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EXISTENCE OF GROUNDSTATES FOR A CLASS OF NONLINEAR CHOQUARD EQUATIONS

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ABSTRACT. We prove the existence of a nontrivial solution $u \in H^1(\mathbb{R}^N)$ to the nonlinear Choquard equation

$$-\Delta u + u = \big(I_\alpha * F(u)\big)F'(u) \quad \text{in } \mathbb{R}^N,$$

where I_α is a Riesz potential, under almost necessary conditions on the nonlinearity F in the spirit of Berestycki and Lions. This solution is a groundstate and has additional local regularity properties; if moreover F is even and monotone on $(0,\infty)$, then u is of constant sign and radially symmetric.

CONTENTS

1. INTRODUCTION

We consider the problem

$$(\mathcal{P}) \qquad -\Delta u + u = \big(I_\alpha * F(u)\big)f(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in (0,N)$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}},$$

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$F \in C^1(\mathbb{R};\mathbb{R})$ and $f := F'$. Solutions of $(\mathcal{P})$ are formally critical points of the functional defined by

$$\mathcal{I}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 - \frac{1}{2}\int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big)F(u).$$

We are interested in the existence and some qualitative properties of solutions to $(\mathcal{P})$.

Problem $(\mathcal{P})$ is a semilinear elliptic equation with a nonlocal nonlinearity. For $N = 3$, $\alpha = 2$ and $F(s) = \frac{s^2}{2}$ it covers in particular the Choquard–Pekar equation

$$-\Delta u + u = (I_2 * |u|^2)u \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

introduced at least in 1954, in a work by S. I. Pekar describing the quantum mechanics of a polaron at rest [32]. In 1976 P. Choquard used (1.1) to describe an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of one component plasma [22]. In 1996 R. Penrose proposed (1.1) as a model of self-gravitating matter [30]. In this context equation (1.1) is usually called the nonlinear Schrödinger–Newton equation. Note that if u solves (1.1), then the function ψ defined by $\psi(t,x) = e^{it}u(x)$ is a solitary wave of the focusing time-dependent Hartree equation

$$i\psi_t + \Delta\psi = -(I_2 * |\psi|^2)\psi \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N.$$

In this context (1.1) is also known as the stationary nonlinear Hartree equation. The existence of solutions for stationary equation (1.1) was proved by variational methods by E. H. Lieb, P.-L. Lions and G. Menzala [22, 24, 28] and also by ordinary differential equations techniques [11, 30, 38]. In the more general case of equation $(\mathcal{P})$ with $F(s) = \frac{1}{p}|s|^p$, problem $(\mathcal{P})$ is known to have a solution if and only if $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ ([27, p. 457], [31, Theorem 1]; see also [15, Lemma 2.7]).

The existence results for $(\mathcal{P})$ up until now were only available when the nonlinearity F is *homogeneous*. This situation contrasts with the striking existence result for the corresponding local problem

$$-\Delta u + u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which can be considered as a limiting problem of $(\mathcal{P})$ when $\alpha \to 0$, with $g = Ff$. H. Berestycki and P.-L. Lions [6, Theorem 1] have proved that (1.2) has a nontrivial solution if nonlinearity $g \in C(\mathbb{R};\mathbb{R})$ satisfies the assumptions

(g_1) there exists $C > 0$ such that for every $s \in \mathbb{R}$, $sg(s) \le C\big(|s|^2 + |s|^{\frac{2N}{N-2}}\big)$,

$$(g_2) \qquad \lim_{s\to 0} \frac{G(s)}{|s|^2} < \frac{1}{2} \qquad \text{and} \qquad \limsup_{|s|\to\infty} \frac{G(s)}{|s|^{\frac{2N}{N-2}}} \le 0,$$

$$(g_3) \qquad \text{there exists } s_0 \in \mathbb{R}\setminus\{0\} \text{ such that } G(s_0) > \frac{s_0^2}{2},$$

where $G(s) = \int_0^s g(\sigma)\,\mathrm{d}\sigma$ (and if $g = Ff$, then $G = \frac{F^2}{2}$). They also proved that if $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a finite energy solution of (1.2), then u satisfies the Pohožaev identity [6, Proposition 1]

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2}\int_{\mathbb{R}^N} |u|^2 = N\int_{\mathbb{R}^N} G(u). \tag{1.3}$$

This, in particular, implies that assumptions (g_1), (g_2) and (g_3) are "almost necessary" for the existence of nontrivial finite energy solutions of (1.2). Indeed, the

necessity of (g_3) follows directly from (1.3). For (g_1) and (g_2), if $f(s) = s^p$ with $s \notin (1, \frac{N+2}{N-2})$, then (1.3) immediately implies that (1.2) does not have any bounded finite-energy nontrivial solution.

In this spirit, we prove the existence of solutions to Choquard equation $(\mathcal{P})$, assuming that nonlinearity $f \in C(\mathbb{R};\mathbb{R})$ satisfies the *growth assumption*:

(f_1) there exists $C > 0$ such that for every $s \in \mathbb{R}$, $|sf(s)| \le C\big(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}\big)$,

its antiderivative $F : s \in \mathbb{R} \mapsto \int_0^s f(\sigma)\,\mathrm{d}\sigma$ is *subcritical*:

$$(f_2) \qquad \lim_{s\to 0} \frac{F(s)}{|s|^{\frac{N+\alpha}{N}}} = 0 \qquad \text{and} \qquad \lim_{|s|\to\infty} \frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0,$$

and *nontrivial*:

$$(f_3) \qquad \text{there exists } s_0 \in \mathbb{R}\setminus\{0\} \text{ such that } F(s_0) \neq 0.$$

It is standard to check using Hardy–Littlewood–Sobolev inequality that if $f \in C(\mathbb{R};\mathbb{R})$ satisfies growth assumption (f_1), then $\mathcal{I}$ defines on the Sobolev space $H^1(\mathbb{R}^N)$ a continuously differentiable functional and critical points of $\mathcal{I}$ are weak solutions of equation $(\mathcal{P})$. In what follows, solutions of $(\mathcal{P})$ are always understood in the weak sense.

We say $u \in H^1(\mathbb{R}^N)\setminus\{0\}$ is a *groundstate* of $(\mathcal{P})$ if u is a solution of $(\mathcal{P})$ and

$$\mathcal{I}(u) = c := \inf\big\{\mathcal{I}(v) \ : \ v \in H^1(\mathbb{R}^N)\setminus\{0\} \text{ is a solution of } (\mathcal{P})\big\}. \tag{1.4}$$

Our main result in this paper is the following.

Theorem 1 (Existence of a groundstate)**.** *Assume that $N \ge 3$ and $\alpha \in (0, N)$. If $f \in C(\mathbb{R};\mathbb{R})$ satisfies (f_1), (f_2) and (f_3), then $(\mathcal{P})$ has a groundstate.*

We also prove that any weak solution of $(\mathcal{P})$ has additional regularity properties.

Theorem 2 (Local regularity)**.** *Assume that $N \ge 3$ and $\alpha \in (0, N)$. If $f \in C(\mathbb{R};\mathbb{R})$ satisfies (f_1) and $u \in H^1(\mathbb{R}^N)$ solves $(\mathcal{P})$, then for every $q \ge 1$, $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$.*

In particular, Theorem 2 with the Morrey–Sobolev embeddings implies that solutions of $(\mathcal{P})$ are locally Hölder continuous. If f has additional smoothness, then regularity of u could be further improved via Schauder estimates. Let us emphasize that Theorem 2 is established only under the growth assumption (f_1) and does not require additional subcriticality assumption (f_2).

The regularity information of Theorem 2 allows us to establish a Pohožaev integral identity for all finite energy solutions of $(\mathcal{P})$.

Theorem 3 (Pohožaev identity)**.** *Assume that $N \ge 3$ and $\alpha \in (0, N)$. If $f \in C(\mathbb{R};\mathbb{R})$ satisfies (f_1) and $u \in H^1(\mathbb{R}^N)$ solves $(\mathcal{P})$, then*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2}\int_{\mathbb{R}^N} |u|^2 = \frac{N+\alpha}{2}\int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big)F(u). \tag{1.5}$$

In particular, (1.5) implies that if $u \neq 0$ is a solution of $(\mathcal{P})$, then

$$\mathcal{I}(u) = \frac{\alpha+2}{2(N+\alpha)}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N} |u|^2 > 0.$$

Pohožaev identity (1.5) shows that our assumptions (f_1), (f_2) and (f_3) are "almost necessary" for the existence of nontrivial solutions to $(\mathcal{P})$. Indeed, if $F(s) = \frac{1}{p}|s|^p$, then (1.5) implies that problem $(\mathcal{P})$ does not have nontrivial weak

solutions in $H^1(\mathbb{R}^N)$ if $p \notin \left(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right)$ (see also [31, Theorem 2] where supercritical ranges of p are included). If (f_3) fails, then a solution $u \in H^1(\mathbb{R}^N)$ would satisfy $-\Delta u + u = 0$ and would then necessarily be trivial.

Whereas the upper critical exponent $(N+\alpha)/(N-2)$ appears as a natural extension of the critical Sobolev exponent $2N/(N-2)$ for the local problem (1.2) with $G = F^2$, the lower critical exponent $(N+\alpha)/N$ in assumptions (f_1) and (f_2) is a new phenomenon. It is due to the effect of the nonlocal term in $(\mathcal{P})$ and has no analogues in (1.2). The growth restriction $|sf(s)| \leq c|s|^{(N+\alpha)/N}$ for $|s| < 1$ occurs naturally in the application of the Hardy–Littlewood–Sobolev inequality to verify that $\mathcal{I} \in C^1(H^1(\mathbb{R}^N); \mathbb{R})$. In fact, Pohožaev identity confirms that the power $(N+\alpha)/N$ is optimal for the existence of solutions, and in this respect it plays the role of the lower critical exponent for $(\mathcal{P})$.

Finally, we obtain qualitative properties of groundstates of $(\mathcal{P})$, which are the counterpart of the properties obtained for solutions of the corresponding local equation [10, 14, 16].

Theorem 4 (Qualitative properties of groundstates)**.** *Assume that $N \geq 3$ and $\alpha \in (0, N)$. If $f \in C(\mathbb{R}; \mathbb{R})$ satisfies (f_1) and, in addition, f is odd and has constant sign on $(0, \infty)$, then every groundstate of $(\mathcal{P})$ has constant sign and is radially symmetric with respect to some point in $\mathbb{R}^N$.*

Before explaining the proofs of our results, we recall the strategy of H. Berestycki and P.-L. Lions's proof of the existence of solutions to (1.2) [6, §3]. They consider the constrained minimization problem

$$\min\Big\{\int_{\mathbb{R}^N} |\nabla u|^2 \ : \ u \in H^1(\mathbb{R}^N) \ \text{ and } \int_{\mathbb{R}^N} G(u) - \frac{|u|^2}{2} = 1\Big\}; \tag{1.6}$$

they first show that by the Pólya–Szegő inequality for the Schwarz symmetrization, the minimum can be taken on radial and radially nonincreasing functions. Then they show the existence of a minimum $v \in H^1(\mathbb{R}^N)$ by the direct method of the calculus of variations. This minimum v satisfies the equation

$$-\Delta v = \theta\big(g(v) - v\big) \quad \text{in } \mathbb{R}^N,$$

with a Lagrange multiplier $\theta > 0$. They conclude by noting that $u \in H^1(\mathbb{R}^N)$ defined for $x \in \mathbb{R}^N$ by $u(x) = v(x/\sqrt{\theta})$ solves (1.2).

The approach of H. Berestycki and P.-L. Lions fails for nonlocal problem $(\mathcal{P})$ for two different reasons. First, the nonlocal term will not be preserved or controlled under Schwarz symmetrization unless the nonlinearity f satisfies the more restrictive assumption of Theorem 4. Second, the final scaling argument fails: the three terms in $(\mathcal{P})$ scale differently in space, so one cannot hope to get rid of a Lagrange multiplier by scaling in space. In general, a constrained minimization of type (1.6) cannot be used for the study of *solutions of equations* with multiple scaling rates.

Similar issues of multiple scaling rates arise, for instance, in the study of nonlocal nonlinear Schrödinger–Maxwell or Schrödinger–Poisson equations. For instance, the existence of a radial groundstate solution to a class of Schrödinger–Maxwell equations under general Berestycki–Lions type assumptions on the nonlinear term was established in [2] by applying the mountain–pass theorem to a family of truncated functionals and then by proving the convergence of the obtained sequence of radially symmetric critical points using the radial compactness lemma of Strauss. Despite some similarities, the structure of Schrödinger–Maxwell equations is very

different with Choquard equations and thus new techniques are required for the study of $(\mathcal{P})$. Moreover, such results establish the existence of radial groundstates while we are interested in the construction of *global groundstates*.

In the present work, in order to prove the existence of solutions of $(\mathcal{P})$, instead of the constrained minimization problem of type (1.6), we consider in section 2 the mountain pass level

$$b = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{I}\big(\gamma(t)\big), \tag{1.7}$$

where the set of paths is defined as

$$\Gamma = \big\{\gamma \in C\big([0,1]; H^1(\mathbb{R}^N)\big) \ : \ \gamma(0) = 0 \text{ and } \mathcal{I}(\gamma(1)) < 0\big\}. \tag{1.8}$$

Classically, in order to show that b is a critical level of the functional $\mathcal{I}$, one constructs a *Palais–Smale sequence* at the level b, that is, a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^N)$ such that $\mathcal{I}(u_n) \to b$ and $\mathcal{I}'(u_n) \to 0$ as $n \to \infty$. Then one proves that the sequence $(u_n)_{n\in\mathbb{N}}$ converges up to translations and extraction of a subsequence [37, 43]. The first step of this approach is to establish the *boundedness* of the sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^N)$. Usually this involves an Ambrosetti–Rabinowitz type superlinearity assumption, which in our setting would require the existence of $\mu > 1$ such that $s \in \mathbb{R}^+ \mapsto F(s)/s^\mu$ is nondecreasing.

In order to avoid an Ambrosetti–Rabinowitz type condition, in section 2 we employ a scaling technique introduced by L. Jeanjean. It consists in constructing a Palais–Smale sequence that *satisfies asymptotically the Pohožaev identity* [18] (see also [1,2,4,17]). This improvement is related to the monotonicity trick of M. Struwe [37, §II.9] and L. Jeanjean [19]. Next, we prove with a concentration compactness argument the existence of a nontrivial solution u to $(\mathcal{P})$ under the assumptions (f_1), (f_2) and (f_3) only. This combination of the scaling technique with a concentration-compactness argument which does not rely on the radial compactness and a priori radial symmetry of the solution is a novelty in our proof.

To conclude that such a constructed solution u is a groundstate, we first show that $\mathcal{I}(u) = b$. This is a straightforward computation if u satisfies the Pohožaev identity (1.5) proved in section 3.3. This however brings a regularity issue, as the proof of the identity (1.5) requires a little more regularity than $u \in H^1(\mathbb{R}^N)$. The growth assumption (f_1) allows a *critical growth* of f and is too weak for a direct bootstrap argument. We study the delicate question of regularity of u in section 3.1 by introducing a new regularity result which can be thought of as a *nonlocal counterpart of the critical Brezis–Kato regularity result* [8]. Once additional regularity of the solution u is established, the Pohožaev identity (1.5) follows and can be employed to estimate the critical level $\mathcal{I}(u)$. This is done using the construction of paths associated to critical points in section 4.1 following L. Jeanjean and K. Tanaka [20].

The qualitative properties of the groundstate of Theorem 4 are established in section 5. We show that the absolute value of a groundstate and its polarization are also groundstates. This leads to contradiction with the strong maximum principle if the solution is not invariant under these transformations.

Finally in section 6 we explain how the proof of Theorem 1 can be simplified under the assumptions of Theorem 4 using symmetric mountain pass [40], adapting the original argument of Berestycki and Lions for $(\mathcal{P})$.

2. Construction of a solution

2.1. **Construction of a Pohožaev–Palais–Smale sequence.** We first prove that there is a sequence of almost critical points at the level b defined in (1.7) that satisfies asymptotically (1.5). We define the Pohožaev functional $\mathcal{P} : H^1(\mathbb{R}^N) \to \mathbb{R}$ for $u \in H^1(\mathbb{R}^N)$ by

$$\mathcal{P}(u) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2}\int_{\mathbb{R}^N} |u|^2 - \frac{N+\alpha}{2}\int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big)F(u).$$

Proposition 2.1 (Construction of a Pohožaev–Palais–Smale sequence)**.** *If $f \in C(\mathbb{R};\mathbb{R})$ satisfies (f_1) and (f_3), then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^N)$ such that, as $n \to \infty$,*

$$\begin{aligned} &\mathcal{I}(u_n) \to b > 0, \\ &\mathcal{I}'(u_n) \to 0 \qquad\qquad \textit{strongly in } \big(H^1(\mathbb{R}^N)\big)', \\ &\mathcal{P}(u_n) \to 0. \end{aligned}$$

Proof. Our strategy consists in first proving in claims 1 and 2 that the functional $\mathcal{I}$ has the mountain pass geometry before concluding by a minimax principle.

Claim 1. The critical level satisfies

$$b < \infty.$$

Proof of the claim. We need to show that the set of paths Γ is nonempty. In view of the definition of Γ, it is sufficient to construct $u \in H^1(\mathbb{R}^N)$ such that $\mathcal{I}(u) < 0$. If we choose s_0 of assumption (f_3) so that $F(s_0) \neq 0$ and set $w = s_0\chi_{B_1}$, we obtain

$$\int_{\mathbb{R}^N} \big(I_\alpha * F(w)\big)F(w) = F(s_0)^2 \int_{B_1}\int_{B_1} I_\alpha(x-y) > 0.$$

By (f_1) the left-hand side is continuous in $L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. Since $H^1(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, there exists $v \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \big(I_\alpha * F(v)\big)F(v) > 0.$$

We will take the function u in the family of functions $u_\tau \in H^1(\mathbb{R}^N)$ defined for $\tau > 0$ and $x \in \mathbb{R}^N$ by $u_\tau(x) = v\big(\frac{x}{\tau}\big)$. On this family, we compute for every $\tau > 0$,

$$\mathcal{I}(u_\tau) = \frac{\tau^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{\tau^N}{2}\int_{\mathbb{R}^N} |v|^2 - \frac{\tau^{N+\alpha}}{2}\int_{\mathbb{R}^N} \big(I_\alpha * F(v)\big)F(v),$$

and observe that for $\tau > 0$ large enough, $\mathcal{I}(u_\tau) < 0$. □

Claim 2. The critical level satisfies

$$b > 0.$$

Proof of the claim. Recall the Hardy–Littlewood–Sobolev inequality [23, theorem 4.3]: if $s \in (1, \frac{N}{\alpha})$, then for every $v \in L^s(\mathbb{R}^N)$, $I_\alpha * v \in L^{Ns/(N-\alpha s)}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |I_\alpha * v|^{\frac{Ns}{N-\alpha s}} \le C\Big(\int_{\mathbb{R}^N} |v|^s\Big)^{\frac{N}{N-\alpha s}}, \tag{2.1}$$

where $C > 0$ depends only on α, N and s. By the upper bound (f_1) on F, for every $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big)F(u) &\le C\Big(\int_{\mathbb{R}^N} |F(u)|^{\frac{2N}{N+\alpha}}\Big)^{1+\frac{\alpha}{N}} \\ &\le C'\Big(\int_{\mathbb{R}^N} |u|^2 + |u|^{\frac{2N}{N-2}}\Big)^{1+\frac{\alpha}{N}} \\ &\le C''\bigg(\Big(\int_{\mathbb{R}^N} |u|^2\Big)^{1+\frac{\alpha}{N}} + \Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^{1+\frac{\alpha+2}{N-2}}\bigg).\end{aligned}$$

Hence there exists $\delta > 0$ such that if $\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \le \delta$, then

$$\int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big)F(u) \le \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2,$$

and therefore

$$\mathcal{I}(u) \ge \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2.$$

In particular, if $\gamma \in \Gamma$, then $\int_{\mathbb{R}^N} |\nabla\gamma(0)|^2 + |\gamma(0)|^2 = 0 < \delta < \int_{\mathbb{R}^N} |\nabla\gamma(1)|^2 + |\gamma(1)|^2$ and by the intermediate value theorem there exists $\bar\tau \in (0,1)$ such that $\int_{\mathbb{R}^N} |\nabla\gamma(\bar\tau)|^2 + |\gamma(\bar\tau)|^2 = \delta$. At the point $\bar\tau$,

$$\frac{\delta}{4} \le \mathcal{I}\big(\gamma(\bar\tau)\big) \le \sup_{\tau\in[0,1]} \mathcal{I}(\gamma(\tau)).$$

Since $\gamma \in \Gamma$ is arbitrary, this implies that $b \ge \frac{\delta}{4} > 0$. □

Conclusion. Following L. Jeanjean [18, §2] (see also [17, §4]), we define the map $\Phi : \mathbb{R} \times H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ for $\sigma \in \mathbb{R}$, $v \in H^1(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ by

$$\Phi(\sigma, v)(x) = v(e^{-\sigma}x).$$

For every $\sigma \in \mathbb{R}$ and $v \in H^1(\mathbb{R}^N)$, the functional $\mathcal{I} \circ \Phi$ is computed as

$$\mathcal{I}\big(\Phi(\sigma, v)\big) = \frac{e^{(N-2)\sigma}}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{e^{N\sigma}}{2}\int_{\mathbb{R}^N} |v|^2 - \frac{e^{(N+\alpha)\sigma}}{2}\int_{\mathbb{R}^N} \big(I_\alpha * F(v)\big)F(v).$$

In view of (f_1), $\mathcal{I} \circ \Phi$ is continuously Fréchet–differentiable on $\mathbb{R} \times H^1(\mathbb{R}^N)$. We define the family of paths

$$\tilde\Gamma = \Big\{\tilde\gamma \in C\big([0,1]; \mathbb{R} \times H^1(\mathbb{R}^N)\big) \ : \ \tilde\gamma(0) = (0,0) \text{ and } (\mathcal{I} \circ \Phi)\big(\tilde\gamma(1)\big) < 0\Big\}.$$

As $\Gamma = \{\Phi \circ \tilde\gamma \ : \ \tilde\gamma \in \tilde\Gamma\}$, the mountain pass levels of $\mathcal{I}$ and $\mathcal{I} \circ \Phi$ coincide:

$$b = \inf_{\tilde\gamma\in\tilde\Gamma} \sup_{\tau\in[0,1]} (\mathcal{I} \circ \Phi)\big(\tilde\gamma(\tau)\big).$$

By the minimax principle [43, theorem 2.9], there exists a sequence $\big((\sigma_n, v_n)\big)_{n\in\mathbb{N}}$ in $\mathbb{R} \times H^1(\mathbb{R}^N)$ such that as $n \to \infty$,

$$\begin{aligned}&(\mathcal{I} \circ \Phi)(\sigma_n, v_n) \to b, \\ &(\mathcal{I} \circ \Phi)'(\sigma_n, v_n) \to 0 \qquad\qquad \text{in } \big(\mathbb{R} \times H^1(\mathbb{R}^N)\big)^*.\end{aligned}$$

Since for every $(h, w) \in \mathbb{R} \times H^1(\mathbb{R}^N)$,

$$(\mathcal{I} \circ \Phi)'(\sigma_n, v_n)[h, w] = \mathcal{I}'\big(\Phi(\sigma_n, v_n)\big)[\Phi(\sigma_n, w)] + \mathcal{P}\big(\Phi(\sigma_n, v_n)\big)h,$$

we reach the conclusion by taking $u_n = \Phi(\sigma_n, v_n)$. □

2.2. **Convergence of Pohožaev–Palais–Smale sequences.** We will now show how a solution of problem $(\mathcal{P})$ can be constructed from the sequence given by Proposition 2.1.

Proposition 2.2 (Convergence of Pohožaev–Palais–Smale sequences)**.** *Let $f \in C(\mathbb{R};\mathbb{R})$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\mathbb{R}^N)$. If f satisfies (f_1) and (f_2), $\big(\mathcal{I}(u_n)\big)_{n\in\mathbb{N}}$ is bounded and, as $n \to \infty$,*

$$\begin{aligned} &\mathcal{I}'(u_n) \to 0 && \text{strongly in } (H^1(\mathbb{R}^N))', \\ &\mathcal{P}(u_n) \to 0, \end{aligned}$$

then

- *either up to a subsequence $u_n \to 0$ strongly in $H^1(\mathbb{R}^N)$,*
- *or there exists $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\mathcal{I}'(u) = 0$ and a sequence $(a_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that up to a subsequence $u_n(\cdot - a_n) \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ as $n \to \infty$.*

Proof. Assume that the first part of the alternative does not hold, that is,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^n} |\nabla u_n|^2 + |u_n|^2 > 0. \tag{2.2}$$

We first establish in claim 1 the boundedness of the sequence and then the nonvanishing of the sequence in claim 2.

Claim 1. The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$.

Proof of claim 1. For every $n \in \mathbb{N}$,

$$\frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |u_n|^2 = \mathcal{I}(u_n) - \frac{1}{N+\alpha}\mathcal{P}(u_n).$$

As the right-hand side is bounded by our assumptions, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. □

Claim 2. For every $p \in (2, \frac{2N}{N-2})$,

$$\liminf_{n\to\infty} \sup_{a\in\mathbb{R}^N} \int_{B_1(a)} |u_n|^p > 0.$$

Proof of claim 2. First, by (2.2) and the definition of the Pohožaev functional $\mathcal{P}$ we have

$$\begin{aligned} \liminf_{n\to\infty} &\int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big) F(u_n) \\ &= \liminf_{n\to\infty} \frac{N-2}{N+\alpha} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{N+\alpha} \int_{\mathbb{R}^N} |u|^2 - \frac{2}{N+\alpha}\mathcal{P}(u_n) > 0. \end{aligned} \tag{2.3}$$

For every $n \in \mathbb{N}$, the function u_n satisfies the inequality ([26, lemma I.1], [43, lemma 1.21], [31, lemma 2.3])

$$\int_{\mathbb{R}^N} |u_n|^p \le C\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2\Big)\Big(\sup_{a\in\mathbb{R}^N} \int_{B_1(a)} |u_n|^p\Big)^{1-\frac{2}{p}}.$$

As F is continuous and satisfies (f_2), for every $\epsilon > 0$, there exists C_ϵ such that for every $s \in \mathbb{R}$,

$$|F(s)|^{\frac{2N}{N+\alpha}} \le \epsilon\big(|s|^2 + |s|^{\frac{2N}{N-2}}\big) + C_\epsilon |s|^p.$$

Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ and hence, by the Sobolev embedding, in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we have

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}|F(u_n)|^{\frac{2N}{N+\alpha}} \le C''\epsilon + C'_\epsilon\Big(\liminf_{n\to\infty}\sup_{a\in\mathbb{R}^N}\int_{B_1(a)}|u_n|^p\Big)^{1-\frac{2}{p}}.$$

Since $\epsilon > 0$ is arbitrary, if $\liminf_{n\to\infty}\sup_{a\in\mathbb{R}^N}\int_{B_1(a)}|u_n|^p = 0$, then

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}|F(u_n)|^{\frac{2N}{N+\alpha}} = 0,$$

and the Hardy–Littlewood–Sobolev inequality implies that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}\big(I_\alpha * F(u_n)\big)F(u_n) = 0,$$

in contradiction with (2.3). □

Conclusion. Up to a translation, we can now assume that for some $p \in (2, \frac{2N}{N-2})$,

$$\liminf_{n\to\infty}\int_{B_1}|u_n|^p > 0.$$

By Rellich's theorem, this implies that up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges weakly in $H^1(\mathbb{R}^N)$ to $u \in H^1(\mathbb{R}^N)\setminus\{0\}$.

As the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, by the Sobolev embedding, it is also bounded in $L^2(\mathbb{R}^N)\cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. By (f_1), the sequence $(F\circ u_n)_{n\in\mathbb{N}}$ is therefore bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Since the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly to u in $H^1(\mathbb{R}^N)$, it converges up to a subsequence to u almost everywhere in $\mathbb{R}^N$. By continuity of F, $(F\circ u_n)_{n\in\mathbb{N}}$ converges almost everywhere to $F\circ u$ in $\mathbb{R}^N$. This implies that the sequence $(F\circ u_n)_{n\in\mathbb{N}}$ converges weakly to $F\circ u$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. As the Riesz potential defines a linear continuous map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, the sequence $(I_\alpha * (F\circ u_n))_{n\in\mathbb{N}}$ converges weakly to $I_\alpha * (F\circ u)$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$.

On the other hand, in view of (f_1) and by Rellich's theorem, the sequence $(f\circ u_n)_{n\in\mathbb{N}}$ converges strongly to $f\circ u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for every $p\in[1,\frac{2N}{\alpha+2})$. We conclude that

$$\big(I_\alpha * (F\circ u_n)\big)(f\circ u_n) \rightharpoonup \big(I_\alpha * (F\circ u)\big)(f\circ u) \qquad \text{weakly in } L^p(\mathbb{R}^N),$$

for every $p\in[1,\frac{2N}{N+2})$. This implies in particular that for every $\varphi\in C^1_c(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N}\nabla u\cdot\nabla\varphi + u\varphi - \int_{\mathbb{R}^N}\big(I_\alpha * (F\circ u)\big)\big((f\circ u)\varphi\big)\\ = \lim_{n\to\infty}\int_{\mathbb{R}^N}\nabla u\cdot\nabla\varphi + u\varphi - \int_{\mathbb{R}^N}\big(I_\alpha * (F\circ u_n)\big)(f\circ u_n)\varphi = 0;\end{aligned}$$

that is, u is a weak solution of $(\mathcal{P})$. □

We point out that the assumption (f_2) is only used in the proof of claim 2. Note also that without the additional assumptions of Theorem 4 we cannot rely on the Strauss radial compactness lemma [6, theorem A1] which is equivalent to the compactness of the embedding of the Sobolev subspace of radial functions $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for $2<p<2N/(N-2)$. Instead, our proof of convergence of Pohožaev–Palais–Smale sequences uses a direct concentration–compactness type argument of

Proposition 2.2. Such an approach could be useful for the study of other problems where radial symmetry of solutions either fails or is not readily available.

Observe that in the limit $\alpha \to 0$, the assumptions (f_1), (f_2) and (f_3) do not allow us to recover exactly (g_1), (g_2) and (g_3). The gap between (f_1) when $\alpha \to 0$ and (g_1) is purely technical. When $\alpha \to 0$, (f_2) gives the assumptions $\lim_{s\to 0} F(s)^2/|s|^2 = 0$ and $\lim_{|s|\to\infty} F(s)^2/|s|^{2N/(N-2)} = 0$, which is stronger than (g_2). The first assumption is not really surprising, as it can be observed that in (1.2) both $g(u)$ and u have the same spatial homogeneity and therefore by scaling it could always be assumed that $\lim_{s\to 0} G(s)/s^2 = 0$. The second assumption is equivalent to $\limsup_{|s|\to\infty} F(s)^2/|s|^{2N/(N-2)} \le 0$. Finally (f_3) gives $G(s) = F(s)^2 \ge 0$, which is actually weaker than (g_3). This weakening of the condition can also be explained by the difference between the various scalings of the problem $(\mathcal{P})$.

3. Regularity of solutions and Pohožaev identity

The assumption (f_1) is too weak for the standard bootstrap method as in [12, lemma A.1], [31, proposition 4.1]. Instead, in order to prove regularity of solutions of $(\mathcal{P})$ we shall rely on a nonlocal version of the Brezis–Kato estimate.

3.1. A nonlocal Brezis–Kato type regularity estimate.

A special case of the regularity result of Brezis and Kato [8, theorem 2.3] states that if $u \in H^1(\mathbb{R}^N)$ is a solution of the linear elliptic equation

$$-\Delta u + u = Vu \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

and $V \in L^\infty(\mathbb{R}^N) + L^{\frac{N}{2}}(\mathbb{R}^N)$, then $u \in L^p(\mathbb{R}^N)$ for every $p \ge 1$. We extend this result to a class of nonlocal linear equations.

Proposition 3.1 (Improved integrability of solution of a nonlocal critical linear equation)**.** *If $H, K \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$ and $u \in H^1(\mathbb{R}^N)$ solves*

$$-\Delta u + u = (I_\alpha * Hu)K, \tag{3.2}$$

then $u \in L^p(\mathbb{R}^N)$ for every $p \in [2, \frac{N}{\alpha}\frac{2N}{N-2})$. Moreover, there exists a constant C_p independent of u such that

$$\Big(\int_{\mathbb{R}^N} |u|^p\Big)^{\frac{1}{p}} \le C_p\Big(\int_{\mathbb{R}^N} |u|^2\Big)^{\frac{1}{2}}.$$

Note that the space $L^{2N/(\alpha+2)}(\mathbb{R}^N)$ is *critical* in this statement: starting from the information that $u \in H^1(\mathbb{R}^N) \subset L^{2N/(N-2)}(\mathbb{R}^N)$, a standard Hardy–Littlewood–Sobolev estimate would just show that $u \in L^{2N/(N-2)}(\mathbb{R}^N)$ and would thus give no additional regularity information. Instead, our proof of Proposition 3.1 follows the strategy of Brezis and Kato (see also Trudinger [39, Theorem 3]). The adaptation of the argument is complicated by the nonlocal effect of u on the right-hand side.

Our main new tool for the proof of Proposition 3.1 is the following lemma, which is a nonlocal counterpart of the estimate [8, lemma 2.1]: if $V \in L^\infty(\mathbb{R}^N) + L^{\frac{N}{2}}(\mathbb{R}^N)$, then for every $\epsilon > 0$, there exists C_ϵ such that

$$\int_{\mathbb{R}^N} V|u|^2 \le \epsilon^2 \int_{\mathbb{R}^N} |\nabla u|^2 + C_\epsilon \int_{\mathbb{R}^N} |u|^2. \tag{3.3}$$

Lemma 3.2. *Let $N \geq 2$, $\alpha \in (0,2)$ and $\theta \in (0,2)$. If $H, K \in L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ and $\frac{\alpha}{N} < \theta < 2 - \frac{\alpha}{N}$, then for every $\epsilon > 0$, there exists $C_{\epsilon,\theta} \in \mathbb{R}$ such that for every $u \in H^1(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \big(I_\alpha * (H|u|^\theta)\big) K|u|^{2-\theta} \leq \epsilon^2 \int_{\mathbb{R}^N} |\nabla u|^2 + C_{\epsilon,\theta} \int_{\mathbb{R}^N} |u|^2.$$

In the limit $\alpha = 0$, this result is consistent with (3.3); the parameter θ only plays a role in the nonlocal case.

In order to prove Lemma 3.2, we shall use several times the following inequality.

Lemma 3.3. *Let $q, r, s, t \in [1, \infty)$ and $\lambda \in [0,2]$ such that*

$$1 + \frac{\alpha}{N} - \frac{1}{s} - \frac{1}{t} = \frac{\lambda}{q} + \frac{2-\lambda}{r}.$$

If $\theta \in (0,2)$ satisfies

$$\min(q,r)\Big(\frac{\alpha}{N} - \frac{1}{s}\Big) < \theta < \max(q,r)\Big(1 - \frac{1}{s}\Big),$$
$$\min(q,r)\Big(\frac{\alpha}{N} - \frac{1}{t}\Big) < 2 - \theta < \max(q,r)\Big(1 - \frac{1}{t}\Big),$$

then for every $H \in L^s(\mathbb{R}^N)$, $K \in L^t(\mathbb{R}^N)$ and $u \in L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * \big(H|u|^\theta\big)) K|u|^{2-\theta} \leq C\Big(\int_{\mathbb{R}^N} |H|^s\Big)^{\frac{1}{s}} \Big(\int_{\mathbb{R}^N} |K|^t\Big)^{\frac{1}{t}} \Big(\int_{\mathbb{R}^N} |u|^q\Big)^{\frac{\lambda}{q}} \Big(\int_{\mathbb{R}^N} |u|^r\Big)^{\frac{2-\lambda}{r}}.$$

Proof. First observe that if $\tilde{s} > 1$, $\tilde{t} > 1$ satisfy $\frac{1}{\tilde{t}} + \frac{1}{\tilde{s}} = 1 + \frac{\alpha}{N}$, the Hardy–Littlewood–Sobolev inequality is applicable and

$$\int_{\mathbb{R}^N} \big(I_\alpha * (H|u|^\theta)\big) K|u|^{2-\theta} \leq C\Big(\int_{\mathbb{R}^N} \big|Hu^\theta\big|^{\tilde{s}}\Big)^{\frac{1}{\tilde{s}}} \Big(\int_{\mathbb{R}^N} \big|Ku^{2-\theta}\big|^{\tilde{t}}\Big)^{\frac{1}{\tilde{t}}}.$$

Let $\mu \in \mathbb{R}$. Note that if

$$0 \leq \mu \leq \theta \qquad \text{and} \qquad \frac{1}{\tilde{s}} := \frac{\mu}{q} + \frac{\theta - \mu}{r} + \frac{1}{s} < 1, \tag{3.4}$$

then by Hölder's inequality

$$\Big(\int_{\mathbb{R}^N} \big|Hu^\theta\big|^{\tilde{s}}\Big)^{\frac{1}{\tilde{s}}} \leq \Big(\int_{\mathbb{R}^N} |H|^s\Big)^{\frac{1}{s}} \Big(\int_{\mathbb{R}^N} |u|^q\Big)^{\frac{\mu}{q}} \Big(\int_{\mathbb{R}^N} |u|^r\Big)^{\frac{\theta-\mu}{r}}.$$

Similarly, if

$$\lambda - (2-\theta) \leq \mu \leq \lambda \qquad \text{and} \qquad \frac{1}{\tilde{t}} := \frac{\lambda - \mu}{q} + \frac{(2-\theta) - (\lambda - \mu)}{r} + \frac{1}{t} < 1, \tag{3.5}$$

then

$$\Big(\int_{\mathbb{R}^N} \big|Ku^{2-\theta}\big|^{\tilde{t}}\Big)^{\frac{1}{\tilde{t}}} \leq \Big(\int_{\mathbb{R}^N} |K|^t\Big)^{\frac{1}{t}} \Big(\int_{\mathbb{R}^N} |u|^q\Big)^{\frac{\lambda-\mu}{q}} \Big(\int_{\mathbb{R}^N} |u|^r\Big)^{\frac{2-\theta-(\lambda-\mu)}{r}}.$$

It can be checked that (3.4) and (3.5) can be satisfied for some $\mu \in \mathbb{R}$ if and only if the assumptions of the lemma hold. In particular, $\frac{1}{\tilde{t}} + \frac{1}{\tilde{s}} = \frac{1}{s} + \frac{1}{t} = \frac{\lambda}{q} + \frac{2-\lambda}{r} = 1 + \frac{\alpha}{N}$, so that we can conclude. $\square$

Proof of Lemma 3.2. Let $H = H^* + H_*$ and $K = K^* + K_*$ with $H^*, K^* \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ and $H_*, K_* \in L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$. Applying Lemma 3.3, with $q = r = \frac{2N}{N-2}$, $s = t = \frac{2N}{\alpha+2}$ and $\lambda = 0$, we have since $|\theta - 1| < \frac{N-\alpha}{N-2}$,

$$\int_{\mathbb{R}^N} \big(I_\alpha * (H_*|u|^\theta)\big)(K_*|u|^{2-\theta}) \leq C\Big(\int_{\mathbb{R}^N} |H_*|^{\frac{2N}{\alpha+2}}\Big)^{\frac{\alpha+2}{2N}} \Big(\int_{\mathbb{R}^N} |K_*|^{\frac{2N}{\alpha+2}}\Big)^{\frac{\alpha+2}{2N}} \Big(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\Big)^{1-\frac{2}{N}}.$$

Taking now $s = t = \frac{2N}{\alpha}$, $q = r = 2$ and $\lambda = 2$, we have since $|\theta - 1| < \frac{N-\alpha}{N}$,

$$\int_{\mathbb{R}^N} \big(I_\alpha * (H^*|u|^\theta)\big)(K^*|u|^{2-\theta}) \leq C\Big(\int_{\mathbb{R}^N} |H^*|^{\frac{2N}{\alpha}}\Big)^{\frac{\alpha}{2N}} \Big(\int_{\mathbb{R}^N} |K^*|^{\frac{2N}{\alpha}}\Big)^{\frac{\alpha}{2N}} \int_{\mathbb{R}^N} |u|^2.$$

Similarly, with $s = \frac{2N}{\alpha+2}$, $t = \frac{2N}{\alpha}$, $q = 2$, $r = \frac{2N}{N-2}$ and $\lambda = 1$,

$$\int_{\mathbb{R}^N} \big(I_\alpha * (H_*|u|^\theta)\big)(K^*|u|^{2-\theta}) \leq C\Big(\int_{\mathbb{R}^N} |H_*|^{\frac{2N}{\alpha+2}}\Big)^{\frac{\alpha+2}{2N}} \Big(\int_{\mathbb{R}^N} |K^*|^{\frac{2N}{\alpha}}\Big)^{\frac{\alpha}{2N}} \Big(\int_{\mathbb{R}^N} |u|^2\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\Big)^{\frac{1}{2}-\frac{1}{N}}$$

and with $s = \frac{2N}{\alpha}$, $t = \frac{2N}{\alpha+2}$, $q = 2$, $r = \frac{2N}{N-2}$ and $\lambda = 1$,

$$\int_{\mathbb{R}^N} \big(I_\alpha * (H^*|u|^\theta)\big)(K_*|u|^{2-\theta}) \leq C\Big(\int_{\mathbb{R}^N} |H^*|^{\frac{2N}{\alpha}}\Big)^{\frac{\alpha}{2N}} \Big(\int_{\mathbb{R}^N} |K_*|^{\frac{2N}{\alpha+2}}\Big)^{\frac{\alpha+2}{2N}} \Big(\int_{\mathbb{R}^N} |u|^2\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\Big)^{\frac{1}{2}-\frac{1}{N}}.$$

By the Sobolev inequality, we have thus proved that for every $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} \big(I_\alpha * (H|u|^\theta)\big)(K|u|^{2-\theta}) &\leq C\bigg(\Big(\int_{\mathbb{R}^N} |H_*|^{\frac{2N}{\alpha+2}} \int_{\mathbb{R}^N} |K_*|^{\frac{2N}{\alpha+2}}\Big)^{\frac{\alpha+2}{2N}} \int_{\mathbb{R}^N} |\nabla u|^2 \\ &\qquad + \Big(\int_{\mathbb{R}^N} |H^*|^{\frac{2N}{\alpha}} \int_{\mathbb{R}^N} |K^*|^{\frac{2N}{\alpha}}\Big)^{\frac{\alpha}{2N}} \int_{\mathbb{R}^N} |u|^2\bigg).\end{aligned}$$

The conclusion follows by choosing H^* and K^* such that

$$C\Big(\int_{\mathbb{R}^N} |H_*|^{\frac{2N}{\alpha+2}} \int_{\mathbb{R}^N} |K_*|^{\frac{2N}{\alpha+2}}\Big)^{\frac{\alpha+2}{2N}} \leq \epsilon^2. \qquad \square$$

Proof of Proposition 3.1. By Lemma 3.2 with $\theta = 1$, there exists $\lambda > 0$ such that for every $\varphi \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^n} \big(I_\alpha * |H\varphi|\big)|K\varphi| \leq \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\varphi|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^N} |\varphi|^2.$$

Choose sequences $(H_k)_{k\in\mathbb{N}}$ and $(K_k)_{k\in\mathbb{N}}$ in $L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ such that $|H_k| \leq |H|$, $|K_k| \leq |K|$, and $H_k \to H$ and $K_k \to K$ almost everywhere in $\mathbb{R}^N$. For each $k \in \mathbb{N}$, the form $a_k : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ defined for $\varphi \in H^1(\mathbb{R}^N)$ and $\psi \in H^1(\mathbb{R}^N)$ by

$$a_k(\varphi, \psi) = \int_{\mathbb{R}^N} \nabla\varphi \cdot \nabla\psi + \lambda\varphi\psi - \int_{\mathbb{R}^N} (I_\alpha * H_k\varphi)K_k\psi$$

is bilinear and coercive; by the Lax–Milgram theorem [7, corollary 5.8], there exists a unique solution $u_k \in H^1(\mathbb{R}^N)$ of

$$-\Delta u_k + \lambda u_k = \big(I_\alpha * (H_k u_k)\big) K_k + (\lambda - 1)u, \tag{3.6}$$

where $u \in H^1(\mathbb{R}^N)$ is the given solution of (3.2). It can be proved that the sequence $(u_k)_{k\in\mathbb{N}}$ converges weakly to u in $H^1(\mathbb{R}^N)$ as $k \to \infty$.

For $\mu > 0$, we define the truncation $u_{k,\mu} : \mathbb{R}^N \to \mathbb{R}$ for $x \in \mathbb{R}^N$ by

$$u_{k,\mu}(x) = \begin{cases} -\mu & \text{if } u_k(x) \le -\mu, \\ u_k(x) & \text{if } -\mu < u_k(x) < \mu, \\ \mu & \text{if } u_k(x) \ge \mu. \end{cases}$$

Since $|u_{k,\mu}|^{p-2} u_{k,\mu} \in H^1(\mathbb{R}^N)$, we can take it as a test function in (3.6):

$$\begin{aligned} \int_{\mathbb{R}^N} \tfrac{4(p-1)}{p^2} \big|\nabla (u_{k,\mu})^{\frac{p}{2}}\big|^2 + \big||u_{k,\mu}|^{\frac{p}{2}}\big|^2 \\ &\le \int_{\mathbb{R}^N} (p-1)|u_{k,\mu}|^{p-2}\big|\nabla u_{k,\mu}\big|^2 + |u_{k,\mu}|^{p-2} u_{k,\mu} u_k \\ &= \int_{\mathbb{R}^N} \big(I_\alpha * (H_k u_k)\big)\big(K_k |u_{k,\mu}|^{p-2} u_{k,\mu}\big) + (\lambda - 1) u |u_{k,\mu}|^{p-2} u_{k,\mu}. \end{aligned}$$

If $p < \frac{2N}{\alpha}$, by Lemma 3.2 with $\theta = \frac{2}{p}$, there exists $C > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} \big(I_\alpha * |H_k u_{k,\mu}|\big)\big(|K_k||u_{k,\mu}|^{p-2} u_{k,\mu}\big) \\ &\le \int_{\mathbb{R}^N} \big(I_\alpha * (|H||u_{k,\mu}|)\big)\big(|K||u_{k,\mu}|^{p-1}\big) \\ &\le \tfrac{2(p-1)}{p^2} \int_{\mathbb{R}^N} \big|\nabla (u_{k,\mu})^{\frac{p}{2}}\big|^2 + C \int_{\mathbb{R}^N} \big||u_{k,\mu}|^{\frac{p}{2}}\big|^2. \end{aligned}$$

We have thus

$$\tfrac{2(p-1)}{p^2} \int_{\mathbb{R}^N} \big|\nabla (u_{k,\mu})^{\frac{p}{2}}\big|^2 \le C' \int_{\mathbb{R}^N} \big(|u_k|^p + |u|^p\big) + \int_{A_{k,\mu}} \big(I_\alpha * (|K_k||u_k|^{p-1})\big)|H_k u_k|,$$

where

$$A_{k,\mu} = \big\{x \in \mathbb{R}^N : |u_k(x)| > \mu\big\}.$$

Since $p < \frac{2N}{\alpha}$, by the Hardy–Littlewood–Sobolev inequality,

$$\int_{A_{k,\mu}} \big(I_\alpha * (|K_k||u_k|^{p-1})\big)|H_k u_k| \le C\Big(\int_{\mathbb{R}^N} \big||K_k||u_k|^{p-1}\big|^r\Big)^{\frac{1}{r}} \Big(\int_{A_{k,\mu}} |H_k u_k|^s\Big)^{\frac{1}{s}},$$

with $\frac{1}{r} = \frac{\alpha}{2N} + 1 - \frac{1}{p}$ and $\frac{1}{s} = \frac{\alpha}{2N} + \frac{1}{p}$. By Hölder's inequality, if $u_k \in L^p(\mathbb{R}^N)$, then $|K_k||u_k|^{p-1} \in L^r(\mathbb{R}^N)$ and $|H_k u_k| \in L^s(\mathbb{R}^N)$, whence by Lebesgue's dominated convergence theorem

$$\lim_{\mu\to\infty} \int_{A_{k,\mu}} \big(I_\alpha * (|K_k||u_k|^{p-1})\big)|H_k u_k| = 0.$$

In view of the Sobolev estimate, we have proved the inequality

$$\limsup_{k\to\infty} \Big(\int_{\mathbb{R}^N} |u_k|^{\frac{pN}{N-2}}\Big)^{1-\frac{2}{N}} \le C'' \limsup_{k\to\infty} \int_{\mathbb{R}^N} |u_k|^p.$$

By iterating over p a finite number of times we cover the range $p \in [2, \frac{N}{\alpha}\frac{2N}{N-2})$. $\square$

Remark 3.1. A close inspection of the proofs of Lemma 3.2 and of Proposition 3.1 gives a more precise dependence of the constant C_p. Given a function $M : (0,\infty) \to (0,\infty)$ and $p \in (2, \frac{N}{\alpha}\frac{2N}{N-2})$, there exists $C_{p,M}$ such that if for every $\epsilon > 0$, K and H can be decomposed as $K = K_* + K^*$ and $H = H_* + H^*$ with

$$\Big(\int_{\mathbb{R}^N} |K_*|^{\frac{2N}{\alpha+2}}\Big)^{\frac{2N}{\alpha+2}} \le \epsilon \qquad \text{and} \qquad \Big(\int_{\mathbb{R}^N} |K^*|^{\frac{2N}{\alpha}}\Big)^{\frac{2N}{\alpha}} \le M(\epsilon),$$
$$\Big(\int_{\mathbb{R}^N} |H_*|^{\frac{2N}{\alpha+2}}\Big)^{\frac{2N}{\alpha+2}} \le \epsilon \qquad \text{and} \qquad \Big(\int_{\mathbb{R}^N} |H^*|^{\frac{2N}{\alpha}}\Big)^{\frac{2N}{\alpha}} \le M(\epsilon),$$

and if $u \in H^1(\mathbb{R}^N)$ satisfies

$$-\Delta u + u = (I_\alpha * Hu)K,$$

then one has

$$\Big(\int_{\mathbb{R}^N} |u|^p\Big)^{\frac{1}{p}} \le C_{p,M}\Big(\int_{\mathbb{R}^N} |u|^2\Big)^{\frac{1}{2}}.$$

3.2. **Regularity of solutions.** Now we are in a position to establish additional regularity of solutions of the nonlinear nonlocal problem $(\mathcal{P})$.

Proof of Theorem 2. Define $H : \mathbb{R}^N \to \mathbb{R}$ and $K : \mathbb{R}^N \to \mathbb{R}$ for $x \in \mathbb{R}^N$ by $H(x) = F\big(u(x)\big)/u(x)$ and $K(x) = f\big(u(x)\big)$. Observe that for every $x \in \mathbb{R}^N$,

$$|K(x)| \le C\big(|u(x)|^{\frac{\alpha}{N}} + |u(x)|^{\frac{\alpha+2}{N-2}}\big)$$

and

$$|H(x)| \le C\big(\tfrac{N}{N+\alpha}|u(x)|^{\frac{\alpha}{N}} + \tfrac{N-2}{N+\alpha}|u(x)|^{\frac{\alpha+2}{N-2}}\big),$$

so that $K, H \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$. By Proposition 3.1, $u \in L^p(\mathbb{R}^N)$ for every $p \in [2, \frac{N}{\alpha}\frac{2N}{N-2})$. In view of (f_1), $F \circ u \in L^q(\mathbb{R}^N)$ for every $q \in [\frac{2N}{N+\alpha}, \frac{N}{\alpha}\frac{2N}{N+\alpha})$. Since $\frac{2N}{N+\alpha} < \frac{N}{\alpha} < \frac{N}{\alpha}\frac{2N}{N+\alpha}$, we have $I_\alpha * (F \circ u) \in L^\infty(\mathbb{R}^N)$, and thus

$$|-\Delta u + u| \le C\big(|u|^{\frac{\alpha}{N}} + |u|^{\frac{\alpha+2}{N-2}}\big).$$

By the classical bootstrap method for subcritical local problems in bounded domains, we deduce that $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $p \ge 1$. □

3.3. **Pohožaev identity.** The proof of Pohožaev identity (1.5) is a generalization of the argument for $f(s) = s^p$ [31] (see also particular cases [29], [13, lemma 2.1]). The strategy is classical and consists in testing the equation against a suitable cut-off of $x \cdot \nabla u(x)$ and integrating by parts ([21, proposition 6.2.1], [43, appendix B]).

Proof of Theorem 3. By Theorem 2, $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$. Fix $\varphi \in C^1_c(\mathbb{R}^N)$ such that $\varphi = 1$ in a neighbourhood of 0. The function $v_\lambda \in W^{1,2}(\mathbb{R}^N)$ defined for $\lambda \in (0,\infty)$ and $x \in \mathbb{R}^N$ by

$$v_\lambda(x) = \varphi(\lambda x)\, x \cdot \nabla u(x)$$

can be used as a test function in the equation to obtain

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda + \int_{\mathbb{R}^N} u v_\lambda = \int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big)(f(u)v_\lambda).$$

The left-hand side can be computed by integration by parts for every $\lambda > 0$ as

$$\begin{aligned}\int_{\mathbb{R}^N} uv_\lambda &= \int_{\mathbb{R}^N} u(x)\varphi(\lambda x)\, x\cdot\nabla u(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N} \varphi(\lambda x)\, x\cdot\nabla\big(\tfrac{|u|^2}{2}\big)(x)\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^N} \big(N\varphi(\lambda x)+\lambda x\cdot\nabla\varphi(\lambda x)\big)\frac{|u(x)|^2}{2}\,\mathrm{d}x.\end{aligned}$$

Lebesgue's dominated convergence theorem implies that

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N} uv_\lambda = -\frac{N}{2}\int_{\mathbb{R}^N}|u|^2.$$

Similarly, as $u\in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$, the gradient term can be written as

$$\begin{aligned}\int_{\mathbb{R}^N}\nabla u\cdot\nabla v_\lambda &= \int_{\mathbb{R}^N}\varphi(\lambda x)\big(|\nabla u|^2 + x\cdot\nabla\big(\tfrac{|\nabla u|^2}{2}\big)(x)\big)\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^N}\big((N-2)\varphi(\lambda x)+\lambda x\cdot\nabla\varphi(\lambda x)\big)\frac{|\nabla u(x)|^2}{2}\,\mathrm{d}x.\end{aligned}$$

Lebesgue's dominated convergence again is applicable since $\nabla u\in L^2(\mathbb{R}^N)$ and we obtain

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\nabla u\cdot\nabla v_\lambda = -\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2.$$

The last term can be rewritten by integration by parts for every $\lambda > 0$ as

$$\begin{aligned}\int_{\mathbb{R}^N}\big(I_\alpha * F(u)\big)(f(u)v_\lambda) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(F\circ u)(y)I_\alpha(x-y)\varphi(\lambda x)\, x\cdot\nabla(F\circ u)(x)\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y)\Big((F\circ u)(y)\varphi(\lambda x)\, x\cdot\nabla(F\circ u)(x)\\ &\qquad\qquad + (F\circ u)(x)\varphi(\lambda y)\, y\cdot\nabla(F\circ u)(y)\Big)\,\mathrm{d}x\,\mathrm{d}y\\ &= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F\big(u(y)\big)I_\alpha(x-y)\big(N\varphi(\lambda x)+x\cdot\nabla\varphi(\lambda x)\big)F\big(u(x)\big)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + \frac{N-\alpha}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F\big(u(y)\big)I_\alpha(x-y)\\ &\qquad\qquad \frac{(x-y)\cdot\big(x\varphi(\lambda x)-y\varphi(\lambda y)\big)}{|x-y|^2}F\big(u(x)\big)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We can thus apply Lebesgue's dominated convergence theorem to conclude that

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\big(I_\alpha * F(u)\big)f(u)\, v_\lambda = -\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u). \qquad \square$$

4. From solutions to groundstates

4.1. **Solutions and paths.** One of the applications of the Pohožaev identity (1.5) is the possibility to associate to any variational solution of $(\mathcal{P})$ a path, following an argument of L. Jeanjean and K. Tanaka [20].

Proposition 4.1 (Lifting a solution to a path)**.** *If $f \in C(\mathbb{R};\mathbb{R})$ satisfies (f_1) and $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ solves $(\mathcal{P})$, then there exists a path $\gamma \in C\big([0,1]; H^1(\mathbb{R}^N)\big)$ such that*

$$\begin{aligned} \gamma(0) &= 0, \\ \gamma(1/2) &= u, \\ \mathcal{I}\big(\gamma(t)\big) &< \mathcal{I}(u), && \text{for every } t \in [0,1] \setminus \{1/2\}, \\ \mathcal{I}\big(\gamma(1)\big) &< 0. \end{aligned}$$

Proof. The proof follows closely the arguments for the local problem developed by L. Jeanjean and K. Tanaka [20, lemma 2.1]. We define the path $\tilde{\gamma} : [0,\infty) \to H^1(\mathbb{R}^N)$ by

$$\tilde{\gamma}(\tau)(x) = \begin{cases} u(x/\tau) & \text{if } \tau > 0, \\ 0 & \text{if } \tau = 0. \end{cases}$$

The function $\tilde{\gamma}$ is continuous on $(0,\infty)$; for every $\tau > 0$,

$$\int_{\mathbb{R}^N} |\nabla \tilde{\gamma}(\tau)|^2 + |\tilde{\gamma}(\tau)|^2 = \tau^{N-2} \int_{\mathbb{R}^N} |\nabla u|^2 + \tau^N \int_{\mathbb{R}^N} |u|^2,$$

so that $\tilde{\gamma}$ is continuous at 0. By the Pohožaev identity of Theorem 3, the functional can be computed for every $\tau > 0$ as

$$\begin{aligned} \mathcal{I}\big(\tilde{\gamma}(\tau)\big) &= \frac{\tau^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\tau^N}{2} \int_{\mathbb{R}^N} |u|^2 - \frac{\tau^{N+\alpha}}{2} \int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big) F(u) \\ &= \Big(\frac{\tau^{N-2}}{2} - \frac{(N-2)\tau^{N+\alpha}}{2(N+\alpha)}\Big) \int_{\mathbb{R}^N} |\nabla u|^2 + \Big(\frac{\tau^N}{2} - \frac{N\tau^{N+\alpha}}{2(N+\alpha)}\Big) \int_{\mathbb{R}^N} |u|^2. \end{aligned}$$

It can be checked directly that $\mathcal{I} \circ \tilde{\gamma}$ achieves strict global maximum at 1: for every $\tau \in [0,\infty) \setminus \{1\}$, $\mathcal{I}\big(\tilde{\gamma}(\tau)\big) < \mathcal{I}(u)$. Since

$$\lim_{\tau \to \infty} \mathcal{I}\big(\tilde{\gamma}(\tau)\big) = -\infty,$$

the path γ can then be defined by a suitable change of variable. □

4.2. **Minimality of the energy and existence of a groundstate.** We now have all the tools available to show that the mountain-pass critical level b defined in (1.7) coincides with the groundstate energy level c defined in (1.4), which completes the proof of Theorem 1.

Proof of Theorem 1. By Propositions 2.1 and 2.2, there exists a Pohožaev–Palais–Smale sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^N)$ at the mountain-pass level $b > 0$, that converges weakly to some $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ that solves $(\mathcal{P})$. Since $\lim_{n\to\infty} \mathcal{P}(u_n) = 0$, by the weak convergence of the sequence $(u_n)_{n\in\mathbb{N}}$, the weak lower-semicontinuity of the norm and the Pohožaev identity of Theorem 3,

$$\begin{aligned} \mathcal{I}(u) &= \mathcal{I}(u) - \frac{\mathcal{P}(u)}{N+\alpha} \\ &= \frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |u|^2 \\ &\le \liminf_{n\to\infty} \frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |u_n|^2 \\ &= \liminf_{n\to\infty} \mathcal{I}(u_n) - \frac{\mathcal{P}(u_n)}{N+\alpha} = \liminf_{n\to\infty} \mathcal{I}(u_n) = b. \end{aligned} \tag{4.1}$$

Since u is a nontrivial solution of $(\mathcal{P})$, we have $\mathcal{I}(u) \geq c$ by definition of the groundstate energy level c, and hence $c \leq b$.

Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ be another solution of $(\mathcal{P})$ such that $\mathcal{I}(v) \leq \mathcal{I}(u)$. If we lift v to a path (Proposition 4.1) and recall the definition (1.7) of the mountain-pass level b, we conclude that $\mathcal{I}(v) \geq b \geq \mathcal{I}(u)$. We have thus proved that $\mathcal{I}(v) = \mathcal{I}(u) = b = c$, and this concludes the proof of Theorem 1. □

4.3. **Compactness of the set of groundstates.** As a byproduct of the proof of Theorem 1, the weak convergence of the translated subsequence of Proposition 2.2 can be upgraded into strong convergence.

Corollary 4.2 (Strong convergence of translated Pohožaev–Palais–Smale sequences)**.** *Under the assumptions of Proposition* 2.2*, if*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 > 0,$$

and if

$$\liminf_{n\to\infty} \mathcal{I}(u_n) \leq c,$$

then there exists $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\mathcal{I}'(u) = 0$ and a sequence $(a_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that up to a subsequence $u_n(\cdot - a_n) \to u$ strongly in $H^1(\mathbb{R}^N)$ as $n \to \infty$.

Proof. By Proposition 2.2, up to a subsequence and translations, we can assume that the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly to u. Since equality holds in (4.1),

$$\begin{aligned}\frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |u|^2 \\ = \liminf_{n\to\infty} \frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |u_n|^2,\end{aligned}$$

and hence $(u_n)_{n\in\mathbb{N}}$ converges strongly to u in $H^1(\mathbb{R}^N)$. □

As a direct consequence we have some information on the set of groundstates:

Proposition 4.3 (Compactness of the set of groundstates)**.** *The set of groundstates*

$$\mathcal{S}_c = \left\{u \in H^1(\mathbb{R}^N) \,:\, \mathcal{I}(u) = c \text{ and } u \text{ is a weak solution of } (\mathcal{P})\right\}$$

is compact in $H^1(\mathbb{R}^N)$ endowed with the strong topology up to translations in $\mathbb{R}^N$.

Proof. This is a direct consequence of Theorem 3 and Corollary 4.2. □

Remark 4.1 (Uniform regularity of groundstates). By the uniform regularity of solutions (Remark 3.1) and the compactness of the set of groundstates (Proposition 4.3), for every $p \in [2, \frac{N}{\alpha}\frac{2N}{N-2})$, $\mathcal{S}_c$ is bounded in $L^p(\mathbb{R}^N)$.

5. Qualitative properties of groundstates

5.1. **Paths achieving the mountain-pass level.** Arguments in this section will use the following elementary property of the paths in the construction of the mountain-pass critical level b.

Lemma 5.1 (Optimal paths yield critical points)**.** *Let $f \in C(\mathbb{R};\mathbb{R})$ satisfy (f_1) and $\gamma \in \Gamma$, where Γ is defined in* (1.8)*. If for every $t \in [0,1] \setminus \{t_*\}$, one has*

$$b = \mathcal{I}\big(\gamma(t_*)\big) > \mathcal{I}\big(\gamma(t)\big),$$

then $\mathcal{I}'\big(\gamma(t_*)\big) = 0$.

Proof. This can be deduced from the quantitative deformation lemma of M. Willem (see [43, lemma 2.3]). Assume that $\mathcal{I}'(\gamma(t_*)) \neq 0$. By continuity, it is possible to choose $\delta > 0$ and $\epsilon > 0$ such that $\inf\{\|\mathcal{I}'(v)\| \ : \ \|v - \gamma(t_*)\| \leq \delta\} > 8\epsilon/\delta$. With Willem's notation, take $X = H^1(\mathbb{R}^N)$, $S = \{\gamma(t_*)\}$ and $c = b$. By the deformation lemma, there exists $\eta \in C([0,1]; H^1(\mathbb{R}^N))$ such that $\eta(1,\gamma) \in \Gamma$ and $\mathcal{I}\big(\eta(1,\gamma(t_*))\big) \leq b - \epsilon < b$ and for every $t \in [0,1]$, we have $\mathcal{I}\big(\eta(1,\gamma(t))\big) \leq \mathcal{I}\big(\gamma(t)\big) < b$. Since $[0,1]$ is compact, we conclude with the contradiction that $\sup_{t\in[0,1]} \mathcal{I}\big(\eta(1,\gamma(t))\big) < b$. □

5.2. **Positivity of groundstates.** We now prove that when f is odd, groundstates do not change sign.

Proposition 5.2 (Groundstates do not change sign)**.** *Let $f \in C(\mathbb{R};\mathbb{R})$ satisfy (f_1). If f is odd and does not change sign on $(0,\infty)$, then any groundstate $u \in H^1(\mathbb{R}^N)$ of $(\mathcal{P})$ has constant sign.*

Proof. Without loss of generality, we can assume that $f \geq 0$ on $(0,\infty)$. By Proposition 4.1, there exists an optimal path $\gamma \in \Gamma$ on which the functional $\mathcal{I}$ achieves its maximum at $1/2$. Since f is odd, F is even and thus for every $v \in H^1(\mathbb{R}^N)$,

$$\mathcal{I}(|v|) = \mathcal{I}(v).$$

Hence, for every $t \in [0,1] \setminus \{1/2\}$,

$$\mathcal{I}(|\gamma(t)|) = \mathcal{I}(\gamma(t)) = \mathcal{I}(\gamma(\tfrac{1}{2})) = \mathcal{I}(|\gamma(\tfrac{1}{2})|).$$

By Lemma 5.1, $|u| = |\gamma(1/2)|$ is also a groundstate. It satisfies the equation

$$-\Delta|u| + |u| = \big(I_\alpha * F(|u|)\big) f(|u|).$$

Since u is continuous by Theorem 2, by the strong maximum principle we conclude that $|u| > 0$ on $\mathbb{R}^N$ and thus u has constant sign. □

5.3. **Symmetry of groundstates.** In this section, we now prove that groundstates are radial.

Proposition 5.3 (Groundstates are symmetric)**.** *Let $f \in C(\mathbb{R};\mathbb{R})$ satisfies (f_1). If f is odd and does not change sign on $(0,\infty)$, then any groundstate $u \in H^1(\mathbb{R}^N)$ of $(\mathcal{P})$ is radially symmetric about a point.*

The argument relies on polarizations. It is intermediate between the argument based on equality cases in polarization inequalities [31] and the argument based on the Euler-Lagrange equation satisfied by polarizations [5, 42].

Before proving Proposition 5.3, we recall some elements of the theory of polarization ([3], [41], [9], [44, §8.3]).

Assume that $H \subset \mathbb{R}^N$ is a closed half-space and that σ_H is the reflection with respect to ∂H. The polarization $u^H : \mathbb{R}^N \to \mathbb{R}$ of $u : \mathbb{R}^N \to \mathbb{R}$ is defined for $x \in \mathbb{R}^N$ by

$$u^H(x) = \begin{cases} \max\big(u(x), u(\sigma_H(x))\big) & \text{if } x \in H, \\ \min\big(u(x), u(\sigma_H(x))\big) & \text{if } x \notin H. \end{cases}$$

We will use the following standard property of polarizations [9, lemma 5.3].

Lemma 5.4 (Polarization and Dirichlet integrals)**.** *If $u \in H^1(\mathbb{R}^N)$, then $u^H \in H^1(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} |\nabla u^H|^2 = \int_{\mathbb{R}^N} |\nabla u|^2.$$

We shall also use a polarization inequality with equality cases [31, lemma 5.3] (without the equality cases, see [3, corollary 4], [41, proposition 8]).

Lemma 5.5 (Polarization and nonlocal integrals)**.** *Let $\alpha \in (0, N)$, $u \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and $H \subset \mathbb{R}^N$ be a closed half-space. If $u \geq 0$, then*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)\, u(y)}{|x-y|^{N-\alpha}}\, dx\, dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^H(x)\, u^H(y)}{|x-y|^{N-\alpha}}\, dx\, dy,$$

with equality if and only if either $u^H = u$ or $u^H = u \circ \sigma_H$.

The last tool that we need is a characterization of symmetric functions by polarizations ([42, proposition 3.15], [31, lemma 5.4]).

Lemma 5.6 (Symmetry and polarization)**.** *Assume that $u \in L^2(\mathbb{R}^N)$ is nonnegative. There exist $x_0 \in \mathbb{R}^N$ and a nonincreasing function $v : (0, \infty) \to \mathbb{R}$ such that for almost every $x \in \mathbb{R}^N$, $u(x) = v(|x - x_0|)$ if and only if for every closed half-space $H \subset \mathbb{R}^N$, $u^H = u$ or $u^H = u \circ \sigma_H$.*

Proof of Proposition 5.3. The strategy consists in proving that u^H is also a groundstate (see corresponding results for the local problem [5, 42] and a weaker abstract result [35]) and to deduce therefrom that $u = u^H$ or $u^H = u \circ \sigma_H$.

Without loss of generality, we can assume that $f \geq 0$ on $(0, \infty)$. By Proposition 5.2, we can assume that $u > 0$. In view of Proposition 4.1, there exists an optimal path γ such that $\gamma(1/2) = u$ and $\gamma(t) \geq 0$ for every $t \in [0, 1]$. For every half-space H define the path $\gamma^H : [0, 1] \to H^1(\mathbb{R}^N)$ by $\gamma^H(t) = (\gamma(t))^H$. By Lemma 5.4, $\gamma^H \in C([0, 1]; H^1(\mathbb{R}^N))$. Observe that since F is nondecreasing, $F(u^H) = F(u)^H$, and therefore, for every $t \in [0, 1]$, by Lemmas 5.4 and 5.5,

$$\mathcal{I}\big(\gamma^H(t)\big) \leq \mathcal{I}\big(\gamma(t)\big).$$

Observe that $\gamma^H \in \Gamma$ so that

$$\max_{t \in [0,1]} \mathcal{I}\big(\gamma^H(t)\big) \geq b.$$

Since for every $t \in [0, 1] \setminus \{1/2\}$,

$$\mathcal{I}\big(\gamma^H(t)\big) \leq \mathcal{I}\big(\gamma(t)\big) < b,$$

we have

$$\mathcal{I}\big(\gamma^H(\tfrac{1}{2})\big) = \mathcal{I}\big(\gamma(\tfrac{1}{2})\big) = b.$$

By Lemmas 5.4 and 5.5, we have either $F(u)^H = F(u)$ or $F(u^H) = F(u \circ \sigma_H)$ in $\mathbb{R}^N$. Assume that $F(u)^H = F(u)$. Then, we have for every $x \in H$,

$$\int_{u(\sigma_H(x))}^{u(x)} f(s)\, \mathrm{d}s = F\big(u(x)\big) - F\big(u(\sigma_H(x))\big) \geq 0;$$

this implies that either $u(\sigma_H(x)) \leq u(x)$ or $f = 0$ on $[u(x), u(\sigma_H(x))]$. In particular, $f(u^H) = f(u)$ on $\mathbb{R}^N$. By Lemma 5.1 applied to γ^H, we have $\mathcal{I}'(u^H) = 0$; and therefore,

$$-\Delta u^H + u^H = \big(I_\alpha * F(u^H)\big) f(u^H) = \big(I_\alpha * F(u)\big) f(u).$$

Since u satisfies $(\mathcal{P})$, we conclude that $u^H = u$.

If $F(u^H) = F(u \circ \sigma_H)$, we conclude similarly that $u^H = u \circ \sigma_H$. Since this holds for arbitrary H, we conclude by Lemma 5.6 that u is radial and radially decreasing. □

6. Alternative proof of the existence

In this section we sketch an alternative proof of the existence of a nontrivial solution $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $c \leq \mathcal{I}(u) \leq b$, under the additional symmetry assumption of Theorem 4 and in the spirit of the symmetrization arguments of H. Berestycki and P.-L. Lions [6, pp. 325-326]. The advantage of this approach is that it bypasses the concentration compactness argument and delays the Pohožaev identity which is still needed to prove that $b \leq c$.

Proof of Theorem 1 *under the additional assumptions of Theorem* 4. In addition to (f_1), (f_2) and (f_3), assume that f is an odd function which has constant sign on $(0, \infty)$. With this additional assumption,

$$\mathcal{I} \circ \Phi(\sigma, |v|^H) \leq \mathcal{I} \circ \Phi(\sigma, v).$$

Therefore, by the symmetric variational principle [40, theorem 3.2], we can prove as in the proof of Proposition 2.1 the existence of a sequence $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ such that as $n \to \infty$,

$$\begin{aligned}
\mathcal{I}(u_n) &\to b, \\
\mathcal{I}'(u_n) &\to 0 && \text{in } \big(H^1(\mathbb{R}^N)\big)', \\
\mathcal{P}(u_n) &\to 0, \\
u_n - u_n^* &\to 0 && \text{in } L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N),
\end{aligned}$$

where $u_n^* : \mathbb{R}^N \to \mathbb{R}$ is the Schwarz symmetrization of u_n, that is, for every $t > 0$, $\{x \in \mathbb{R}^N : u_n^*(x) > t\}$ is a ball that has the same Lebesgue measure as $\{x \in \mathbb{R}^N : |u_n(x)| > t\}$ ([33, 34], [3, corollary 3], [9, §2], [23, §3.3], [44, definition 8.3.1]).

As previously, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$; by the Pólya–Szegő inequality ([33], [34], [9, theorem 8.2], [23, lemma 7.17], [44, theorem 8.3.14]), $u_n^* \in W^{1,2}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |\nabla u_n^*|^2 \leq \int_{\mathbb{R}^N} |\nabla u_n|^2,$$

and thus the sequence $(u_n^*)_{n\in\mathbb{N}}$ is also bounded in $H^1(\mathbb{R}^N)$. Since u_n^* is radial for every $n \in \mathbb{N}$, the sequence $(u_n^*)_{n\in\mathbb{N}}$ is compact in $L^p(\mathbb{R}^N)$ for every $p \in (2, \frac{2N}{N-2})$ ([36, lemmas 2 and 3], [25, proposition 1.1], [43, Corollary 1.26]).

As $u_n - u_n^* \to 0$ as $n \to \infty$ in $L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, the sequence $(u_n)_{n\in\mathbb{N}}$ is also compact $L^p(\mathbb{R}^N)$ for every $p \in (2, \frac{2N}{N-2})$.

In view of (f_2), this implies that $F(u_n) \to F(u)$ as $n \to \infty$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and thus

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big)F(u_n) = \int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big)F(u) > 0. \tag{6.1}$$

Now one can prove that $u_n(\cdot - a_n)$ converges to a nontrivial solution $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ as in the proof of Proposition 2.2. By (6.1), it also follows that

$$c \le \mathcal{I}(u) \le b.$$

Finally, employing the Pohožaev identity as in the proof of Theorem 1 allows us to conclude that $c = b$. □

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