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SPLIT SUBVARIETIES OF GROUP EMBEDDINGS

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ABSTRACT. Let G be a connected reductive group and X an equivariant compactification of G. In X, we study generalised and opposite generalised Schubert varieties, and their intersections called generalised Richardson varieties and projected generalised Richardson varieties. Any complete G-embedding has a canonical Frobenius splitting, and we prove that the compatibly split subvarieties are the generalised projected Richardson varieties extending a result of Knutson, Lam and Speyer to the situation.

1. INTRODUCTION

Let G be a connected reductive group over a field k of positive characteristic p. A G-embedding is a normal $G \times G$-variety X together with a $G \times G$-equivariant open embedding of G in X.

G-embeddings share many of the nice properties of rational projective homogeneous spaces. For example, any G-embedding has cellular decompositions defined by $B \times B$- and $B^- \times B^-$-orbits for B and B^- opposite Borel subgroups of G. We call these cells and their closures generalised (and opposite generalised) Schubert cells and varieties. As for classical Schubert varieties, generalised Schubert varieties are normal and Cohen-Macaulay (see for example [7, 8, 10] for more details). We study the intersection of two opposite Schubert varieties. We call these intersections generalised Richardson varieties. We also consider generalised projected Richardson varieties which are the images of generalised Richardson varieties under morphisms of G-embeddings.

The existence of Frobenius splittings is another instance of the common features between projective rational homogeneous spaces and G-embeddings. Frobenius splittings were first introduced by Mehta and Ramanathan in [15] for projective rational homogeneous spaces to prove cohomology vanishing results and regularity properties of Schubert varieties. Using this technique, Rittatore [22] obtained regularity results for all G-embeddings, in particular the Cohen-Macaulay property. Brion and Polo [7], Brion and Thomsen [8] and He and Thomsen [10] also obtained regularity results for $B \times B$-orbit closures in group embeddings.

For rational projective homogeneous spaces, Knutson, Lam and Speyer [12] proved that in G/P (with P a parabolic subgroup containing B) the projections of Richardson varieties are all the compatibly split subvarieties for the unique B-canonical splitting. For X a complete G-embedding, X has a unique Frobenius splitting ϕ compatibly splitting the $G \times G$, $B \times B$ and $B^- \times B^-$ divisors (see Proposition 5.1). We introduce projected generalised Richardson varieties (see Definition 4.2) and prove the following result.

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Theorem 1.1. *The projected generalised Richardson varieties of a complete G-embedding are the ϕ-compatibly split subvarieties.*

The fact that projected generalised Richardson varieties are compatibly split follows from results of He and Thomsen [10]. We use techniques of Knutson, Lam and Speyer [12] to prove that these varieties are the only compatibly split subvarieties.

On the way we prove several results on generalised Schubert varieties, generalised Richardson varieties and projected generalised Richardson varieties: the later are normal and form a stratification of X. It is also interesting to note that not all the properties of Schubert varieties extend to G-embeddings. In particular, in general the intersection of two opposite generalised Schubert varieties is neither irreducible nor equidimensional (see Example 3.3), but their irreducible components are projected generalised Richardson varieties and therefore normal (see Remark 5.6). For X toroidal, however, the intersection of two opposite generalised Schubert varieties is irreducible (see Proposition 3.8).

Notation. We work over an algebraically closed field k of positive characteristic p. Varieties are reduced, separated, connected schemes of finite type over k.

Let G be a reductive group over k and let T be a maximal torus of G. Denote by $W = N_G(T)/T$ the Weyl group of T and by Φ the root system associated to (G, T). Let B be a Borel subgroup of G containing T. Denote by Δ the set of simple roots induced by B and by Φ^+ the set of positive roots. For J a subset of Δ, denote by P_J the parabolic subgroup containing B with $\Delta_P = J$ where for P a parabolic subgroup containing B, Δ_P is the set of simple roots of the Levi factor of P containing T. Denote by P_J^- the opposite parabolic subgroup and by L_J the Levi subgroup containing T of both P_J and P_J^-. Write Z_J for the center of L_J. We write U_J and U_J^- for the unipotent radicals of P_J and P_J^-. We write W_J for the Weyl group of P_J and W^J for the set of minimal length representatives of W/W_J. Recall that there exists for $u \in W$ a unique length additive decomposition $u = u^J u_J$ with $u^J \in W^J$ and $u_J \in W_J$. Denote by B_J the intersection $B \cap L_J$ and by B_J^- the intersection $B^- \cap L_J$. We write w_0 for the longest element in W. For L a group, we denote by (L, L) its derived group and $\mathrm{diag}(L)$ the diagonal embedding of L in $L \times L$.

2. G-EMBEDDINGS

2.1. **Toroidal G-embeddings.** Consider the $G \times G$-action on G given by the formula $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. A G-embedding is a normal $G \times G$-variety X together with an open equivariant embedding $G \to X$. A morphism of G-embeddings is a $G \times G$-equivariant morphism between G-embeddings extending the identity on G. These varieties are special cases of spherical varieties. We refer to [13, 16] for an overview on the geometry of spherical varieties.

Definition 2.1. A G-embedding X is called toroidal if any $B \times B$-stable divisor in X containing a $G \times G$-orbit is $G \times G$-stable. A G-embedding X is called simple if X has a unique closed $G \times G$-orbit in X.

2.2. **Description of $G \times G$-orbits.** Let G be a reductive group, X be a G-embedding and $(G \times G) \cdot x$ be a $G \times G$-orbit in X. We describe the stabiliser H of x. The following result, whose proof is essentially due to Brion, generalises in positive characteristic a result of Alexeev and Brion [1, Proposition 3.1].

Proposition 2.2. *1. There exists an element $x' \in (G\times G)\cdot x$ unique up to $T\times T$-action such that* $\mathrm{diag}(T)$ *fixes x'.*

2. Assume that $x = x'$. Then there exists a subset I of Δ, the union of two orthogonal subsets J and K such that the subgroup H is conjugate in $T\times T$ to $(U_I H_J \times U_I^- H_J)\mathrm{diag}L_K$, where $(L_J, L_J) \subset H_J \subset (L_J, L_J)Z_K$ and Z_K is the center of L_K.

Furthermore if X is toroidal, then $J = \emptyset$.

Proof. 1. This follows from [6, Proposition 6.2.3] for toroidal embeddings. The general result follows from the toroidal case. Note that since x' is unique up to $T\times T$-action, it follows that its stabiliser will be unique up to conjugation in $T\times T$.

2. Using results of Sumihiro [23] (see also [16, Theorem 2.3.1]), we may assume that X is equivariantly embedded in $\mathbb{P}(V)$ with V a G-module. Consider $\widehat{X}$ the affine cone over X and $\widehat{G} = G\times\mathbb{G}_m$. The stabiliser of the cone over the orbit $G\cdot x$ is $\widehat{H} \simeq H$. We can thus assume that X is affine. According to a result of Rittatore (see [21, Proposition 1]) the affine G-embedding X is an algebraic monoid. The result is a consequence of the theory of algebraic monoids. For this theory, we refer to [19] although many of the results we use were first proved by Putcha [17, 18].

By [19, Theorem 4.5.(c)], any $G\times G$-orbit is the orbit of an idempotent e contained in the closure of the maximal torus. We may therefore replace x by e. The stabiliser is the subgroup $H = \{(x, y) \in G\times G \mid xey^{-1} = e\}$ of $G\times G$. Let $(x, y) \in H$; then $xe = ey$ and $exe = ey = xe$. Therefore x lies in $P(e) = \{x \in G \mid xe = exe\}$. In the same way, $eye = xe = ey$ and y lies in $P^-(e) = \{y \in G \mid ey = eye\}$. According to [19, Theorem 4.5.(a)], these groups are opposite parabolic subgroups of G and their unipotent radicals $R_uP(e)$ and $R_uP^-(e)$ satisfy $R_uP(e)e = \{e\} = eR_uP^-(e)$. In particular we have the inclusions

$$R_uP(e)\times R_uP^-(e) \subset H \subset P(e)\times P^-(e).$$

Note that the Levi subgroup of both $P(e)$ and $P^-(e)$ is $L(e) = P(e)\cap P^-(e) = \{x \in G \mid xe = ex\} = C_G(e)$.

By [19, Theorem 4.8.(a)], the subset $eXe = \{x \in X \mid x = exe\}$ is an algebraic monoid with unit e and unit group $C_G(e)e = eC_G(e)$. Consider the morphism $p_e : P(e)\times P^-(e) \to C_G(e)e \times eC_G(e)$ defined by $p_e(x, y) = (xe, ey)$. It is a group homomorphism: $p_e(xx', yy') = (xx'e, yy'e) = (xex'e, yey'e)$, whose kernel contains $R_uP(e)\times R_uP^-(e)$. Thus p_e factors through its restriction to $L(e)\times L(e)$. Since $L(e)$ is reductive, the morphism $L(e) \to C_G(e)e,\ x \mapsto xe$ is the quotient of a finite cover of $L(e)$ by some semi-simple factors and a subgroup of the centre.

For $(x, y) \in H$, we have $p_e(x, y) = (xe, ey) = (xe, xe)$; therefore p_e maps H to $\mathrm{diag}(C_G(e)e)$. This mapping is surjective since for $x \in C_G(e)$, we have $xe = ex$; therefore $(x, x) \in H$ and $p_e(x, x) = (xe, xe)$. Furthermore, $R_uP(e)\times R_uP^-(e) \subset \ker(p_e) \subset H$. All this implies our result: let I be such that $P_I = P(e)$ and $P_I^- = P^-(e)$, let $J \subset I$ be maximal such that $(L_J, L_J) \subset H(e) = \ker(L(e) \to C_G(e)e)$ and let K be the complement of J in I. The subsets J and K are orthogonal (since the morphism $L(e) \to C_G(e)e,\ x \mapsto xe$ is the quotient of a finite cover of $L(e)$ by some semi-simple factors and a subgroup of the centre). Furthermore the group $H(e)$ satisfies $(L_J, L_J) \subset H(e) \subset (L_J, L_J)Z_K$, where Z_K is the center of L_K. We have $H = (U_J H(e)\times U_J^- H(e))\mathrm{diag}(C_G(e)e)$. Since $C_G(e)e$ is a quotient by a subgroup contained in $H(e)$ of L_K, this concludes the proof of the first assertion.

For the second assertion, we use [19, Theorem 5.18]: for s a simple reflection, the inclusion $(G \times G) \cdot e \subset \overline{BsB^-}$ holds if and only if $se = es = e$, *i.e.* if and only if $(s, 1)$ and $(1, s)$ are in H. This happens if and only if $s \in J$. $\square$

Let $\pi : \widetilde{X} \to X$ be a morphism of G-embeddings. Let $\widetilde{x} \in \widetilde{X}$ and $x = \pi(\widetilde{x})$. Let $\widetilde{\Omega} = (G \times G) \cdot \widetilde{x}$ and $\Omega = (G \times G) \cdot x$. We denote by $\widetilde{H}$ and H the stabiliser of $\widetilde{x}$ and x respectively. There is an inclusion $\widetilde{H} \subset H$. Let $\widetilde{I}, \widetilde{J}, \widetilde{K}$ and I, J, K be the subsets of Δ corresponding to $\widetilde{H}$ and H according to the previous proposition.

Corollary 2.3. *Let H and $\widetilde{H}$ be as above.*

1. *The groups $\widetilde{H}$ and H are simultaneously conjugate to $(U_{\widetilde{I}} H_{\widetilde{J}} \times U_{\widetilde{I}}^- H_{\widetilde{J}})\mathrm{diag} L_{\widetilde{K}}$ and $(U_I H_J \times U_I^- H_J)\mathrm{diag} L_K$ with $(L_{\widetilde{J}}, L_{\widetilde{J}}) \subset H_{\widetilde{J}} \subset (L_{\widetilde{J}}, L_{\widetilde{J}}) Z_{\widetilde{K}}$ and $(L_J, L_J) \subset H_J \subset (L_J, L_J) Z_K$.*
2. *We have the inclusions $K \subset \widetilde{K}$, $\widetilde{J} \subset J$ and $\widetilde{I} \subset I$.*
3. *Assume that $\widetilde{X}$ is toroidal, that π is proper and that $\widetilde{\Omega}$ is closed in $\pi^{-1}(\Omega)$. Then $\widetilde{I} = \widetilde{K} = K$ and $\widetilde{J} = \emptyset$.*

Proof. 1. Choose $\widetilde{x}$ a $\mathrm{diag}(T)$-fixed point in $\widetilde{\Omega}$. Then $\widetilde{x}$ is unique up to $T \times T$-action and the same holds for $x = \pi(\widetilde{x})$. The result follows from the former proposition since the stabilisers of $\widetilde{x}$ and x are of the desired form up to conjugation in $T \times T$.

2. The inclusions $\widetilde{J} \subset J$ and $\widetilde{I} \subset I$ follow from the inclusion $\widetilde{H} \subset H$. Let α be a positive root not contained in the root system generated by $\widetilde{K}$. Then α is either a root of the root system generated by $\widetilde{J}$ or we have the inclusion $U_\alpha \subset U_{\widetilde{I}}$. In the first case α cannot be a root of the root system generated by K. In the second case, we have the inclusions $U_\alpha \times \{1\} \subset \widetilde{H} \subset H$. It follows that α cannot be a root of the root system generated by K. The inclusion $K \subset \widetilde{K}$ follows.

3. The fact that $\widetilde{J} = \emptyset$ and $\widetilde{K} = \widetilde{I}$ follows from the former proposition. With the above assumption, the map $\pi^{-1}(\Omega) \to \Omega$ is proper and $\widetilde{\Omega}$ is closed in $p^{-1}(\Omega)$. In particular, the map $\widetilde{\Omega} \to \Omega$ is proper. But according to 1. its fibers are isomorphic to the contracted product $(L_J \times L_J) \times^{P_{\widetilde{I}} \cap L_J \times P_{\widetilde{I}}^- \cap L_J} L$, where L is a quotient of $L_{\widetilde{I} \cap J}$ by a central subgroup. Since the fiber is proper it follows that $\widetilde{I} \cap J = \emptyset$. $\square$

3. Generalised Schubert and Richardson varieties

3.1. **Definition and first properties.** Let X be a G-embedding. We describe the $B \times B$-orbits and the $B^- \times B^-$-orbits in any $G \times G$-orbit. Since the $B \times B$-orbits and the $B^- \times B^-$-orbits are contained in $G \times G$-orbits, we may fix such an orbit Ω, and according to Proposition 2.2, there is an element $h_\Omega \in \Omega$ such that the stabiliser H of h_Ω is of the form $(U_I H_J \times U_I^- H_J)\mathrm{diag} L_K$ where $(L_J, L_J) \subset H_J \subset (L_J, L_J) Z_K$ and Z_K is the center of L_K.

Definition 3.1. Let Ω, H and $h = h_\Omega \in \Omega$ be as above. Let $u, v, w, x \in W$.

1. We denote by $p_\Omega : \Omega \to G/P_I \times G/P_I^-$ the morphism induced by the inclusion $H \subset P_I \times P_I^-$.
2. The generalised Schubert cell $\mathring{X}_{u,v}(\Omega)$ is the $B \times B$-orbit $(Bu \times Bv) \cdot h$. The generalised Schubert variety $X_{u,v}(\Omega)$ is the closure of $\mathring{X}_{u,v}(\Omega)$ in X.
3. The generalised opposite Schubert cell $\mathring{X}^{w,x}(\Omega)$ is the $B^- \times B^-$-orbit given by $(w_0, w_0) \cdot \mathring{X}_{w_0 w, w_0 x}(\Omega) = (B^- w \times B^- x) \cdot h$. The generalised opposite Schubert variety $X^{w,x}(\Omega)$ is the closure of $\mathring{X}^{w,x}(\Omega)$.

4. The generalised open Richardson variety $\mathring{X}^{w,x}_{u,v}(\Omega)$ is defined as the intersection $\mathring{X}_{u,v}(\Omega) \cap \mathring{X}^{w,x}(\Omega)$. The generalised Richardson variety $X^{w,x}_{u,v}(\Omega)$ is defined as the intersection $X_{u,v}(\Omega) \cap X^{w,x}(\Omega)$.

Let I, J and K be as above. Recall that $u \in W$ can be written $u = u^I u_I = u^I u_J u_K$.

Lemma 3.2. *Let Ω be a $G \times G$-orbit and let $h \in \Omega$ be as above.*

1. The $B \times B$ orbits in Ω are the generalised Schubert cells.

2. We have $\mathring{X}_{u,v}(\Omega) = \mathring{X}_{u',v'}(\Omega)$ if and only if $u^I = u'^I$, $(v^I)^{-1} = (v'^I)^{-1}$ and $u_K(v_K)^{-1} = u'_K(v'_K)^{-1}$.

3. We have $p_\Omega(\mathring{X}_{u,v}(\Omega)) = BuP_I/P_I \times BvP_I^-/P_I^-$.

4. The $B^- \times B^-$ orbits in Ω are the opposite generalised Schubert cells.

5. We have $\mathring{X}^{w,x}(\Omega) = \mathring{X}^{w',x'}(\Omega)$ if and only if $w^I = w'^I$, $(x^I)^{-1} = (x'^I)^{-1}$ and $w_K(x_K)^{-1} = w'_K(x'_K)^{-1}$.

6. We have $p_\Omega(\mathring{X}^{w,x}(\Omega)) = B^-wP_I/P_I \times B^-xP_I^-/P_I^-$.

Proof. This follows from [2, Lemma 1.2] since the orbit Ω is induced from a quotient L' of L_K in the following sense: $\Omega \simeq (G \times G)^{P_I \times P_I^-} L'$. □

Example 3.3. In general $X^{w,x}_{u,v}(\Omega)$ is neither irreducible nor equidimensional. We will however prove in Proposition 3.8 that for X toroidal, the variety $X^{w,x}_{u,v}(\Omega)$ is irreducible.

Let X be $\mathbb{P}(M_3(\mathsf{k}))$ where $M_3(\mathsf{k})$ is the vector space of 3×3 matrices. The group $G \times G$ with $G = \mathrm{PGL}_3(\mathsf{k})$ acts on X by $(P,Q)\cdot A = PAQ^{-1}$. Let B be the image of the subgroup of upper-triangular matrices in G and B^- be the image in G of lower-triangular matrices. For $A \in M_3(\mathsf{k})$, denote by C_1, C_2 and C_3 the columns of A.

The $G\times G$-orbits are indexed by the rank. Let Ω_2 be the orbit of matrices of rank 2. Denote by $X_{1,2}$ and $X_{2,3}$ the closed subsets given by the equations $C_1 \wedge C_2 = 0$ and $C_2 \wedge C_3 = 0$. The intersections $\overline{\Omega_2} \cap X_{1,2}$ and $\overline{\Omega_2} \cap X_{2,3}$ are easily seen to be $B \times B$- and $B^- \times B^-$-generalised Schubert varieties. Denote them by $X_{u,v}(\Omega_2)$ and $X^{w,x}(\Omega_2)$. Let $X^{w,x}_{u,v}(\Omega_2)$ be the corresponding generalised Richardson variety. One easily checks that $X^{w,x}_{u,v}(\Omega_2)$ is the union

$$X^{w,x}_{u,v}(\Omega_2) = \{[A] \in X \mid C_2 = 0\} \cup \{[A] \in X \mid \mathrm{rk}(A) = 1\}.$$

This is the decomposition of $X^{w,x}_{u,v}(\Omega_2)$ in irreducible components. The dimensions of these components are 5 and 4. Therefore $X^{w,x}_{u,v}(\Omega_2)$ is neither irreducible nor equidimensional.

Proposition 3.4. *Let Ω and $h \in \Omega$ be as above. Let $u, v, w, x \in W$. The variety $\mathring{X}^{w,x}_{u,v}(\Omega)$ is irreducible and smooth.*

Proof. We follow the proof of the same result for rational projective homogeneous spaces. Let I, J, K be such that $H = \mathrm{Stab}(h) = (U_I H_J \times U_I^- H_J)\mathrm{diag}(L_K)$ with $(L_J, L_J) \subset H_J \subset (L_J, L_J)Z_K$. There is an open dense subset of Ω given by $(B^- \times B)\cdot h$. We translate this subset in a neighborhood $(wB^-w^{-1}w \times xBx^{-1}x)\cdot h$ of $(w,x)\cdot h$. This neighborhood contains the $B^- \times B^-$-orbit $(B^-w \times B^-x)\cdot h_K$. In what follows, we write, for E a subset of G and α a root of (G,T), $\alpha \in E$ for $U_\alpha \subset E$. We have an isomorphism given by the action

$$U_{w,x} \times (B^-w \times B^-x)\cdot h \simeq (wB^-w^{-1}w \times xBx^{-1}x)\cdot h$$

with

$$U_{w,x} = \prod_{\alpha>0,\ w^{-1}(\alpha)\in U_I^- \cup B_K^-} U_\alpha \times \prod_{\beta>0,\ x^{-1}(\beta)\in U_I \cup B_K} U_\beta.$$

Intersecting with $\mathring{X}_{u,v}(\Omega)$ which is stable under $U_{w,x}$, we get

$$U_{w,x} \times \mathring{X}^{w,x}_{u,v}(K) \simeq (wB^-w^{-1}w \times xBx^{-1}x) \cdot h \cap \mathring{X}_{u,v}(K).$$

Since $\mathring{X}_{u,v}(K)$ is irreducible and smooth, the same holds for the right hand side (which is an open subset of $\mathring{X}_{u,v}(K)$), and therefore $\mathring{X}^{w,x}_{u,v}(K)$ is irreducible and smooth. $\square$

Lemma 3.5. *The intersection $X^{w,x}_{u,v}(\Omega) \cap \Omega$ is the closure of the cell $\mathring{X}^{w,x}_{u,v}(\Omega)$ and is irreducible.*

Proof. The variety $X^{w,x}_{u,v}(\Omega) \cap \Omega$ is a union of intersections $\mathring{X}_{u',v'}(\Omega) \cap \mathring{X}^{w',x'}(\Omega)$ where $\mathring{X}_{u',v'}(\Omega)$ are the generalised Schubert cells contained in $X_{u,v}(\Omega) \cap \Omega$ and $\mathring{X}^{w',x'}(\Omega)$ are the generalised opposite Schubert cells contained in $X^{w,x}(\Omega) \cap \Omega$.

In the orbit Ω, since these Schubert cells are stable for opposite Borel subgroups of $G \times G$, they are in general position and therefore intersect properly (see [11]). In particular $X^{w,x}_{u,v}(\Omega) \cap \Omega$ contains a unique intersection $\mathring{X}_{u',v'}(\Omega) \cap \mathring{X}^{u',v'}(\Omega)$ of codimension $\operatorname{codim}_\Omega X_{u,v}(\Omega) + \operatorname{codim}_\Omega X^{w,x}(\Omega)$: the generalised open Richardson variety $\mathring{X}^{u,v}_{u,v}(\Omega)$. Since Ω is smooth, it follows from [9, Lemma, p. 108] that the codimension of any irreducible component of $X^{w,x}_{u,v}(\Omega) \cap \Omega$ in Ω is at least $\operatorname{codim}_\Omega X_{u,v}(\Omega) + \operatorname{codim}_\Omega X^{w,x}(\Omega)$. Thus $X^{w,x}_{u,v}(\Omega) \cap \Omega$ is the closure of $\mathring{X}^{w,x}_{u,v}(\Omega)$ and is irreducible. $\square$

Lemma 3.6. *Let Ω and $h \in \Omega$ be as above. Let $u, v, w, x \in W$. The closure of the image $p_\Omega(\mathring{X}^{w,x}_{u,v}(\Omega))$ is a product of projected Richardson varieties in $G/P_I \times G/P_I^-$.*

Proof. We shall see in Section 5 that all the generalised Schubert cells, varieties, opposite cells and opposite varieties are $B \times B$-canonically split for the same splitting. In follows that all the generalised (open) Richardson varieties are also $B \times B$-canonically split and the closure of their images $p_\Omega(\mathring{X}^{w,x}_{u,v}(\Omega))$ are again $B \times B$-canonically split. Applying [12, Theorem 5.1], these varieties are products of projected Richardson varieties. $\square$

Example 3.7. In general $p_\Omega(\mathring{X}^{w,x}_{u,v}(\Omega))$ is not a product of Richardson variety or even the intersection of opposite $B \times B$- and $B^- \times B^-$-orbits. We will however prove in Proposition 3.13 that for X toroidal, the variety $p_\Omega(X^{w,x}_{u,v}(\Omega))$ is a product of Richardson variety.

Let X be $\mathbb{P}(M_4(\mathsf{k}))$, where $M_4(\mathsf{k})$ is the vector space of 4×4 matrices. The group $G \times G$ with $G = \mathrm{PGL}_4(\mathsf{k})$ acts on X by $(P, Q) \cdot A = PAQ^{-1}$. Let B be the image of the subgroup of upper-triangular matrices in G and B^- be the image in G of lower-triangular matrices. For $A \in M_3(\mathsf{k})$, denote by C_1, C_2, C_3 and C_4 the columns of A. Let (e_1, e_2, e_3, e_4) be the canonical basis of k^4.

The $G \times G$-orbits are indexed by the rank. Let Ω_2 be the orbit of matrices of rank 2. We have the structure map $p_{\Omega_2} : \Omega_2 \to \mathbb{G}(2,4) \times \mathbb{G}(2,4)$ defined by $p_{\Omega_2}([A]) = (\ker A, \mathrm{Im} A)$. Here $\mathbb{G}(2,4)$ denotes the Graßmann variety of lines in $\mathbb{P}^3$. The fiber $p_{\Omega_2}^{-1}(V_2, W_2)$ is the open subset of $\mathbb{P}\,\mathrm{Hom}(\mathsf{k}^4/V_2, W_2)$ of invertible elements.

Let Θ be the dense $B\times B$-orbit in $\mathbb{G}(2,4)\times\mathbb{G}(2,4)$ and let Θ^0 be the dense $B^-\times B^-$-orbit. One easily checks that $\{[A]\in\Omega_2 \mid p_{\Omega_2}([A])\in\Theta \text{ and } 0\neq A(e_1)\in\langle e_1,e_2,e_3\rangle\}$ and $\{[A]\in\Omega_2 \mid p_{\Omega_2}([A])\in\Theta^0 \text{ and } 0\neq A(e_4)\in\langle e_2,e_3,e_4\rangle\}$ are irreducible and $B\times B$- resp. $B^-\times B^-$-stable and therefore contain dense $B\times B$- and $B^-\times B^-$-orbits that we denote by $\mathring{X}_{u,v}(\Omega_2)$ and $\mathring{X}^{w,x}(\Omega_2)$.

We claim that $p_{\Omega_2}(\mathring{X}^{w,x}_{u,v}(\Omega_2))$ is dense in but different from $\Theta\cap\Theta^0$.

Let $(V_2,W_2)\in\Theta\cap\Theta^0$ such that $V_2\cap\langle e_1,e_4\rangle=0$ and $W_2\cap\langle e_2,e_3\rangle=0$. Then the classes $(\bar e_1,\bar e_4)$ of e_1 and e_4 in k^4/V_2 form a basis. Furthermore $W_2\cap\langle e_1,e_2,e_3\rangle$ and $W_2\cap\langle e_2,e_3,e_4\rangle$ are in direct sum. Therefore, there is an isomorphism $[f]\in\mathbb{P}\operatorname{Hom}(\mathsf{k}^4/V_2,W_2)$ with $f(\bar e_1)\in W_2\cap\langle e_1,e_2,e_3\rangle$ and $f(\bar e_4)\in W_2\cap\langle e_2,e_3,e_4\rangle$; thus $(V_2,W_2)\in p_{\Omega_2}(\mathring{X}^{w,x}_{u,v}(\Omega_2))$.

Let $(V_2,W_2)=(\langle e_1+e_4,e_2+e_3\rangle),\langle e_1+e_3,e_2+e_4\rangle)\in\Theta\cap\Theta^0$. An element $[A]\in X^{w,x}_{u,v}(\Omega_2)$ with $p_{\Omega_2}([A])=(V_2,W_2)$ should satisfy $0\neq A(e_1)\in\langle e_1+e_3\rangle$, $0\neq A(e_4)\in\langle e_2+e_4\rangle$ and $A(e_1+e_4)=0$. This is impossible.

3.2. **Generalised Richardson varieties in the toroidal case.**

Proposition 3.8. *Let X be toroidal. Generalised Richardson varieties are irreducible and Cohen-Macaulay.*

Proof. Let Ω be a $G\times G$-orbit in X. The variety $X^{w,x}_{u,v}(\Omega)$ is a union of intersections $\mathring{X}_{u',v'}(\Omega')\cap\mathring{X}^{w',x'}(\Omega')$ where Ω' is an $G\times G$-orbit contained in $\overline{\Omega}$, where $\mathring{X}_{u',v'}(\Omega')$ are the generalised Schubert cells contained in $X_{u,v}(\Omega)\cap\Omega'$ and where $\mathring{X}^{w',x'}(\Omega')$ are the generalised opposite Schubert cells contained in $X^{w,x}(\Omega)\cap\Omega'$.

In the orbit Ω', since these Schubert cells are stable for opposite Borel subgroups of $G\times G$, they are in general position and therefore intersect properly (see [11]). In particular $X^{w,x}_{u,v}(\Omega)$ contains a unique intersection $\mathring{X}_{u',v'}(\Omega')\cap\mathring{X}^{u',v'}(\Omega')$ of codimension at most $\operatorname{codim}_{\overline{\Omega}}X_{u,v}(\Omega)+\operatorname{codim}_{\overline{\Omega}}X^{w,x}(\Omega)$: the generalised open Richardson variety $\mathring{X}^{u,v}_{u,v}(\Omega)$.

Assume for the moment that the irreducible components of $X^{w,x}_{u,v}(\Omega)$ are of codimension at most $\operatorname{codim}_{\overline{\Omega}}X_{u,v}(\Omega)+\operatorname{codim}_{\overline{\Omega}}X^{w,x}(\Omega)$. This implies that $X^{w,x}_{u,v}(\Omega)$ is the closure of $\mathring{X}^{w,x}_{u,v}(\Omega)$ and is irreducible. The Cohen-Macaulay property follows from [9, Lemma, p. 108].

It is therefore enough to prove that the irreducible components of $X^{w,x}_{u,v}(\Omega)$ are of codimension at most $\operatorname{codim}_{\overline{\Omega}}X_{u,v}(\Omega)+\operatorname{codim}_{\overline{\Omega}}X^{w,x}(\Omega)$. For this we consider the following diagram:

$$\begin{tikzcd}
G\times G \arrow[d, equal] & \Gamma \arrow[l, "p"'] \arrow[r, "q"] \arrow[d] & X^{w,u}(\Omega) \arrow[d] \\
G\times G & G\times G\times X_{u,v}(\Omega) \arrow[l] \arrow[r, "\mu"] & \overline{\Omega}
\end{tikzcd}$$

where μ is induced by the action of $G\times G$ on X and the right square is a cartesian square.

Lemma 3.9. *The morphism μ is flat.*

Proof. According to [5, (14.4.4) and (15.2.1)] and since $\overline{\Omega}$ is normal (see [22]) it is enough to prove that the fibres of μ are equidimensional and reduced.

Note that the fibre over $h\in\overline{\Omega}$ is a principal bundle over $F=((G\times G)\cdot h)\cap X_{u,v}(\Omega)$. It is thus enough to prove that F is equidimensional and reduced. Let Z

be the closure of $(G\times G)\cdot h$. Since X is toroidal, there exists a chain of $G\times G$-orbit closures $Z=Z_0\subset Z_1\subset\cdots\subset Z_n=\overline{\Omega}$ such that each Z_i has codimension 1 in Z_{i+1}. In particular Z_i consists of smooth points of Z_{i+1} and is toroidal. Intersecting with $X_{u,v}(\Omega)$ which never contains $G\times G$-orbits we get that F is equidimensional.

To prove that the fibres are reduced, it is enough to consider the fiber of a point in a closed G-orbit. The fiber is in that case a multiplicity free subvariety of a flag variety and therefore reduced (see [3]). $\square$

By base change we get that q is flat and by [5, (14.4.6)] and [4, (2.4.6)] its fibres are equidimensional. This in particular implies that Γ is equidimensional of dimension

$$\dim G+\dim X_{u,v}(\Omega)+\dim X^{w,x}(\Omega)-\dim\Omega.$$

Now $X^{w;x}_{u,v}(\Omega)$ is a general fiber of the morphism p and Chevalley's Theorem implies that all its irreducible components are of codimension at most $\operatorname{codim}_{\overline{\Omega}}X_{u,v}(\Omega)+\operatorname{codim}_{\overline{\Omega}}X^{w,x}(\Omega)$, proving the result. $\square$

Proposition 3.10. *Let X be toroidal. Generalised Richardson varieties are normal.*

Proof. Generalised Richardson varieties are Cohen-Macaulay by Proposition 3.8. It remains to prove that they are smooth in codimension one. But by Proposition 3.4 the generalised open Richardson varieties are smooth; therefore the nonsmooth locus is contained in smaller generalised Richardson varieties. The divisorial part of the nonsmooth locus is therefore contained in one of these smaller generalised Richardson varieties. But since all generalised Richardson varieties are Frobenius split for the same splitting (see Section 5), their intersection is reduced and therefore generically smooth. Since the irreducible components of the complement in the generalised Richardson variety of the generalised open Richardson variety is obtained by intersection with the divisorial B and B^- Schubert varieties, it follows that generalised Richardson varieties are smooth in codimension one. $\square$

Definition 3.11. Let $(Y')_{Y'\in\mathcal{Y}}$ be a finite family of closed irreducible subvarieties of an irreducible variety Y. The family $\mathcal{Y}$ is called a stratification if $Y\in\mathcal{Y}$, and for $Y',Y''\in\mathcal{Y}$, the intersection $Y'\cap Y''$ is the union of subvarieties in $\mathcal{Y}$.

Proposition 3.12. *Let X be toroidal. Generalised Richardson varieties form a stratification of X.*

Proof. Since $X^{w;x}_{u,v}(\Omega)$ is irreducible, this follows from the fact that X is the disjoint union of the open generalised Richardson varieties. $\square$

Proposition 3.13. *Let X be toroidal and let Ω and $h\in\Omega$ be as above. Let $u,v,w,x\in W$. The closure of the image $p_\Omega(\mathring{X}^{w;x}_{u,v}(\Omega))$ is a product of Richardson varieties in $G/P_I\times G/P_I^-$.*

Proof. Note that the image $p_\Omega(\mathring{X}^{w;x}_{u,v}(\Omega))$ is contained in the product of Richardson varieties $(\overline{BuP_I}/P_I\cap\overline{B^-wP_I}/P_I)\times(\overline{BvP_I^-}/P_I^-\cap\overline{B^-xP_I^-}/P_I^-)$. Furthermore its closure is a product of projected Richardson varieties, so it is enough to prove that the projections to G/P_I and G/P_I^- of the closure of $p_\Omega(\mathring{X}^{w;x}_{u,v}(\Omega))$ contain the above Richardson varieties $(\overline{BuP_I}/P_I\cap\overline{B^-wP_I}/P_I)$ and $(\overline{BvP_I^-}/P_I^-\cap\overline{B^-xP_I^-}/P_I^-)$.

Let Ω' be a closed $G\times G$-orbit in the closure of Ω. Since X is toroidal, the orbit Ω' is isomorphic to $G/B \times G/B^-$ and we have a commutative diagram (see for example [10, Section 5.5] for the fact that $p_{\widetilde{\Omega}}$ extends to the closure of Ω):

$$\begin{tikzcd} & \Omega' \arrow[r, equal, "p_{\widetilde{\Omega}'}"] \arrow[d, hook] & G/B \times G/B^- \arrow[d] \\ \Omega \arrow[r, hook] & \overline{\Omega} \arrow[r, "p_{\widetilde{\Omega}}"] & G/P_I \times G/P_I^- \end{tikzcd}$$

According to [10, Proposition 6.3], the $B\times B$-orbit $Bu'B/B \times Bv'B^-/B^-$ in Ω' is contained in $X_{u,v}(\Omega)$ if and only if there exists $a \in W_I$ with $u' \leq ua$ and $v' \geq va$. The same argument proves that the $B^- \times B^-$-orbit $B^-w'B/B \times B^-x'B^-/B^-$ in Ω' is contained in $X^{w,x}(\Omega)$ if and only if there exists $b \in W_I$ with $w' \geq wb$ and $x' \leq xb$. Let $\pi : G/B \to G/P_I$ and $\pi_- : G/B^- \to G/P_I^-$. For a such that $u' = ua$ is of maximal length in uW_I and for b such that $x' = xb$ is of maximal length in xW_I, we have $\pi^{-1}(\overline{BuP_I/P_I}) = \overline{Bu'B/B}$ and $\pi_-^{-1}(\overline{B^-xP_I^-/P_I^-}) = \overline{B^-x'B^-/B^-}$. Let $v' = va$ and $w' = wa$; we have that $(\overline{BuP_I}/P_I \cap \overline{B^-wP_I}/P_I) \times (\overline{BvP_I^-}/P_I^- \cap \overline{B^-xP_I-}/P_I^-)$ is equal to $\pi(\overline{Bu'B}/B \cap \overline{B^-w'B}/B) \times \pi_-(\overline{Bv'B^-}/B^- \cap \overline{B^-x'B-}/B-)$, which is contained in the closure of the image $p_\Omega(\mathring{X}^{w,x}_{u,v}(\Omega))$. ☐

4. Generalised projected Richardson varieties

4.1. **Definition and first properties.** Recall the following general result on G-embeddings.

Proposition 4.1. *1. For any G-embedding X, there exists a toroidal G-embedding $\widetilde{X}$ and a $G\times G$-equivariant morphism $\psi : \widetilde{X} \to X$.*

2. For any G-embedding X and toroidal G-embeddings $\widetilde{X}$ and $\widetilde{X}'$ with $G\times G$-equivariant morphisms $\psi : \widetilde{X} \to X$ and $\psi' : \widetilde{X}' \to X$, there exists a toroidal embedding $\widehat{X}$ with $G\times G$-equivariant morphisms $\varphi : \widehat{X} \to \widetilde{X}$ and $\varphi : \widehat{X} \to \widetilde{X}$ such that the following diagram is commutative.

$$\begin{tikzcd} \widehat{X} \arrow[r, "\varphi"] \arrow[d, "\varphi'"'] & \widetilde{X} \arrow[d, "\psi"] \\ \widetilde{X}' \arrow[r, "\psi'"] & X \end{tikzcd}$$

Proof. 1. This result is proved in [6, Theorem 6.2.5].

2. This result is classical for spherical varieties in general (see for example [13]). ☐

Definition 4.2. Let X be a G-embedding; a projected generalised Richardson variety is the image of a generalised Richardson variety $\widetilde{X}^{w,x}_{u,v}(\Omega)$ under an equivariant morphism $\varphi : \widetilde{X} \to X$ with $\widetilde{X}$ toroidal.

Lemma 4.3. *Let $\varphi : Y \to X$ be a $G\times G$-equivariant morphism between toroidal G-embeddings. Let $u, v, w, x \in W$.*

1. Let Ω be a $G\times G$-orbit in X; then there exists a $G\times G$-orbit Ω' in Y such that $\varphi(\mathring{Y}^{w,x}_{u,v}(\Omega')) = \mathring{X}^{w,x}_{u,v}(\Omega)$ and $\varphi(Y^{w,x}_{u,v}(\Omega')) = X^{w,x}_{u,v}(\Omega)$.

2. Let Ω' be a $G\times G$-orbit in Y and $\Omega = \varphi(\Omega')$. Then $\varphi(\mathring{Y}^{w,x}_{u,v}(\Omega')) = \mathring{X}^{w,x}_{u,v}(\Omega)$ and $\varphi(Y^{w,x}_{u,v}(\Omega')) = X^{w,x}_{u,v}(\Omega)$.

Proof. It is enough to prove 2. since for any $G\times G$-orbit in X, there exists a $G\times G$-orbit Ω' in Y such that $\varphi(\Omega')=\Omega$. Write $\Omega'=(G\times G)\cdot\widetilde{x}$ and $\Omega=(G\times G)\cdot x$ and let $\widetilde{H}$ and H the stabilisers of $\widetilde{x}$ and x in $G\times G$. According to Corollary 2.3, we have $H=\widetilde{H}Z$ for Z a subgroup of $T\times T$. In particular, the map $\varphi:\Omega'\to\Omega$ is a quotient by Z, maps $\mathring{Y}_{u,v}(\Omega')$ to $\mathring{X}_{u,v}(\Omega)$ and $\varphi^{-1}(\mathring{X}_{u,v}(\Omega))=\mathring{Y}_{u,v}(\Omega')$. The same holds for the opposite Schubert cells: $\varphi(\mathring{Y}^{w,x}(\Omega'))=\mathring{X}^{w,x}(\Omega)$ and $\varphi^{-1}(\mathring{X}^{w,x}(\Omega))=\mathring{Y}^{w,x}(\Omega')$. Taking closures, the same result holds for generalised Schubert varieties and generalised opposite Schubert varieties. We deduce that $\varphi(\mathring{Y}_{uv}^{w,x}(\Omega'))=\mathring{X}_{uv}^{w,x}(\Omega)$, and taking closures, the result follows (recall that for X toroidal, the variety $X_{u,v}^{w,x}(\Omega)$ is irreducible). ☐

Corollary 4.4. *Let $\psi:\widetilde{X}\to X$ be a morphism of G-embeddings with $\widetilde{X}$ toroidal. Then any projected generalised Richardson variety is the projection of a generalised Richardson variety in $\widetilde{X}$.*

Proof. Let $\psi:\widetilde{X}\to X$ and $\psi':\widehat{X}\to X$ be two toroidal varieties dominating X. Let $\widehat{X}_{u,v}^{w,x}(\Omega')$ be a generalised Richardson variety in $\widetilde{X}$; we prove that $\psi'(\widehat{X}_{u,v}^{w,x}(\Omega'))$ is also the projection of a generalised Richardson variety in $\widetilde{X}$. Let $\widetilde{X}''$ be smooth and toroidal dominating both $\widetilde{X}$ and $\widehat{X}$ as given in Proposition 4.1. Then $\varphi({\varphi'}^{-1}(\widehat{X}_{u,v}^{w,x}(\Omega')))$ is again a generalised Richardson variety in $\widetilde{X}$ and the result follows. ☐

4.2. **Parabolic induction.** In this subsection we consider the following situation. Let $\mathcal{G}$ be a reductive group, $\mathcal{T}$ be a maximal torus and $\mathcal{P}$ be a parabolic subgroup containing $\mathcal{T}$. Let $\mathcal{U}$ be the unipotent radical of $\mathcal{P}$. We denote by $\mathcal{W},\mathcal{W}_P$ the Weyl groups of $(\mathcal{G},\mathcal{T})$ and $(\mathcal{P},\mathcal{T})$ and by $\mathcal{L}$ the Levi subgroup of $\mathcal{P}$ containing $\mathcal{T}$. Let $\mathcal{B}$ be a Borel subgroup of $\mathcal{G}$ with $\mathcal{T}\subset\mathcal{B}\subset\mathcal{P}$ and let $\mathcal{B}^-$ be the opposite Borel subgroup with respect to $\mathcal{T}$. We write $\mathcal{W}^{\mathcal{P}}$ for the set of minimal length representatives of $\mathcal{W}/\mathcal{W}_P$.

Let $\mathcal{H}$ be a spherical subgroup of $\mathcal{G}$ contained in $\mathcal{P}$ such that $\mathcal{U}\subset\mathcal{H}$, and let $\mathcal{X}=\mathcal{G}/\mathcal{H}$ and $\mathcal{Y}=\mathcal{G}/\mathcal{P}$. We have $\mathcal{X}\simeq\mathcal{G}\times^{\mathcal{P}}\mathcal{P}/\mathcal{H}$ and $\mathcal{P}/\mathcal{H}\simeq\mathcal{L}/\mathcal{L}\cap\mathcal{H}$. The quotient $\mathcal{P}/\mathcal{H}$ is thus a $\mathcal{L}$-spherical variety. Let $p:\mathcal{X}\to\mathcal{Y}$ be the natural projection. By [2, Lemma 1.2]), any $\mathcal{B}$-orbit of $\mathcal{X}$ is of the form $\mathcal{B}\lambda\mathcal{O}$ for $\lambda\in\mathcal{W}^P$ and $\mathcal{O}$ a $\mathcal{B}_{\mathcal{L}}=\mathcal{B}\cap\mathcal{L}$-orbit in $\mathcal{P}/\mathcal{H}$. Any $\mathcal{B}^-$-orbit of $\mathcal{X}$ is of the form $\mathcal{B}^-\mu\mathcal{O}^-$ for $\mu\in\mathcal{W}^P$ and $\mathcal{O}^-$ a $\mathcal{B}_{\mathcal{L}}^-=\mathcal{B}^-\cap\mathcal{L}$-orbit in $\mathcal{P}/\mathcal{H}$.

Lemma 4.5. *Let $b\in\mathcal{B}$ and $b_-\in\mathcal{B}^-$, let $\lambda,\mu\in\mathcal{W}^{\mathcal{P}}$, let $\mathcal{O}=\mathcal{B}_{\mathcal{L}}\nu\cdot\mathcal{H}$ with $\nu\in\mathcal{L}$ a $\mathcal{B}_{\mathcal{L}}$-orbit in $\mathcal{P}/\mathcal{H}$ and let $\mathcal{O}^-=\mathcal{B}_{\mathcal{L}}^-\xi\cdot\mathcal{H}$ with $\xi\in\mathcal{L}$ a $\mathcal{B}_{\mathcal{L}}^-$-orbit in $\mathcal{P}/\mathcal{H}$.*

1. *We have $\mathcal{B}\lambda\mathcal{O}=\mathcal{B}\lambda\nu\cdot\mathcal{H}$ and $\mathcal{B}^-\mu\mathcal{O}^-=\mathcal{B}^-\mu\xi\cdot\mathcal{H}$.*
2. *The intersection $p^{-1}(b\lambda\nu\cdot\mathcal{P})\cap\mathcal{B}\lambda\mathcal{O}$ is $b\lambda\nu(\mathcal{P}\cap\mathcal{B}^{(\lambda\nu)^{-1}})\cdot\mathcal{H}$.*
3. *The intersection $p^{-1}(b_-\mu\xi\cdot\mathcal{P})\cap\mathcal{B}^-\mu\mathcal{O}^-$ is $b_-\mu\xi(\mathcal{P}\cap(\mathcal{B}^-)^{(\mu\xi)^{-1}})\cdot\mathcal{H}$.*
4. *Assume that $b\lambda\nu\cdot\mathcal{P}=b_-\mu\xi\cdot\mathcal{P}$. Let $\zeta=(b\lambda\nu)^{-1}(b_-\mu\xi)$. Then we have*

$$p^{-1}(b\lambda\nu\cdot\mathcal{P})\cap\mathcal{B}\lambda\mathcal{O}\cap\mathcal{B}^-\mu\mathcal{O}^-=b\lambda\nu(\mathcal{P}\cap\mathcal{B}^{(\lambda\nu)^{-1}})\cdot\mathcal{H}\cap b\lambda\nu(\mathcal{P}\cap\zeta(\mathcal{B}^-)^{(\mu\xi)^{-1}})\cdot\mathcal{H}.$$

5. *Under the isomorphism $b\lambda\nu\mathcal{P}\cdot\mathcal{H}\simeq\mathcal{P}/\mathcal{H}\simeq\mathcal{L}/(\mathcal{L}\cap\mathcal{H})$, we have*

$$p^{-1}(b\lambda\nu\cdot\mathcal{P})\cap\mathcal{B}\lambda\mathcal{O}\cap\mathcal{B}^-\mu\mathcal{O}^-\simeq\Big({\mathcal{B}_{\mathcal{L}}}^{\nu^{-1}}\cdot(\mathcal{L}\cap\mathcal{H})\Big)\cap\Big(\zeta_l({\mathcal{B}_{\mathcal{L}}}^-)^{\xi^{-1}})\cdot(\mathcal{L}\cap\mathcal{H})\Big),$$

where $\zeta=\zeta_l\zeta_u$ with $\zeta_l\in\mathcal{L}$ and $\zeta_u\in\mathcal{U}$.

Proof. 1. Let $b_1\lambda b_2\nu\cdot\mathcal{H}\in\mathcal{B}\lambda\mathcal{O}$ with $b_1\in\mathcal{B}$ and $b_2\in\mathcal{B}_{\mathcal{L}}$. Since $\lambda\in\mathcal{W}^{\mathcal{P}}$, we have $\lambda b_2\lambda^{-1}\in\mathcal{B}$ and $b_1\lambda b_2\lambda^{-1}\lambda\nu\cdot\mathcal{H}\in\mathcal{B}\lambda\nu\cdot\mathcal{H}$. This proves $\mathcal{B}\lambda\mathcal{O}=\mathcal{B}\lambda\nu\cdot\mathcal{H}$. A similar argument proves the second equality.

2. We have $p^{-1}(b\lambda\nu\cdot\mathcal{P})\cap\mathcal{B}\lambda\nu\cdot\mathcal{H}=b\lambda\nu\mathcal{P}\cdot\mathcal{H}\cap\mathcal{B}\lambda\nu\cdot\mathcal{H}$ and the equality follows.

3. A similar argument as in 2. proves the result.

4. Follows from 2., 3. and the equality

$$b_-\mu\xi(\mathcal{P}\cap(\mathcal{B}^{-1})^{(\mu\xi)^{-1}})\cdot\mathcal{H}=b\lambda\nu(\mathcal{P}\cap\zeta(\mathcal{B}^{-1})^{(\mu\xi)^{-1}})\cdot\mathcal{H}.$$

5. Let $p\in\mathcal{P}$; then there is a unique decomposition $p=p_lp_u$ with $p_l\in\mathcal{L}$ and $p_u\in\mathcal{U}$ and the map $\mathcal{P}/\mathcal{H}\to\mathcal{L}/\mathcal{L}\cap\mathcal{H}$ is given by $p\cdot\mathcal{H}\mapsto p_l\cdot(\mathcal{L}\cap\mathcal{H})$. Furthermore, the map $p\mapsto p_l$ is multiplicative and maps $\mathcal{H}$ to $\mathcal{L}\cap\mathcal{H}$.

Since $\lambda,\mu\in\mathcal{W}^{\mathcal{P}}$, we have $\mathcal{B}^{\lambda^{-1}}\cap\mathcal{L}=\mathcal{B}_{\mathcal{L}}$ and $(\mathcal{B}^-)^{\mu^{-1}}\cap\mathcal{L}=\mathcal{B}_{\mathcal{L}}^-$. Furthermore, since $\nu,\xi\in\mathcal{L}$, we have

$$\mathcal{B}^{(\lambda\nu)^{-1}}\cap\mathcal{L}=\mathcal{B}_{\mathcal{L}}{}^{\nu^{-1}}\text{ and }(\mathcal{B}^-)^{(\mu\xi)^{-1}}\cap\mathcal{L}=\mathcal{B}_{\mathcal{L}}^-{}^{\xi^{-1}}.$$

Now for $p\in\mathcal{P}\cap\mathcal{B}^{(\lambda\nu)^{-1}}$ and $q\in\mathcal{P}\cap\mathcal{B}^{-(\mu\xi)^{-1}}$ we have $p_l\in\mathcal{B}^{(\lambda\nu)^{-1}}\cap\mathcal{L}=\mathcal{B}_{\mathcal{L}}{}^{\nu^{-1}}$ and $q_l\in\mathcal{B}^{-(\mu\xi)^{-1}}\cap\mathcal{L}=\mathcal{B}_{\mathcal{L}}^-{}^{\xi^{-1}}$.

Let $b\lambda\nu p\cdot\mathcal{H}\in p^{-1}(b\lambda\nu\cdot\mathcal{P})\cap\mathcal{B}\lambda\mathcal{O}\cap\mathcal{B}^-\mu\mathcal{O}^-$. Then $b\lambda\nu p\cdot\mathcal{H}$ is mapped to $p_l\cdot(\mathcal{L}\cap\mathcal{H})$ in $\mathcal{L}/(\mathcal{L}\cap\mathcal{H})$. Furthermore, according to 4., there are elements $h_1,h_2\in\mathcal{H}$ such that $ph_1\in(\mathcal{B})^{(\lambda\nu)^{-1}}$ and $\zeta^{-1}ph_2\in\mathcal{B}^{-(\mu\xi)^{-1}}$. Then $p_l(h_1)_l=(ph_1)_l\in\mathcal{B}_{\mathcal{L}}{}^{\nu^{-1}}$ and $\zeta_l^{-1}p_l(h_2)_l=(ph_2)_l\in\mathcal{B}_{\mathcal{L}}^-{}^{\xi^{-1}}$. Since $(h_1)_l,(h_2)_l\in(\mathcal{L}\cap\mathcal{H})$, we have $p_l\cdot(\mathcal{L}\cap\mathcal{H})=p_l(h_1)_l\cdot(\mathcal{L}\cap\mathcal{H})\in\mathcal{B}_{\mathcal{L}}{}^{\nu^{-1}}\cdot(\mathcal{L}\cap\mathcal{H})$ and $p_l\cdot(\mathcal{L}\cap\mathcal{H})=p_l(h_2)_l\cdot(\mathcal{L}\cap\mathcal{H})\in\zeta_l(\mathcal{B}_{\mathcal{L}}{}^-)^{\xi^{-1}}\cdot(\mathcal{L}\cap\mathcal{H})$. Therefore

$$p_l\cdot(\mathcal{L}\cap\mathcal{H})\in\mathcal{B}_{\mathcal{L}}{}^{\nu^{-1}}\cdot((\mathcal{L}\cap\mathcal{H})\cap\zeta_l(\mathcal{B}_{\mathcal{L}}{}^-)^{\xi^{-1}})\cdot(\mathcal{L}\cap\mathcal{H}).$$

The converse inclusion is easy. □

We apply the above result to the following situation. Let X be a G-embedding and $\Omega=(G\times G)\cdot x$ such that the stabiliser H of x is as given in Proposition 2.2: $H=(U_IH_J\times U_I^-H_J)\mathrm{diag}(L_K)$. Let $\mathcal{G}=G\times G$, $\mathcal{T}=T\times T$, $\mathcal{B}=B\times B$, $\mathcal{H}=H$ and $\mathcal{P}=P_I\times P_I^-$. We have $\mathcal{U}=U_I\times U_I^-\subset\mathcal{H}$ and $\mathcal{L}=L_I\times L_I=(L_J\times L_K)\times(L_J\times L_K)$.

Corollary 4.6. *1. The fibers of the map $p_\Omega:\Omega\to G/P_I\times G/P_I^-$ are isomorphic to a quotient L of L_K by a central subgroup.*

2. The fibers of the restriction $p_\Omega:\mathring{X}_{u,v}^{w,x}(\Omega)\to p_\Omega(\mathring{X}_{u,v}^{w,x}(\Omega))$ are isomorphic to the intersection of $(B_K\times B_K)^{(u_K,v_K)^{-1}}\cdot 1_L$ and a translate $\zeta(B_K^-\times B_K^-)^{(w_K,x_K)^{-1}}\cdot 1_L$ for some $\zeta\in L$ (depending on the fiber).

Proof. 1. The fibers are isomorphic to $\mathcal{P}/\mathcal{H}\simeq\mathcal{L}/(\mathcal{L}\cap\mathcal{H})=(L_J\times L_K)\times(L_J\times L_K)/(H_J\times H_J)\mathrm{diag}(L_K)$. The last term is isomorphic to a quotient of L_K by a central subgroup (contained in H_J).

2. The $B\times B$-orbit $\mathring{X}_{u,v}(\Omega)$ is of the form $\mathcal{B}(u,v)\cdot\mathcal{H}=\mathcal{B}(u^I,v^I)\mathcal{B}_L(u_I,v_I)\cdot\mathcal{H}$. A similar statement holds for $\mathring{X}^{w,x}(\Omega)$. Applying the above lemma, we get that the fibers are isomorphic to the intersection of $(B_I\times B_I)^{(u_I,v_I)^{-1}}\cdot 1_L$ and a translate $\zeta(B_I^-\times B_I)^{(w_I,x_I)^{-1}}\cdot 1_L$ for some $\zeta\in L$. Since L is a quotient of $L_J\times L_K$ by $(H_J\times H_J)\mathrm{diag}(L_K)$, the result follows. □

Remark 4.7. Note that in L we have $(B_K \times B_K)^{(u_K,v_K)^{-1}} \cdot 1_L = B_K^{u_K^{-1}} B_K^{v_K^{-1}}$ and $\zeta(B_K^- \times B_K^-)^{(w_K,x_K)^{-1}} \cdot 1_L = \zeta \cdot (B_K^{-\,w_K^{-1}} B_K^{-\,x_K^{-1}})$.

Let $\varphi : \widetilde{X} \to X$ be a morphism of G-embeddings with $\widetilde{X}$ toroidal. According to Corollary 2.3 and Proposition 2.2, there exists $\widetilde{x} \in \widetilde{X}$ and $x = \varphi(\widetilde{x})$ such that if $\widetilde{H}$ and H are the stabilisers of $\widetilde{x}$ and x, then

$$\widetilde{H} = (U_{\widetilde{I}} H_{\widetilde{J}} \times U_{\widetilde{I}}^- H_{\widetilde{J}})\mathrm{diag} L_{\widetilde{K}} \text{ and } H = (U_I H_J \times U_I^- H_J)\mathrm{diag} L_K$$

with $(L_{\widetilde{J}}, L_{\widetilde{J}}) \subset H_{\widetilde{J}} \subset (L_{\widetilde{J}}, L_{\widetilde{J}}) Z_{\widetilde{K}}$, $(L_J, L_J) \subset H_J \subset (L_J, L_J) Z_K$ and $K \subset \widetilde{K} = \widetilde{I} \subset I$. Note that $\widetilde{I} = K \cup (\widetilde{I} \cap J)$ and that K and $\widetilde{I} \cap J$ are orthogonal. Applying the above result we get the following corollary.

Corollary 4.8. *Let $\widetilde{\Omega} = (G \times G) \cdot \widetilde{x}$ and $\Omega = (G \times G) \cdot x$.*

1. There is a commutative diagram

$$\begin{tikzcd} \widetilde{\Omega} \arrow[r, "\varphi"] \arrow[d, "p_{\widetilde{\Omega}}"'] & \Omega \arrow[d, "p_\Omega"] \\ G/P_{\widetilde{I}} \times G/P_{\widetilde{I}}^- \arrow[r] & G/P_I \times G/P_I. \end{tikzcd}$$

The fibers of $p_{\widetilde{\Omega}}$ and p_Ω are isomorphic to quotients of $L_K \times L_{\widetilde{I} \cap J}$ and L_K by central subgroups. The morphism between these fibers induced by φ is the morphism induced by the the first projection.

2. Let $u, v, w, x \in W$. There is a commutative diagram

$$\begin{tikzcd} \mathring{\widetilde{X}}{}_{u,v}^{w,x}(\widetilde{\Omega}) \arrow[r, "\varphi"] \arrow[d, "p_{\widetilde{\Omega}}"'] & \mathring{X}_{u,v}^{w,x}(\Omega) \arrow[d, "p_\Omega"] \\ p_{\widetilde{\Omega}}(\mathring{\widetilde{X}}{}_{u,v}^{w,x}(\widetilde{\Omega})) \arrow[r] & p_\Omega(\mathring{X}_{u,v}^{w,x}(\Omega)), \end{tikzcd}$$

with vertical fibers isomorphic to

$$\left(B_K^{u_K^{-1}} B_K^{v_K^{-1}} \cap \widetilde{\zeta}_K \cdot B_K^{-\,w_K^{-1}} B_K^{-\,x_K^{-1}}\right) \times \left(B_{\widetilde{I}\cap J}^{u_{\widetilde{I}\cap J}^{-1}} B_{\widetilde{I}\cap J}^{v_{\widetilde{I}\cap J}^{-1}} \cap \widetilde{\zeta}_{\widetilde{I}\cap J} \cdot B_{\widetilde{I}\cap J}^{-\,w_{\widetilde{I}\cap J}^{-1}} B_{\widetilde{I}\cap J}^{-\,x_{\widetilde{I}\cap J}^{-1}}\right)$$

and

$$B_K^{u_K^{-1}} B_K^{v_K^{-1}} \cap \zeta_K \cdot B_K^{-\,w_K^{-1}} B_K^{-\,x_K^{-1}}.$$

Furthermore, the morphism between these fibers induced by φ is the morphism induced by the first projection.

4.3. **Stratification.** Let X be a proper G-embedding. In this subsection, we prove that the projected generalised Richardson varieties in X form a stratification. For this, according to Corollary 4.4, we can fix a toroidal variety $\widetilde{X}$ together with a proper $G \times G$-equivariant morphism $\varphi : \widetilde{X} \to X$. All the projected generalised Richardson varieties are of the form $\varphi(\widetilde{X}_{u,v}^{w,x}(\widetilde{\Omega}))$ for some orbit $\widetilde{\Omega}$ and elements $u, v, w, x \in W$.

Definition 4.9. 1. For each $G \times G$-orbit in X, we choose a $G \times G$-orbit $\widetilde{\Omega}$ in $\widetilde{X}$ such that $\widetilde{\Omega}$ is minimal in $\varphi^{-1}(\Omega)$. We define $I, J, K, \widetilde{I}, \widetilde{J}, \widetilde{K}$ as the subsets of simple roots such that

- $\Omega \simeq (G \times G)/H$ with $H = (U_I H_J \times U_I^- H_J)\mathrm{diag} L_K$ and $(L_J, L_J) \subset H_J \subset (L_J, L_J) Z_K$.
- $\widetilde{\Omega} \simeq (G \times G)/\widetilde{H}$ with $\widetilde{H} = (U_{\widetilde{I}} H_{\widetilde{J}} \times U_{\widetilde{I}}^- H_{\widetilde{J}})\mathrm{diag} L_{\widetilde{I}}$ and $H_{\widetilde{J}} \subset Z_{\widetilde{I}}$.

Recall from Corollary 2.3 that we have $\widetilde{J} = \emptyset$ and $K = \widetilde{K} = \widetilde{I}$ and that the roots in J and K are orthogonal. We write $\psi : G/P_K \times G/P_K^- \to G/P_I \times G/P_I^-$.

2. The set $\mathfrak{R}_X$ is the set of tuples (Ω, u, v, w, x) with Ω a $G \times G$-orbit of X and $u, v, w, x \in W$ with $u = u^I$ and $x = x^I$.

3. For Ω a $G \times G$-orbit in X and $u, v, w, x \in W$, we define

- $\mathring{\widetilde{R}}{}^{w,x}_{u,v}(\widetilde{\Omega}) = (BuP_K \cap B^- wP_K)/P_K \times (BvP_K^- \cap B^- xP_K^-)/P_K^-$ and $\widetilde{R}^{w,x}_{u,v}(\widetilde{\Omega})$ its closure.
- $\mathring{R}^{w,x}_{u,v}(\Omega) = (BuP_I \cap B^- wP_I)/P_I \times (BvP_I^- \cap B^- xP_I^-)/P_I^-$ and $R^{w,x}_{u,v}(\Omega)$ its closure.
- $\mathring{PR}{}^{w,x}_{u,v}(\Omega) = \psi(\mathring{\widetilde{R}}{}^{w,x}_{u,v}(\widetilde{\Omega}))$ and $PR^{w,x}_{u,v}(\Omega) = \psi(\widetilde{R}^{w,x}_{u,v}(\widetilde{\Omega}))$ its closure.
- $\Pi^{w,x}_{u,v}(\Omega) = \varphi(\widetilde{X}^{w,x}_{u,v}(\widetilde{\Omega}))$ and $\mathring{\Pi}^{w,x}_{u,v}(\Omega) = \varphi(\mathring{\widetilde{X}}{}^{w,x}_{u,v}(\widetilde{\Omega}))$.

Lemma 4.10. *Let $(\Omega, u, v, w, x) \in \mathfrak{R}_X$ and $\widetilde{\Omega}$ be as above.*

1. Consider the commutative diagram and $\widetilde{y} \in G/P_{\widetilde{I}} \times G/P_{\widetilde{I}}^-$:

$$\begin{tikzcd}
\widetilde{\Omega} \arrow[r, "\varphi"] \arrow[d, "p_{\widetilde{\Omega}}"'] & \Omega \arrow[d, "p_\Omega"] \\
G/P_{\widetilde{I}} \times G/P_{\widetilde{I}}^- \arrow[r, "\psi"] & G/P_I \times G/P_I.
\end{tikzcd}$$

The map $p_{\widetilde{\Omega}}^{-1}(\widetilde{y}) \to p_\Omega^{-1}(\psi(\widetilde{y}))$ induced by φ is an isomorphism.

2. Consider the commutative diagram and $\widetilde{y} \in \mathring{\widetilde{R}}{}^{w,x}_{u,v}(\widetilde{\Omega})$:

$$\begin{tikzcd}
\mathring{\widetilde{X}}{}^{w,x}_{u,v}(\widetilde{\Omega}) \arrow[r, "\varphi"] \arrow[d, "p_{\widetilde{\Omega}}"'] & \mathring{X}^{w,x}_{u,v}(\Omega) \arrow[d, "p_\Omega"] \\
p_{\widetilde{\Omega}}(\mathring{\widetilde{X}}{}^{w,x}_{u,v}(\widetilde{\Omega})) \arrow[r] & p_\Omega(\mathring{X}^{w,x}_{u,v}(\Omega)).
\end{tikzcd}$$

The map $p_{\widetilde{\Omega}}^{-1}(\widetilde{y}) \to p_\Omega^{-1}(\psi(\widetilde{y}))$ induced by φ is an isomorphism.

Proof. 1. Since φ is surjective, the map $p_{\widetilde{\Omega}}^{-1}(\widetilde{y}) \to p_\Omega^{-1}(\psi(\widetilde{y}))$ is surjective. According to Proposition 2.2 and Corollary 2.3, we can write $\widetilde{\Omega} = G \times G/\widetilde{H}$ and $\Omega = G \times G/H$ such that $\widetilde{H} = (U_K \widetilde{H}_J \times U_K^- \times \widetilde{H}_J)\mathrm{diag}(L_K)$ and $H = (U_I H_J \times U_I^- \times H_J)\mathrm{diag}(L_K)$ with $\widetilde{H}_J \subset Z_K$, $(L_J, L_J) \subset H_J \subset (L_J, L_J) Z_K$, $I = J \cup K$ and J orthogonal to K. The fibers of $p_{\widetilde{\Omega}}$ and p_Ω are therefore isomorphic to $L_K/\widetilde{H}_J$ and L_K/H' for H' some subgroup of Z_K. It follows that the map $p_{\widetilde{\Omega}}^{-1}(\widetilde{y}) \to p_\Omega^{-1}(\psi(\widetilde{y}))$ induced by φ is surjective with fiber isomorphic to the subgroup $H'/\widetilde{H}_J$ of Z_K. It also follows that for $\mathring{\widetilde{X}}_{u',v'}(\widetilde{\Omega})$ a $B \times B$-orbit in $\widetilde{\Omega}$, the fiber of the map $\mathring{\widetilde{X}}_{u',v'}(\widetilde{\Omega}) \to \mathring{X}_{u',v'}(\Omega)$ contains $H'/\widetilde{H}_J$. We prove that this subgroup must be trivial.

Recall from [14, Corollary 3.3] that if a homogeneous spherical variety is such that the stabiliser in a Borel subgroup of a general point is connected, then so is the stabiliser in a Borel subgroup of any point. In particular, this holds for G-embeddings and their $G \times G$-orbits. Let $x \in \mathring{X}_{u,v}(\Omega)$ and let $(B \times B)_x$ be its stabiliser in $B \times B$. The later is connected. Since it is solvable, it therefore acts with a fixed point $\widetilde{x}$ on the fiber $\varphi^{-1}(x) \cap \widetilde{\Omega}$ which is closed. The $B \times B$-orbit of $\widetilde{x}$, which is of the form $\mathring{\widetilde{X}}_{u',v'}(\widetilde{\Omega})$, is therefore isomorphic to $\mathring{X}_{u,v}(\Omega)$ via φ. In particular $H'/\widetilde{H}_J$ is trivial.

2. Follows from 1. and Corollary 4.8. $\square$

We will need the following result generalising Theorem 3.6 in [12] (see also Theorem 7.1 in [20]).

Lemma 4.11. *Let $Q \subset P \subset G$ be parabolic subgroups containing B and let $p_{Q,P} : G/Q \to G/P$ be the projection. If $\mathring{R}^w_u(Q)$ and $\mathring{R}^w_u(P)$ denote the open Richardson variety $(BuQ/Q) \cap (B^-wQ/Q)$ and $BuP/P \cap B^-wP/P$, then for $u \in W^P$ and $w \in W$, we have the equality*

$$\mathring{R}^w_u(P) = \coprod_{\substack{w' \in W^Q \\ (w')^P = w^P}} p_{Q,P}(\mathring{R}^{w'}_u(Q)).$$

Proof. Since $p_{Q,P}(R^{w'}_{u'}(Q)) \subset R^{w'}_{u'}(P)$ and $R^{w'}_{u'}(P) = R^{w'}_{u'}(P)$ for ${u'}^P = u^P$ and ${w'}^P = w^P$, we have the inclusion:

$$\bigcup_{\substack{w' \in W^Q \\ (w')^P = w^P}} p_{Q,P}(\mathring{R}^{w'}_u(Q)) \subset \mathring{R}^w_u(P).$$

Consider the commutative diagram

$$\begin{tikzcd} G/B \arrow[r, "p_{B,Q}"] \arrow[dr, "p_{B,P}"'] & G/Q \arrow[d, "p_{Q,P}"] \\ & G/P \end{tikzcd}$$

and denote by $\mathring{R}^w_u(B)$ the open Richardson variety $BuB/B \cap B^-wB/B$ in G/B. The same argument as above together with [12, Theorem 3.6] and the fact that $u = u^P = u^Q$ gives

$$\mathring{R}^{w'}_u(Q) = \coprod_{\substack{w'' \in W \\ {w''}^Q = {w'}^Q}} p_{B,Q}(\mathring{R}^{w''}_u(B)) \text{ and } \mathring{R}^w_u(P) = \coprod_{\substack{w''' \in W \\ {w'''}^P = w^P}} p_{B,P}(\mathring{R}^{w'''}_u(B)).$$

Note also that $p_{B,P}(\mathring{R}^{w''}_u(B)) = p_{Q,P}(p_{B,Q}(\mathring{R}^{w''}_u(B)))$ and that these locally closed subvarieties are disjoint for $u \in W^P$ fixed (see [12, Theorem 3.6] again). This implies

$$p_{Q,P}(\mathring{R}^{w'}_u(Q)) = \coprod_{\substack{w'' \in W \\ (w'')^Q = {w'}^Q}} p_{B,P}(\mathring{R}^{w''}_u(B)).$$

We get

$$\mathring{R}_u^w(P) = \coprod_{\substack{w'''\in W\\ w'''^P=w^P}} p_{B,P}(\mathring{R}_u^{w'''}(B)) = \coprod_{\substack{w'\in W^Q\\ w'^P=w^P}} \coprod_{\substack{w'''\in W\\ w'''^Q=w'^Q}} p_{B,P}(\mathring{R}_u^{w'''}(B)),$$

and the result follows. ☐

Proposition 4.12. *The family* $(\Pi_{u,v}^{w,x}(\Omega))_{(\Omega,u,v,w,x)\in\mathfrak{R}_X}$ *is a stratification of* X.

Proof. We prove the equality

$$X = \coprod_{(\Omega,u,v,w,x)\in\mathfrak{R}_X} \mathring{\Pi}_{u,v}^{w,x}(\Omega).$$

Let $x\in X$ and let Ω contain x. Fix a $G\times G$-orbit $\widetilde{\Omega}$ minimal in $\varphi^{-1}(\Omega)$. There exist $u,v,w,x\in W$ such that $(\Omega,u,v,w,x)\in\mathfrak{R}_X$ and $x\in\mathring{X}_{u,v}^{w,x}(\Omega)$. Let $y=p_\Omega(x)$. We have $y\in\mathring{R}_{u,v}^{w,x}(\Omega)$ and by the former lemma there exist uniquely determined elements $v',w'\in W^K$ with $v'^I=v^I$ and $w'^I=w^I$ such that $y\in\psi(\mathring{\widetilde{R}}_{u,v'}^{w',x}(\widetilde{\Omega}))$. Let $\widetilde{y}\in\mathring{\widetilde{R}}_{u,v'}^{w',x}(\widetilde{\Omega})$ with $\psi(\widetilde{y})=y$. Note that we also have $\mathring{R}_{u,v}^{w,x}(\Omega)=\mathring{R}_{u,v'}^{w',x}(\Omega)$. Let $v''=v'v_K$ and $w''=w'w_K$. We have $\mathring{X}_{u,v}^{w,x}(\Omega)=\mathring{X}_{u,v''}^{w'',x}(\Omega)$ and $\mathring{\widetilde{R}}_{u,v'}^{w',x}(\widetilde{\Omega})=\mathring{\widetilde{R}}_{u,v''}^{w'',x}(\widetilde{\Omega})$. By Lemma 4.10, there exists an element $\widetilde{x}\in\mathring{\widetilde{X}}_{u,v''}^{w'',x}(\widetilde{\Omega})$ with $p_{\widetilde{\Omega}}(\widetilde{x})=\widetilde{y}$ and $\varphi(\widetilde{x})=x$. It follows that $x\in\mathring{\Pi}_{u,v''}^{w'',x}(\Omega)$ with $(\Omega,u,v'',w'',x)\in\mathfrak{R}_X$ uniquely determined. ☐

Proposition 4.13. *For* $(\Omega,u,v,w,x)\in\mathfrak{R}_X$, *the variety* $\mathring{\Pi}_{u,v}^{w,x}(\Omega)$ *is smooth.*

Proof. Since $u=u^I$ and $x=x^I$, in the commutative diagram

$$\begin{tikzcd}
\mathring{\widetilde{X}}_{u,v}^{w,x}(\widetilde{\Omega}) \arrow[r, "\varphi"] \arrow[d, "p_{\widetilde{\Omega}}"'] & \mathring{X}_{u,v}^{w,x}(\Omega) \arrow[d, "p_\Omega"] \\
\mathring{\widetilde{R}}_{u,v}^{w,x}(\widetilde{\Omega}) \arrow[r, "\psi"] & P\mathring{R}_{u,v}^{w,x}(\Omega),
\end{tikzcd}$$

the map ψ is an isomorphism. By Lemma 4.10 so is the map φ on its image $\mathring{\Pi}_{u,v}^{w,x}(\Omega)$. Since $\mathring{\widetilde{X}}_{u,v}^{w,x}(\widetilde{\Omega})$ is smooth the result follows. ☐

5. Frobenius splittings

5.1. **Existence of a splitting.** Let X be a G-embedding; then X admits a $B\times B$-canonical splitting (see [6, Theorem 6.2.7]). In [10], He and Thomsen exhibit many compatibly split subvarieties of a particular splitting. We recall their results. Write D_α for $X_{w_o s_\alpha,1}(G)$ and $\widetilde{D}_\alpha$ for $X^{w_o s_\alpha,1}(G)$ (recall that G is the dense orbit in X).

Proposition 5.1. *There exists a splitting of* X *compatibly splitting the irreducible* $G\times G$*-divisors, the divisors* $(D_\alpha)_{\alpha\in I}$ *and the divisors* $(\widetilde{D}_\alpha)_{\alpha\in I}$. *For* X *complete, this splitting is unique.*

This splitting is a $(p-1)$*-th power of a global section of* ω_X^{-1}. *It compatibly splits the projected generalised Richardson varieties.*

Proof. We start with X toroidal. The existence of this splitting (and the fact that it is a $(p-1)$-th power of a global section of ω_X^{-1}) is proved in [6, Theorem 6.2.7]. The unicity follows from general arguments: let ϕ be a Frobenius splitting compatibly splitting the irreducible $G \times G$-divisors, the divisors $(D_\alpha)_{\alpha\in I}$ and the divisors $(\widetilde{D}_\alpha)_{\alpha\in I}$. This splitting is given by a global section σ of ω_X^{1-p}. From [6, Theorem 1.4.10] it follows that σ is a global section of

$$\mathcal{L} = \omega_X^{1-p}\left(-\sum_j (p-1)X(j) - \sum_{\alpha\in I}(p-1)(D_\alpha + \widetilde{D}_\alpha)\right),$$

where the $X(j)$ are the irreducible $G \times G$-divisors on X. By [6, Proposition 6.2.6] we have $\mathcal{L} \simeq \mathcal{O}_X$. The uniqueness follows.

The second part follows from He and Thomsen's results in [10]. By [10, Proposition 6.5], all generalised Schubert varieties and opposite Schubert varieties are compatibly split as irreducible components of intersections of the compatibly split generalised Schubert divisors. We conclude that all generalised Richardson varieties are compatibly split.

By projection, using [6, Lemma 1.1.8], the result follows for any G-embedding X and any generalised projected Richardson variety. □

5.2. **Normality of projected generalised Richardson varieties.**

Proposition 5.2. *Let X be a toroidal G-embedding and let $X_{u,v}^{w;x}(\Omega)$ be a generalised Richardson variety. Let $\mathcal{L}$ be a globally generated line bundle on X. Then the map $H^0(X,\mathcal{L}) \to H^0(X_{u,v}^{w;x}(\Omega),\mathcal{L})$ is surjective and the groups $H^i(X_{u,v}^{w;x}(\Omega),\mathcal{L})$ vanish for $i > 0$.*

Proof. We may assume that X is projective. Let $X_{u,v}(\Omega)$ be the Schubert variety and let D be an ample $B\times B$-divisor. Then D is a union of irreducible components of $\partial X_{u,v}(\Omega)$, the union of proper generalised Schubert subvarieties in $X_{u,v}(\Omega)$, it does not contain $X_{u,v}^{w;x}(\Omega)$ and is compatibly split. In particular $X_{u,v}(\Omega)$ is $(p-1)D$-split compatibly splitting $X_{u,v}^{w;x}(\Omega)$. By [6, Theorem 1.4.8], we get that the map in cohomology $H^0(X_{u,v}(\Omega),\mathcal{L}) \to H^0(X_{u,v}^{w;x}(\Omega),\mathcal{L})$ is surjective and the cohomology groups $H^i(X_{u,v}^{w;x}(\Omega),\mathcal{L})$ vanish for $i > 0$. By [10, Corollary 8.5], we have that the map $H^0(X,\mathcal{L}) \to H^0(X_{u,v}(\Omega),\mathcal{L})$ is surjective concluding the proof. □

Corollary 5.3. *The projected generalised Richardson varieties are normal.*

Proof. Let $\varphi : \widetilde{X} \to X$ be a morphism of G-embeddings with $\widetilde{X}$ smooth and toroidal. It suffices to prove that the map $\varphi : \widetilde{X}_{u,v}^{w;x}(\Omega) \to \varphi(\widetilde{X}_{u,v}^{w;x}(\Omega))$ is cohomologically trivial. Let $\mathcal{L}$ be an ample line bundle on X. We have the following commutative diagram:

$$\begin{array}{ccc}
H^i(X,\mathcal{L}) & \longrightarrow & H^i(\widetilde{X},\varphi^*\mathcal{L}) \\
\downarrow & & \downarrow \\
H^i(\varphi(X_{u,v}^{w;x}(\Omega)),\mathcal{L}) & \longrightarrow & H^i(X_{u,v}^{w;x}(\Omega),\varphi^*\mathcal{L}).
\end{array}$$

The top horizontal map is an isomorphism because X has rational singularities, while the right vertical map is surjective by the previous proposition. This implies that the bottom horizontal map is surjective (between trivial groups for $i > 0$). By [6, Lemma 3.3.3] we get the result. ☐

5.3. **Compatibly split subvarieties.** Let X be a complete G-embedding.

Theorem 5.4. *The compatible split subvarieties for the splitting obtained in Proposition* 5.1 *are the projected generalised Richardson varieties.*

Proof. We use the following result of Knutson, Lam and Speyer (see [12, Theorem 5.3]): Let X be complete, normal and Frobenius split and $\mathcal{Y}$ a finite collection of compatibly split subvarieties of X defining a stratification and satisfying:

1. each closed stratum $Y \in \mathcal{Y}$ is normal,
2. each open stratum $Y \setminus \bigcup_{Z\in\mathcal{Y}, Z\subsetneq Y} Z$ is regular, and
3. $\partial X = \bigcup_{Y\in\mathcal{Y}, \mathrm{codim}_X Y=1} Y$ is an anticanonical divisor.

Then $\mathcal{Y}$ contains all the compatibly split subvarieties in X, and for each $Y \in \mathcal{Y}$, the union $\bigcup_{Z\in\mathcal{Y}, Z\subsetneq Y} Z$ is an anticanonical divisor.

Let $\mathcal{Y}$ be the family $(\Pi^{w;x}_{u,v}(\Omega))_{(\Omega,u,v,w,x)\in\mathfrak{R}_X}$ of projected generalised Richardson varieties. By Proposition 4.12 the family $\mathcal{Y}$ is a stratification. By Corollary 5.3 projected generalised Richardson varieties are normal and by Proposition 4.13 the open strata are smooth. Furthermore, the divisorial strata are the divisorial generalised Richardson varieties, *i.e.* the divisors stable under $G \times G$, $B \times B$ or $B^- \times B^-$. This is exactly ∂X and it is an anticanonical divisor by [6, Proposition 6.2.6]. The result follows. ☐

Remark 5.5. Note that as a corollary of the above proof we have that any projected generalised Richardson variety is of the form $\Pi^{w;x}_{u,v}(\Omega)$ for $(\Omega, u, v, w, x) \in \mathfrak{R}_X$.

Remark 5.6. A nonirreducible generalised Richardson variety is not a projected generalised Richardson variety. However its irreducible components are projected generalised Richardson varieties.

Corollary 5.7. *The divisor* $\sum[\Pi^{w',x'}_{u',v'}(\Omega')]$*, where the sum runs over all codimension one projected generalised Richardson subvarieties of* $\Pi^{w;x}_{u,v}(\Omega)$*, is an anticanonical divisor in* $\Pi^{w,x}_{u,v}(\Omega)$*.*

Proof. Follows from the above result and [12, Theorem 5.3]. ☐

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