

REGULAR SINGULAR STRATIFIED BUNDLES AND TAME RAMIFICATION

LARS KINDLER

ABSTRACT. Let X be a smooth variety over an algebraically closed field k of positive characteristic. We define and study a general notion of regular singularities for stratified bundles (i.e. \mathcal{O}_X -coherent $\mathcal{D}_{X/k}$ -modules) on X without relying on resolution of singularities. The main result is that the category of regular singular stratified bundles with finite monodromy is equivalent to the category of continuous representations of the tame fundamental group on finite dimensional k -vector spaces. As a corollary we obtain that a stratified bundle with finite monodromy is regular singular if and only if it is regular singular along all curves mapping to X .

1. INTRODUCTION

If X is a smooth, connected, complex variety, then it is an elementary fact that a vector bundle with flat connection and finite monodromy at a closed point $x \in X(\mathbb{C})$ is automatically regular singular. This implies, together with the Riemann-Hilbert correspondence as developed in [Del70], that the category of flat connections with finite monodromy is equivalent to the category $\text{Rep}_{\mathbb{C}}^{\text{cont}} \pi_1^{\text{ét}}(X, x)$ of continuous representations of the étale fundamental group of X on finite dimensional complex vector spaces equipped with the discrete topology. The goal of this article is to establish an analogous statement over an algebraically closed field k of positive characteristic p .

If X is a smooth, connected k -variety, then both the category of vector bundles with flat connection and the category of coherent \mathcal{O}_X -modules with flat connection lack many of the nice features that they have over the complex numbers. In particular they are not Tannakian categories over k . To remedy this fact we pass to the perspective of modules over the ring of differential operators $\mathcal{D}_{X/k}$ on X : Over \mathbb{C} , the category of vector bundles with flat connection is equivalent to the category of left- $\mathcal{D}_{X/\mathbb{C}}$ -modules which are coherent as \mathcal{O}_X -modules. It turns out that if X is a smooth k -variety with k of positive characteristic, then the category of \mathcal{O}_X -coherent $\mathcal{D}_{X/k}$ -modules is Tannakian over k . This was already worked out in [SR72]. Following Grothendieck and Saavedra we call an \mathcal{O}_X -coherent $\mathcal{D}_{X/k}$ -module a *stratified bundle* (see Remark 2.2).

The content of this article is the definition and study of a sensible notion of regular singularity for stratified bundles in positive characteristic. The main result is the following.

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Theorem 1.1. *Let k be an algebraically closed field of positive characteristic. If X is a smooth, connected k -variety, and $x \in X(k)$ a rational point, then a stratified bundle E on X is regular singular with finite monodromy at x if and only if E is trivialized on a finite tame covering.*

The notion of tameness used here is due to Wiesend and extensively studied in [KS10]. From Theorem 1.1 we easily obtain the following, pleasantly conceptual statement:

Corollary 1.2. *If X is a smooth, connected k -variety, then after choice of a base point $x \in X(k)$, the fiber $E|_x$ of a regular singular stratified bundle E with finite monodromy carries a functorial $\pi_1^{\text{tame}}(X, x)$ -action, and this functor induces an equivalence of categories between the category of regular singular stratified bundles with finite monodromy at x and the category $\text{Rep}_k^{\text{cont}} \pi_1^{\text{tame}}(X, x)$ of finite dimensional continuous k -representations of $\pi_1^{\text{tame}}(X, x)$.*

From the main result of [KS10] we obtain our third main result:

Corollary 1.3. *If X is a smooth, connected k -variety, then a stratified bundle E on X with finite monodromy is regular singular, if and only if it is regular singular along all curves on X .*

We make a few remarks about the definition of the notion of regular singularity and give a brief outline of the article: If the k -variety X has a good compactification, i.e. if there exists a smooth, proper k -variety \overline{X} , such that X is a dense open subset of \overline{X} with a strict normal crossings divisor as complement, then the notion of regular singularity is formally the same as in characteristic 0: A stratified bundle E is regular singular if and only if it extends to an $\mathcal{O}_{\overline{X}}$ -coherent, $\mathcal{O}_{\overline{X}}$ -torsion free $\mathcal{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$ -module. This definition was explored in [Gie75]. Unfortunately, in characteristic $p > 0$, it is not known whether every X admits a good compactification. One first approach to a general definition of regular singularity could be to use [dJ96] to replace X by an alteration $Y \rightarrow X$, such that Y admits a good compactification. Unfortunately, de Jong’s theorem and its refinements (e.g. Gabber’s prime-to- ℓ alterations) do not provide control over the wild part of the alteration $Y \rightarrow X$, which implies that one cannot use Y to check stratified bundles on X for regular singularities.¹ For example, an Artin-Schreier covering $f : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ is an alteration and $f^* f_* \mathcal{O}_{\mathbb{A}_k^1} = \mathcal{O}_{\mathbb{A}_k^1}^{\oplus p}$ is regular singular, but $f_* \mathcal{O}_{\mathbb{A}_k^1}$ is not.

Instead, we use the fact that regular singularity should be a notion local in the codimension 1 points at “infinity”, and we define a stratified bundle to be regular singular, if it is regular singular with respect to all good *partial* compactifications of X ; see Section 4.

In Section 2 we establish the basic facts about stratified bundles and monodromy groups. Section 3 provides the technical basis for the discussion of regular singularities, which starts in Section 4. In Section 5 we prove an extension result for regular singular stratified bundles, which is exploited in Section 6. Here we prove Theorem 1.1 with respect to a fixed good partial compactification. In Sections 7 and 8 we finally complete the proofs of Theorem 1.1 and Corollary 1.3.

To close this introduction, we make a remark about the assumption that the base field k be algebraically closed. The argument at the end of the proof of [EM10,

¹Purely inseparable alterations on the other hand could be very helpful; see [AO00, Question 2.9].

Prop. 2.4] shows that for X proper over k , a stratified bundle E on X is trivial if and only if $E \otimes_k \bar{k}$ is a trivial stratified bundle (with respect to \bar{k}) on $X \times_k \bar{k}$. This shows that Corollary 1.2 is false even for proper X , for example for $X = \text{Spec } k$, or $X = \mathbb{P}_k^r$, if k is not algebraically closed (this was also noted in [dS07, 2.4]). Since establishing Corollary 1.2 as direct analogue to the classical situation over the complex numbers is the main motivation behind writing this article, we allow ourselves to always assume k to be algebraically closed.

Notation. If k is a field and G an affine k -group scheme, then we write Vect_k (resp. $\text{Rep}_k G$) for the category of *finite dimensional* k -vector spaces (resp. representations of G on finite dimensional k -vector spaces). Similarly, Vect_k (resp. $\text{Rep}_k G$) denotes the category of all k -vector spaces (resp. all representations of G on k -vector spaces). If \mathcal{T} is a Tannakian category over k , and $\omega : \mathcal{T} \rightarrow \text{Vect}_k$ a fiber functor, then we write $\pi_1(\mathcal{T}, \omega)$ for the affine k -group scheme such that ω induces an equivalence $\mathcal{T} \rightarrow \text{Rep}_k \pi_1(\mathcal{T}, \omega)$ ([DM82, Thm. 2.11]).

If k is a field and X an affine k -scheme, then we say that global sections $x_1, \dots, x_n \in H^0(X, \mathcal{O}_X)$ are *coordinates* for X if the morphism $X \rightarrow \mathbb{A}_k^n$ defined by them is étale.

2. STRATIFIED BUNDLES AND MONODROMY

In this section we collect some facts about the category of stratified bundles and its Tannakian properties, and we recall the notion of the monodromy group of a stratified bundle. In all of this section, we fix an algebraically closed field k of positive characteristic p , and X will always denote a smooth, connected, separated k -scheme of finite type.

Definition 2.1. We write $\text{Strat}(X)$ for the category of *stratified bundles* on k , i.e. for the category of left- $\mathcal{D}_{X/k}$ -modules which are coherent as \mathcal{O}_X -modules (when considering sections of \mathcal{O}_X as differential operators of order 0), together with $\mathcal{D}_{X/k}$ -linear morphisms. Here $\mathcal{D}_{X/k}$ is the sheaf of rings of differential operators of X relative to k , as developed in [EGA4, §16]. A stratified bundle is called *trivial*, if it is isomorphic to \mathcal{O}_X^n together with its canonical diagonal left- $\mathcal{D}_{X/k}$ -action.

If E is a stratified bundle, we write E^∇ for the sheaf of *horizontal sections* of E , defined by

$$E^\nabla(U) = \{s \in E(U) \mid \theta(s) = 0 \text{ for all } \theta \in \mathcal{D}_{X/k}(U) \text{ with } \theta(1) = 0\}.$$

Note for example that $(\mathcal{O}_X^n)^\nabla(U) = k^n$ for all $U \subseteq X$ open.

Remark 2.2. Since X is smooth over k , a stratified bundle in our sense is precisely a coherent \mathcal{O}_X -module equipped with a “stratification” in the sense of [Gro68]. Moreover a stratified bundle is automatically a locally free \mathcal{O}_X -module (see e.g. [BO78, 2.17]), so the name “stratified bundle” is at least historically appropriate.

Proposition 2.3 ([SR72, §VI.1]). *The category $\text{Strat}(X)$ is a k -linear Tannakian category, and a rational point $x \in X(k)$ gives a neutral fiber functor*

$$\omega_x : \text{Strat}(X) \rightarrow \text{Vect}_k, E \mapsto E|_x := x^* E.$$

Definition 2.4. Write $\langle E \rangle_\otimes$ for the smallest full sub-Tannakian subcategory of $\text{Strat}(X)$ containing a stratified bundle $E \in \text{Strat}(X)$. The objects of $\langle E \rangle_\otimes$ are subquotients of objects of the form $P(E, E^\nabla)$, with $P(r, s) \in \mathbb{N}[r, s]$.

We analyze how the above categories behave when passing to an open subscheme of X :

Lemma 2.5. *Let U be an open dense subscheme of X . The following statements are true:*

- (a) *If $E \in \text{Strat}(X)$, then the restriction functor $\rho_{U,E} : \langle E \rangle_{\otimes} \rightarrow \langle E|_U \rangle_{\otimes}$ is an equivalence.*
- (b) *The restriction functor $\rho_U : \text{Strat}(X) \rightarrow \text{Strat}(U)$ is fully faithful.*
- (c) *If $\text{codim}_X(X \setminus U) \geq 2$, then $\rho_U : \text{Strat}(X) \rightarrow \text{Strat}(U)$ is an equivalence.*

Proof. By our standing assumptions, X is connected. Clearly (b) follows directly from (a). We first prove (c). Assume $\text{codim}_X(X \setminus U) \geq 2$. Denote by $j : U \hookrightarrow X$ the open immersion. Let E be a stratified bundle on U , in particular a locally free, finite rank \mathcal{O}_U -module. Then j_*E is \mathcal{O}_X -coherent by the assumption on the codimension ([SGA2, Exp. VIII, Prop. 3.2]), and it carries a $\mathcal{D}_{X/k}$ -action, since $j_*\mathcal{D}_U/k = \mathcal{D}_{X/k}$. Thus it is also locally free. If \overline{E}' is any other locally free extension of E to X , then we get a short exact sequence

$$0 \rightarrow \overline{E}' \hookrightarrow j_*E \rightarrow G \rightarrow 0,$$

with G supported on $X \setminus U$. Since X is smooth, we have $\mathcal{H}om_{\mathcal{O}_X}(G, \mathcal{O}_X) = 0$ and $\mathcal{E}xt^1_{\mathcal{O}_X}(G, \mathcal{O}_X) = 0$ by [SGA2, Exp. III, Prop. 3.3]. This implies that there is a canonical isomorphism $\overline{E}' \cong ((j_*E)^\vee)^\vee = j_*E$, compatible with the $\mathcal{D}_{X/k}$ -structures. It follows that the functors j_* and j^* are quasi-inverse to each other, which proves that j^* is an equivalence.

To prove (a), let $\text{codim}_X(X \setminus U)$ be arbitrary. Let us first show that $\rho_{U,E} : \langle E \rangle_{\otimes} \rightarrow \langle E|_U \rangle_{\otimes}$ is essentially surjective. If F is an object of $\langle E \rangle_{\otimes}$, and F' a subobject of $\rho_U(F) = F|_U$, then $j_*F' \subseteq j_*(F|_U)$. The quasi-coherent \mathcal{O}_X -modules j_*F' and $j_*(F|_U)$ carry $\mathcal{D}_{X/k}$ -actions, such that $j_*F' \subseteq j_*(F|_U)$ and $F \subseteq j_*(F|_U)$ are sub- $\mathcal{D}_{X/k}$ -modules. Then $F'_X := j_*F' \cap F$ is an \mathcal{O}_X -coherent $\mathcal{D}_{X/k}$ -submodule of F extending F' . Thus we have seen that the essential image of ρ_U is closed with respect to taking subobjects.

But this also shows that $F|_U/F'$ can be extended to X : We just saw that F' extends to $F'_X \subseteq F$, and since ρ_U is exact, this means that $(F/F'_X)|_U = F|_U/F'$. It follows that the essential image of ρ_U is closed with respect to taking subquotients.

The objects of $\langle E|_U \rangle_{\otimes}$ are subquotients of stratified bundles of the form $P(E|_U, (E|_U)^\vee)$, with $P(r, s) \in \mathbb{N}[r, s]$. We can lift all of the objects $P(E|_U, (E|_U)^\vee)$ to X , since $(E^\vee)|_U = (E|_U)^\vee$, so ρ_U is essentially surjective.

It remains to check that $\rho_{U,E} : \langle E \rangle_{\otimes} \rightarrow \langle E|_U \rangle_{\otimes}$ is fully faithful. Let $F_1, F_2 \in \langle E \rangle_{\otimes}$. Since $\text{hom}_{\text{Strat}(X)}(F_1, F_2) = \text{hom}_{\text{Strat}(X)}(\mathcal{O}_X, F_1^\vee \otimes F_2)$, and similarly over U , we may replace F_1 by \mathcal{O}_X . Moreover, by (c), we may remove closed subsets of codimension ≥ 2 from X , so without loss of generality we may assume that $X \setminus U$ is the support of a smooth irreducible divisor D with generic point η . To finish the proof, we may shrink X around η . Thus we may assume that $X = \text{Spec } A$ for some finite type k -algebra A , that F_2 corresponds to a free A -module, say with basis e_1, \dots, e_n , and that we have global coordinates x_1, \dots, x_n , such that $D = (x_1)$. Then $U = \text{Spec } A[x_1^{-1}]$. Finally assume that $\phi : \mathcal{O}_U \rightarrow F_2|_U$ is a morphism of

stratified bundles given by $\phi(1) = \sum_{i=1}^n \phi_i e_i$, $\phi_i \in A[x_1^{-1}]$. We get

$$0 = \partial_{x_1}(\phi(1)) = \sum_{i=1}^n \partial_{x_1}(\phi_i)e_i + \underbrace{\phi_i}_{\in \text{im}(F_2 \rightarrow F_2 \otimes A[x_1^{-1}])}$$

and in particular that $\partial_{x_1}(\phi_i) \in \phi_i A \subseteq A[x_1^{-1}]$. By induction, we assume $\partial_{x_1}^{(a)}(\phi_i) \in \phi_i A$ for every $a \leq m$. But then we compute

$$0 = \partial_{x_1}^{(m)}(\phi(1)) = \sum_{i=1}^n \sum_{\substack{a+b=m \\ a, b \geq 0}} \partial_{x_1}^{(a)}(\phi_i)\partial_{x_1}^{(b)}(e_i),$$

to see that $\partial_{x_1}^{(m)}(\phi_i) \in \phi_i A$, so this holds for all $m \geq 1$. In particular we see that the pole order of $\partial_{x_1}^{(m)}(\phi_i)$ along D is at most the pole order of ϕ_i along D , for all $m \geq 0$. This shows that $\phi_i \in A$ for every i , so ϕ is defined over all of X .

It also extends uniquely: If ψ_1 and ψ_2 are morphisms $\mathcal{O}_X \rightarrow F_2$, such that $(\psi_1 - \psi_2) \otimes A[x_1^{-1}] = 0$, then $\psi_1 = \psi_2$, as A is an integral domain and F_2 is torsion-free. This finishes the proof of (a). □

Corollary 2.6. *If E is a stratified bundle, then E^∇ is the constant sheaf associated with a finite dimensional k -vector space. We will usually identify E^∇ with this vector space.*

Proof. A horizontal section of E over some open U spans a trivial substratified bundle of $E|_U$, which lifts to X by the lemma. □

Now we proceed to the notion of monodromy, which is completely analogous to the classical notion in characteristic 0.

Definition 2.7. If $\omega : \langle E \rangle_\otimes \rightarrow \text{Vectf}_k$ is a fiber functor, we denote by $G(E, \omega) := \pi_1(\langle E \rangle_\otimes, \omega)$ the *monodromy group of E with respect to ω* , i.e. the affine, finite type k -group scheme associated via Tannaka duality with the Tannakian category $\langle E \rangle_\otimes \subseteq \text{Strat}(X)$ and the fiber functor ω .

From Lemma 2.5 we immediately derive (using [DM82, Prop. 2.21]):

Lemma 2.8. *Let $U \subseteq X$ be an open dense subscheme; then*

- (a) *If $\omega : \text{Strat}(U) \rightarrow \text{Vectf}_k$ is a fiber functor, then the induced morphism $\pi_1(\text{Strat}(U), \omega) \rightarrow \pi_1(\text{Strat}(X), \omega((-)|_U))$ is faithfully flat.*
- (b) *If E is a stratified bundle, and $\omega : \langle E|_U \rangle_\otimes \rightarrow \text{Vectf}_k$ a fiber functor, then the induced morphism $G(E|_U, \omega) \rightarrow G(E, \omega((-)|_U))$ is an isomorphism of k -group schemes.*

The following theorem is essential for what is to follow:

Theorem 2.9 ([dS07, Cor. 12]). *If $E \in \text{Strat}(X)$ and $\omega : \text{Strat}(X) \rightarrow \text{Vectf}_k$ a fiber functor, then the k -group scheme $G(E, \omega)$ is smooth.*

Remark 2.10. In [dS07, Cor. 12], the above theorem is only formulated in the case that $G(E, \omega)$ is finite over k , but the given proof holds more generally. Indeed, by [dS07, Thm. 11], the affine group scheme $\pi_1(\text{Strat}(X), \omega)$ is perfect and thus reduced. This means that any quotient of $\pi_1(\text{Strat}(X), \omega)$ is also reduced.

Definition 2.11. An object $E \in \text{Strat}(X)$ is said to be *finite* or to have *finite monodromy* if there is a fiber functor $\omega : \langle E \rangle_{\otimes} \rightarrow \text{Vectf}_k$, such that $G(E, \omega)$ is finite over k . Since k is algebraically closed, Theorem 2.9 implies that this is equivalent to $G(E, \omega)$ being a constant k -group scheme associated with a finite group.

Remark 2.12. Note that in our situation $\langle E \rangle_{\otimes}$ always has k -linear fiber functors by [Del90, Cor. 6.20]. Thus E is finite if and only if every object of $\langle E \rangle_{\otimes}$ is isomorphic to a subquotient of $E^{\oplus n}$ for some n ; see [DM82, Prop. 2.20]. In particular, E has finite monodromy if and only if $G(E, \omega)$ is finite for every fiber functor $\omega : \langle E \rangle_{\otimes} \rightarrow \text{Vectf}_k$.

Proposition 2.13. *If E is a stratified bundle on X , then the following statements are equivalent:*

- (a) E is finite,
- (b) $E|_U$ is finite for some open dense $U \subseteq X$,
- (c) $E|_U$ is finite for every open dense $U \subseteq X$.

Proof. This follows directly from Lemma 2.8. □

Remark 2.14. Before we can state the next proposition, we have to recall some facts about Tannakian categories over k ; general references are [SR72], [DM82], and [Del90].

- (a) The category Vectf_k of finite dimensional k -vector spaces is the Tannakian category associated with the trivial group scheme.
- (b) If \mathcal{T} is a Tannakian category, then there exists a unique strictly full sub-Tannakian category $\mathcal{T}_{\text{triv}} \cong \text{Vectf}_k$ whose objects are the trivial objects of \mathcal{T} , and there exists a functor $H^0(\mathcal{T}, -) : \mathcal{T} \rightarrow \text{Vectf}_k$ assigning to every object of \mathcal{T} its maximal trivial subobject (which exists, because every object of \mathcal{T} has finite length by [Del90, 2.13]). The functor $H^0(\mathcal{T}, -)$ extends to a functor $H^0(\text{Ind}(\mathcal{T}), -) : \text{Ind}(\mathcal{T}) \rightarrow \text{Vect}_k$ on the associated Ind-categories.
- (c) Affine k -group schemes are always projective limits of finite type k -group schemes ([DM82, Cor. 2.4]), and if G is an affine k -group scheme, then $\text{Ind}(\text{Repf}_k G) = \text{Rep}_k G$ ([DM82, Cor. 2.7]). We have $H^0(\text{Repf}_k G, -) = (-)^G$ and $H^0(\text{Rep}_k G, -) = (-)^G$.

In the Ind-category $\text{Ind}(\text{Repf}_k G) = \text{Rep}_k G$ there is the right-regular representation (\mathcal{O}_G, Δ) , where $\Delta : \mathcal{O}_G \rightarrow \mathcal{O}_G \otimes_k \mathcal{O}_G$ is the diagonal comultiplication. This representation is an algebra over the trivial representation, which also is its maximal trivial object (because G acting on itself does not have any invariants), and it has the property that there exists a functorial isomorphism

$$(\mathcal{O}_G, \Delta) \otimes_k V \cong (\mathcal{O}_G, \Delta)^{\text{rank } V}$$

for every representation $V \in \text{Repf}_k G$ ([dS07, 2.3.2 (d)]). In other words, the composition of functors $V \mapsto (\mathcal{O}_G \otimes V)^G$ is the forgetful functor $\text{Repf}_k G \rightarrow \text{Vectf}_k$.

- (d) If \mathcal{T} is a Tannakian category over k , and $\omega : \mathcal{T} \rightarrow \text{Vectf}_k$ a fiber functor, then ω induces a \otimes -equivalence $\mathcal{T} \xrightarrow{\cong} \text{Repf}_k \pi_1(\mathcal{T}, \omega)$ with $G := \pi_1(\mathcal{T}, \omega)$ an affine k -group scheme, and such that ω is naturally isomorphic to the composition $\mathcal{T} \xrightarrow{\cong} \text{Repf}_k G \xrightarrow{\text{forget}} \text{Vectf}_k$ ([DM82, Thm. 2.11]). The right-regular representation (\mathcal{O}_G, Δ) corresponds via ω to an object $A_{\omega} \in \text{Ind}(\mathcal{T})$,

which is an algebra over the unit object of \mathcal{T} , and such that the functor $\omega : \mathcal{T} \rightarrow \text{Vectf}_k$ can be written as $T \mapsto H^0(\text{Ind}(\mathcal{T}), T \otimes A_\omega)$.

Proposition 2.15. *Let $E \in \text{Strat}(X)$ be a stratified bundle, and $\omega : \langle E \rangle_\otimes \rightarrow \text{Vectf}_k$ a fiber functor. Write $G := G(E, \omega)$. Then there exists a smooth G -torsor $h_{E, \omega} : X_{E, \omega} \rightarrow X$ with the following properties:*

- (a) *Every object of $\langle E \rangle_\otimes$ has finite monodromy if and only if $h_{E, \omega}$ is finite étale.*

From now on assume that G is a finite (thus constant by Theorem 2.9) group scheme, and hence $h_{E, \omega}$ finite étale. Then $h_{E, \omega}$ has the following properties:

- (b) *An object $E' \in \text{Strat}(X)$ is contained in $\langle E \rangle_\otimes$ if and only if $h_{E', \omega}^* E' \in \text{Strat}(X_{E, \omega})$ is trivial.*
- (c) *If \mathcal{S} is a strictly full sub-Tannakian subcategory of $\text{Strat}(X)$ such that $\mathcal{S} \subseteq \langle E \rangle_\otimes$, then there exists $E' \in \langle E \rangle_\otimes$ such that $\mathcal{S} = \langle E' \rangle_\otimes$, and a finite étale morphism g such that the diagram*

$$\begin{array}{ccc} X_{E, \omega} & \xrightarrow{g} & X_{E', \omega} \\ h_{E, \omega} \downarrow & \swarrow h_{E', \omega} & \\ X & & \end{array}$$

commutes.

- (d) *If ω' is a second fiber functor on $\langle E \rangle_\otimes$, then there is an isomorphism of X -schemes $X_{E, \omega} \xrightarrow{\cong} X_{E, \omega'}$.*
- (e) *For $E' \in \langle E \rangle_\otimes$ we have a functorial isomorphism*

$$\omega(E') = H^0(\text{Strat}(X_{E, \omega}), h_{E', \omega}^* E') = (h_{E', \omega}^* E')^\nabla.$$

Conversely, if G is a finite constant group scheme and $f : Y \rightarrow X$ a G -torsor, i.e. a finite étale Galois covering with group $G(k)$, and \mathcal{Y} the sub-Tannakian subcategory of $\text{Strat}(X)$ generated by those objects which become trivial after pullback along f , then:

- (f) $\mathcal{Y} = \langle f_* \mathcal{O}_Y \rangle_\otimes$ *as strictly full subcategories of $\text{Strat}(X)$, and there is a fiber functor $\omega_f : \mathcal{Y} \rightarrow \text{Vectf}_k$, such that $f = h_{Y, \omega_f}$, and such that*

$$G(\langle f_* \mathcal{O}_Y \rangle_\otimes, \omega_f) = G.$$

- (g) *If $\mathcal{S} \subseteq \text{Strat}(X)$ is a strictly full sub-Tannakian subcategory such that $f^* E$ is trivial for every object $E \in \mathcal{S}$, then $\mathcal{S} \subseteq \mathcal{Y}$, so $\mathcal{S} = \langle E \rangle_\otimes$ for some $E \in \mathcal{Y}$, $G(E, \omega_f|_{\mathcal{S}})$ is finite constant and there is a morphism $g : Y \rightarrow X_{E, \omega}$, such that $f = h_{E, \omega} \circ g$.*

Proof. This proposition is fairly well known, but the author does not know of a complete reference. Certainly all the ideas from the theory of Tannakian categories are contained in [Del90] (particularly [Del90, §9.]) and [DM82].

We use the the facts recalled in Remark 2.14. The main ingredient not intrinsic to the theory of Tannakian categories is the fact that if $E \in \text{Strat}(X)$ is a stratified bundle, then $G(\langle E \rangle_\otimes, \omega)$ is a smooth k -group scheme by dos Santos' Theorem 2.9. In particular, if $G(\langle E \rangle_\otimes, \omega)$ is finite, then it is finite étale and hence constant if k is algebraically closed.

Back to the notation of the proposition: Let $\rho : \text{Strat}(X) \rightarrow \text{Coh}(X)$ denote the forgetful functor. With the fiber functor $\omega : \langle E \rangle_\otimes \rightarrow \text{Vectf}_k$ we associate in Remark

2.14 an object $A_{E,\omega}$ of $\text{Ind}(\text{Strat}(X))$. Via ρ we can consider $A_{E,\omega}$ as a quasi-coherent \mathcal{O}_X -algebra with $\mathcal{D}_{X/k}$ -action, and such that $A_{E,\omega}$ corresponds to the right-regular representation of G in $\text{Rep}_k G$. Write $h_{E,\omega} : X_{E,\omega} := \text{Spec } A_{E,\omega} \rightarrow X$ for this G -torsor. The property from Remark 2.14 that $(\mathcal{O}_G, \Delta) \otimes V = (\mathcal{O}_G, \Delta)^{\text{rank } V}$ for $V \in \text{Rep}_k G$ translates into the property that

$$X_{E,\omega} := \text{Spec } A_{E,\omega} \cong \underline{\text{Isom}}_k^\otimes(\omega(-) \otimes_k \mathcal{O}_X, \rho|_{\langle E \rangle_\otimes}),$$

where $\underline{\text{Isom}}_k^\otimes$ is defined as in [Del90, 1.11].

Since G is smooth over k , $h_{E,\omega}$ is smooth, and it is finite if and only if $A_{E,\omega}$ is coherent, if and only if $\langle E \rangle_\otimes = \langle A_{E,\omega} \rangle_\otimes$ (because $E \subseteq A_{E,\omega}^m$ for some $m > 0$), so (a) follows.

Now assume that G is a finite étale group scheme on k , hence constant. Then $h_{E,\omega}$ is a finite étale morphism; in particular, $A_{E,\omega}$ is an \mathcal{O}_X -algebra in the category $\text{Strat}(X)$. Moreover, the $\mathcal{D}_{X_{E,\omega}/k}$ -action on $h_{E,\omega}^* E = E \otimes_{\mathcal{O}_X} A_{E,\omega}$ agrees with the tensor product $\mathcal{D}_{X/k}$ -action on $E \otimes_{\mathcal{O}_X} A_{E,\omega}$ via the isomorphism $\mathcal{D}_{X_{E,\omega}/k} \xrightarrow{\cong} h_{E,\omega}^* \mathcal{D}_{X/k}$. In other words, the pull-back functor $h_{E,\omega}^*$ “is”

$$\text{Strat}(X) \rightarrow \text{Strat}(X_{E,\omega}), E \mapsto E \otimes A_{E,\omega}.$$

Now everything follows fairly directly from general theory: If $E' \in \langle E \rangle_\otimes$, then $E' \otimes A_{E,\omega}$ is trivial, and conversely if $E' \otimes A_{E,\omega} = A_{E,\omega}^n$, then the canonical map $E' \rightarrow E' \otimes A_{E,\omega}$ exhibits E' as an object of $\langle A_{E,\omega} \rangle_\otimes = \langle E \rangle_\otimes$. This proves (b).

For (c), first note that $\pi_1(\mathcal{S}', \omega)$ is a quotient of $G(E, \omega)$, so \mathcal{S}' can be written as $\langle E' \rangle_\otimes$ for some $E' \in \langle E \rangle_\otimes$. The morphism $g : X_{E,\omega} \rightarrow X_{E',\omega}$ then comes from the morphism of right-regular representations $(\mathcal{O}_{G(E')}, \Delta) \rightarrow (\mathcal{O}_{G(E)}, \Delta)$.

(d) follows from the fact that $\underline{\text{Isom}}_k^\otimes(\omega, \omega')$ is an fpqc-torsor on k , but k is algebraically closed, so the torsor is trivial, hence $\omega \cong \omega'$.

Statement (e), follows from Remark 2.14, (d).

For (f), note that $f_* \mathcal{O}_Y$ is a stratified bundle on X ; more precisely, it is an \mathcal{O}_X -algebra in $\text{Strat}(X)$. As above we see that the $\mathcal{D}_{Y/k}$ -structure on $f^* E$ agrees with the $\mathcal{D}_{X/k}$ -structure on $E \otimes_{\mathcal{O}_X} f_* \mathcal{O}_Y$ via the isomorphism $\mathcal{D}_{Y/k} \xrightarrow{\cong} f^* \mathcal{D}_{X/k}$. Since f is Galois, $f_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} f_* \mathcal{O}_Y \cong f_* \mathcal{O}_Y^{\text{deg } f}$, so $f_* \mathcal{O}_Y$ is trivialized on Y . Conversely, if $f^* E = E \otimes f_* \mathcal{O}_Y$ is trivial, i.e. $E \otimes f_* \mathcal{O}_Y \cong f_* \mathcal{O}_Y^n$ for some n , then E is a $\mathcal{D}_{X/k}$ -submodule of $f_* \mathcal{O}_Y^n$, so we have proven that the subcategory of $\text{Strat}(X)$ spanned by bundles E trivialized on Y is precisely $\langle f_* \mathcal{O}_Y \rangle_\otimes$. We define the functor $\omega_f : \langle f_* \mathcal{O}_Y \rangle_\otimes \rightarrow \text{Vect}_k$ by $\omega_f(E) = H^0(\text{Strat}(Y), f^* E) = (E \otimes f_* \mathcal{O}_Y)^\nabla$, and this functor is faithful and exact since $f^* E$ is trivial for all E . Finally, the group scheme $G(\langle f_* \mathcal{O}_Y \rangle_\otimes, \omega)$ is the constant group scheme associated with G : Since $Y \times_X Y \cong Y \times_k G$, we see that $\omega_f(f_* \mathcal{O}_Y)$ is the right-regular representation of G .

Lastly, (g) follows immediately from (c). □

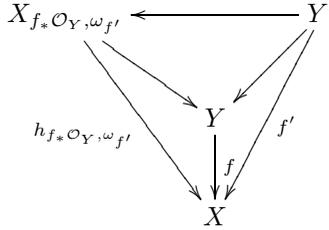
Corollary 2.16. *If $f : Y \rightarrow X$ is a finite étale morphism, and $f' : Y' \rightarrow X$ its Galois closure with Galois group G , then*

$$\mathcal{Y} := \langle f_* \mathcal{O}_Y \rangle_\otimes = \langle f'_* \mathcal{O}_{Y'} \rangle_\otimes \subseteq \text{Strat}(X),$$

$f' = h_{\mathcal{Y}, \omega_{f'}}$, and $f'_* \mathcal{O}_{Y'}$ is finite with monodromy group the constant k -group scheme associated with G .

Proof. By Proposition 2.15, (b), it follows that there is an inclusion $\mathcal{Y} \subseteq \langle f'_* \mathcal{O}_{Y'} \rangle_\otimes$, so if $h_{f_* \mathcal{O}_Y, \omega_{f'}} : X_{f_* \mathcal{O}_Y, \omega_{f'}} \rightarrow X$ is the associated Galois étale morphism, then

$h_{f_*\mathcal{O}_Y, \omega_{f'}}$ factors through f (because $h_{f_*\mathcal{O}_Y, \omega_{f'}}^*(f_*\mathcal{O}_Y)$ is trivial), and then by Proposition 2.15, (g), there is a morphism $Y' \rightarrow X_{f_*\mathcal{O}_Y, \omega_{f'}}$, such that the diagram



commutes. But since f' is the Galois closure of f , it follows that $Y' \rightarrow X_{f_*\mathcal{O}_Y, \omega_{f'}}$ is an isomorphism. This shows that $G(f_*\mathcal{O}_Y, \omega_f) = G$. □

Corollary 2.17. *Let $f : Y \rightarrow X$ be a finite étale morphism of smooth, connected, separated k -varieties. Then f is the trivial covering if and only if the stratified bundle $f_*\mathcal{O}_Y \in \text{Strat}(X)$ is trivial.*

Proof. Let $f' : Y' \rightarrow X$ be the Galois closure of f . Then $G(f_*\mathcal{O}_Y, \omega) \cong G(f'_*\mathcal{O}_{Y'}, \omega) \cong \text{Gal}(Y'/X)_k$, so the claim follows. □

Definition 2.18. Let $E \in \text{Strat}(X)$ be a stratified bundle and $\omega : \langle E \rangle_\otimes \rightarrow \text{Vect}_k$ a fiber functor. Write $h_{E, \omega} : X_{E, \omega} \rightarrow X$ for the smooth $G(\langle E \rangle_\otimes, \omega)$ -torsor associated with $\langle E \rangle_\otimes$ and ω in Proposition 2.15. We call $h_{E, \omega}$ the *Picard-Vessiot torsor associated with E and ω* .

Corollary 2.19. *A stratified bundle E on X has finite monodromy if and only if there exists a finite étale covering $f : Y \rightarrow X$, such that $f^*E \in \text{Strat}(Y)$ is trivial.*

Proof. If E has finite monodromy, then an associated Picard-Vessiot torsor is finite étale and trivializes E . Conversely, if $h : Y \rightarrow X$ is finite étale and h^*E trivial in $\text{Strat}(Y)$, then $E \subseteq h_*h^*E = h_*\mathcal{O}_Y^{\text{rank } E}$ in $\text{Strat}(X)$. But $h_*\mathcal{O}_Y$ has finite monodromy, so E has finite monodromy. □

Remark 2.20. A caution is in order: If E is a stratified bundle with infinite monodromy, then it is true that the $\mathcal{D}_{X/k}$ -module $E \otimes_{\mathcal{O}_X} A_{E, \omega}$ is isomorphic to $A_{E, \omega}^{\text{rank } E}$, but $h_{E, \omega}^*E \in \text{Strat}(X_{E, \omega})$ is not trivial, because it also carries an action of the differential operators relative to $X_{E, \omega} \rightarrow X$, which were trivial in the étale case.

3. LOGARITHMIC DIFFERENTIAL OPERATORS

We continue to denote by k an algebraically closed field of positive characteristic p .

Definition 3.1.

- (a) Let \overline{X} be a smooth, separated, finite type k -scheme, and $X \subseteq \overline{X}$ an open subscheme such that the boundary divisor $D_X := \overline{X} \setminus X$ has strict normal crossings. We denote such a datum by (X, \overline{X}) and call it a *good partial compactification*.
- (b) If (X, \overline{X}) is a good partial compactification, then $\mathcal{D}_{\overline{X}/k}(\log D_X)$ denotes the sheaf of subalgebras of $\mathcal{D}_{\overline{X}/k}$ generated by those differential operators which locally fix all powers of the ideal of the boundary divisor.

Remark 3.2. (a) A good partial compactification (X, \overline{X}) gives rise to a log-scheme over $\text{Spec } k$ with its trivial log-structure, in the sense of [Kat89]. The sheaf $\mathcal{D}_{\overline{X}/k}(\log D_X)$ is an invariant of this relative log-scheme, which can be constructed using an appropriate notion of thickenings in the category of fine saturated log-schemes. For details see e.g. [Kin12, Ch. 2].

(b) If $\overline{U} \subseteq \overline{X}$ is an open affine subset and $x_1, \dots, x_n \in H^0(\overline{U}, \mathcal{O}_{\overline{U}})$ coordinates (see the notational conventions at the end of the introduction), then $\mathcal{D}_{\overline{U}/k}$ is generated by operators $\partial_{x_i}^{(m)}$, $i = 1, \dots, n$, $m \geq 0$; see [BO78, Ch. 2]. Here $\partial_{x_i}^{(m)}$ can be thought of as

$$\frac{1}{m!} \frac{\partial^m}{\partial x_i^m},$$

even though this does not make sense as written. The point is that evaluation of $\partial^m / \partial x_i^m$ on powers of x_i always gives coefficients which are divisible by sufficiently high powers of p .

Moreover, if the strict normal crossings divisor $\overline{U} \setminus X$ is defined by $x_1 \cdot \dots \cdot x_r = 0$, $r \leq n$, then $\mathcal{D}_{\overline{U}/k}(\log D_X \cap \overline{U})$ is generated by operators of the form

$$x_i^m \partial_{x_i}^{(m)}, \text{ for } i = 1, \dots, r, m \geq 0,$$

and

$$\partial_{x_i}^{(m)} \text{ for } i > r, m \geq 0.$$

For notational convenience we write $\delta_{x_i}^{(m)} := x_i^m \partial_{x_i}^{(m)}$.

To proceed, we need to recall a few elementary facts about congruences for binomial coefficients:

Lemma 3.3. *Let p be a prime number.*

(a) Lucas' Theorem: *For $a_0, \dots, a_n, b_0, \dots, b_n$ integers in $[0, p - 1]$, $a := a_0 + a_1 p + \dots + a_n p^n$, $b := b_0 + b_1 p + \dots + b_n p^n$ we have*

$$\binom{a}{b} \equiv \prod_k \binom{a_k}{b_k} \pmod{p}.$$

(b) *If $\alpha \in \mathbb{Z}_p$, then*

$$\alpha = \sum_{n=0}^{\infty} \overline{\binom{\alpha}{p^n}} p^n,$$

where \bar{a} means the unique integer in $[0, \dots, p - 1]$ congruent to a .

(c) *If $\alpha, \beta \in \mathbb{Z}_p$, $d \geq 0$, then*

$$\binom{\alpha\beta}{p^d} \equiv \sum_{\substack{a+b=d \\ a, b \geq 0}} \binom{\alpha}{p^a} \binom{\beta}{p^b} \pmod{p}.$$

Proof. (a) This is easily proven by computing the coefficient of x^b in

$$\sum_{k=0}^a \binom{a}{k} x^k = (1+x)^a \equiv \prod_{k=0}^n (1+x^{p^k})^{a_k} \pmod{p}.$$

(b) This is a consequence of the continuity of $x \mapsto \binom{x}{k}$, and (a).

(c) This follows directly from (b). □

We have the similar functoriality results for $\mathcal{D}_{\bar{X}/k}(\log D_X)$ as for $\mathcal{D}_{\bar{X}/k}$:

Proposition 3.4. *Let (X, \bar{X}) and (Y, \bar{Y}) be good partial compactifications with boundary divisors D_X and D_Y , and $\bar{f} : \bar{Y} \rightarrow \bar{X}$ a morphism such that $\bar{f}(Y) \subseteq X$, i.e. such that \bar{f} induces a morphism of the associated log-schemes. Write $f := \bar{f}|_Y$. Then the following statements are true:*

(a)

$$\bar{f}^* \mathcal{D}_{\bar{X}/k}(\log D_X) := \mathcal{O}_{\bar{Y}} \otimes_{\bar{f}^{-1} \mathcal{O}_{\bar{X}}} \bar{f}^{-1} \mathcal{D}_{\bar{X}/k}(\log D_X)$$

is a $(\mathcal{D}_{\bar{Y}/k}(\log D_Y), \bar{f}^{-1} \mathcal{D}_{\bar{X}/k}(\log D_X))$ -bialgebra.

(b) *There exists a canonical morphism*

$$\mathcal{D}_{\bar{Y}/k}(\log D_Y) \xrightarrow{\bar{f}^\sharp} \bar{f}^* \mathcal{D}_{\bar{X}/k}(\log D_X)$$

fitting in the commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{\bar{Y}/k}(\log D_Y) & \xrightarrow{\bar{f}^\sharp} & \bar{f}^* \mathcal{D}_{\bar{X}/k}(\log D_X) \\ \downarrow & & \downarrow \\ \mathcal{D}_{\bar{Y}/k} & \longrightarrow & \bar{f}^* \mathcal{D}_{\bar{X}/k} \end{array}$$

where the lower horizontal morphism is the classical one arising from the functoriality of the diagonal morphism and its thickenings.

Now assume that \bar{f} is finite, and f étale. Then \bar{f} is faithfully flat. Moreover,

(c) \bar{f}^\sharp is an isomorphism if \bar{f} is tamely ramified with respect to the strict normal crossings divisor D_X .

Proof. Everything follows fairly easily from the general point of view of logarithmic structures of [Kat89], since f being tamely ramified implies that the induced morphism of log-schemes is log-étale.

For the sake of self-containedness, we give an explicit proof for the case that \bar{f} is finite and f étale, which is the only case needed in the sequel. In this case the proof is essentially a question about finite extensions of discrete valuation rings, by localizing at the generic points of the components of the boundary divisor. Let $A \hookrightarrow B$ be such an extension, $x \in A$ and $y \in B$ uniformizers. Then $x = uy^e$ for some $e \geq 1$ and $u \in B^\times$. We know that $K(B) \otimes \mathcal{D}_{B/k}(\log(y)) \xrightarrow{\cong} K(B) \otimes_A \mathcal{D}_{A/k}(\log(x))$ is an isomorphism. Statement (a) is clear, and for (b) we need to show that the above isomorphism maps $\mathcal{D}_{B/k}(\log(y))$ to $B \otimes_A \mathcal{D}_{A/k}(\log(x))$. We claim that

$$(1) \quad \delta_y^{(p^m)} = \sum_{\substack{d+c=m \\ c, d \geq 0}} \binom{e}{p^c} \delta_x^{(p^d)} + y(B \otimes_A \mathcal{D}_{A/k}(\log(x)))$$

where $\delta_y^{(p^m)} = y^{p^m} \partial_y^{(p^m)}$ as usual. Clearly (1) implies (b).

Let us now prove that (1) holds. We compute:

$$\begin{aligned}
 \delta_y^{(p^m)}(x^r) &= \delta_y^{(p^m)}(u^r y^{er}) \\
 &= \sum_{\substack{a+b=p^m \\ a,b \geq 0}} \delta_y^{(a)}(u^r) \delta_y^{(b)}(y^{er}) \\
 &= \binom{er}{p^m} x^r + \sum_{\substack{a+b=p^m \\ a>0,b \geq 0}} \delta_y^{(a)}(u^r) \binom{er}{b} y^{er} \\
 (2) \qquad &= \binom{er}{p^m} x^r + x^r \underbrace{\sum_{\substack{a+b=p^m \\ a>0,b \geq 0}} \binom{er}{b} \frac{\delta_y^{(a)}(u^r)}{u^r}}_{\in(y)}.
 \end{aligned}$$

Note that $\binom{er}{p^m} x^r = \sum_{c+d=m} \binom{e}{p^c} \delta_x^{(p^d)}(x^r)$ by Lemma 3.3, so (2) shows that $\delta_y^{(p^m)} - \sum_{c+d=m} \binom{e}{p^c} \delta_x^{(p^d)} \in y(B \otimes \mathcal{D}_{A/k}(\log(x)))$ as claimed.

For (c) assume that $A \hookrightarrow B$ is tamely ramified. It suffices to show that $\delta_x^{(p^m)}$ is in the image of $\bar{f}^\sharp : \mathcal{D}_{B/k}(\log(y)) \rightarrow \bar{f}^* \mathcal{D}_{A/k}(\log(x))$ for every $m \geq 0$, because \bar{f}^\sharp is injective by the separability of the residue extensions. Consider the completions of A and B : $\widehat{A} \hookrightarrow \widehat{B}$. Replacing \widehat{B} by an étale extension does not change differential operators, so we may assume that $u = v^e$ in \widehat{B} . Indeed, by Hensel’s Lemma, u has an e -th root in \widehat{B} , if and only if it has an e -th root modulo y , and since e is prime to p by assumption, the extension of the residue fields obtained by adjoining an e -th root is separable. Replacing y by vy , we may assume that $x = y^e$. Then (2) shows

$$\delta_y^{(p^m)} = \sum_{\substack{c+d=m \\ c,d \geq 0}} \binom{e}{p^c} \delta_x^{(p^d)}.$$

In particular, $\delta_y^{(1)} = e \delta_x^{(1)}$, so $\bar{\delta}_x^{(1)}$ is in the image of \bar{f}^\sharp . We proceed inductively:

$$\delta_y^{(p^m)} = e \delta_x^{(p^m)} + \underbrace{\sum_{\substack{c+d=m \\ c>1,d \geq 0}} \binom{e}{p^c} \delta_x^{(p^d)}}_{\in \text{im } \bar{f}^\sharp}$$

which completes the proof. □

Corollary 3.5. *We continue to use the notation from Proposition 3.4. If E is a $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module, then $\bar{f}^* E$ is a $\mathcal{D}_{\overline{Y}/k}(\log D_Y)$ -module.*

If \bar{f} is finite étale and tamely ramified with respect to $\overline{X} \setminus X$, and if F is a $\mathcal{D}_{\overline{Y}/k}(\log D_Y)$ -module, then $\bar{f}_ F$ is a $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module.*

4. (X, \overline{X}) -REGULAR SINGULAR STRATIFIED BUNDLES

Definition 4.1. If (X, \overline{X}) is a good partial compactification, then a stratified bundle E is called (X, \overline{X}) -regular singular if it extends to an $\mathcal{O}_{\overline{X}}$ -torsion-free, $\mathcal{O}_{\overline{X}}$ -coherent $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module \overline{E} on \overline{X} .

We define $\text{Strat}^{\text{rs}}((X, \overline{X}))$ to be the full subcategory of $\text{Strat}(X)$ consisting of (X, \overline{X}) -regular singular bundles.

Remark 4.2. The notion of an (X, \overline{X}) -regular singular stratified bundle with (X, \overline{X}) an actual good compactification (i.e. \overline{X} proper) was studied in [Gie75]. Many of his arguments do not use the properness of \overline{X} and carry over to our setup.

The following proposition shows that to check that a stratified bundle is (X, \overline{X}) -regular singular, we may always assume that $\overline{X} \setminus X$ is a smooth divisor, and we may shrink \overline{X} around the generic points of $\overline{X} \setminus X$.

Proposition 4.3. *Let (X, \overline{X}) be a good partial compactification, η_1, \dots, η_r the generic points of $\overline{X} \setminus X$, and E a stratified bundle on X . Then E is (X, \overline{X}) -regular singular, if and only if there are open neighborhoods \overline{U}_i of η_i , $i = 1, \dots, r$, such that $E|_{\overline{U}_i \cap X}$ is $(\overline{U}_i \cap X, \overline{U}_i)$ -regular singular.*

Proof. Only the “if” direction is interesting. Given open neighborhoods \overline{U}_i of η_i , $i = 1, \dots, r$, as in the proposition, we may assume, by shrinking the \overline{U}_i if necessary, that $\eta_j \notin \overline{U}_i$ if $i \neq j$ and then that $X \subseteq \overline{U}_i$. Hence, if \overline{E}_i is an $\mathcal{O}_{\overline{X}}$ -torsion-free $\mathcal{O}_{\overline{U}_i}$ -coherent extension of E to \overline{U}_i as $\mathcal{D}_{\overline{U}_i/k}(\log D_X \cap \overline{U}_i)$ -module, then we can glue to obtain an extension \overline{E}' on $\bigcup_i \overline{U}_i$. Write $\overline{U} := \bigcup_i \overline{U}_i$; then $\overline{X} \setminus \overline{U}$ has codimension ≥ 2 in \overline{X} . If $j : \overline{U} \hookrightarrow \overline{X}$ is the open immersion, then $j_* \overline{E}'$ is an $\mathcal{O}_{\overline{X}}$ -torsion-free, $\mathcal{O}_{\overline{X}}$ -coherent $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module. \square

We also have the notion of a pullback of (X, \overline{X}) -regular singular stratified bundles:

Proposition 4.4. *Let $(X, \overline{X}), (Y, \overline{Y})$ be good partial compactifications and $\bar{f} : \overline{X} \rightarrow \overline{Y}$ a k -morphism, such that $\bar{f}(X) \subseteq Y$. Then for every (X, \overline{X}) -regular singular stratified bundle on X , $(\bar{f}|_Y)^* E$ is (Y, \overline{Y}) -regular singular.*

Proof. This is a direct consequence of Corollary 3.5. \square

Proposition 4.5. *Let (X, \overline{X}) be a good partial compactification and E, E' be (X, \overline{X}) -regular singular bundles. Then the following stratified bundles are also (X, \overline{X}) -regular singular:*

- (a) Every substratified bundle $F \subseteq E$.
- (b) Every quotient-stratified bundle E/F of E .
- (c) $E \otimes_{\mathcal{O}_X} E'$.
- (d) $\mathcal{H}om_{\mathcal{O}_X}(E, E')$.

It follows that $\text{Strat}^{\text{rs}}((X, \overline{X}))$ is a sub-Tannakian subcategory of $\text{Strat}(X)$, and, if $\iota : \text{Strat}^{\text{rs}}((X, \overline{X})) \hookrightarrow \text{Strat}(X)$ denotes the inclusion functor, then for (X, \overline{X}) -regular singular bundle E , restriction of ι gives an equivalence $\langle E \rangle_{\otimes} \rightarrow \langle \iota(E) \rangle_{\otimes}$.

Proof. Again we write $D_X := \overline{X} \setminus X$, $j : X \hookrightarrow \overline{X}$. For (a), let \overline{E} be an $\mathcal{O}_{\overline{X}}$ -torsion-free $\mathcal{O}_{\overline{X}}$ -coherent $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module extending E . Then $j_* F$ and \overline{E} are both $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -submodules of $j_* E$; let \overline{F} be their intersection. Then \overline{F} and $\overline{E}/\overline{F}$ are $\mathcal{O}_{\overline{X}}$ -coherent, $\mathcal{O}_{\overline{X}}$ -torsion free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -modules extending F , resp. E/F .

If $\overline{E}, \overline{E}'$ are $\mathcal{O}_{\overline{X}}$ -coherent $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -modules extending E and E' , then (c) and (d) follow from the fact that $\mathcal{H}om_{\mathcal{O}_{\overline{X}}}(\overline{E}, \overline{E}')$ and $\overline{E} \otimes_{\mathcal{O}_{\overline{X}}} \overline{E}'$ carry natural $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -actions extending those coming from E and E' . \square

Proposition 4.6 ([Gie75, Lemma 3.8]). *Let (X, \overline{X}) be a good partial compactification such that $D_X := \overline{X} \setminus X$ is a smooth divisor, and $i : D_X \hookrightarrow \overline{X}$ the closed immersion (D_X reduced). Write*

$$\overline{\mathcal{D}} := \ker \left(\mathcal{D}_{\overline{X}/k}(\log D_X) \rightarrow i_* i^* \mathcal{D}_{\overline{X}/k} \right).$$

If \overline{E} is a locally free $\mathcal{O}_{\overline{X}}$ -coherent $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module, then $\overline{\mathcal{D}}$ acts \mathcal{O}_{D_X} -linearly on $\overline{E}|_{D_X}$, and there exists a decomposition

$$(3) \quad \overline{E}|_{D_X} = \bigoplus_{\alpha \in \mathbb{Z}_p} F_\alpha,$$

such that $\theta \in \overline{\mathcal{D}}$ acts on F_α via $\theta(s) = \bar{\alpha}s$, where $\bar{\alpha}$ is defined as follows: If y is a local defining equation for D_X , $p^N > \text{ord } \theta$ and $\beta \in \mathbb{N}$, such that $\beta \equiv \alpha \pmod{p^N}$, then

$$(4) \quad \theta(y^\beta) = \bar{\alpha}y^\beta \pmod{y^{\beta+1}}.$$

Remark 4.7. (a) Let us unravel the definition of the F_α after the choice of local coordinates x_1, \dots, x_n as in Remark 3.2 (b), such that D_X is locally cut out by, say, x_1 : A section s of E has the property that $s + x_1 E \in F_\alpha$ if and only if $\delta_{x_1}^{(m)}(s) = \binom{\alpha}{m} s + x_1 E$. The more complicated description (4) from the proposition shows that the decomposition (3) does not depend on the choice of coordinates and exists on all of D_X .

(b) The existence of the decomposition (3) is the reason for a rather striking difference of our setup from the characteristic 0 situation: Proposition 4.6 has as a consequence (see Corollary 5.4) that in our setup there are no objects analogous to regular singular flat connections with nilpotent but nontrivial residues.

Definition 4.8. Let \overline{E} be an $\mathcal{O}_{\overline{X}}$ -locally free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module with $D_X := \overline{X} \setminus X$ smooth. The elements $\alpha \in \mathbb{Z}_p$ such that $F_\alpha \neq 0$ in the decomposition (3), are called *exponents of \overline{E} along D_X* . If D_X is not smooth, but $D_X = \bigcup_{i=1}^r D_i$, D_i smooth divisors, then *the exponents of \overline{E} along D_i* are defined by restricting \overline{E} to an open set $U \subseteq \overline{X}$ intersecting D_i , but not D_j , for $j \neq i$. Finally, if \overline{E} is $\mathcal{O}_{\overline{X}}$ -torsion-free, but not locally free, then the exponents are defined by restricting to an open subset $\overline{U} \subseteq \overline{X}$ such that $\text{codim}_{\overline{X}} \overline{X} \setminus \overline{U} \geq 2$ and such that $E|_{\overline{U}}$ is locally free.

Proposition 4.6 shows that this definition does not depend on the choices made.

We state another analogy to the characteristic 0 situation (see [GL76]):

Theorem 4.9. *Let (X, \overline{X}) be a good partial compactification, and E an $\mathcal{O}_{\overline{X}}$ -torsion-free, $\mathcal{O}_{\overline{X}}$ -coherent $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module. If the exponents of E do not differ by integers, then E is locally free if and only if it is reflexive.*

Proof. This is a direct consequence of [Gie75, Thm. 3.5]. □

Remark 4.10. We will see later on (Theorem 5.2) that an (X, \overline{X}) -regular singular stratified bundle always has an extension to an $\mathcal{O}_{\overline{X}}$ -locally free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module.

Proposition 4.11. *Let (Y, \overline{Y}) and (X, \overline{X}) be good partial compactifications with boundary divisors D_Y and D_X , and $\bar{f} : \overline{Y} \rightarrow \overline{X}$ a finite morphism, such that $\bar{f}(Y) \subseteq X$, and such that $f := \bar{f}|_Y$ is étale. Let D'_Y be a component of D_Y mapping to the component D'_X of D_X . If E is an (X, \overline{X}) -regular singular stratified bundle, then the exponents of the (Y, \overline{Y}) -regular singular bundle f^*E along D'_Y are the exponents of E along D'_X multiplied by the ramification index of D'_Y over D'_X .*

Proof. This is again a question about discrete valuation rings. Let $A \hookrightarrow B$ be a finite extension of discrete valuation rings, such that the extension of fraction fields $K(A) \hookrightarrow K(B)$ is separable. Let x be a uniformizer for A and y a uniformizer for B , $x = uy^e$ with $u \in B^\times$. We use the computation from Proposition 3.4. Let E be an A -module with $\mathcal{D}_{A/k}(\log(x))$ -action. If $a \in E$ is such that $\delta_x^{(p^m)}(a) = \binom{\alpha}{p^m}a + xE$ for some $\alpha \in \mathbb{Z}_p$, then (1) shows that

$$\delta_y^{(p^m)}(a \otimes 1) = \sum_{\substack{c+d=m \\ c,d \geq 0}} \binom{e}{p^c} \binom{\alpha}{p^d} a \otimes 1 + y(E \otimes B) = \binom{e\alpha}{p^m} a \otimes 1 + y(E \otimes B),$$

which proves the proposition. □

Proposition 4.12. *Let (X, \overline{X}) be a good partial compactification such that $D_X := \overline{X} \setminus X$ is smooth. Let E be an (X, \overline{X}) -regular singular stratified bundle, and let \overline{E} and \overline{E}' be two locally free $\mathcal{O}_{\overline{X}}$ -coherent $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -modules, such that $\overline{E}'|_X = \overline{E}|_X = E$ as $\mathcal{D}_{X/k}$ -modules. Write*

$$\overline{E}|_{D_X} = \bigoplus_{\alpha \in \mathbb{Z}_p} F_\alpha \quad \text{and} \quad \overline{E}'|_{D_X} = \bigoplus_{\alpha \in \mathbb{Z}_p} F'_\alpha,$$

as in Proposition 4.6, and define $\text{Exp}(\overline{E}) := \{\alpha \in \mathbb{Z}_p | F_\alpha \neq 0\}$ and $\text{Exp}(\overline{E}') := \{\alpha \in \mathbb{Z}_p | F'_\alpha \neq 0\}$. Then the images of $\text{Exp}(\overline{E})$ and $\text{Exp}(\overline{E}')$ in \mathbb{Z}_p/\mathbb{Z} are identical.

Proof. Let η be the generic point of $\overline{X} \setminus X$. To prove the proposition, we may shrink \overline{X} around η , so that we can assume that $\overline{X} = \text{Spec } A$ is affine, $\overline{E}, \overline{E}'$ are free, and that there are coordinates $x_1, \dots, x_n \in H^0(\overline{X}, \mathcal{O}_{\overline{X}})$ such that $D_X = (x_1)$. Write $j : X \hookrightarrow \overline{X}$. Then $\overline{E} \cap \overline{E}', \overline{E}, \overline{E}' \subseteq j_*E$ are $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -submodules, and replacing \overline{E} by $\overline{E} \cap \overline{E}'$, we may assume that $\overline{E} \subseteq \overline{E}'$ is a $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -submodule.

We may now consider the situation over $\mathcal{O}_{\overline{X}, \eta}$, which is a discrete valuation ring with uniformizer x_1 . For some $n \in \mathbb{N}$ we have $x_1^n \overline{E}'_\eta \subseteq \overline{E}_\eta \subseteq \overline{E}'_\eta$, and it is not difficult to compute that $\text{Exp}(x_1^n \overline{E}'_\eta) = \{\alpha + n | \alpha \in \text{Exp}(\overline{E}'_\eta)\}$. Thus it suffices to show that if $\alpha \in \mathbb{Z}_p$ is an exponent of \overline{E}_η , then $\alpha + N_\alpha$ is an exponent of \overline{E}'_η for some $N_\alpha \in \mathbb{Z}$.

Let $\alpha_1, \dots, \alpha_r$ be the exponents of \overline{E}_η . If $e \in \overline{E}_\eta \setminus x_1 \overline{E}'_\eta$ is an element such that $\delta_{x_1}^{(m)}(e) = \binom{\alpha_1}{m}e + x_1 \overline{E}_\eta$, then α_1 also is an exponent of \overline{E}'_η . If there is no such e , let e_1, \dots, e_r be a lift of a basis of \overline{E}_η/x_1 , such that $\delta_{x_1}^{(m)}(e_i) = \binom{\alpha_i}{m}e_i + x_1 \overline{E}_\eta$ for all $m \geq 0$, and define \overline{E}_{-1} as the submodule of \overline{E}'_η spanned by $x_1^{-1}e_1, e_2, \dots, e_r$. Note that $\overline{E}_\eta \subsetneq \overline{E}_{-1}$. It is readily checked that \overline{E}_{-1} is stable under the $\delta_{x_1}^{(m)}$, and we show that $\alpha_1 - 1$ is an exponent of \overline{E}_{-1} . Since $\mathcal{O}_{\overline{X}, \eta}/(x_1)$ is a field, and since

$e_i \notin x_1 \overline{E}_{-1}$ for $i > 1$, one finds $t \in \overline{E}_{-1}$, such that t, e_2, \dots, e_r is a lift of a basis of $\overline{E}_{-1}/x_1 \overline{E}_{-1}$, and such that $\delta_{x_1}^{(m)}(t) = \binom{\beta}{m} t + x_1 \overline{E}_{-1}$, for some $\beta \in \mathbb{Z}_p$. We compute:

$$\delta_{x_1}^{(m)}(x_1^{-1} e_1) = \binom{\alpha_1 - 1}{m} x_1^{-1} e_1 + f + x_1 \overline{E}_{-1}$$

for some $f \in \langle e_2, \dots, e_r \rangle$. On the other hand, writing $x_1^{-1} e_1 = \lambda_1 t + \sum_{j>1} \lambda_j e_j$ with $\lambda_i \in \mathcal{O}_{\overline{X}, \eta}$, we get

$$\delta_{x_1}^{(m)}(x_1^{-1} e_1) = \binom{\beta}{m} \lambda_1 t + \sum_{j>1} \binom{\alpha_j}{m} \lambda_j e_j + x_1 \overline{E}_{-1}.$$

Since $x_1^{-1} e_1 \notin \langle e_2, \dots, e_r \rangle$, we have $\lambda_1 \notin (x_1)$. Hence, comparing coefficients shows that $\beta = \alpha_1 - 1$, which shows that $\alpha_1 - 1$ is an exponent of \overline{E}_{-1} .

If there is $e \in \overline{E}_{-1} \setminus x_1 \overline{E}'_\eta$ such that $\delta_{x_1}^{(m)}(e) = \binom{\alpha_1 - 1}{m} e + x_1 \overline{E}_{-1}$, then $\alpha_1 - 1$ is an exponent of \overline{E}'_η . Otherwise we construct $\overline{E}_{-2} := (\overline{E}_{-1})_{-1} \supsetneq \overline{E}_{-1}$ contained in \overline{E}'_η as above, with exponent $\alpha_1 - 2$. Since $\overline{E}'_\eta/\overline{E}_\eta$ has finite length, this process has to terminate, so there is some n such that $\alpha_1 - n$ is an exponent of \overline{E}'_η . \square

Definition 4.13. Let (X, \overline{X}) be a good partial compactification, such that $D_X := \overline{X} \setminus X = \bigcup_{i=1}^r D_i$, with D_i smooth divisors. If E is an (X, \overline{X}) -regular singular bundle on X , let \overline{E} be an $\mathcal{O}_{\overline{X}}$ -torsion-free $\mathcal{O}_{\overline{X}}$ -coherent $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module extending E . Then write $\text{Exp}_i(E)$ for the image of the set of exponents of \overline{E} along D_i in \mathbb{Z}_p/\mathbb{Z} . The set $\text{Exp}_i(E)$ is independent of the choice of \overline{E} and it is called the *set of exponents of E along D_i* . Finally, write $\text{Exp}_{(X, \overline{X})}(E) = \bigcup_i \text{Exp}_i(E)$.

Remark 4.14. We emphasize that by definition the exponents of an (X, \overline{X}) -regular singular bundle are elements of \mathbb{Z}_p/\mathbb{Z} , while the exponents of an $\mathcal{O}_{\overline{X}}$ -locally free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module are elements of \mathbb{Z}_p .

5. τ -EXTENSIONS OF (X, \overline{X}) -REGULAR SINGULAR STRATIFIED BUNDLES

In this section, we study in which ways a given (X, \overline{X}) -regular singular bundle E can extend to an $\mathcal{O}_{\overline{X}}$ -locally free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module.

Definition 5.1. Let (X, \overline{X}) be a good partial compactification, D_X the boundary divisor, and $\tau : \mathbb{Z}_p/\mathbb{Z} \rightarrow \mathbb{Z}_p$ a set-theoretical section of the projection $\mathbb{Z}_p \rightarrow \mathbb{Z}_p/\mathbb{Z}$. If E is an (X, \overline{X}) -regular singular bundle, then a τ -extension of E is a finite rank, $\mathcal{O}_{\overline{X}}$ -locally free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module \overline{E}^τ such that the exponents of \overline{E}^τ lie in the image of τ .

Theorem 5.2. *If (X, \overline{X}) is a good partial compactification, $D_X := \overline{X} \setminus X$, E an (X, \overline{X}) -regular singular bundle on X , and $\tau : \mathbb{Z}_p/\mathbb{Z} \rightarrow \mathbb{Z}_p$ a section of the projection, then a τ -extension of E exists and is unique up to isomorphisms which restrict to the identity $E \rightarrow E$ on X .*

Proof. This proof is an extension of the method of [Gie75, Lemma 3.10].

From Proposition 4.3 and Theorem 4.9 it follows easily that we may assume without loss of generality that D_X is a smooth divisor with generic point η , and to prove the proposition we may shrink \overline{X} around η . Hence, we assume that \overline{X} is

affine with coordinates x_1, \dots, x_n , such that $D_X = (x_1)$, that E is free of rank r , and that there exists an $\mathcal{O}_{\overline{X}}$ -free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -extension \overline{E} of E .

Let $\text{Exp}(\overline{E}) = \{\alpha_1, \dots, \alpha_r\}$ be the set of exponents of \overline{E} along D_X . To prove the existence of a τ -extension, we proceed in two steps:

- (a) If $a \in \mathbb{Z}_+$, then there exists an $\mathcal{O}_{\overline{X}}$ -free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module $\overline{E}^{(-a)}$ with exponents

$$\text{Exp}(\overline{E}^{(-a)}) = \{\alpha_1 - a, \dots, \alpha_r - a\},$$

such that $\overline{E}^{(-a)}|_X = E$. Indeed, we can take $\overline{E}^{(-a)} := \overline{E}(aD_X) = \overline{E} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}}(aD_X)$, because $\mathcal{O}_{\overline{X}}(aD_X)$ is defined by x_1^{-a} .

- (b) From \overline{E} we construct an $\mathcal{O}_{\overline{X}}$ -free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module \overline{E}_i extending E , such that

$$\text{Exp}(\overline{E}_i) = \{\alpha_j | \alpha_j \neq \alpha_i\} \cup \{\alpha_i + 1\}.$$

Applying step (a) for an appropriate $a \in \mathbb{Z}_+$, and then step (b) repeatedly for various i , we obtain a τ -extension.

We construct \overline{E}_i as in step (b) for $i = 1$. Assume that $\alpha_1 = \dots = \alpha_\ell$ and $\alpha_j \neq \alpha_1$ for $j > \ell$. After perhaps shrinking \overline{X} around η there exists a lift $e_1, \dots, e_r \in \overline{E}$ of a basis of $\overline{E}/x_1\overline{E}$, such that $\delta_{x_1}^{(m)}(e_i) = \binom{\alpha_i}{m}e_i + x_1\overline{E}$ for all $m \geq 0$. If $j : X \hookrightarrow \overline{X}$ denotes the inclusion, define \overline{E}_1 to be the $\mathcal{O}_{\overline{X}}$ -submodule of j_*E spanned by $x_1e_1, \dots, x_1e_\ell, e_{\ell+1}, \dots, e_r$. Then $(\overline{E}_1)|_X = \overline{E}|_X$. Note that $\delta_{x_1}^{(m)}(e_i) \in \overline{E}_1$ for $i > \ell$. For $i \leq \ell$ we compute:

$$\delta_{x_1}^{(m)}(x_1e_i) = \binom{\alpha_1 + 1}{m}x_1e_i + x_1^2\overline{E}.$$

As $x_1^2\overline{E} \subseteq x_1\overline{E}_1$, we see that $\delta_{x_1}^{(m)}(x_1e_i) \in \overline{E}_1$, and hence \overline{E}_1 is stable under the $\delta_{x_1}^{(m)}$. Moreover, since $x_1e_i \notin x_1\overline{E}_1$, it follows that $\delta_{x_1}^{(m)}(x_1e_i) = \binom{\alpha_1+1}{m}x_1e_i \neq 0 \pmod{x_1\overline{E}_1}$, so $\alpha_1 + 1$ is an exponent of \overline{E}_1 .

To compute the other exponents, note that for every $i > \ell$ there exists $f_i \in \langle x_1e_1, \dots, x_1e_\ell \rangle$, such that

$$(5) \quad \delta_{x_1}^{(m)}(e_i) = \binom{\alpha_i}{m}e_i + f_i + x_1\overline{E}_1.$$

If g_1, \dots, g_r is a lift of a basis of \overline{E}_1/x_1 such that $\delta_{x_1}^{(m)}(g_i) = \binom{\beta_i}{m}g_i + x_1\overline{E}_1$, for some $\beta_i \in \mathbb{Z}_p$, and $\beta_1 = \dots = \beta_h = \alpha_1 + 1$ for $\ell \leq h \leq r$, then for $i > \ell$ we write $e_i = \sum_{j=1}^r \lambda_j g_j$, $\lambda_j \in \mathcal{O}_{\overline{X}}$. There are two cases: If $\lambda_j \in (x_1)$ for all $j > h$, then $\alpha_i = \alpha_1 + 1$. Otherwise, if $j_0 > h$ is such that $\lambda_{j_0} \notin (x_1)$, we compute

$$\delta_{x_1}^{(m)}(e_i) = \binom{\alpha_1 + 1}{m} \sum_{j \leq h} \lambda_j g_j + \sum_{j > h} \binom{\beta_j}{m} \lambda_j g_j + x_1\overline{E}_1,$$

and compare coefficients with (5). It follows that $\beta_{j_0} = \alpha_i$, since $f_i \in \langle g_1, \dots, g_h \rangle$.

This finishes the proof of the existence of a τ -extension. Its unicity follows from the following lemma:

Lemma 5.3. *Let $\overline{X} = \text{Spec } A$ be an affine k -variety and $x_1, \dots, x_n \in A$ coordinates such that $D_X = (x_1)$. Let X be $\overline{X} \setminus D_X = \text{Spec } A[x_1^{-1}]$ and $\tau : \mathbb{Z}_p/\mathbb{Z} \rightarrow \mathbb{Z}_p$ a section of the canonical projection. If E is a free \mathcal{O}_X -module of rank r with $\mathcal{D}_{X/k}$ -action,*

and if $\overline{E}_1, \overline{E}_2$ are free τ -extensions of E , via $\phi : \overline{E}_1|_X \xrightarrow{\sim} \overline{E}_2|_X$, then there is a $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -isomorphism $\overline{E}_1 \rightarrow \overline{E}_2$ extending ϕ .

Proof. This argument is an adaption of [AB01, Prop 4.7]. Let M be the free A -module corresponding to \overline{E}_1 and \overline{E}_2 . Denote by

$$\nabla_i : \mathcal{D}_{\overline{X}/k}(\log D_X) \rightarrow \text{End}_k(M)$$

the two $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -actions on M , coming from the actions on $\overline{E}_1, \overline{E}_2$. By Proposition 4.12 we know that ∇_1 and ∇_2 have the same exponents $\alpha_1, \dots, \alpha_r$ along D_X . Let s_1, \dots, s_r be a basis of M such that

$$\nabla_1 \left(\delta_{x_1}^{(k)} \right) (s_i) \equiv \binom{\alpha_i}{k} s_i + x_1 M,$$

and let s'_1, \dots, s'_r be a basis of M , such that

$$\nabla_2 \left(\delta_{x_1}^{(k)} \right) (s'_i) \equiv \binom{\alpha_i}{k} s'_i + x_1 M.$$

We need to check that $\phi(s_i) \in M \subseteq M \otimes_A A[x_1^{-1}]$ for all i . Let $k(x_1)$ be the residue field $A_{(x_1)}/x_1 A_{(x_1)}$. Fix $i \in \{1, \dots, r\}$, and write $\phi(s_i) = \sum_{j=1}^r f_{ij} s'_j$ with $f_{ij} \in A[x_1^{-1}]$. Assume that there is an integer $\ell_i > 0$, such that the maximal pole order of the f_{ij} along x_1 is ℓ_i . Then $x_1^{\ell_i} \phi(s_i) \in M \subseteq M \otimes_A A[x_1^{-1}]$, and also

$$x_1^{\ell_i} \nabla_2 \left(\delta_{x_1}^{(k)} \right) (\phi(s_i)) \in M \subseteq M \otimes_A A[x_1^{-1}].$$

Tensoring the equality

$$x_1^{\ell_i} \phi \left(\nabla_1 \left(\delta_{x_1}^{(k)} \right) (s_i) \right) = x_1^{\ell_i} \nabla_2 \left(\delta_{x_1}^{(k)} \right) (\phi(s_i))$$

with $k(x_1)$, we obtain

$$(6) \quad x_1^{\ell_i} \binom{\alpha_i}{k} \sum_{j=1}^m f_{ij} s'_j = x_1^{\ell_i} \sum_{j=1}^m \nabla_2 \left(\delta_{x_1}^{(k)} \right) (f_{ij} s'_j).$$

In the completion of the discrete valuation ring $A_{(x_1)}$, we can write $f_{ij} = \sum_{s=-\ell_i}^{d_{ij}} a_{ijs} x_1^s$, with x_1 not dividing in the a_{ijs} , and $a_{ijs} \neq 0$ for $s = -\ell_i$ and some j . But then we can compute

$$x_1^{\ell_i} \nabla_2 \left(\delta_{x_1}^{(k)} \right) (f_{ij} s'_j) = \sum_{s=-\ell_i}^{d_{ij}} a_{ijs} x_1^{s+\ell_i} \binom{\alpha_j + s}{k} s'_j.$$

Then (6) gives in $k(x_1)$ the equation

$$x_1^{\ell_i+s} a_{ijs} \binom{\alpha_i}{k} = a_{ijs} x_1^{\ell_i+s} \binom{\alpha_j + s}{k},$$

both sides of which are 0, except when $s = -\ell_i$. But this implies $\binom{\alpha_i}{k} = \binom{\alpha_j - \ell_i}{k}$ for all k , and hence $\alpha_i = \alpha_j - \ell_i$ by Lucas' Theorem (Lemma 3.3 (a)), which is impossible, as $\alpha_1, \dots, \alpha_r$ lie in the image of τ , and thus map injectively to \mathbb{Z}_p/\mathbb{Z} . Thus $\ell_i = 0$ and hence $\phi(s_i) \in M$. \square

Corollary 5.4. *The essential image of the fully faithful restriction functor*

$$\text{Strat}(\overline{X}) \rightarrow \text{Strat}^{\text{rs}}((X, \overline{X}))$$

is the full subcategory of (X, \overline{X}) -regular singular bundles with exponents equal to 0 in \mathbb{Z}_p/\mathbb{Z} .

Proof. By the τ -extension Theorem 5.2, we have to show that a finite rank $\mathcal{O}_{\overline{X}}$ -locally free $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -module \overline{E} with exponents 0 has a canonical $\mathcal{D}_{\overline{X}/k}$ -action. For this we may assume that \overline{X} is affine with coordinates x_1, \dots, x_n such that $D_X = (x_1)$, and that \overline{E} is free. Then having exponents 0 means that for every $e \in \overline{E}$, and for all $m \geq 1$,

$$\delta_{x_1}^{(m)}(e) = 0 \cdot e + x_1 \overline{E}.$$

Thus we can define $\partial_{x_1}^{(1)}(e) := \frac{\delta_{x_1}^{(1)}(e)}{x_1}$. In particular, the $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -action defines an honest flat connection with p -curvature 0 on \overline{E} . Then, by Cartier’s Theorem ([Kat70, Thm. 5.1]), if $(-)^{(1)}$ denotes Frobenius twist, then $\overline{E} = F_{X/k}^* \overline{E}_1$, where \overline{E}_1 is the $\mathcal{D}_{\overline{X}^{(1)}/k}(\log D_X^{(1)})$ -module obtained as the sheaf of sections s of \overline{E} such that $\partial_{x_i}^{(1)}(s) = 0$ for all i . Moreover, \overline{E}_1 also has exponents 0, and $\delta_{x_1}^{(p)}$ acts as $\delta_{x_1^{(1)}}^{(1)}$ on \overline{E}_1 . We reapply the argument to give meaning to the action of $\partial_{x_1}^{(1)} = \partial_{x_1^{(1)}}^{(p)}$ on \overline{E} . Then we apply Cartier’s Theorem again, etc. □

Remark 5.5. Corollary 5.4 reveals a big difference to the classical situation over the complex numbers: If (X, \overline{X}) is a good partial compactification over \mathbb{C} , then a flat connection on X can be (X, \overline{X}) -regular singular with all exponents 0 (i.e. with nilpotent residues), but still not extend to a flat connection on \overline{X} .

6. (X, \overline{X}) -REGULAR SINGULAR STRATIFIED BUNDLES WITH FINITE MONODROMY

We are ready to prove Main Theorem 1.1 with respect to a fixed good partial compactification. As before, let k denote an algebraically closed field of characteristic $p > 0$.

Theorem 6.1 ((X, \overline{X}) -Main Theorem). *Let (X, \overline{X}) be a good partial compactification (Definition 3.1). Then for a stratified bundle $E \in \text{Strat}(X)$, the following statements are equivalent:*

- (a) *E is (X, \overline{X}) -regular singular and has finite monodromy.*
- (b) *There exists a finite étale covering $f : Y \rightarrow X$, tamely ramified with respect to $\overline{X} \setminus X$, such that $f^*E \in \text{Strat}(Y)$ is trivial.*

Remark 6.2. Recall that in the situation of Theorem 6.1 the morphism f is tamely ramified with respect to $\overline{X} \setminus X$, if the discrete rank 1 valuations of $k(X)$ associated with the codimension 1 points of \overline{X} are tamely ramified in $k(Y)$.

The theorem will be deduced from the following lemma, which is the technical heart of the proof:

Lemma 6.3 (Main Lemma). *Let (X, \overline{X}) be a good partial compactification and $f : Y \rightarrow X$ a finite Galois étale morphism. Then the stratified bundle $f_*\mathcal{O}_Y$ is (X, \overline{X}) -regular singular, if and only if f is tamely ramified with respect to $\overline{X} \setminus X$.*

Proof of Theorem 6.1 (assuming Lemma 6.3). Let E be an (X, \overline{X}) -regular singular stratified bundle with finite monodromy. Let $\omega : \langle E \rangle_{\otimes} \rightarrow \text{Vect}_k$ be a fiber functor, and $h_{E,\omega} : X_{E,\omega} = \text{Spec } A_{E,\omega} \rightarrow X$ the Picard-Vessiot torsor associated with E and ω (Definition 2.18). Then $A_{E,\omega} \in \langle E \rangle_{\otimes}$ is (X, \overline{X}) -regular singular, so the Main Lemma 6.3 implies that $h_{E,\omega}$ is tamely ramified with respect to $\overline{X} \setminus X$. By construction $h_{E,\omega}^* E$ is trivial.

Conversely, if $f : Y \rightarrow X$ is finite étale and tamely ramified with respect to $\overline{X} \setminus X$, then $E \subseteq f_* f^* E = f_* \mathcal{O}_Y^{\text{rank } E}$ as stratified bundles, and $f_* \mathcal{O}_Y$ is (X, \overline{X}) -regular singular by the Main Lemma 6.3. □

Now to the proof of the Main Lemma 6.3:

Proof. Corollary 3.5 implies that $f_* \mathcal{O}_Y$ is (X, \overline{X}) -regular singular if f is tamely ramified with respect to $\overline{X} \setminus X$. Indeed, we may assume that $D_X := \overline{X} \setminus X$ is a smooth divisor and that $\overline{X} = \text{Spec } A$ is affine, such that $\overline{X} \setminus X$ is cut out by a regular element $t \in A$. Then $Y = \text{Spec } B$ is affine, and we can write \overline{Y} for the normalization of \overline{X} in $k(Y)$. After shrinking \overline{X} around the generic point of D_X if necessary, and after replacing \overline{Y} by the preimage of the smaller \overline{X} , $D_Y := \overline{Y} \setminus Y$ is a strict normal crossings divisor, and we get the following commutative diagram:

$$\begin{array}{ccc} \overline{Y} & \xrightarrow{\bar{f}} & \overline{X} \\ \uparrow & & \uparrow \\ Y & \xrightarrow{f} & X \end{array}$$

Then Corollary 3.5 applies and shows that $f_* \mathcal{O}_Y$ is (X, \overline{X}) -regular singular.

The converse is more involved: Again we may assume without loss of generality that D_X is a smooth irreducible divisor with generic point η , and in the construction we may shrink \overline{X} around η . We proceed in five steps:

- (a) Note that the exponents of $f_* \mathcal{O}_Y$ are torsion in \mathbb{Z}_p/\mathbb{Z} , because by Proposition 4.11, pulling back a $\mathcal{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$ -extension of $f_* \mathcal{O}_Y$ along \bar{f} multiplies the exponents by the ramification indices of \bar{f} along D_X , and clearly $f^* f_* \mathcal{O}_Y = \mathcal{O}_Y^{\text{deg } f}$ is trivial.
- (b) By Theorem 5.2 we find an $\mathcal{O}_{\overline{X}}$ -coherent, $\mathcal{O}_{\overline{X}}$ -torsion-free extension \overline{E} of $f_* \mathcal{O}_Y$ with $\mathcal{D}_{\overline{X}/k}(\log D_X)$ -action and exponents in $\mathbb{Z} \cap \mathbb{Q}$; say $\frac{a_1}{b}, \dots, \frac{a_r}{b}$ with $(b, p) = 1$.
- (c) Shrinking \overline{X} around η , if necessary, we may assume that $\overline{X} = \text{Spec } A$, with local coordinates x_1, \dots, x_n such that $D_X = (x_1)$. Then define $\overline{Z}_1 := \text{Spec } A[x_1^{1/b}]$, and let $\bar{h} : \overline{Z}_1 \rightarrow \overline{X}$ be the associated covering. Let $Z_1 := \bar{h}^{-1}(X)$ and $h = \bar{h}|_{Z_1}$. Then h is étale, \bar{h} finite and tamely ramified with respect to $\overline{X} \setminus X$, and $h^* f_* \mathcal{O}_Y$ has exponents equal to 0 in \mathbb{Z}_p/\mathbb{Z} , which means that by Corollary 5.4 there exists a stratified bundle $\overline{E}_1 \in \text{Strat}(\overline{Z}_1)$ extending $h^* f_* \mathcal{O}_Y$.
- (d) Now we claim that there exists a finite étale covering $\bar{g} : \overline{Z} \rightarrow \overline{Z}_1$ such that $\bar{g}^* \overline{E}_1$ is trivial. Indeed, this is true for $\overline{E}_1|_{Z_1} = h^* f_* \mathcal{O}_Y$ because it is true for $f_* \mathcal{O}_Y$, and by Proposition 2.13 restriction functor $\langle \overline{E}_1 \rangle_{\otimes} \rightarrow \langle h^* f_* \mathcal{O}_Y \rangle_{\otimes}$ is an equivalence.

- (e) We can finish up: Write $Z := \bar{g}^{-1}(Z_1)$, $g = \bar{g}|_Z$ and $h = h_1g$. Then we have the diagram of finite étale maps

$$\begin{array}{ccc}
 Y \times_X Z & \xrightarrow{f_Z} & Z \\
 \downarrow h_Y & & \downarrow g \\
 Y & \xrightarrow{f} & X \\
 & & \downarrow h_1 \\
 & & Z_1
 \end{array}
 \quad \begin{array}{c}
 \curvearrowright \\
 h \\
 \curvearrowleft
 \end{array}$$

and h is tamely ramified with respect to $\bar{X} \setminus X$ by construction. But also by construction $h^*f_*\mathcal{O}_Y$ is trivial, and since $h^*f_*\mathcal{O}_Y = f_{Z,*}\mathcal{O}_{Y \times_X Z}$, Corollary 2.17 shows that f_Z is the trivial covering. But this means that the covering $h : Z \rightarrow X$ dominates $f : Y \rightarrow X$, so f is tamely ramified with respect to $\bar{X} \setminus X$. \square

Now write $D_X := \bar{X} \setminus X$ and denote by $\pi_1^{D_X}(X, x)$ the profinite group associated with the Galois category of finite étale coverings of X which are tamely ramified with respect to D_X .

Corollary 6.4. *Let (X, \bar{X}) be a good partial compactification, and $D_X := \bar{X} \setminus X$. Let $x \in X(k)$ be a rational point. Then the fiber functor $\omega_x : \text{Strat}^{\text{rs}}((X, \bar{X})) \rightarrow \text{Vect}_k$ induces an equivalence of the category of (X, \bar{X}) -regular singular stratified bundles with finite monodromy with the category $\text{Rept}_k^{\text{cont}} \pi_1^{D_X}(X, x)$.*

In other words: If $\pi_1^{D_X}(X, x) = \varprojlim_i G_i$ with G_i finite, then the maximal étale quotient of $\pi_1(\text{Strat}^{\text{rs}}((X, \bar{X})), \omega_x)$ is $\pi_1^{D_X}(X, x)_k := \varprojlim_i (G_i)_k$, where ω_x is the neutral fiber functor associated with x .

7. REGULAR SINGULAR STRATIFIED BUNDLES IN GENERAL

Since resolution of singularities is not available in positive characteristic, we unfortunately cannot use good compactifications to define regular singularity of stratified bundles in positive characteristic. In this section we present a definition which works in general, and we generalize the results from the previous sections to this new notion of regular singularity.

We continue to denote by k an algebraically closed field of characteristic $p > 0$, and by X a smooth, connected, separated k -scheme of finite type.

Definition 7.1. A stratified bundle E on X is called *regular singular* if it is (X, \bar{X}) -regular singular for all good partial compactifications (X, \bar{X}) of X . The category $\text{Strat}^{\text{rs}}(X)$ is defined to be the full subcategory of $\text{Strat}(X)$ with objects the regular singular stratified bundles.

This is inspired by the following definition:

Definition 7.2 (Kerz-Schmidt, Wiesend, [KS10]). A finite étale morphism $f : Y \rightarrow X$ is called *tame* if the induced extension $k(X) \hookrightarrow k(Y)$ is tamely ramified with respect to every *geometric discrete rank 1 valuation of $K(X)$* . Here a discrete rank 1 valuation of $K(X)$ is called *geometric* if its valuation ring appears as the local ring of a codimension 1 point on some model of the function field $K(X)$ of X .

Remark 7.3. It is not difficult to see that $f : Y \rightarrow X$ is tame if and only if it is tamely ramified with respect to $\overline{X} \setminus X$ for all good partial compactifications (X, \overline{X}) of X .

Proposition 4.5 immediately implies:

Proposition 7.4. *The category $\text{Strat}^{\text{rs}}(X)$ is a sub-Tannakian subcategory of $\text{Strat}(X)$.*

Let's see that the above definition of regular singularity agrees with the one from [Gie75] in the presence of a good compactification:

Proposition 7.5. *Assume that X admits a good compactification \overline{X} , i.e. a good partial compactification (X, \overline{X}) with \overline{X} proper. Then a stratified bundle E is regular singular, if and only if it is (X, \overline{X}) -regular singular.*

Proof. If E is regular singular, then E is (X, \overline{X}) -regular singular by definition, so it remains to prove the converse. Assume that E is (X, \overline{X}) -regular singular, and let (X, \overline{X}') be any good partial compactification. To show that E is (X, \overline{X}') -regular singular, we may remove a closed subset of codimension ≥ 2 from \overline{X}' . Hence, by the properness of \overline{X} we may assume that there exists a morphism $g : \overline{X}' \rightarrow \overline{X}$, such that $g|_X = \text{id}_X$. Then g satisfies the assumptions of Proposition 3.4, and thus if \overline{E} is an $\mathcal{O}_{\overline{X}}$ -torsion-free, $\mathcal{O}_{\overline{X}}$ -coherent $\mathcal{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$ -module \overline{E} extending E , then $g^* \overline{E}$ is an $\mathcal{O}_{\overline{X}'}$ -torsion-free, $\mathcal{O}_{\overline{X}'}$ -coherent $\mathcal{D}_{\overline{X}'/k}(\log \overline{X}' \setminus X)$ -module extending E , so E is (X, \overline{X}') -regular singular. \square

The proof of the Main Theorem 1.1 now is simple:

Theorem 7.6. *For a stratified bundle $E \in \text{Strat}(X)$, the following statements are equivalent:*

- (a) *E is regular singular and has finite monodromy.*
- (b) *E is trivialized by a finite étale tame morphism.*

Proof. Let E be a stratified bundle with finite monodromy. Then E is regular singular, if and only if it is (X, \overline{X}) -regular singular for every good partial compactification (X, \overline{X}) . As in the proof of Theorem 6.1, we see that this is equivalent to the Picard-Vessiot torsor of E associated to some fiber functor being tamely ramified with respect to every good partial compactification of X , and hence tame. \square

Now write $\pi_1^{\text{tame}}(X, x)$ for the profinite group associated with the Galois category of all finite étale tame coverings of X , as defined in [KS10].

Corollary 7.7. *Let $x \in X(k)$ be a rational point, and $\omega_x : \text{Strat}^{\text{rs}}(X) \rightarrow \text{Vect}_k$ the associated fiber functor. Then ω_x induces an equivalence of the full subcategory of $\text{Strat}^{\text{rs}}(X)$ with objects the regular singular stratified bundles with finite monodromy with the category $\text{Rep}_k^{\text{cont}} \pi_1^{\text{tame}}(X, x)$.*

In other words, if $\pi_1^{\text{tame}}(X, x) = \varprojlim_i G_i$, then the maximal pro-étale quotient of $\pi_1(\text{Strat}^{\text{rs}}(X), \omega_x)$ is $\pi_1^{\text{tame}}(X, x)_k := \varprojlim_i (G_i)_k$.

8. TESTING FOR REGULAR SINGULARITIES ON CURVES

If X is a smooth complex variety, then one of the basic facts about a flat connection (E, ∇) on X is that (E, ∇) is regular singular, if and only if for all regular \mathbb{C} -curves C and all \mathbb{C} -morphisms $\phi : C \rightarrow X$, the flat connection $\phi^*(E, \nabla)$ on C is regular singular. We prove an analogue for stratified bundles with finite monodromy with respect to our notion of regular singularity in positive characteristic. For information about stratified bundles with arbitrary monodromy, see Remark 8.4.

Let k denote an algebraically closed field of characteristic $p > 0$. Again we first work with respect to a fixed good partial compactification.

We start with a lemma:

Lemma 8.1. *Let X be a smooth, separated, finite type k -scheme, E a stratified bundle on X with finite monodromy, and $\omega : \langle E \rangle_{\otimes} \rightarrow \text{Vect}_k$ a fiber functor. Let $\phi : C \rightarrow X$ be a nonconstant morphism with C a regular k -curve. Then the following statements are true:*

- (a) *There exists a fiber functor $\omega_{\phi} : \langle E|_C \rangle_{\otimes} \rightarrow \text{Vect}_k$, such that the diagram*

$$\begin{array}{ccc}
 \langle E \rangle_{\otimes} & \xrightarrow{\omega} & \text{Vect}_k \\
 \text{restriction} \downarrow & \nearrow \omega_{\phi} & \\
 \langle E|_C \rangle_{\otimes} & &
 \end{array}$$

*commutes. Here $E|_C := \phi^*E$.*

- (b) *If $h_{E,\omega} : X_{E,\omega} \rightarrow X$ is the Picard-Vessiot torsor associated with E and ω , then $h_{E,\omega} \times_X \phi : X_{E,\omega} \times_X C \rightarrow C$ is isomorphic to a disjoint union of copies of the Picard-Vessiot torsor $h_{E|_C,\omega_{\phi}} : C_{E|_C,\omega_{\phi}} \rightarrow C$.*

Proof. Recall that $G(E, \omega)$ is a finite, constant k -group scheme by Theorem 2.9; write G for the associated finite group. Then $h_{E,\omega}$ is Galois étale with group G . Consider the fiber product $X_{E,\omega} \times_X C \rightarrow C$. This is a finite étale covering, and it is the disjoint union $\coprod_j C_j \rightarrow C$, with C_j connected regular curves. Moreover, all C_j are C -isomorphic, say to $f : C' \rightarrow C$, as they are permuted by the action of G . To summarize notation, we have the commutative diagram

$$(7) \quad \begin{array}{ccccc}
 C' & \xrightarrow{i} & X_{E,\omega} \times_X C = \coprod C' & \xrightarrow{\text{pr}} & X_{E,\omega} \\
 & \searrow f & \downarrow & & \downarrow h_{E,\omega} \\
 & & C & \xrightarrow{\phi} & X,
 \end{array}$$

where i is one of the natural inclusions $C' \hookrightarrow \coprod C'$. Define $\omega_{\phi} : \langle E|_C \rangle_{\otimes} \rightarrow \text{Vect}_k$ by $F \mapsto H^0(\text{Strat}(C'), f^*F)$; see Remark 2.14. This is a k -linear fiber functor since $f^*(E|_C)$ is trivial, and we obtain the following commutative diagram:

$$\begin{array}{ccc}
 \langle E \rangle_{\otimes} & \xrightarrow{|_C} & \langle E|_C \rangle_{\otimes} \\
 \omega \searrow & & \swarrow \omega_{\phi} \\
 & \text{Vect}_k &
 \end{array}$$

Indeed, again by Proposition 2.15, we know that (with notation from (7))

$$\begin{aligned}\omega(N) &= H^0(\text{Strat}(X_{E,\omega}), \underbrace{h_{E,\omega}^* N}_{\text{trivial}}) \\ &= H^0(\text{Strat}(C'), i^* \text{pr}^* h_{E,\omega}^* N) \\ &= H^0(\text{Strat}(C'), f^* N|_C) \\ &= \omega_\phi(N|_C)\end{aligned}$$

for every object $N \in \langle E \rangle_{\otimes}$. By [DM82, Prop. 2.21b] this implies that $G(\langle E|_C \rangle_{\otimes}, \omega_\phi) \hookrightarrow G(\langle E \rangle, \omega)$ is a closed immersion, and that $f : C' \rightarrow C$ is a $G(\langle E|_C \rangle, \omega_\phi)$ -torsor; in fact it is the Picard-Vessiot torsor associated with $E|_C$ and $\omega|_C$, according to the following elementary lemma:

Lemma 8.2. *Let $H \subseteq G$ be finite groups, and $R \subseteq G$ be a set of representatives for G/H . Then $k[G] = \bigoplus_{r \in R} k[H]$ in the category of $k[H]$ -modules. \square*

From Lemma 8.1 and the results of [KS10] it follows that regular singularity (at least for stratified bundles with finite monodromy) is a property determined on the family of curves mapping to X .

Theorem 8.3. *Let X be a smooth, finite type k -scheme, Then a stratified bundle E on X with finite monodromy is regular singular, if and only if $E|_C := \phi^* E$ is regular singular for every k -morphism $\phi : C \rightarrow X$, with C a regular k -curve.*

Proof. Let ω be a neutral fiber functor for $\langle E \rangle_{\otimes}$, and let $h_{E,\omega} : X_{E,\omega} \rightarrow X$ be the Picard-Vessiot torsor for E and ω . Clearly E is regular singular if and only if $(h_{E,\omega})_* \mathcal{O}_{X_{E,\omega}}$ is regular singular, which by Theorem 7.6 is equivalent to $h_{E,\omega}$ being tame. By [KS10, Thm. 4.4], $h_{E,\omega}$ is tame if and only if $h_{E,\omega} \times_X \phi : X_{E,\omega} \times_X C \rightarrow C$ is tame for all $\phi : C \rightarrow X$ as in the claim. But by Lemma 8.1, $h_{E,\omega} \times_X \phi$ is isomorphic to (a disjoint union of copies of) the Picard-Vessiot torsor associated with $E|_C$ and the fiber functor ω_ϕ constructed in Lemma 8.1. This shows that E is regular singular if and only if $h_{E|_C, \omega_\phi}$ is tame for all $\phi : C \rightarrow X$, if and only if $E|_C$ is regular singular for all $\phi : C \rightarrow X$. \square

Remark 8.4. It is unknown to the author whether Theorem 8.3 remains true without the finiteness assumption on the monodromy of E . There are partial results assuming resolution of singularities; see [Kin12, Sec. 3.4].

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