k^{(i)}$ will be important for our analysis.

(i) *Dependence on* q_j, ω_j χ_j: $D_k^{(0)}$ depends on q_j, ω_j, $j \leq (k-1)/2$, and $D_k^{(i)}$, $i = 1,2$, depend on q_j, ω_j, $j \leq (k-2)/2$, and on χ_p, $p \leq k-1$, only:

$$\begin{aligned} &D_k^{(0)} = D_k^{(0)}(\rho; q_j, \omega_j, j \leq (k-1)/2), \\ &D_k^{(i)} = D_k^{(i)}(\rho; q_j, \omega_j, j \leq (k-2)/2; \chi_p, p \leq k-1), \quad i = 1,2. \end{aligned}$$

In addition, one has

$$t^{1/3}D_1^{(0)} = 2q_0(t)\varphi_\rho - iq_0'(t)q_0^3(t)(\varphi + 2\rho\varphi_\rho) - \frac{1}{2}q_0''(t)q_0^6(t)\rho\varphi, \quad q_0(t) = q_0 t^{1/3}, \tag{2.13}$$

and for $k = 2l+1$, $l \geq 1$:

$$D_{2l+1}^{(0)} = D_{2l+1}^{(0,0)} + D_{2l+1}^{(0,1)}, \tag{2.14}$$

where $D_{2l+1}^{(0,1)} = D_{2l+1}^{(0,1)}(\rho; q_j, \omega_j, j \le l-1)$ depends on q_j, ω_j, $j \le l-1$, only and $D_{2l+1}^{(0,0)}$ is given by

(2.15)

$$\begin{aligned} &t^{(2l+1)/3} D_{2l+1}^{(0,0)} \\ &\quad = 2(q_l(t) + q_0(t)\omega_l(t))\varphi_\rho - \frac{1}{2}(q_0^6(t)q_l''(t) + 6q_0''(t)q_0^5(t)q_l(t) + 3q_0''(t)q_0^6(t)\omega_l(t))\rho\varphi \\ &\qquad - i(q_0^3(t)q_l'(t) + 3q_0'(t)q_0^2(t)q_l(t) + 2q_0'(t)q_0^3(t)\omega_l(t))(\varphi + 2\rho\varphi_\rho) \\ &\qquad - iq_0^4(t)\omega_l'(t)(\varphi + \rho\varphi_\rho), \end{aligned}$$

$q_l(t) = q_l t^{(2l+1)/3}$, $\omega_l(t) = \omega_l t^{2l/3}$.

(ii) *Smoothness and decay*: if all χ_j, $j \le k-1$, verify (2.9), then a similar estimate holds for D_k:

$$|e^{\mu|\rho|}\partial_\rho^l D_k(\rho)| \le C_{l,\mu,k}, \quad \rho \in \mathbb{R}, \tag{2.16}$$

for any $l \ge 0$ and $\mu < 1$.

(iii) *Parity*: if for any $j \le k-1$, $v_j \stackrel{\text{def}}{=} \operatorname{Re}\chi_j$ and $u_j \stackrel{\text{def}}{=} \operatorname{Im}\chi_j$ satisfy

$$v_j(-\rho) = (-1)^j v_j(\rho), \quad u_j(-\rho) = (-1)^{j+1} u_j(\rho), \tag{2.17}$$

then one has the same property for $\mathcal{G}_k^+ \stackrel{\text{def}}{=} \operatorname{Re} D_k$, $\mathcal{G}_k^- \stackrel{\text{def}}{=} \operatorname{Im} D_k$:

$$\mathcal{G}_k^+(-\rho) = (-1)^k \mathcal{G}_k^+(-\rho), \quad \mathcal{G}_k^-(-\rho) = (-1)^{k+1}\mathcal{G}_k^-(-\rho). \tag{2.18}$$

Our goal now is to show that we can find coefficients $q_0, q_1, \dots, q_N$, $\omega_1, \dots, \omega_N$ (q_0 being as in Theorem 1.1 and q_2 arbitrary) and functions $\chi_1, \dots, \chi_{2N+2}$ satisfying (2.9), (2.10), (2.17) such that

$$\mathcal{H}\vec{\chi}_k = \vec{D}_k, \quad k = 1, \dots, 2N+2. \tag{2.19}$$

Estimate (2.11) will then follow from (2.12). As we see below, the equations with $k = 5$ require some special care, therefore, we will go explicitly through $k = 1, \dots, 5$ and then we will proceed by induction.

It will be convenient for us to rewrite (2.19) as

$$L_+ v_k = \mathcal{G}_k^+, \quad L_- u_k = \mathcal{G}_k^-, \quad k = 1, \dots, 2N+2, \tag{2.20}$$

where $L_+ = -\partial_\rho^2 + 1 - 6\varphi^2$, $L_- = -\partial_\rho^2 + 1 - 2\varphi^2$ are self-adjoint operators in $L^2(\mathbb{R})$ with the spectrum $\sigma(L_+) = \{-3, 0\} \cup [1, \infty)$, $\sigma(L_-) = \{0\} \cup [1, \infty)$. One has

$$L_-\varphi = 0, \quad L_+\varphi_\rho = 0.$$

For $k = 1$ the equations (2.20) give

$$L_+ v_1 = \mathcal{G}_1^+, \quad L_- u_1 = \mathcal{G}_1^-, \tag{2.21}$$

with

$$\mathcal{G}_1^+ = 2q_0\partial_\rho\varphi + \frac{1}{9}q_0^7\rho\varphi, \quad \mathcal{G}_1^- = -\frac{1}{3}q_0^4(\varphi + 2\rho\varphi_\rho);$$

see (2.13).

Since $(\mathcal{G}_1^-, \varphi)_{L^2(\mathbb{R})} = -\frac{1}{3}q_0^4(\varphi + 2\rho\varphi_\rho, \varphi)_{L^2(\mathbb{R})} = 0$, the equation $L_- u_1 = \mathcal{G}_1^-$ has a unique solution u_1 in L^2 satisfying $(u_1, \varphi)_{L^2(\mathbb{R})} = 0$. One can compute it explicitly:

$$u_1 = \frac{1}{6}q_0^4(\rho^2\varphi + c_0\varphi), \quad c_0 = -\frac{\|\rho\varphi\|_{L^2(\mathbb{R})}^2}{\|\varphi\|_{L^2(\mathbb{R})}^2}.$$

The solvability of the equation $L_+v_1 = \mathcal{G}_1^+$ is subjected to the condition

$$(\mathcal{G}_1^+, \varphi_\rho)_{L^2(\mathbb{R})} = 0,$$

which gives

$$q_0 - Aq_0^7 = 0, \quad A = \frac{\|\varphi\|^2_{L^2(\mathbb{R})}}{36\|\varphi_\rho\|^2_{L^2(\mathbb{R})}} = \frac{1}{12}.$$

Therefore, we find

$$q_0 = 12^{1/6}. \tag{2.22}$$

Under this solvability condition, the equation $L_+v_1 = \mathcal{G}_1^+$ has a unique solution v_1 satisfying $(v_1, \varphi_\rho)_{L^2(\mathbb{R})} = 0$, which is given by

$$v_1 = -\frac{q_0}{3}(\partial_\rho(\rho^2\varphi) + c_1\varphi_\rho), \quad c_1 = -\frac{(\partial_\rho(\rho^2\varphi), \varphi_\rho)_{L^2(\mathbb{R})}}{\|\varphi_\rho\|_{L^2(\mathbb{R})}}.$$

Note that v_1 is an odd and u_1 is an even function of ρ, in agreement with (2.17), and they both satisfy (2.9).

Consider (2.20) with $k = 2$. We have

$$L_+v_2 = \mathcal{G}_2^+, \quad L_-u_2 = \mathcal{G}_2^-, \tag{2.23}$$

where $\mathcal{G}_2^+ = \mathcal{G}_2^+(\rho; q_0)$, $\mathcal{G}_2^- = \mathcal{G}_2^-(\rho; q_0)$ are smooth exponentially decaying functions of ρ, verifying (2.16), (2.18). Since $\mathcal{G}_2^+$ is even and $\mathcal{G}_2^-$ is odd, the solvability conditions $(\mathcal{G}_2^-, \varphi)_{L^2(\mathbb{R})} = (\mathcal{G}_2^+, \varphi_\rho)_{L^2(\mathbb{R})} = 0$ are satisfied and (2.23) has a unique solution v_2, u_2 with v_2 even and u_2 odd, verifying (2.9).

Next consider $k = 3, 4$. For $k = 3$ we have

$$L_+v_3 = \mathcal{G}_3^+(q_0, q_1, \omega_1, \chi_1, \chi_2), \quad L_-u_3 = \mathcal{G}_3^-(q_0, q_1, \omega_1, \chi_1, \chi_2), \tag{2.24}$$

with $\mathcal{G}_3^+(q_0, q_1, \omega_1, \chi_1, \chi_2)$, $\mathcal{G}_3^-(q_0, q_1, \omega_1, \chi_1, \chi_2)$ verifying (2.16), (2.18). Using (2.14), (2.15), (2.22), one can write $\mathcal{G}_3^+$, $\mathcal{G}_3^-$ as

$$\begin{aligned} \mathcal{G}_3^+ &= 2(q_1 + q_0\omega_1)\varphi_\rho + 4(2q_1 + q_0\omega_1)\rho\varphi + \tilde{\mathcal{G}}_3^+, \\ \mathcal{G}_3^- &= -(2q_0^3q_1 + \frac{2}{3}q_0^4\omega_1)(\varphi + 2\rho\varphi_\rho) - \frac{2}{3}q_0^4\omega_1(\varphi + \rho\varphi_\rho) + \tilde{\mathcal{G}}_3^-, \end{aligned} \tag{2.25}$$

where $\tilde{\mathcal{G}}_3^+ = \tilde{\mathcal{G}}_3^+(q_0, \chi_1, \chi_2)$, $\tilde{\mathcal{G}}_3^- = \tilde{\mathcal{G}}_3^-(q_0, \chi_1, \chi_2)$ depend on q_0, χ_1, χ_2 only, and therefore, are determined by now.

Using (2.25), one can write the solvability conditions $(\mathcal{G}_3^-, \varphi)_{L^2(\mathbb{R})} = (\mathcal{G}_3^+, \varphi_\rho)_{L^2(\mathbb{R})} = 0$ as

$$-\frac{q_0^4\|\varphi\|^2_{L^2(\mathbb{R})}}{3}\omega_1 + (\tilde{\mathcal{G}}_3^-, \varphi)_{L^2(\mathbb{R})} = 0,$$

$$2(q_1 + q_0\omega_1)\|\varphi_\rho\|^2_{L^2(\mathbb{R})} - 2(2q_1 + q_0\omega_1)\|\varphi\|^2_{L^2(\mathbb{R})} + (\tilde{\mathcal{G}}_3^+, \varphi_\rho)_{L^2(\mathbb{R})} = 0,$$

which gives,

$$\omega_1 = \frac{3(\tilde{\mathcal{G}}_3^-, \varphi)_{L^2(\mathbb{R})}}{q_0^4\|\varphi\|^2_{L^2(\mathbb{R})}},$$

$$q_1 = -\frac{2}{5}q_0\omega_1 + \frac{(\tilde{\mathcal{G}}_3^+, \varphi_\rho)_{L^2(\mathbb{R})}}{10\|\varphi_\rho\|^2_{L^2(\mathbb{R})}}.$$

With this choice of ω_1, q_1, (2.24) has a unique solution v_3, u_3 verifying (2.9), (2.10) with v_3, u_3 being of the same parity as $\mathcal{G}_3^+$, $\mathcal{G}_3^-$: v_3 is odd and u_3 is even.

The case $k=4$ is similar to that of $k=2$: we have

$$L_+ v_4 = \mathcal{G}_4^+, \quad L_- u_4 = \mathcal{G}_4^-, \tag{2.26}$$

with $\mathcal{G}_4^+$ even and $\mathcal{G}_4^-$ odd, depending on $q_0, q_1, \omega_1, \chi_1, \chi_2, \chi_3$ only. Thus, (2.26) has a unique solution v_4, u_4 with v_4 even and u_4 odd, verifying (2.9).

Consider the case $k=5$. We have

$$L_+ v_5 = \mathcal{G}_5^+, \quad L_- u_5 = \mathcal{G}_5^-, \tag{2.27}$$

with

$$\mathcal{G}_5^+ = \mathcal{G}_5^+(q_0, q_1, q_2, \omega_1, \omega_2, \chi_1, \dots, \chi_4) \text{ odd}$$

and

$$\mathcal{G}_5^- = \mathcal{G}_5^-(q_0, q_1, q_2, \omega_1, \omega_2, \chi_1, \dots, \chi_4) \text{ even.}$$

It follows from (2.15), (2.22) that

$$\begin{aligned} \mathcal{G}_5^+ &= 2(q_2 + q_0\omega_2)\varphi_\rho + \frac{4}{3}(q_2 + 3q_0\omega_2)\rho\varphi + \tilde{\mathcal{G}}_5^+, \\ \mathcal{G}_5^- &= -\frac{2}{3}(4q_0^3 q_2 + q_0^4\omega_2)(\varphi + 2\rho\varphi_\rho) - \frac{4}{3}q_0^4\omega_2(\varphi + \rho\varphi_\rho) + \tilde{\mathcal{G}}_5^-, \end{aligned} \tag{2.28}$$

where $\tilde{\mathcal{G}}_5^\pm = \tilde{\mathcal{G}}_5^\pm(q_0, q_1, \omega_1, \chi_1, \dots, \chi_4)$ are determined by now. Substituting this representation into the solvability conditions $(\mathcal{G}_5^-, \varphi)_{L^2(\mathbb{R})} = 0$, $(\mathcal{G}_5^+, \varphi_\rho)_{L^2(\mathbb{R})} = 0$, we obtain

$$\omega_2 = \frac{3(\tilde{\mathcal{G}}_5^-, \varphi)_{L^2(\mathbb{R})}}{2q_0^4\|\varphi\|^2_{L^2(\mathbb{R})}}, \tag{2.29}$$

$$-4q_0\omega_2\|\varphi_\rho\|^2_{L^2(\mathbb{R})} + (\tilde{\mathcal{G}}_5^+, \varphi_\rho)_{L^2(\mathbb{R})} = 0, \tag{2.30}$$

which means that one has to have

$$q_0^3(\tilde{\mathcal{G}}_5^+, \varphi_\rho)_{L^2(\mathbb{R})} = 2(\tilde{\mathcal{G}}_5^-, \varphi)_{L^2(\mathbb{R})}. \tag{2.31}$$

Note that (2.29), (2.30) do not contain q_2, therefore, it can be chosen arbitrarily (as soon as we can show that (2.31) holds). To check (2.31), we proceed as follows. Consider $\omega^{(1)}(t) = 1 + \omega_1 t^{2/3}$, $q^{(1)}(t) = q_0 t^{1/3} + q_1 t$, $\chi^{(1)}(\rho, t) = \sum_{k=1}^4 t^{k/3}\chi_k(\rho)$. By our construction, they verify

$$\mathcal{H}\vec{\chi}^{(1)} + l(q^{(1)}, \omega^{(1)})\vec{\chi}^{(1)} + \mathcal{F}(q^{(1)}, \omega^{(1)}) + F^{\geq 2}(\vec{\chi}^{(1)}) = t^{5/3}\vec{\mathcal{D}}(\rho) + \frac{1}{1 + \omega^{(1)}q^{(1)}\rho}\mathcal{S}(\rho, t), \tag{2.32}$$

where $\vec{\mathcal{D}} = \binom{\mathcal{D}}{-\mathcal{D}}$, $\mathcal{D} = -\tilde{\mathcal{G}}_5^+ - i\tilde{\mathcal{G}}_5^-$, and $\mathcal{S}(\rho, t)$ admits the estimate

$$|e^{\mu|\rho|}\partial_\rho^l \mathcal{S}(\rho, t)| \leq C_{l,\mu} t^2, \quad \rho \in \mathbb{R}, \tag{2.33}$$

for any $l \geq 0$ and $\mu < 1$. Define

$$z(x,t) \stackrel{\text{def}}{=} e^{i\theta_1(t) + iv_1(t)r/2}\lambda_1(t)(\varphi + \chi^{(1)})(\lambda_1(t)(r - q^{(1)}(t)), t), \tag{2.34}$$

where λ_1, v_1, θ_1 are given by (2.3), (2.5) with $q = q^{(1)}$ and $\omega = \omega^{(1)}$. Then $z(t)$ solves

$$iz_t + \Delta z + 2|z|^2 z = R_0(x,t) + R_1(x,t), \quad x \in \mathbb{R}^3, \tag{2.35}$$

where (by (2.32) and (2.33))

$$R_0(x,t) = -t^{5/3}e^{i\theta_1(t) + iv_1(t)r/2}\lambda_1^3(t)\mathcal{D}(\lambda_1(t)(r - q^{(1)}(t))),$$

and R_1 satisfies

$$\|R_1\|_{L^2(\mathbb{R}^3)} \le C\, t^{2/3}. \tag{2.36}$$

Consider the energy $E(z(t)) = \int(|\nabla z(x,t)|^2 - |z(x,t)|^4)dx$. It follows from (2.35), (2.36), (2.34) that

$$\frac{d}{dt}E(z(t)) = \alpha t^{-1} + O(t^{-2/3}), \quad t \to 0, \tag{2.37}$$

with

$$\alpha = -\frac{2}{3q_0^8}\left(q_0^3(\tilde{\mathcal{G}}_5^+, \varphi_\rho)_{L^2(\mathbb{R})} - 2(\tilde{\mathcal{G}}_5^-, \varphi)_{L^2(\mathbb{R})}\right). \tag{2.38}$$

On the other hand, using the definition of $z(t)$, we can write an expansion of $E(z(t))$ in powers of $t^{1/3}$:

$$E(z(t)) = \sum_{k=-4}^{0} t^{k/3}c_k + O(t^{1/3}), \quad t \to 0, \tag{2.39}$$

with some constants c_k. Comparing this expansion to (2.37) one gets $c_{-4} = \cdots = c_{-1} = 0$ and $\alpha = 0$, which is precisely (2.31).

Verifying the solvability conditions (2.29), (2.30) results in the fact that the system (2.27) has a unique solution v_5, u_5 satisfying (2.9), (2.10) with v_5 being odd and u_5 being even.

To finish the proof of Proposition 2.1, we proceed by induction. Suppose we have solved (2.20) with $k = 1, \dots, 2l-1$, $l \ge 3$, and have found q_j, ω_j, $j \le l-1$, and χ_p, $p \le 2l-1$, verifying (2.9), (2.10), and (2.17). Consider $k = 2l, 2l+1$. For $k = 2l$ we have

$$\begin{aligned} L_+ v_{2l} &= \mathcal{G}_{2l}^+(q_j, \omega_j, j \le l-1; \chi_p, p \le 2l-1), \\ L_- u_{2l} &= \mathcal{G}_{2l}^-(q_j, \omega_j, j \le l-1; \chi_p, p \le 2l-1), \end{aligned} \tag{2.40}$$

with $\mathcal{G}_{2l}^+$ being even and $\mathcal{G}_{2l}^-$ being odd, verifying (2.16). Therefore, the equation (2.40) has a unique solution v_{2l}, u_{2l} with v_{2l} even and u_{2l} odd, that satisfies (2.9).

Consider $k = 2l+1$. We have

$$\begin{aligned} L_+ v_{2l+1} &= \mathcal{G}_{2l+1}^+(q_j, \omega_j, j \le l; \chi_p, p \le 2l), \\ L_- u_{2l+1} &= \mathcal{G}_{2l+1}^-(q_j, \omega_j, j \le l; \chi_p, p \le 2l), \end{aligned} \tag{2.41}$$

where $\mathcal{G}_{2l+1}^+$ is odd and $\mathcal{G}_{2l+1}^-$ is even. From (2.15), (2.22) one has

$$\begin{aligned} \mathcal{G}_{2l+1}^+ &= 2(q_l + q_0\omega_l)\varphi_\rho - \frac{4}{3}((2l^2 - l - 7)q_l - 3q_0\omega_l)\rho\varphi + \tilde{\mathcal{G}}_{2l+1}^+, \\ \mathcal{G}_{2l+1}^- &= -\frac{2q_0^3}{3}((l+2)q_l + q_0\omega_l)(\varphi + 2\rho\varphi_\rho) - \frac{2lq_0^4}{3}\omega_l(\varphi + \rho\varphi_\rho) + \tilde{\mathcal{G}}_{2l+1}^-, \end{aligned} \tag{2.42}$$

where $\tilde{\mathcal{G}}_{2l+1}^\pm$ depends only on $q_j, \omega_j, j \le l-1; \chi_p, p \le 2l$. Using this representation one can write the solvability conditions $(\mathcal{G}_{2l+1}^-, \varphi)_{L^2(\mathbb{R})} = 0$, $(\mathcal{G}_{2l+1}^+, \varphi_\rho)_{L^2(\mathbb{R})} = 0$ as

$$\begin{aligned} &-\frac{q_0^4 l\omega_l}{3}\|\varphi\|_{L^2(\mathbb{R})}^2 + (\tilde{\mathcal{G}}_{2l+1}^-, \varphi)_{L^2(\mathbb{R})} = 0, \\ &2q_l(2l^2 - l - 6)\|\varphi_\rho\|_{L^2(\mathbb{R})}^2 - 4q_0\omega_l\|\varphi_\rho\|_{L^2(\mathbb{R})}^2 + (\tilde{\mathcal{G}}_{2l+1}^+, \varphi_\rho)_{L^2(\mathbb{R})} = 0, \end{aligned}$$

which gives

$$\omega_l = \frac{3(\tilde{\mathcal{G}}^-_{2l+1},\varphi)_{L^2(\mathbb{R})}}{lq_0^4\|\varphi\|^2_{L^2(\mathbb{R})}},$$

$$q_l = \frac{1}{2(2l^2-l-6)}\left(\frac{4(\tilde{\mathcal{G}}^-_{2l+1},\varphi)_{L^2(\mathbb{R})}}{lq_0^3\|\varphi_\rho\|^2_{L^2(\mathbb{R})}} - \frac{(\tilde{\mathcal{G}}^+_{2l+1},\varphi_\rho)_{L^2(\mathbb{R})}}{\|\varphi_\rho\|^2_{L^2(\mathbb{R})}}\right).$$

After verifying the solvability conditions, one finds v_{2l+1}, u_{2l+1} as the unique solution of (2.41) satisfying (2.10). □

Define truncation functions $\theta_j \in C_c^\infty(\mathbb{R})$ by

$$\theta_0(\zeta) = \begin{cases} 1, & |\zeta| \le \frac{1}{200} \\ 0, & |\zeta| \ge \frac{1}{100} \end{cases}, \qquad \theta_1(\zeta) = \begin{cases} 1 & \text{for } |\zeta| \le \frac{1}{20} \\ 0 & \text{for } |\zeta| \ge \frac{1}{10} \end{cases}. \tag{2.43}$$

For each $N \in \mathbb{N}$, introduce

$$U^{(N)}(\rho,t) = \left(\varphi + \sum_{k=1}^{2N+2} t^{k/3}\chi_k(\rho)\right)\theta_0(\rho\omega q). \tag{2.44}$$

Note that

$$\operatorname{supp} U^{(N)} \subset \left\{|\rho| \le \frac{1}{100q\omega}\right\}. \tag{2.45}$$

By construction each approximation $U^{(N)}$ solves (see equation (2.4))

$$\begin{aligned} i\frac{1}{\lambda^2}U_t^{(N)} + U^{(N)}_{\rho\rho} - U^{(N)} + 2|U^{(N)}|^2U^{(N)} &= -\frac{2}{\rho+\lambda q}U^{(N)}_\rho \\ &+ \frac{q''\rho}{2\lambda^3}U^{(N)} - i\frac{\lambda'}{\lambda^3}\big(1+\rho\,\partial_\rho\big)U^{(N)} - i\frac{q'}{\lambda(\rho+\lambda q)}U^{(N)} + \mathcal{R}_N(\rho,t), \end{aligned} \tag{2.46}$$

where the remainder $\mathcal{R}_N$ is supported in $\{|\rho| \le \frac{1}{100q\omega}\}$ by (2.43) and (2.45). By (2.11) for each $N \in \mathbb{N}$ there exists a time $t_0(N) > 0$ such that $\mathcal{R}_N$ satisfies the following bounds:

$$|\partial_\rho^k \mathcal{R}_N(\rho,t)| \le C_k\, e^{-\mu|\rho|}\, t^{(2N+3)/3} \quad \text{for} \quad 0 < t \le t_0(N), \quad k = 0,1,2,..., \tag{2.47}$$

with the constants μ and C_k independent of N.

Observe also that from (2.44) (the first term is time-independent φ), we have by (2.9) that

$$|\partial_\rho^k U^{(N)}(\rho,t)| \le C\, e^{-\mu|\rho|} \quad \text{for} \quad 0 < t < t_0(N), \quad k = 0,1,2,\dots. \tag{2.48}$$

For each $N \in \mathbb{N}$ we set $\psi^{(N)}$ by

$$\psi^{(N)}(r,t) = e^{i\theta(t)}e^{iv(t)r/2}\lambda(t)\,U^{(N)}\big(\lambda(t)(r-q(t)),t\big). \tag{2.49}$$

Then $\psi^{(N)}$ solves

$$i\partial_t\psi^{(N)} + \Delta\psi^{(N)} + 2|\psi^{(N)}|^2\psi^{(N)} = iH^{(N)}(x,t), \tag{2.50}$$

where

$$H^{(N)}(r,t) = ie^{i\theta(t)}e^{iv(t)r/2}\lambda^3\mathcal{R}_N(\lambda(r-q(t)),t). \tag{2.51}$$

By (2.47), it follows that

$$\|\partial_r^k H^{(N)}(t)\|_{L^2(\mathbb{R}^3)} \le C_k\lambda^{3+k}t^{(2N+3)/3}\|1_{|r-q|\le\lambda^{-1}}\|_{L^2(\mathbb{R}^3)},$$

and hence,

(2.52)
$$t^{2k/3}\|\partial_r^k H^{(N)}(t)\|_{L^2(\mathbb{R}^3)} \le C_k t^{(2N-1)/3} \quad \text{for } 0 < t \le t_0(N), \quad k = 0, 1, 2, \dots .$$

Proposition 2.2. *Given $e \in \mathbb{R}$, we can choose q_2 in the construction in Proposition 2.1 such that the energy of the approximate solution $E(\psi^{(N)})$ estimates as*

$$|E(\psi^{(N)})(t) - e| \le C t^{\frac{2N}{3}-1} \quad \textit{for } 0 < t < t_0(N). \tag{2.53}$$

Proof. The following two estimates are straightforward consequences of (2.50) and (2.52):

$$\left|\frac{d}{dt}\|\psi^{(N)}(t)\|^2_{L^2(\mathbb{R}^3)}\right| \le C\, t^{\frac{2N-2}{3}} \quad \text{for } 0 < t \le t_0(N), \tag{2.54}$$

and

$$\left|\frac{d}{dt}E(\psi^{(N)})(t)\right| \le C\, \lambda^2\, t^{\frac{2N-2}{3}} \quad \text{for } 0 < t \le t_0(N). \tag{2.55}$$

Consider $q^{-2}(t)\lambda^{-3}(t)E(\psi^{(N)})(t)$. By construction of $\psi^{(N)}$, it admits an expansion in positive powers of $t^{1/3}$ (compare to (2.37), (2.38)):

$$q^{-2}(t)\lambda^{-3}(t)E(\psi^{(N)})(t) = \sum_{k=0}^{K(N)} \beta_k t^{k/3} + O(t^\infty), \quad t \to 0, \tag{2.56}$$

where β_k depends on χ_p, $p \le k$, and on ω_j, q_j, $j \le k/2$. More precisely, for $k = 2l$ one has

$$\beta_{2l} = \frac{4}{3}(\omega_l + (2l+3)\frac{q_l}{q_0}) + \tilde{\beta}_{2l}, \quad l \ge 0, \tag{2.57}$$

with $\tilde{\beta}_{2l} = \tilde{\beta}_{2l}(\chi_p, p \le 2l; \omega_j, q_j, j \le l-1)$ depending on χ_p, $p \le 2l$, and on ω_j, q_j, $j \le l-1$, only. From (2.55), (2.56), we deduce $\beta_0 = \dots = \beta_3 = 0$ and

$$|E(\psi^{(N)})(t) - q_0^{-4}\beta_4| \le C t^{\frac{2N}{3}-1} \quad \text{for } 0 < t \le t_0(N).$$

Therefore, if we fix q_2 by requiring $\frac{4}{3}(\omega_2 + 7\frac{q_2}{q_0}) + \tilde{\beta}_4 = e q_0^4$, then we get (2.53). □

3. Comparison to exact solution

Let us begin with some notational conventions for this section. We exclusively use the *real* inner product

$$\langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{R}^3)} = \operatorname{Re} \int_{\mathbb{R}^3} \psi_1(x)\overline{\psi_2(x)}\, dx \tag{3.1}$$

on $L^2(\mathbb{R}^3)$ as well as the related 1D version

$$\langle \psi_1, \psi_2 \rangle_{L^2(0<r<\infty)} = \operatorname{Re} \int_{r=0}^{\infty} \psi_1(r)\overline{\psi_2(r)}\, dr.$$

Note that

$$\langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{R}^3)} = 4\pi \langle r\psi_1(r), r\psi_2(r) \rangle_{L^2(0<r<\infty)}.$$

For a (densely defined) functional $A : L^2(\mathbb{R}^3) \to \mathbb{R}$, via the inner product (3.1) we identify $A'(\psi)$ with a function and $A''(\psi)$ with an operator, which is self-adjoint with respect to (3.1). Let

$$J \stackrel{\text{def}}{=} -\tfrac{1}{2}i$$

and note that, with respect to the inner product (3.1), $J^* = -J$. Define the Poisson bracket

$$\{A, B\}(\psi) = \langle JA'(\psi), B'(\psi)\rangle,$$

which yields a functional $\{A, B\} : L^2(\mathbb{R}^3) \to \mathbb{R}$.

We find it convenient to state estimates in terms of the time-dependent scaled Sobolev norms:

$$\|h\|_{X^k(\mathbb{R}^3)} \stackrel{\text{def}}{=} \sum_{j=0}^{k} t^{2j/3} \|\nabla^j h\|_{L^2(\mathbb{R}^3)}. \tag{3.2}$$

Most frequently, we use the case $k = 1$:

$$\|h\|_{X^1(\mathbb{R}^3)} \stackrel{\text{def}}{=} t^{2/3} \|\nabla h\|_{L^2(\mathbb{R}^3)} + \|h\|_{L^2(\mathbb{R}^3)}.$$

By default $X^1 = X^1(\mathbb{R}^3)$, although we shall have occasion to use the variant $X^1(0 < r < \infty)$, which has the expected definition.

We shall frequently need the following radial Sobolev inequality. For any radial function f,

$$\|f \cdot 1_{r \sim t^{1/3}}\|_{L^\infty(\mathbb{R}^3)} \lesssim t^{-1/3} \|\nabla f\|^{1/2}_{L^2(\mathbb{R}^3)} \|f\|^{1/2}_{L^2(\mathbb{R}^3)} \lesssim t^{-2/3} \|f\|_{X^1(\mathbb{R}^3)}. \tag{3.3}$$

Recall the function $\psi^{(N)}(t)$ defined on $0 < t \le t_0(N)$ by (2.49). For any $0 < \epsilon \le t_0(N)$, let $\psi(t)$ be the solution to (1.1) such that

$$\psi(\epsilon) = \psi^{(N)}(\epsilon)$$

(so ψ depends on ϵ and N but this is suppressed in our notation). Let

$$h(t) \stackrel{\text{def}}{=} \psi(t) - \psi^{(N)}(t).$$

Each of the Propositions 3.2, 3.3, 3.4, 3.5, and 3.6 in this section references ψ and h defined as above. All bounds stated will be valid for $0 < t \le t_0(N)$. All implicit constants (indicated through the notation $\lesssim$) will be independent of ϵ and N.

Note that (1.1) can be written as

$$\partial_t \psi = JE'(\psi), \tag{3.4}$$

while (2.50) can be written as

$$\partial_t \psi^{(N)} = JE'(\psi^{(N)}) + H^{(N)}, \tag{3.5}$$

where $H^{(N)}$ is defined in (2.51) and satisfies (2.52).

Recalling (2.43), define the radial localized momentum

$$\mathcal{P}_q(\psi) = \operatorname{Im} \int_{\mathbb{R}^3} \left(\theta_1\Big(\frac{r-q}{q}\Big)\right)^2 \bar{\psi}\, \partial_r \psi \, dx. \tag{3.6}$$

Define the functional

$$W(\psi) = \underbrace{\left(1 + \frac{v^2}{4\lambda^2}\right)}_{O(1)} M(\psi) - \underbrace{\frac{v}{\lambda^2}}_{O(t^{2/3})} \mathcal{P}_q(\psi) + \underbrace{\frac{1}{\lambda^2}}_{O(t^{4/3})} E(\psi), \tag{3.7}$$

where $M(\psi)$, $\mathcal{P}_q(\psi)$, and $E(\psi)$ are the mass, localized radial momentum, and energy functionals defined in (1.2), (3.6), and (1.3). Define the Lyapunov functional

$$G(h) \stackrel{\text{def}}{=} W(\psi) - W(\psi^{(N)}) - \langle W'(\psi^{(N)}), h\rangle. \tag{3.8}$$

The statement and proof of the following lemma are based on Holmer-Lin [4, Lemma 5.1].

Lemma 3.1. *For $G(h)$ as defined in* (3.8)*, we have*

$$\partial_t G(h) = \{E, W\}(\psi) - \{E, W\}(\psi^{(N)}) - \langle \{E, W\}'(\psi^{(N)}), h\rangle - \mathcal{E}_1 + \mathcal{E}_2, \tag{3.9}$$

where

$$\mathcal{E}_1 \stackrel{\text{def}}{=} \langle W''(\psi^{(N)})H^{(N)}, h\rangle + \langle W'(\psi^{(N)}), JE'(\psi) - JE'(\psi^{(N)}) - JE''(\psi^{(N)})h\rangle,$$

$\mathcal{E}_2$ is the result of the time derivative landing on any of the parameters in W, and $H^{(N)}$ is as in (3.5)*.*

Explicitly,

$$\begin{aligned}\mathcal{E}_2 = {} & (\frac{1}{2}v\dot{v}\lambda^{-2} - \frac{1}{2}v^2\lambda^{-3}\dot{\lambda})(M(\psi) - M(\psi^{(N)}) - \langle M'(\psi^{(N)}), h\rangle) \\ & + (-\dot{v}\lambda^{-2} - 2v\lambda^{-3}\dot{\lambda})(\mathcal{P}_q(\psi) - \mathcal{P}_q(\psi^{(N)}) - \langle P_q'(\psi^{(N)}), h\rangle) \\ & - \lambda^{-2}\dot{\lambda}(E(\psi) - E(\psi^{(N)}) - \langle E'(\psi^{(N)}), h\rangle).\end{aligned} \tag{3.10}$$

Proof. We write expressions for the time derivatives of each term in (3.8), dropping the terms that lead to (3.10). First, from (3.4),

$$\begin{aligned}\partial_t W(\psi) &= \langle W'(\psi), \partial_t\psi\rangle \\ &= \langle W'(\psi), JE'(\psi)\rangle.\end{aligned} \tag{3.11}$$

Next, from (3.5),

$$\begin{aligned}\partial_t W(\psi^{(N)}) &= \langle W'(\psi^{(N)}), \partial_t\psi^{(N)}\rangle \\ &= \langle W'(\psi^{(N)}), JE'(\psi^{(N)}) + H^{(N)}\rangle.\end{aligned} \tag{3.12}$$

Finally, from (3.4) and (3.5),

$$\begin{aligned}\partial_t\langle W'(\psi^{(N)}), h\rangle &= \langle W''(\psi^{(N)})\partial_t\psi^{(N)}, h\rangle + \langle W'(\psi^{(N)}), \partial_t\psi - \partial_t\psi^{(N)}\rangle \\ &= \langle W''(\psi^{(N)})(JE'(\psi^{(N)}) + H^{(N)}), h\rangle \\ &\quad + \langle W'(\psi^{(N)}), JE'(\psi) - JE'(\psi^{(N)}) - H^{(N)}\rangle.\end{aligned} \tag{3.13}$$

Taking (3.11) minus (3.12) minus (3.13), and noting the cancellation of $\langle W'(\psi^{(N)}), H^{(N)}\rangle$ in (3.12) and (3.13), we obtain

$$\begin{aligned}\partial_t G(h) = {} & \langle W'(\psi), JE'(\psi)\rangle - \langle W'(\psi^{(N)}), JE'(\psi^{(N)})\rangle \\ & - \langle W''(\psi^{(N)})(JE'(\psi^{(N)}) + H^{(N)}), h\rangle \\ & - \langle W'(\psi^{(N)}), JE'(\psi) - JE'(\psi^{(N)})\rangle.\end{aligned} \tag{3.14}$$

Since

$$\{E, W\}(\psi) = \langle JE'(\psi), W'(\psi)\rangle,$$

we compute

$$\langle \{E, W\}'(\psi^{(N)}), h\rangle = \langle JE''(\psi^{(N)})h, W'(\psi^{(N)})\rangle + \langle JE'(\psi^{(N)}), W''(\psi^{(N)})h\rangle. \tag{3.15}$$

Now, (3.9) follows from (3.14) and (3.15). □

Recall the definition of the X^k norm in (3.2). By processing the terms in Lemma 3.1, we obtain the following upper bound on $\partial_t G$.

Proposition 3.2. *Let $G(h)$ be defined as in* (3.8) *and suppose*[3] $\|h\|_{X^1} \leq 1$. *Then*

$$|\partial_t G(h)| \lesssim t^{-1}\|h\|_{X^1}^2 + t\|H^{(N)}\|_{X^1}^2. \tag{3.16}$$

This bound is valid for $\epsilon \leq t \leq t_0(N)$, and the implicit constants are independent of ϵ and N.

Proof. To apply Lemma 3.1, we need to bound $\mathcal{E}_1$, $\mathcal{E}_2$, and to compute the Poisson bracket $\{E, W\}$.

Computation yields (here, θ_1 is short for $\theta_1((r-q)/q)$),

$$\begin{aligned} E'(\psi) &= -2\Delta\psi - 4|\psi|^2\psi \\ &= -2\partial_r^2\psi - 4r^{-1}\partial_r\psi - 4|\psi|^2\psi, \end{aligned} \tag{3.17}$$

and, for convenience, taking $\alpha = \partial_r(\theta_1^2) + 2r^{-1}\theta_1^2$, we have

$$\mathcal{P}_q'(\psi) = -i(2\theta_1^2\partial_r\psi + \alpha\psi). \tag{3.18}$$

First, we address $\mathcal{E}_1$. We have

$$\begin{aligned} W'(\psi^{(N)}) = {} & 2\underbrace{(1+\frac{1}{4}v^2\lambda^{-2})}_{\mathcal{O}(1)}\psi^{(N)} - i\underbrace{v\lambda^{-2}}_{\mathcal{O}(t^{2/3})}(2\theta_1^2\partial_r\psi^{(N)} + \alpha\psi^{(N)}) \\ & + \underbrace{\lambda^{-2}}_{\mathcal{O}(t^{4/3})}(-2\Delta\psi^{(N)} - 4|\psi^{(N)}|^2\psi^{(N)}). \end{aligned} \tag{3.19}$$

By (3.19) and (3.3), we obtain

$$\|W'(\psi^{(N)})\|_{L^2} \lesssim \|\psi^{(N)}\|_{X^2} \lesssim 1.$$

However, this ignores the fact that the leading order term in $\psi^{(N)}$ is the push-forward of $\varphi(\rho) = \operatorname{sech}\rho$, which leads to a vanishing term in (3.19). Thus, we in fact have

$$\|W'(\psi^{(N)})\|_{L^2} \lesssim t^{1/3}. \tag{3.20}$$

We have

$$E'(\psi) - E'(\psi^{(N)}) - E''(\psi^{(N)})h = \psi^{(N)}h^2 + h^3 \tag{3.21}$$

(ignoring complex conjugates). In the pairing $\langle W'(\psi^{(N)}), JE'(\psi) - JE'(\psi^{(N)}) - JE''(\psi^{(N)})h\rangle$, each term will have a factor of $\psi^{(N)}$, which localizes to $r \sim t^{1/3}$.

By (3.21), (3.3), and (3.20), it follows that

$$\begin{aligned} & |\langle W'(\psi^{(N)}), JE'(\psi) - JE'(\psi^{(N)}) - JE''(\psi^{(N)})h\rangle| \\ & \quad \lesssim \|W'(\psi^{(N)})\|_{L^2}(\|\psi^{(N)}\|_{L^\infty}\|h\cdot 1_{r\sim t^{1/3}}\|_{L^\infty}\|h\|_{L^2} + \|h\cdot 1_{r\sim t^{1/3}}\|_{L^\infty}^2\|h\|_{L^2}) \\ & \quad \lesssim t^{-1}\|h\|_{X^1}^2. \end{aligned} \tag{3.22}$$

From (3.19), we find that $W''(\psi^{(N)})$ is an operator of the form

$$W''(\psi^{(N)}) = \mathcal{O}(1) + \mathcal{O}(t^{2/3})(\partial_r + \alpha) + \mathcal{O}(t^{4/3})(\Delta + (\psi^{(N)})^2). \tag{3.23}$$

[3]This is part of the bootstrap hypothesis. In fact, the argument ultimately yields the much stronger estimate $\|h\|_{X^1} \leq t^{2N/3}$.

In the expression, $\langle W''(\psi^{(N)})H^{(N)}, h\rangle$, the presence of the function $H^{(N)}$ gives localization to $r \sim t^{1/3}$, and hence, (3.3) is applicable. It follows from (3.23) and (3.3) that

$$|\langle W''(\psi^{(N)})H^{(N)}, h\rangle| \lesssim \|H^{(N)}\|_{X^1}\|h\|_{X^1} \lesssim t^{-1}\|h\|_{X^1}^2 + t\|H^{(N)}\|_{X^1}^2. \tag{3.24}$$

Combining (3.22) and (3.24), we obtain the bound

$$|\mathcal{E}_1| \lesssim t^{-1}\|h\|_{X^1}^2 + t\|H^{(N)}\|_{X^1}^2. \tag{3.25}$$

Next, we compute the bound on $\mathcal{E}_2$. Working with (3.10), we obtain

$$|\mathcal{E}_2| \lesssim t^{-1}\|h\|_{X^1}^2. \tag{3.26}$$

Next, we compute the Poisson bracket $\{E, W\}$. However, since $\{E, E\} = 0$ and $\{E, M\} = 0$, we have $\{E, W\} = -\frac{v}{\lambda^2}\{E, \mathcal{P}_q\}$. Substituting (3.17) and (3.18) and performing numerous integrations by parts, we obtain

$$\begin{aligned}
\{E, \mathcal{P}_q\}(\psi) &= \operatorname{Re}\int (-i)E'(\psi)\overline{\mathcal{P}_q'(\psi)}\,dx \\
&= 8\pi \operatorname{Re}\int_{r=0}^{\infty} (-\partial_r^2\psi - 2r^{-1}\partial_r\psi - 2|\psi|^2\psi)(2\theta_1^2\partial_r\bar{\psi} + \alpha\bar{\psi})\, r^2\, dr \\
&= 8\pi \int_{r=0}^{\infty} |\partial_r\psi|^2 \underbrace{(\partial_r(\theta_1^2 r^2) - 4\theta_1^2 r + \alpha r^2)}_{\sim t^{-1/3}r^2\, 1_{r\sim t^{1/3}}}\, dr \\
&\quad + 8\pi \int_{r=0}^{\infty} |\psi|^4 \underbrace{(-\partial_r(\theta_1^2 r^2) + 2\alpha r^2)}_{\sim t^{-1/3}r^2\, 1_{r\sim t^{1/3}}}\, dr \\
&\quad + 8\pi \int_{r=0}^{\infty} |\psi|^2 \underbrace{(-\frac{1}{2}\partial_r(\alpha r^2) + \partial_r(r\alpha))}_{\sim t^{-1}r^2\, 1_{r\sim t^{1/3}}}\, dr.
\end{aligned}$$

From this, it is apparent that the quadratic (in h) part of $\{E, \mathcal{P}_q\}(\psi)$ satisfies

$$\begin{aligned}
&|v\lambda^{-2}||\{E, \mathcal{P}_q\}(\psi) - \{E, \mathcal{P}_q\}(\psi^{(N)}) - \langle\{E, \mathcal{P}_q\}'(\psi^{(N)}), h\rangle| \\
&\quad \lesssim t^{1/3}\|\nabla h\|_{L^2}^2 + t^{1/3}\|\psi^{(N)}\|_{L^\infty}^2\|h\|_{L^2}^2 + t^{-1/3}\|h\|_{L^2}^2 \\
&\quad \lesssim t^{-1}\|h\|_{X^1}^2.
\end{aligned}$$

This furnishes a bound on the main term in (3.9), while $\mathcal{E}_1$ and $\mathcal{E}_2$ have been bounded above in (3.25) and (3.26). Thus, the bound (3.16) follows from Lemma 3.1. □

We next write the remainder h as the push-forward of a function f:

$$h(r,t) = e^{i\theta(t)}e^{iv(t)r/2}\lambda(t)\, f\big(\lambda(t)(r - q(t)), t\big) \tag{3.27}$$

and recalling that $\rho = \lambda(r - q)$, define f_1 as the truncation

$$f_1(\rho, t) \stackrel{\text{def}}{=} \theta_1\left(\frac{\rho}{\lambda q}\right)\lambda^{1/2}\left(\frac{\rho}{\lambda} + q\right) f(\rho, t). \tag{3.28}$$

Note that due to the support of θ_1 (which restricts $|\rho| \ll t^{-1/3}$), we have that $\lambda^{1/2}(\rho/\lambda + q) \sim 1$.

Let

$$\xi_0 = i\varphi\,, \quad \xi_1 = -\partial_\rho\varphi\,, \quad \xi_2 = \varphi + \rho\partial_\rho\varphi\,, \quad \xi_3 = i\rho\varphi,$$

where $\varphi(\rho) = \operatorname{sech}\rho$, and, recalling that $J = -\frac{1}{2}i$ and $J^{-1} = 2i$, define

$$\kappa_j \stackrel{\text{def}}{=} \langle f_1, \tfrac{1}{2}J^{-1}\xi_j\rangle_{L^2(-\infty<\rho<\infty)}. \tag{3.29}$$

Define the projection operator

$$Pf_1 = -\kappa_2(f_1)\,\xi_0 + \kappa_0(f_1)\,\xi_2 - \kappa_1(f_1)\,\xi_3 + \kappa_3(f_1)\,\xi_1. \tag{3.30}$$

The following proposition establishes the coercivity of the functional $G(h)$.

Proposition 3.3. *Let $G(h)$ be defined as in* (3.8)*, and suppose that $\|h\|_{X^1} \le t^{2/3}$. Then*

$$\|h\|_{X^1}^2 \lesssim G(h) + \sum_{i=0}^{3} |\kappa_i|^2, \tag{3.31}$$

where κ_j, $j = 0, 1, 2, 3$, are defined in (3.29)*. This bound is valid for $\epsilon \le t \le t_0(N)$. The implicit constant is independent of ϵ and N.*

Proof. From (3.8),

$$G(h) = \langle W''(\psi^{(N)})h, h\rangle + \mathcal{E}_3, \tag{3.32}$$

where $\mathcal{E}_3$ takes the form

$$\mathcal{E}_3 = \mathcal{O}(t^{4/3})\int \psi^{(N)}h^3 + \mathcal{O}(t^{4/3})\int h^4.$$

Hence,

$$\begin{aligned} |\mathcal{E}_3| &\lesssim t^{4/3}\|\psi^{(N)}\|_{L^\infty}\|h\cdot 1_{r\sim t^{1/3}}\|_{L^\infty}\|h\|_{L^2}^2 + t^{4/3}\|h\|_{L^4}^4 \\ &\lesssim \|\psi^{(N)}\|_{X^1}\|h\|_{X^1}^3 + t^{4/3}\|\nabla h\|_{L^2}^2\|h\|_{L^2} \\ &\lesssim \|h\|_{X^1}^3 + t^{-2/3}\|h\|_{X^1}^4 \\ &\le \frac{1}{4}\delta\|h\|_{X^1}^2, \end{aligned} \tag{3.33}$$

where in the second line we have applied (3.3) in the first term and used the 3D Gagliardo-Nirenberg inequality (since there is no spatial localization to $r \sim t^{1/3}$ in this term). From (3.7), we have that $W''(\psi^{(N)})$ is the operator

$$\begin{aligned} W''(\psi^{(N)}) = {} & 2\left(1 + \frac{1}{4}v^2\lambda^{-2}\right) + 2iv\lambda^{-2}\theta_1^2\partial_r \\ & + 2\lambda^{-2}(-\partial_r^2 - 2r^{-1}\partial_r - 4|\psi^{(N)}|^2 - 2(\psi^{(N)})^2\mathcal{C}), \end{aligned}$$

where $\mathcal{C}$ is the operator of complex conjugation. For convenience, let

$$\begin{aligned} B \stackrel{\text{def}}{=} {} & 2\left(1 + \frac{1}{4}v^2\lambda^{-2}\right) + 2iv\lambda^{-2}\theta_1^2\partial_r \\ & + 2\lambda^{-2}(-\partial_r^2 - 4|\psi^{(N)}|^2 - 2(\psi^{(N)})^2\mathcal{C}) \end{aligned}$$

and

$$D \stackrel{\text{def}}{=} 2\left(1 + \frac{1}{4}v^2\lambda^{-2}\right) - 2\lambda^{-2}\partial_r^2.$$

Then using that $\partial_r^2\, r = r\partial_r^2 + 2\partial_r$, we obtain

$$\langle W''(\psi^{(N)})h, h\rangle_{L^2(\mathbb{R}^3)} = \langle B(rh), rh\rangle_{L^2(0\le r\le\infty)}. \tag{3.34}$$

Let $\theta_2 = (1-\theta_1^2)^{1/2}$ so that $h = \theta_1^2 h + \theta_2^2 h$. Substituting this decomposition into the right side of (3.34), we get

(3.35)
$$\langle W''(\psi^{(N)})h, h\rangle_{L^2(\mathbb{R}^3)} = \langle B(r\theta_1 h), r\theta_1 h\rangle_{L^2(0<r<\infty)} + \langle D(r\theta_2 h), r\theta_2 h\rangle_{L^2(0<r<\infty)} + \mathcal{E}_4,$$

where

$$\mathcal{E}_4 = \sum_{j=1}^{2} \left(2\lambda^{-2}(\langle 2(\partial_r\theta_j)\partial_r(\theta_j r h), rh\rangle + \langle(\partial_r^2\theta_j) r\theta_j h, rh\rangle)\right)$$

arises from the commutator of ∂_r^2 and θ_j. These terms are lower order, however, since an r-derivative on h "costs" $t^{-2/3}$ in the definition of the X^1 norm, whereas ∂_r landing on θ_j gives only the penalty of $t^{-1/3}$. Specifically,

$$|\mathcal{E}_4| \lesssim t^{1/3}\|h\|_{X^1}^2 \leq \frac{1}{4}\delta\|h\|_{X^1}^2. \tag{3.36}$$

By (3.35) and (3.36),

$$G(h) = \langle B(r\theta_1 h), r\theta_1 h\rangle_{L^2(0<r<\infty)} + \langle D(r\theta_2 h), r\theta_2 h\rangle_{L^2(0<r<\infty)} + \mathcal{E}_3 + \mathcal{E}_4. \tag{3.37}$$

Define the operator

$$A \stackrel{\text{def}}{=} (1 + \partial_\rho^2 + 4\varphi^2 + 2\varphi^2\mathcal{C}).$$

Substituting (3.27), (3.28), we obtain

$$\langle B(r\theta_1 h), r\theta_1 h\rangle_{L^2(0<r<\infty)} = \langle Af_1, f_1\rangle_{L^2(-\infty<\rho<\infty)}.$$

Recalling (3.30), it is a classical fact (see for example [7, §4]) that there exists $\delta > 0$ such that

$$\delta\|f_1 - Pf_1\|_{L^2(\mathbb{R})}^2 \leq \langle A(f_1 - Pf_1), (f_1 - Pf_1)\rangle_{L^2(\mathbb{R})}. \tag{3.38}$$

As a projection operator, P satisfies $P^2 = P$ and $P^* = P$ (adjoint with respect to the $\langle\cdot,\cdot\rangle$ inner product defined by (3.1)). Hence,

$$\|f_1 - Pf_1\|_{L^2(\mathbb{R})}^2 = \|f_1\|_{L^2(\mathbb{R})}^2 - \|Pf_1\|_{L^2(\mathbb{R})}^2.$$

Also,

$$\|APf_1\|_{L^2(\mathbb{R})} + \|Pf_1\|_{L^2(\mathbb{R})} \lesssim \sum_{k=0}^{3} |\kappa_j|.$$

Applying this to (3.38), we obtain

$$\delta\|f_1\|_{L^2(\mathbb{R})}^2 \leq \langle Af_1, f_1\rangle_{L^2(\mathbb{R})}^2 + \sum_{j=0}^{3} |\kappa_j|^2.$$

Substituting back (3.27), (3.28), we obtain

$$\delta\|r\theta_1 h\|_{L^2(0<r<\infty)}^2 \leq \langle B(r\theta_1 h), (r\theta_1 h)\rangle_{L^2(0<r<\infty)} + \sum_{j=0}^{3} |\kappa_j|^2. \tag{3.39}$$

Directly from the definition of B, using integration by parts, we get

(3.40)
$$t^{4/3}\|\partial_r(r\theta_1 h)\|_{L^2(0<r<\infty)}^2 \lesssim \langle B(r\theta_1 h), (r\theta_1 h)\rangle_{L^2(0<r<\infty)} + \|r\theta_1 h\|_{L^2(0<r<\infty)}^2.$$

The inequalities (3.39) and (3.40) together yield

$$\delta\|r\theta_1 h\|^2_{X^1(0<r<\infty)} \leq \langle B(r\theta_1 h), (r\theta_1 h)\rangle_{L^2(0<r<\infty)} + \sum_{j=0}^{3} |\kappa_j|^2. \tag{3.41}$$

Directly from the definition of D via integration by parts,

$$\delta\|r\theta_2 h\|^2_{X^1(0<r<\infty)} \leq \langle D(r\theta_2 h), (r\theta_2 h)\rangle_{L^2(0<r<\infty)}. \tag{3.42}$$

Summing (3.41) and (3.42), we obtain

$$\begin{aligned}\delta(\|r\theta_1 h\|^2_{X^1(0<r<\infty)} &+ \|r\theta_2 h\|^2_{X^1(0<r<\infty)}) \\ &\lesssim \langle B(r\theta_1 h), (r\theta_1 h)\rangle_{L^2(0<r<\infty)} + \langle D(r\theta_2 h), (r\theta_2 h)\rangle_{L^2(0<r<\infty)} + \sum_{j=0}^{3} |\kappa_j|^2.\end{aligned} \tag{3.43}$$

Combining (3.37) and (3.43), we obtain

$$\delta(\|r\theta_1 h\|^2_{X^1(0<r<\infty)} + \|r\theta_2 h\|^2_{X^1(0<r<\infty)}) \lesssim G(h) + \sum_{j=0}^{3} |\kappa_j|^2 - \mathcal{E}_3 - \mathcal{E}_4. \tag{3.44}$$

Again, by the fact that commutators are of lower order, we have

$$\delta(\|r\theta_1 h\|^2_{X^1(0<r<\infty)} + \|r\theta_2 h\|^2_{X^1(0<r<\infty)}) = \delta\|h\|^2_{X^1(\mathbb{R}^3)} + \mathcal{E}_5, \tag{3.45}$$

where

$$|\mathcal{E}_5| \lesssim t^{1/3}\|h\|^2_{X^1}. \tag{3.46}$$

Combining (3.44) and (3.45), and making use of error estimates (3.33), (3.36), and (3.46), we obtain (3.31). □

We now come back to our original substitution (2.1) and the equation (2.2) for U (i.e., to our solution before the conditions (2.3) were enforced) and the equation (3.49) for $U^{(N)}$. Consider a four parameter $(\lambda_1, \theta_1, q_1, v_1)$-family of profiles:

$$\phi^{(N)}(\lambda_1, q_1, \theta_1, v_1, r, t) \stackrel{\text{def}}{=} e^{i\Theta(r,t)}\tilde{\lambda}(t)\, U^{(N)}(\tilde{\lambda}(t)\big(r - \tilde{q}(t)\big), t) \tag{3.47}$$

with

$$\Theta(r,t) = \theta(t) + \theta_1 + (v(t) + v_1)r/2, \quad \tilde{\lambda}(t) = \lambda_1\lambda(t), \quad \tilde{q}(t) = q(t) + q_1.$$

Here, the parameters λ_1, q_1, θ_1, and v_1 are assumed time-*independent*, and recall that the (time-dependent) parameters $\theta(t), \lambda(t), q(t), v(t)$ satisfy conditions (2.3). For the sake of brevity, we write $\rho = \tilde{\lambda}(r - \tilde{q})$, $\tilde{v} = v + v_1$, and $U^{(N)}$, which stands for $U^{(N)}(\rho, t)$. Let

$$F^{(N)}(r,t) = i\partial_t\phi^{(N)} + \Delta\phi^{(N)} + 2|\phi^{(N)}|^2\phi^{(N)}. \tag{3.48}$$

Using the definition (3.47), the above is equivalent to

$$\begin{aligned}F^{(N)}(r,t) = {} & e^{i\Theta(r,t)}\tilde{\lambda}(t)\left(iU_t^{(N)} + \tilde{\lambda}^2 U_{\rho\rho}^{(N)}\right) + 2|\phi^{(N)}|^2\phi^{(N)} - \left(\dot{\theta} + \frac{\dot{v}r}{2} + \frac{\tilde{v}^2}{4}\right)\phi^{(N)} \\ & + i\, e^{i\Theta(r,t)}\left(\dot{\tilde{\lambda}}U^{(N)} + \tilde{v}\tilde{\lambda}^2 U_\rho^{(N)} + \tilde{\lambda}(\dot{\tilde{\lambda}}(r - \tilde{q}) - \tilde{\lambda}\dot{\tilde{q}})U_\rho^{(N)}\right) \\ & + e^{i\Theta(r,t)}\frac{2\tilde{\lambda}^3}{\rho + \tilde{\lambda}\tilde{q}}\, U_\rho^{(N)} + i\frac{\tilde{\lambda}\tilde{v}}{\rho + \tilde{\lambda}\tilde{q}}\,\phi^{(N)}.\end{aligned}$$

Recalling (2.46), multiplying it by $\tilde{\lambda}^3$ and substituting into the second and third terms in the above, we obtain

(3.49)
$$\begin{aligned} F^{(N)}(r,t) = {} & e^{\Theta(r,t)}\tilde{\lambda}^3(t)\left[i\left(\frac{1}{\tilde{\lambda}^2}-\frac{1}{\lambda^2}\right)U_t^{(N)} + \left(1-\frac{\tilde{\gamma}}{\tilde{\lambda}^2}\right)U^{(N)} + i\frac{v_1}{\tilde{\lambda}}U_\rho^{(N)}\right. \\ & + \frac{\dot{v}}{2}\rho\left(\frac{1}{\lambda^3}-\frac{1}{\tilde{\lambda}^3}\right)U^{(N)} + i\frac{\dot{\lambda}}{\lambda}\left(\frac{1}{\tilde{\lambda}^2}-\frac{1}{\lambda^2}\right)\left(1+\rho\,\partial_\rho\right)U^{(N)} \\ & + i\left(\frac{\tilde{v}}{\tilde{\lambda}(\rho+\tilde{\lambda}\tilde{q})}-\frac{v}{\lambda(\rho+\lambda\,q)}\right)U^{(N)} \\ & \left. + \left(\frac{2}{\rho+\tilde{\lambda}\tilde{q}}-\frac{2}{\rho+\lambda\,q}\right)U_\rho^{(N)} + \mathcal{R}_N(\rho,t)\right], \end{aligned}$$

where $\tilde{\gamma} = \dot{\theta} + \tilde{v}^2/4 + \dot{v}\tilde{q}/2$ and $\rho = \tilde{\lambda}(r-\tilde{q})$.

Define

$$\begin{aligned} \eta_0(r,t) &= \partial_{\theta_1}\,\phi^{(N)}\Big|_{\lambda_1=1,\theta_1=q_1=v_1=0}, \\ \eta_1(r,t) &= \frac{1}{\lambda(t)}\,\partial_{q_1}\phi^{(N)}\Big|_{\lambda_1=1,\theta_1=q_1=v_1=0}, \\ \eta_2(r,t) &= \partial_{\lambda_1}\phi^{(N)}\Big|_{\lambda_1=1,\theta_1=q_1=v_1=0}, \\ \eta_3(r,t) &= (2\lambda(t)\,\partial_{v_1} - q(t)\lambda(t)\,\partial_{\theta_1})\phi^{(N)}\Big|_{\lambda_1=1,\theta_1=q_1=v_1=0}. \end{aligned} \tag{3.50}$$

Solving for $i\partial_t\phi^{(N)}$ in (3.48) and using (3.49) (and commutation of derivatives), we obtain

$$i\partial_t\eta_0 = E''(\psi^{(N)})\eta_0 + H^{(N)}, \tag{3.51}$$

$$i\lambda^{-1}\partial_t(\lambda\eta_1) = E''(\psi^{(N)})\eta_1 + i\frac{1}{2}\lambda^{-1}\dot{v}\,\eta_0 + O_X(t^{-2/3}), \tag{3.52}$$

$$i\partial_t(\eta_2) = E''(\psi^{(N)})\eta_2 - 2i\lambda^2\,\eta_0 + O_X(t^{-1}), \tag{3.53}$$

$$i\lambda\,\partial_t(\lambda^{-1}\eta_3) = E''(\psi^{(N)})\eta_3 - 2i\lambda^2\,\eta_1 + O_X(t^{-1}). \tag{3.54}$$

Proposition 3.4. *Suppose that $\|h\|_{X^1} \le t^{2/3}$. There exists σ_j satisfying*

$$|\kappa_j - \sigma_j| \lesssim t^{1/3}\|h\|_{X^1} \tag{3.55}$$

so that

$$|\partial_t\sigma_0| \lesssim t^{-4/3}\|h\|_{X^1}^2 + \|H^{(N)}\|_{X^1}, \tag{3.56}$$

$$|\lambda^{-1}\partial_t\lambda\sigma_1| \lesssim t^{-1}|\sigma_0| + t^{-2/3}\|h\|_{X^1} + \|H^{(N)}\|_{X^1}, \tag{3.57}$$

$$|\lambda\partial_t\lambda^{-1}\sigma_2| \lesssim t^{-4/3}|\sigma_0| + t^{-1}\|h\|_{X^1} + \|H^{(N)}\|_{X^1}, \tag{3.58}$$

$$|\lambda\partial_t\lambda^{-1}\sigma_3| \lesssim t^{-4/3}|\sigma_1| + t^{-1}\|h\|_{X} + \|H^{(N)}\|_{X^1}. \tag{3.59}$$

These bounds are valid for $\epsilon \le t \le t_0(N)$, and the implicit constants are independent of ϵ and N.

Proof. Let

$$\mu(r) \stackrel{\text{def}}{=} \theta_1((r-q)/q)\lambda^{-1/2}r^{-1}$$

(note that $\mu(r) = O(1)$) and also let

$$\Xi_j(r,t) \stackrel{\text{def}}{=} \mu(r)e^{i\theta}e^{ivr/2}\lambda\,\xi_j(\lambda(r-q)).$$

Recall η_j, $j = 0,1,2,3$, from (3.50) and observe that

$$\Xi_j = \eta_j + O_{L^2(\mathbb{R}^3)}(t^{1/3}), \tag{3.60}$$

where the error term results from φ being replaced by $U^{(N)}$ (see (2.44)). A change of variables calculation gives

$$\kappa_j = \tfrac{1}{2}\langle f_1, J^{-1}\xi_j\rangle_{L^2(-\infty<\rho<\infty)} = \tfrac{1}{2}\langle h, J^{-1}\Xi_j\rangle_{L^2(\mathbb{R}^3)}.$$

Define

$$\sigma_j = \tfrac{1}{2}\langle h, J^{-1}\eta_j\rangle_{L^2(\mathbb{R}^3)}. \tag{3.61}$$

Then by (3.60) $\kappa_j - \sigma_j$ satisfies the bound (3.55). Then

$$\partial_t\sigma_0 = \tfrac{1}{2}\langle \partial_t h, J^{-1}\eta_0\rangle + \tfrac{1}{2}\langle h, J^{-1}\partial_t\eta_0\rangle.$$

Substituting (3.71) and (3.51), we obtain

$$\begin{aligned}\partial_t\sigma_0 = {} & \tfrac{1}{2}\langle J(E''(\psi^{(N)})h + 4\psi^{(N)}|h|^2 + 2\overline{\psi^{(N)}}h^2 + 2|h|^2h) - H^{(N)}, J^{-1}\eta_0\rangle \\ & + \tfrac{1}{2}\langle h, J^{-1}(JE''(\psi^{(N)})\eta_0 + H^{(N)})\rangle.\end{aligned}$$

Since $J^* = -J$ and $E''(\psi^{(N)})$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle$, we have the cancellation $\langle JE''(\psi^{(N)})h, J^{-1}\eta_0\rangle + \langle h, J^{-1}JE''(\psi^{(N)})\eta_0\rangle = 0$. Estimating the remaining terms, we obtain (3.56). Similarly, using the estimates (3.52) - (3.54), we obtain the rest of (3.57)-(3.59). □

Proposition 3.5. *For a sufficiently large N, independent of $0 < \epsilon < t_0(N)$, we have*

$$\|h\|_{X^1} \le t^{2N/3}, \quad \epsilon \le t \le t_0(N). \tag{3.62}$$

Proof. This will be proved by a bootstrap argument invoking the estimates obtained in Propositions 3.2, 3.3, and 3.4. Recall that

$$\|H^{(N)}\|_{X^1} \le t^{(2N-1)/3}. \tag{3.63}$$

We make the following bootstrap assumption:

$$\forall\, 0 < t \le t_0(N), \qquad \|h\|_{X^1} \le 2t^{2N/3}. \tag{3.64}$$

Assumption (3.64) will be validated provided we can show, using Propositions 3.2, 3.3, and 3.4, that (3.64) reinforces itself – specifically that (3.62) holds as a consequence.

By (3.64), (3.63) inserted into (3.16)

$$|\partial_t G(h)| \lesssim t^{4N/3-1}.$$

Integrating, we obtain

$$G(h) = G(h(t)) - G(h(\epsilon)) \lesssim \frac{1}{N}t^{4N/3}. \tag{3.65}$$

By (3.64), (3.63), inserted into (3.56),

$$|\partial_t\sigma_0| \lesssim t^{(2N-1)/3}.$$

Integrating over $[\epsilon, t]$ using that $\sigma_j(\epsilon) = 0$, we obtain

$$|\sigma_0| \lesssim t^{(2N+2)/3}. \tag{3.66}$$

By (3.64), (3.63), (3.66) inserted into (3.57) and (3.58), we obtain

$$|\lambda^{-1}\partial_t\lambda\sigma_1| \lesssim t^{(2N-2)/3}, \qquad |\lambda\partial_t\lambda^{-1}\sigma_2| \lesssim t^{(2N-3)/3}.$$

Integrating, we obtain

$$|\sigma_1| \lesssim t^{(2N+1)/3}, \qquad |\sigma_2| \lesssim \frac{1}{N}t^{2N/3}. \tag{3.67}$$

Inserting (3.64), (3.63), (3.67) into (3.59), we obtain

$$|\lambda\partial_t\lambda^{-1}\sigma_3| \lesssim t^{(2N-3)/3}.$$

Integrating, we obtain

$$|\sigma_3| \lesssim \frac{1}{N}t^{2N/3}. \tag{3.68}$$

By (3.55), κ_j can be replaced by σ_j in (3.31). By (3.65), (3.66), (3.67), (3.68) inserted into (3.31),

$$\|h\|_{X^1}^2 \lesssim \frac{1}{N}t^{4N/3},$$

and hence, (3.62) holds by taking N sufficiently large. □

In order to construct the solution advertised in Theorem 1.1, we will carry out a compactness argument. For this compactness argument, we need higher regularity (H^2) control and tighter localization:

Proposition 3.6. *For N taken large enough so that Proposition* 3.5 *holds, we have the further bounds*

$$t^{4/3}\|\nabla^2 h\|_{L^2(\mathbb{R}^3)} \lesssim t^{\frac{2N}{3}-1} \tag{3.69}$$

and

$$\|x\psi(t)\|_{L^2(\mathbb{R}^3)} \lesssim t^{1/3} \tag{3.70}$$

for $\epsilon \le t \le t_0(N)$, with implicit constants independent of N and $0 < \epsilon \le t_0(N)$.

Proof. Recall

$$\partial_t h = J(E''(\psi^{(N)})h + 8\psi^{(N)}|h|^2 + 4\overline{\psi^{(N)}}h^2 + 4|h|^2 h) - H^{(N)}, \tag{3.71}$$

where

$$JE''(\psi^{(N)}) = +i(\Delta + 4|\psi^{(N)}|^2 + 2(\psi^{(N)})^2\mathcal{C})$$

and $\mathcal{C}$ denotes the operator of complex conjugation. We have available H^1_x control of h and thus seek a "higher order energy estimate". Applying ∂_j ($j = 1, 2, 3$) to (3.71), we obtain

$$\partial_t\partial_j h = JE''(\psi^{(N)})\partial_j h + A(h) + B(h, \partial_j h) - \partial_j H^{(N)}, \tag{3.72}$$

where $A(h)$ contains terms of the form $\psi_N \cdot \partial_j\psi_N \cdot h$ (up to complex conjugation) and $B(h, \partial_j h)$ denotes terms that contain one power of $\partial_j h$ or $\partial_j \bar{h}$ and at least one power of h. Thus $\partial_j h$, modulo error terms, satisfies at the linear level the same equation satisfied by h itself. Let

$$\tilde{\psi} \stackrel{\text{def}}{=} \psi^{(N)} + \partial_j h\,.$$

Then by (3.5) and (3.72),

$$\partial_t \tilde{\psi} = JE'(\psi^{(N)}) + JE''(\psi^{(N)})\partial_j h + A(h) + B(h, \partial_j h) + H^{(N)} - \partial_j H^{(N)}. \tag{3.73}$$

Note the expansion

$$JE'(\tilde{\psi}) = JE'(\psi^{(N)} + \partial_j h) = JE'(\psi^{(N)}) + JE''(\psi^{(N)})\partial_j h + JF, \tag{3.74}$$

where

$$\begin{aligned} F &\stackrel{\text{def}}{=} E'(\tilde{\psi}) - E'(\psi^{(N)}) - E''(\psi^{(N)})\partial_j h \\ &= 4\psi^{(N)}|\partial_j h|^2 + 2\overline{\psi^{(N)}}(\partial_j h)^2 + 2|\partial_j h|^2 \partial_j h. \end{aligned}$$

Using the expansion (3.74), the equation (3.73) becomes

$$\partial_t \tilde{\psi} = JE'(\tilde{\psi}) + D - JF, \tag{3.75}$$

where

$$D \stackrel{\text{def}}{=} A(h) + B(h, \partial_j h) + H^{(N)} - \partial_j H^{(N)}.$$

We now use the functional G that appeared in Lemma 3.1 and Propositions 3.2, 3.3, where we replace ψ by $\tilde{\psi}$. Specifically, in place of (3.8) we take

$$G(\partial_j h) \stackrel{\text{def}}{=} W(\tilde{\psi}) - W(\psi^{(N)}) - \langle W'(\psi^{(N)}), \partial_j h\rangle.$$

A slightly modified version of Lemma 3.1 follows, in which (3.75) is applied in place of (3.4), and the following identity is obtained in place of (3.9):

$$\partial_t G(\partial_j h) = \{E, W\}(\tilde{\psi}) - \{E, W\}(\psi^{(N)}) - \langle \{E, W\}'(\psi^{(N)}), \partial_j h\rangle - \mathcal{E}_1 + \mathcal{E}_2,$$

where

$$\begin{aligned} \mathcal{E}_1 = {} & \langle W''(\psi^{(N)})H^{(N)}, \partial_j h\rangle - \langle W'(\tilde{\psi}) - W'(\psi^{(N)}), D - JF\rangle \\ & + \langle W'(\psi^{(N)}), JF\rangle \end{aligned}$$

and $\mathcal{E}_2$ is the same as in (3.10), but with ψ replaced by $\tilde{\psi}$. The proof of Proposition 3.2 modifies accordingly, to yield in place of (3.16)

$$|\partial_t G(\partial_j h)| \lesssim t^{-1}\|\partial_j h\|_{X^1}^2 + t^{\frac{4N}{3}-3}. \tag{3.76}$$

Indeed, $\mathcal{E}_1$ is estimated using $\|W'(\psi^{(N)})\|_{L^2} \lesssim t^{1/3}$, and

$$\|F\|_{L^2} + t^{2/3}\|1_{r\sim t^{1/3}}\nabla F\|_{L^2} \lesssim t^{-4/3}\|\partial_j h\|_{X^1}^2$$

$$\|D\|_{L^2} + t^{2/3}\|1_{r\sim t^{1/3}}\nabla D\|_{L^2} \lesssim t^{-2}\|h\|_{X^1} + t^{-4/3}\|h\|_{X^1}\|\partial_j h\|_{X^1}.$$

By an argument following the proof of Proposition 3.3,

$$\|\partial_j h\|_{X^1}^2 \lesssim G(\partial_j h) + \sum_{i=0}^{3} |\tilde{\kappa}_i|^2, \tag{3.77}$$

where $\tilde{\kappa}_i = \langle \partial_j h, \frac{1}{2}J^{-1}\Xi_i\rangle$ and Ξ_i is given in (3.60). The equations (3.76) and (3.77) can be combined to yield (3.69).

Next, we will establish (3.70). The pseudoconformal conservation law (see Strauss [10, p. 13]) is

$$\partial_t \left(\frac{1}{2}\|(x + 2it\nabla)\psi\|_{L^2(\mathbb{R}^3)}^2 - 2t^2\|\psi\|_{L^4(\mathbb{R}^3)}^4 \right) = 2t\|\psi\|_{L^4(\mathbb{R}^3)}^4. \tag{3.78}$$

By energy conservation,

$$\|\psi\|_{L^4(\mathbb{R}^3)}^4 = \|\nabla\psi\|_{L^2(\mathbb{R}^3)}^2 - E(\psi) \lesssim t^{-4/3}. \tag{3.79}$$

By (2.53), $E(\psi) = E(\psi^{(N)}(\epsilon)) \sim 1$ (independently of ϵ), and hence, (3.79) yields

$$\|\psi\|_{L^4(\mathbb{R}^3)} \lesssim t^{-1/3}.$$

Applying this to (3.78), we obtain

$$\|(x + 2it\nabla)\psi\|_{L^2(\mathbb{R}^3)} \lesssim \|(x + 2it\nabla)\psi^{(N)}\big|_{t=\epsilon}\|_{L^2(\mathbb{R}^3)} + t^{1/3}.$$

By the localization of $\psi^{(N)}$ to $|x| \sim t^{1/3}$, we have

$$\|(x + 2it\nabla)\psi^{(N)}(t)\|_{L^2} \sim t^{1/3},$$

from which (3.70) follows. □

4. Proof of Theorem 1.1

Fix N sufficiently large so that Propositions 3.5 and 3.6 are applicable. Let $t_0 = t_0(N)$. For any sequence $\epsilon_j \to 0$, let ψ_j be the solution to (1.1) as described at the beginning of §3 with $\epsilon = \epsilon_j$. By Propositions 3.5 and 3.6, we have

$$\|\psi_j(t_0)\|_{X^2(\mathbb{R}^3)} + t^{-1/3}\|x\psi_j(t_0)\|_{L^2} \lesssim 1$$

with implicit constant independent of j. By the Rellich compactness theorem, we can pass to a subsequence such that $\psi_j(t_0) \to \psi_0$ in $X^1(\mathbb{R}^3)$. Let ψ be the solution to (1.1) with $\psi(t_0) = \psi_0$. By Proposition 3.5

$$\sup_{\epsilon_j \le t \le t_0} \|\psi_j(t) - \psi^{(N)}(t)\|_{X^1(\mathbb{R}^3)} \le t^{2N/3}. \tag{4.1}$$

By continuous dependence on initial conditions in the Cauchy problem, for each fixed $0 < t \le t_0$, we have $\psi_j(t) \to \psi(t)$ in $H^1(\mathbb{R}^3)$ as $j \to \infty$. Hence, we can send $j \to \infty$ in (4.1) to obtain

$$\sup_{0 < t \le t_0} \|\psi(t) - \psi^{(N)}(t)\|_{X^1(\mathbb{R}^3)} \le t^{2N/3}.$$

This yields the bound on the first two terms in (1.9). To deduce the bound on the third term in (1.9), we follow the argument in the proof of Proposition 3.6 utilizing the pseudoconformal conservation law.

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