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WEAK MULTIPLIER BIALGEBRAS

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Abstract. A non-unital generalization of weak bialgebra is proposed with a multiplier-valued comultiplication. Certain canonical subalgebras of the multiplier algebra (named the 'base algebras') are shown to carry coseparable co-Frobenius coalgebra structures. Appropriate modules over a nice enough weak multiplier bialgebra are shown to constitute a monoidal category via the (co)module tensor product over the base (co)algebra. The relation to Van Daele and Wang's (regular and arbitrary) weak multiplier Hopf algebra is discussed.

Introduction

The most well-known examples of *Hopf algebras* are the linear spans of (arbitrary) *groups*. Dually, also the vector space of functions on a *finite* group carries the structure of a Hopf algebra. In the case of *infinite* groups, however, the vector space of functions — with finite support — possesses no unit. Consequently, it is no longer a Hopf algebra but, more generally, a *multiplier Hopf algebra* [10]. Replacing groups with *finite groupoids*, both their linear spans and the dual vector spaces of functions carry *weak Hopf algebra* structures [2]. Finally, removing the finiteness constraint in this situation, both the linear spans of arbitrary groupoids and the vector spaces of functions with finite support on them are examples of *weak multiplier Hopf algebras* as introduced in the recent papers [12, 13].

Van Daele's approach to multiplier Hopf algebras is based on the principle of using minimal input data. That is, one starts with a non-unital algebra A with an appropriately well-behaving multiplication and a multiplicative map Δ from A to the multiplier algebra of $A \otimes A$. This allows one to define maps T_1 and T_2 from $A \otimes A$ to the multiplier algebra of $A \otimes A$ as

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b),$$

where 1 stands for the unit of the multiplier algebra of A. (If A is a usual, unital bialgebra over a field k, then these maps are the left and right Galois maps for the A-extension $k \to A$ provided by the unit of A.) The axioms of multiplier Hopf algebras assert first that T_1 and T_2 establish isomorphisms from $A \otimes A$ to $A \otimes A$ regarded as a two-sided ideal in the multiplier algebra. Second, T_1 and T_2 are required to obey $(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})$ (replacing the coassociativity of Δ in the unital case). These axioms are in turn equivalent to the existence of a

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counit and an antipode with the expected properties. In particular, if A has a unit, then it is a multiplier Hopf algebra if and only if it is a Hopf algebra.

A similar philosophy is applied in [13] to define weak multiplier Hopf algebras. Recall that if A is a weak Hopf algebra over a field k with a unit 1, then its comultiplication Δ is not required to preserve 1 (i.e. $\Delta(1)$ may differ from $1 \otimes 1$). Consequently, the maps T_1 and T_2 are no longer linear automorphisms of $A \otimes A$. Instead, they induce isomorphisms between some canonical vector subspaces determined by the element $\Delta(1)$. In the situation when A is allowed to possess no unit, in [13] the role of $\Delta(1)$ is played by an idempotent element in the multiplier algebra of $A \otimes A$, which is meant to be part of the structure. It is used to single out some canonical vector subspaces of $A \otimes A$. The maps T_1 and T_2 are required to induce isomorphisms between these vector subspaces and the same (coassociativity) axiom $(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})$ is imposed. In contrast to the case of non-weak multiplier Hopf algebras, however, these axioms do not seem to imply the existence and the expected properties of the counit and the antipode. Therefore in [13] also the existence of a counit $\epsilon : A \to k$ is assumed (in the sense that $(\epsilon \otimes \mathsf{id})T_1$ and $(\mathsf{id} \otimes \epsilon)T_2$ are equal to the multiplication on A). Adding these counit axioms, the existence of the antipode and *most* of the expected properties of the counit and the antipode do follow. However — at least without requiring that the opposite algebra obeys the same set of axioms, called the *regularity* condition in [13] — some crucial properties seem to be missing (see [13] for several discussions on this issue). Most significantly, in a usual, unital weak Hopf algebra, the counit ϵ is required to obey two symmetrical conditions:

(wm)
$$(\epsilon \otimes \epsilon)((a \otimes 1)\Delta(b)(1 \otimes c)) = \epsilon(abc) = (\epsilon \otimes \epsilon)((a \otimes 1)\Delta^{\mathsf{op}}(b)(1 \otimes c)), \qquad \forall a, b, c \in A.$$

Interestingly enough, the axioms of weak multiplier Hopf algebras in [13] imply the second equality but apparently not the first one (unless regularity is assumed). In this way, even if a weak multiplier Hopf algebra has a unit, it may not be a usual, unital weak Hopf algebra.

One aim of this paper is to identify an intermediate class between regular and arbitrary weak multiplier Hopf algebras in [13]. This class should be big enough to contain any usual weak Hopf algebra. On the other hand, its members should have the expected properties like the (separable Frobenius type) structure of the base algebras.

In fact we take a broader perspective in getting to this goal. If considering monoids instead of groups, their linear spans (and vector spaces of functions in the finite case) are only *bialgebras*, no longer Hopf algebras. Similarly, the linear spans of small categories with finitely many objects (and the vector spaces of functions in the case when also the number of arrows is finite) are only *weak bialgebras* but not weak Hopf algebras. So with the ultimate aim to describe the analogous structures associated to categories without any (or at least with milder) finiteness assumption, we study *weak multiplier bialgebras.* In this case the existence and the appropriate properties of the counit have to be assumed. In Section 2 we propose a set of axioms defining a weak multiplier bialgebra and we present it in several equivalent forms. We show that any *regular* weak multiplier Hopf algebra obeys these axioms and so does any weak bialgebra (with a unit). By generalizing to the multiplier setting several equivalent properties that distinguish bialgebras among weak bialgebras, we also propose a notion of *multiplier bialgebra* (which is, however, different from both notions in [6] and [9] occurring under the same name). In Section 3 and Section 4 we study some distinguished subalgebras of the multiplier algebra of a weak multiplier bialgebra. They generalize the 'source' and 'target' (also called 'right' and 'left') base algebras of a unital weak bialgebra. Whenever the comultiplication is 'full' (in the sense of [13]), they are shown to carry firm Frobenius algebra structures arising from a coseparable co-Frobenius coalgebra in the sense of [1]. In Section 5 we study an appropriate category of modules over a regular weak multiplier bialgebra with a full comultiplication. It is shown to be a monoidal category equipped with a strict monoidal and faithful (in some sense 'forgetful') functor to the category of firm bimodules over the base algebra. In Section 6 we introduce the notion of antipode on a regular weak multiplier bialgebra. Whenever the comultiplication is full, the antipode axioms are shown to be equivalent to the projections of the maps T_1 and T_2, to maps between relative tensor products over the base algebras, being isomorphisms. The resulting structure is equivalent to a weak multiplier Hopf algebra in the sense of [13] such that in addition both conditions in (wm) hold. We claim that this is the desired 'intermediate' class between regular and arbitrary weak multiplier Hopf algebras in which one can answer the questions left open in [13] and which is big enough to contain any unital weak Hopf algebra. Some preliminary information on multiplier algebras is collected in Section 1.

1. Preliminaries on multiplier algebras

For a field k, we denote by $\otimes$ the tensor product of k-vector spaces. Let A be a vector space over k. If there is an associative multiplication

$$\mu : A \otimes A \to A, \qquad a \otimes b \mapsto ab,$$

then we say that A is a *non-unital* algebra. If in addition μ is surjective, then A is said to be an *idempotent* algebra. Furthermore, if any of the conditions ($ab = 0$, $\forall a \in A$) and ($ba = 0$, $\forall a \in A$) implies $b = 0$, then the multiplication μ is termed *non-degenerate*. Clearly, also the algebra A^{op} on the same vector space A with the opposite multiplication is idempotent and non-degenerate whenever A is so. If A and B are idempotent and non-degenerate algebras, then so is $A \otimes B$ with the factorwise multiplication; see e.g. [6, Lemma 1.11].

Let A be a non-unital algebra with a non-degenerate multiplication. A *multiplier* on A [5] is a pair (λ, ρ) of k-linear maps $A \to A$ such that $a\lambda(b) = \rho(a)b$ for all $a, b \in A$. Then it follows that λ is a morphism of right A-modules and ρ is a map of left A-modules. The vector space of multipliers on A — via the componentwise linear structure — is denoted by $\mathbb{M}(A)$. It is an associative algebra via the multiplication $(\lambda', \rho')(\lambda, \rho) = (\lambda'\lambda, \rho\rho')$ (where juxtaposition means composition) and unit $1 = (\mathsf{id}, \mathsf{id})$. Any element a of A can be regarded as a multiplier as $(b \mapsto ab, b \mapsto ba)$. This allows us to regard A as a dense two-sided ideal in $\mathbb{M}(A)$: Indeed, for $(\lambda, \rho) \in \mathbb{M}(A)$ and $a \in A$, $a(\lambda, \rho) = \rho(a)$ and $(\lambda, \rho)a = \lambda(a)$, and — by non-degeneracy of the multiplication — $\rho = 0$ if and only if $\lambda = 0$. Clearly, $\mathbb{M}(A)^{\mathsf{op}} \cong \mathbb{M}(A^{\mathsf{op}})$. If B denotes a second non-unital algebra with a non-degenerate multiplication, then we have algebra embeddings $A \otimes B \subseteq \mathbb{M}(A) \otimes \mathbb{M}(B) \subseteq \mathbb{M}(A \otimes B)$. None of these inclusions will be explicitly denoted throughout the paper. The multiplication in $\mathbb{M}(A)$ will also be denoted by $\mu : \mathbb{M}(A) \otimes \mathbb{M}(A) \to \mathbb{M}(A)$.

Throughout this paper, if X is a set of vectors of some vector space, we will denote by $\langle X \rangle$ the vector subspace linearly spanned by X.

Theorem 1.1 ([12, Proposition A.3]). *Let A and B be non-unital algebras with non-degenerate multiplications and let $\gamma : A \to \mathbb{M}(B)$ be a multiplicative linear map. Assume that there is an idempotent element $e \in \mathbb{M}(B)$ such that*

$$\langle \gamma(a)b \mid a \in A,\ b \in B\rangle = \{eb \mid b \in B\} \quad \textit{and} \quad \langle b\gamma(a) \mid a \in A,\ b \in B\rangle = \{be \mid b \in B\}.$$

Then there is a unique multiplicative linear map $\overline{\gamma} : \mathbb{M}(A) \to \mathbb{M}(B)$ such that $\overline{\gamma}(1) = e$ and $\overline{\gamma}(a) = \gamma(a)$, for all $a \in A$.

If for some map γ there exists an idempotent element e as in Theorem 1.1, then it is clearly unique (cf. [13, Proposition 1.6]).

Let A be a non-unital idempotent algebra with a non-degenerate multiplication and let $\Delta : A \to \mathbb{M}(A \otimes A)$ be a multiplicative linear map. Assume that there is an idempotent element $E \in \mathbb{M}(A \otimes A)$ such that

$$\begin{aligned}&\langle \Delta(a)(b \otimes b') \mid a, b, b' \in A\rangle = \langle E(b \otimes b') \mid b, b' \in A\rangle \quad \text{and}\\ &\langle (b \otimes b')\Delta(a) \mid a, b, b' \in A\rangle = \langle (b \otimes b')E \mid b, b' \in A\rangle\end{aligned}$$

as vector spaces. Then by Theorem 1.1, there exist the extended multiplicative maps $\overline{\Delta} : \mathbb{M}(A) \to \mathbb{M}(A \otimes A)$, $\overline{\Delta \otimes \mathsf{id}} : \mathbb{M}(A \otimes A) \to \mathbb{M}(A \otimes A \otimes A)$ and $\overline{\mathsf{id} \otimes \Delta} : \mathbb{M}(A \otimes A) \to \mathbb{M}(A \otimes A \otimes A)$. If the ranges of the maps

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b), \quad \text{for } a \otimes b \in A \otimes A,$$

belong to $A \otimes A$ and they satisfy $(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})$, then it follows by [12, Proposition A.8] that $(\overline{\mathsf{id} \otimes \Delta})(E) = (\overline{\Delta \otimes \mathsf{id}})(E)$. This allows us to define the idempotent element

$$E^{(3)} := (\overline{\mathsf{id} \otimes \Delta})(E) = (\overline{\Delta \otimes \mathsf{id}})(E) \tag{1.1}$$

in $\mathbb{M}(A \otimes A \otimes A)$.

For any k-vector space A, denote by tw the flip map $A \otimes A \to A \otimes A$, $a \otimes b \mapsto b \otimes a$. For a non-unital algebra A with a non-degenerate multiplication, and for a multiplicative linear map $\Delta : A \to \mathbb{M}(A \otimes A)$, define a multiplicative linear map $\Delta^{\mathsf{op}} : A \to \mathbb{M}(A \otimes A)$ via

$$\Delta^{\mathsf{op}}(a)(b \otimes c) := \mathsf{tw}(\Delta(a)(c \otimes b)) \quad \text{and} \quad (b \otimes c)\Delta^{\mathsf{op}}(a) := \mathsf{tw}((c \otimes b)\Delta(a))$$

and define $\Delta_{13} : A \to \mathbb{M}(A \otimes A \otimes A)$ by

$$\begin{aligned}&\Delta_{13}(a)(b \otimes c \otimes d) := (\mathsf{id} \otimes \mathsf{tw})(\Delta(a)(b \otimes d) \otimes c) \quad \text{and}\\ &(b \otimes c \otimes d)\Delta_{13}(a) := (\mathsf{id} \otimes \mathsf{tw})((b \otimes d)\Delta(a) \otimes c).\end{aligned}$$

2. The weak multiplier bialgebra axioms

The central notion of the paper, weak multiplier bialgebra, is introduced in this section. Several equivalent forms of the axioms are presented, and their first consequences are drawn. Some illustrative examples are collected.

Definition 2.1. A *weak multiplier bialgebra* A over a field k is given by

- an idempotent k-algebra with non-degenerate multiplication $\mu : A \otimes A \to A$,
- an idempotent element E in $\mathbb{M}(A \otimes A)$,
- a multiplicative linear map $\Delta : A \to \mathbb{M}(A \otimes A)$ (called the *comultiplication*),
- and a linear map $\epsilon : A \to k$ (called the *counit*),

which are subject to the axioms below:

(i) For any elements $a, b \in A$, the elements

$$T_1(a \otimes b) := \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) := (a \otimes 1)\Delta(b)$$

of $\mathbb{M}(A \otimes A)$ belong to the two-sided ideal $A \otimes A$.

(ii) The comultiplication is coassociative in the sense that

$$(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id}).$$

(iii) The counit obeys

$$(\epsilon \otimes \mathsf{id})T_1 = \mu = (\mathsf{id} \otimes \epsilon)T_2.$$

(iv) In terms of the idempotent element E,

$$\langle \Delta(a)(b \otimes b') \mid a, b, b' \in A \rangle = \langle E(b \otimes b') \mid b, b' \in A \rangle \quad \text{and}$$
$$\langle (b \otimes b')\Delta(a) \mid a, b, b' \in A \rangle = \langle (b \otimes b')E \mid b, b' \in A \rangle.$$

(v) The idempotent element E satisfies the equality

$$(E \otimes 1)(1 \otimes E) = E^{(3)} = (1 \otimes E)(E \otimes 1)$$

in $\mathbb{M}(A \otimes A \otimes A)$; cf. (1.1).

(vi) For any $a, b, c \in A$,

$$(\epsilon \otimes \mathsf{id})((1 \otimes a)E(b \otimes c)) = (\epsilon \otimes \mathsf{id})(\Delta(a)(b \otimes c)) \quad \text{and}$$
$$(\epsilon \otimes \mathsf{id})((a \otimes b)E(1 \otimes c)) = (\epsilon \otimes \mathsf{id})((a \otimes b)\Delta(c)).$$

It follows immediately from axiom (iv) and the idempotency of E that $E\Delta(a) = \Delta(a) = \Delta(a)E$, for all $a \in A$.

Remark 2.2. In a weak multiplier bialgebra, the idempotent element E and the counit ϵ are uniquely determined in fact by the multiplication μ and the comultiplication Δ. The uniqueness of E follows by the uniqueness of the idempotent element in Theorem 1.1. We will come back to the uniqueness of ϵ later in this section (cf. Theorem 2.8).

Definition 2.3. A weak multiplier bialgebra A is said to be *regular* if also the elements

$$T_3(a \otimes b) := (1 \otimes b)\Delta(a) \quad \text{and} \quad T_4(a \otimes b) := \Delta(b)(a \otimes 1)$$

of $\mathbb{M}(A \otimes A)$ belong to the two-sided ideal $A \otimes A$, for all $a, b \in A$.

For a regular weak multiplier bialgebra A, some of the axioms can be rewritten in the following equivalent forms:

(ii) $\Leftrightarrow (T_4 \otimes \mathsf{id})(\mathsf{id} \otimes T_3) = (\mathsf{id} \otimes T_3)(T_4 \otimes \mathsf{id})$,

(iii) $\Leftrightarrow (\epsilon \otimes \mathsf{id})T_3 = \mu^{\mathsf{op}} = (\mathsf{id} \otimes \epsilon)T_4$.

So a weak multiplier bialgebra A over a field is regular if and only if the opposite algebra A^{op} is a weak multiplier bialgebra too, via the same comultiplication Δ, counit ϵ and idempotent element E.

Below we shall provide some equivalent forms of axiom (vi) in Definition 2.1. In particular, this will allow us to prove the uniqueness of the counit.

Proposition 2.4. *For any weak multiplier bialgebra A over a field, and for any $a \in A$, the linear maps $A \to A$,*

$$b \mapsto (\epsilon \otimes \mathsf{id})T_2(a \otimes b) \quad \textit{and} \quad b \mapsto (\epsilon \otimes \mathsf{id})((a \otimes b)E) \tag{2.1}$$

define a multiplier $\overline{\sqcap}^L(a)$ on A, giving rise to a linear map $\overline{\sqcap}^L : A \to \mathbb{M}(A)$.

Proof. For any $a, b, c \in A$,

$$\begin{aligned} c((\epsilon \otimes \mathsf{id})T_2(a \otimes b)) &= (\epsilon \otimes \mathsf{id})((a \otimes c)\Delta(b)) \\ &\overset{(vi)}{=} (\epsilon \otimes \mathsf{id})((a \otimes c)E(1 \otimes b)) = ((\epsilon \otimes \mathsf{id})((a \otimes c)E))b. \end{aligned}$$

□

Proposition 2.5. *Let A be a weak multiplier bialgebra over a field. For any $a, b \in A$, the following assertions hold.*

(1) $(\mathsf{id} \otimes \overline{\sqcap}^L)T_2(a \otimes b) = (ab \otimes 1)E$ *as elements of* $\mathbb{M}(A \otimes A)$.
(2) $(a \otimes 1)E$ *belongs to the (non-unital) subalgebra* $A \otimes \mathbb{M}(A)$ *of* $\mathbb{M}(A \otimes A)$.
(3) $(a \otimes 1)E(1 \otimes b)$ *belongs to the (non-unital) subalgebra* $A \otimes A$ *of* $\mathbb{M}(A \otimes A)$.

Proof. (1) For any $a, b, p, q \in A$,

$$\begin{aligned} (p \otimes q)((\mathsf{id} \otimes \overline{\sqcap}^L)T_2(a \otimes b)) &= (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})[(T_2(pa \otimes b) \otimes q)(1 \otimes E)] \\ &\overset{(iv)}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})[(T_2(pa \otimes b) \otimes q)(E \otimes 1)(1 \otimes E)] \\ &\overset{(v)}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})[(T_2 \otimes \mathsf{id})(pa \otimes b \otimes q)E^{(3)}] \\ &= (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(T_2 \otimes \mathsf{id})[(pa \otimes b \otimes q)(1 \otimes E)] \\ &\overset{(iii)}{=} (p \otimes q)(ab \otimes 1)E. \end{aligned}$$

In the first equality we used the definition of $\overline{\sqcap}^L$ in Proposition 2.4 and the left A-module map property of T_2. The fourth equality follows by

$$\begin{aligned} ((T_2 \otimes \mathsf{id})(a \otimes b \otimes c))E^{(3)} &\overset{(1.1)}{=} (a \otimes 1 \otimes 1)(\Delta \otimes \mathsf{id})(b \otimes c)(\overline{\Delta \otimes \mathsf{id}})(E) \\ &= (a \otimes 1 \otimes 1)((\Delta \otimes \mathsf{id})((b \otimes c)E)) \\ &= (T_2 \otimes \mathsf{id})((a \otimes b \otimes c)(1 \otimes E)), \end{aligned}$$

for any $a, b, c \in A$.

(2) In the equality in (1), the left hand side belongs to $A \otimes \mathbb{M}(A)$; hence so does the right hand side. Since A is an idempotent algebra by assumption, this proves (2).

(3) follows immediately from (2), since A is an ideal of $\mathbb{M}(A)$. □

Proposition 2.6. *Let A be an idempotent algebra over a field k with a non-degenerate multiplication, $\Delta : A \to \mathbb{M}(A \otimes A)$ a multiplicative linear map, $\epsilon : A \to k$ a linear map and E an idempotent element in $\mathbb{M}(A \otimes A)$. Assume that the axioms (i)-(v) — but not necessarily (vi) — in Definition* 2.1 *hold. The following assertions are equivalent.*[1]

(1) $(\epsilon \otimes \mathsf{id})((a \otimes b)E(1 \otimes c)) = (\epsilon \otimes \mathsf{id})((a \otimes b)\Delta(c))$ *for all* $a, b, c \in A$.
(2) $(\mathsf{id} \otimes \epsilon)((a \otimes 1)E(b \otimes c)) = (\mathsf{id} \otimes \epsilon)(\Delta(a)(b \otimes c))$ *for all* $a, b, c \in A$.
(3) $(a \otimes 1)E(1 \otimes c) \in A \otimes A$ *and* $(\epsilon \otimes \epsilon)((a \otimes 1)E(1 \otimes c)) = \epsilon(ac)$ *for all* $a, c \in A$.
(4) $(\epsilon \otimes \epsilon)((a \otimes 1)\Delta(b)(1 \otimes c)) = \epsilon(abc)$ *for all* $a, b, c \in A$.

[1]The proof of (4) ⇒ (1) was kindly communicated to us by A. Van Daele.

Proof. (1)⇒(3). Note that (1) is in fact the second axiom in Definition 2.1 (vi). Hence the same reasoning used to prove Proposition 2.5 (3) shows that for all $a, b \in A$, $(a \otimes 1)E(1 \otimes b) \in A \otimes A$, so that (1) is equivalent to $(\epsilon \otimes \mathsf{id})((a \otimes 1)E(1 \otimes c)) = (\epsilon \otimes \mathsf{id})T_2(a \otimes c)$ for all $a, c \in A$. Applying ϵ to both sides of this equality and using the counitality axiom (iii), we obtain the equality in (3).

(2)⇒(3). Symmetrically, if (2) holds, then for any $a \in A$ the maps

$$b \mapsto (\mathsf{id} \otimes \epsilon)(E(b \otimes a)) \quad \text{and} \quad b \mapsto (\mathsf{id} \otimes \epsilon)T_1(b \otimes a) \tag{2.2}$$

define a multiplier $\overline{\sqcap}^R(a)$ on A, for which

$$(\overline{\sqcap}^R \otimes \mathsf{id})T_1(a \otimes b) = E(1 \otimes ab), \qquad \forall a, b \in A. \tag{2.3}$$

This proves $E(1 \otimes b) \in \mathbb{M}(A) \otimes A$, hence $(a \otimes 1)E(1 \otimes b) \in A \otimes A$. Then (2) is equivalent to

$$(\mathsf{id} \otimes \epsilon)((a \otimes 1)E(1 \otimes c)) = (\mathsf{id} \otimes \epsilon)T_1(a \otimes c) \tag{2.4}$$

for all $a, c \in A$. Applying ϵ to both sides of this equality and using the counitality axiom (iii), we obtain the equality in (3).

(3)⇒(1). For any $a, b, c \in A$,

$$\begin{aligned}
(\epsilon \otimes \mathsf{id})((a \otimes 1)\ E\ (1 \otimes b))c &\overset{(iii)}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})(\mathsf{id} \otimes T_1)((a \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes b \otimes c)) \\
&\overset{(v)}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})((a \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes E)(1 \otimes T_1(b \otimes c))) \\
&\overset{(iv)}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})((a \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes T_1(b \otimes c))) \\
&\overset{(3)}{=} (\epsilon \otimes \mathsf{id})((a \otimes 1)T_1(b \otimes c)) = ((\epsilon \otimes \mathsf{id})T_2(a \otimes b))c,
\end{aligned}$$

so we conclude by the non-degeneracy of the multiplication in A.

(3)⇒(2) is proven symmetrically.

(1) (and (3)) ⇒(4). For any $a, b, c \in A$,

$$(\epsilon \otimes \epsilon)((a \otimes 1)\Delta(b)(1 \otimes c)) \overset{(1)}{=} (\epsilon \otimes \epsilon)((a \otimes 1)E(1 \otimes bc)) \overset{(3)}{=} \epsilon(abc).$$

(4)⇒(1). For the idea of the reasoning below, we are grateful to Fons Van Daele. In view of axiom (iv) in Definition 2.1, (1) is equivalent to

$$(\epsilon \otimes \mathsf{id})((a \otimes b)\Delta(c)(1 \otimes d)) = (\epsilon \otimes \mathsf{id})((a \otimes b)\Delta(cd)), \ \forall a, b, c, d \in A,$$

hence by the non-degeneracy of the multiplication, also to

$$(\epsilon \otimes \mathsf{id})T_2(a \otimes c)d = (\epsilon \otimes \mathsf{id})T_2(a \otimes cd), \ \forall a, c, d \in A.$$

So we will prove it in this last form. For any $c, d \in A$, denote $c' \otimes d' := T_1(c \otimes d)$ (allowing for implicit summation). Then for any $b \in A$,

$$T_1(bc \otimes d) = \Delta(b)(c' \otimes d') = T_1(b \otimes d')(c' \otimes 1). \tag{2.5}$$

With this information in mind, for any $a, b, c, d \in A$,

$$
\begin{aligned}
((\epsilon \otimes \mathsf{id})T_2(a \otimes b))cd &= (\epsilon \otimes \mathsf{id})[T_2(a \otimes b)(1 \otimes cd)] \\
&\overset{(iii)}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})(\mathsf{id} \otimes T_1)[(T_2(a \otimes b)(1 \otimes c)) \otimes d] \\
&\overset{(2.5)}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})(a \otimes b \otimes d'))(1 \otimes c' \otimes 1)] \\
&\overset{(ii)}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})[((T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(a \otimes b \otimes d'))(1 \otimes c' \otimes 1)] \\
&\overset{(4)}{=} (\epsilon \otimes \mathsf{id})[(a \otimes 1)T_1(b \otimes d')(c' \otimes 1)] \\
&\overset{(2.5)}{=} (\epsilon \otimes \mathsf{id})[(a \otimes 1)T_1(bc \otimes d)] = ((\epsilon \otimes \mathsf{id})T_2(a \otimes bc))d,
\end{aligned}
$$

so we conclude by the non-degeneracy of the multiplication. □

The following symmetric version is immediate.

Proposition 2.7. *Let A be an idempotent algebra over a field k with a non-degenerate multiplication, $\Delta : A \to \mathbb{M}(A \otimes A)$ a multiplicative linear map, $\epsilon : A \to k$ a linear map and E an idempotent element in $\mathbb{M}(A \otimes A)$. Assume that also the ranges of the maps T_3 and T_4 in Definition 2.3 are in the ideal $A \otimes A$ and that the axioms (i)-(v) — but not necessarily (vi) — in Definition 2.1 hold. The following assertions are equivalent.*

1. $(\epsilon \otimes \mathsf{id})((1 \otimes a)E(b \otimes c)) = (\epsilon \otimes \mathsf{id})(\Delta(a)(b \otimes c))$, *for any $a, b, c \in A$.*
2. $(\mathsf{id} \otimes \epsilon)((a \otimes b)E(c \otimes 1)) = (\mathsf{id} \otimes \epsilon)((a \otimes b)\Delta(c))$, *for any $a, b, c \in A$.*
3. $(1 \otimes a)E(c \otimes 1) \in A \otimes A$ *and* $(\epsilon \otimes \epsilon)((1 \otimes a)E(c \otimes 1)) = \epsilon(ac)$, *for any $a, c \in A$.*
4. $(\epsilon \otimes \epsilon)((1 \otimes a)\Delta(b)(c \otimes 1)) = \epsilon(abc)$, *for any $a, b, c \in A$.*

Theorem 2.8. *The counit of a weak multiplier bialgebra A over a field k is uniquely determined by the multiplication and the comultiplication.*

Proof. We have seen in Remark 2.2 that the idempotent element E is uniquely fixed. Let $\epsilon, \epsilon' : A \to k$ be counits for A. Then for all $a, b, c \in A$,

$$
\begin{aligned}
(\epsilon \otimes \epsilon')[(a \otimes 1)\, \Delta(b)\, (1 \otimes c)] &= (\epsilon \otimes \epsilon')[(a \otimes 1)T_1(b \otimes c)] \\
&= (\epsilon \otimes \epsilon \otimes \epsilon')[(a \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes T_1(b \otimes c))] \\
&\overset{(iv)}{=} (\epsilon \otimes \epsilon \otimes \epsilon')[(a \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes E)(1 \otimes T_1(b \otimes c))] \\
&\overset{(v)}{=} (\epsilon \otimes \epsilon \otimes \epsilon')[(a \otimes 1 \otimes 1)(\overline{\mathsf{id} \otimes \Delta})(E(1 \otimes b))(1 \otimes 1 \otimes c)] \\
&= (\epsilon \otimes \epsilon \otimes \epsilon')[(a \otimes 1 \otimes 1)(\mathsf{id} \otimes T_1)(E(1 \otimes b) \otimes c)] \\
&\overset{(iii)}{=} (\epsilon \otimes \epsilon')[(a \otimes 1)E(1 \otimes bc)] = (\epsilon \otimes \epsilon')T_1(a \otimes bc) \overset{(iii)}{=} \epsilon'(abc).
\end{aligned}
$$

In the second and the penultimate equalities we used Proposition 2.6 (3) and (2) (in its alternative form (2.4)) for ϵ and for ϵ', respectively. In the fifth equality we used from the proof of Proposition 2.6 (2)⇒(3) the fact that $E(1 \otimes b) \in \mathbb{M}(A) \otimes A$.

Symmetrically, using Proposition 2.6 (3) for ϵ' in the second equality, Proposition 2.5 (2) in the fifth equality and Proposition 2.6 (1) for ϵ in the penultimate equality,

$$\begin{aligned}
(\epsilon\otimes\epsilon')[(a\otimes 1)\ \Delta(b)\ (1\otimes c)] &= (\epsilon\otimes\epsilon')[T_2(a\otimes b)(1\otimes c)]\\
&= (\epsilon\otimes\epsilon'\otimes\epsilon')[(T_2(a\otimes b)\otimes 1)(1\otimes E)(1\otimes 1\otimes c)]\\
&\overset{(iv)}{=} (\epsilon\otimes\epsilon'\otimes\epsilon')[(T_2(a\otimes b)\otimes 1)(E\otimes 1)(1\otimes E)(1\otimes 1\otimes c)]\\
&\overset{(v)}{=} (\epsilon\otimes\epsilon'\otimes\epsilon')[(a\otimes 1\otimes 1)(\overline{\Delta\otimes \mathsf{id}})((b\otimes 1)E)(1\otimes 1\otimes c)]\\
&= (\epsilon\otimes\epsilon'\otimes\epsilon')[(T_2\otimes \mathsf{id})(a\otimes (b\otimes 1)E)(1\otimes 1\otimes c)]\\
&\overset{(iii)'}{=} (\epsilon\otimes\epsilon')[(ab\otimes 1)E(1\otimes c)] = (\epsilon\otimes\epsilon')T_2(ab\otimes c) \overset{(iii)'}{=} \epsilon(abc),
\end{aligned}$$

where the label (iii)' refers to the application of axiom (iii) to ϵ'. So we conclude by the idempotency of A that $\epsilon = \epsilon'$. $\square$

Two main sources of examples of weak multiplier bialgebras are regular weak multiplier Hopf algebras in [13] and weak bialgebras [2, 7] (possessing units), as we shall see in the next two theorems.

Theorem 2.9. *If an idempotent algebra A over a field with a non-degenerate multiplication possesses a regular weak multiplier Hopf algebra structure in the sense of* [13]*, then A is also a (regular) weak multiplier bialgebra via the same structure maps.*

Proof. Axioms (i), (ii), (iii) and (v) in Definition 2.1 are parts of the definition of regular weak multiplier Hopf algebras in [13]. Since A is an idempotent algebra by assumption, the axioms $E(A\otimes A) = T_1(A\otimes A)$ and $(A\otimes A)E = T_2(A\otimes A)$ in [13] imply our axiom (iv). It remains to prove that axiom (vi) holds true. By [13, Proposition 2.3], for any weak multiplier Hopf algebra A over a field, there exists a linear map $R_1 : A\otimes A\to A\otimes A$ such that $T_1R_1(a\otimes b) = E(a\otimes b)$ for all $a, b\in A$. Then applying $\epsilon\otimes \mathsf{id}$ to both sides and using the counitality axiom (iii) in Definition 2.1, it follows that

$$\mu R_1(a\otimes b) = (\epsilon\otimes \mathsf{id})[E(a\otimes b)],\qquad \forall a, b\in A. \tag{2.6}$$

For any $a, b, c\in A$,

$$T_1[(a\otimes 1)R_1(b\otimes c)] = \Delta(a)(T_1R_1(b\otimes c)) = \Delta(a)E(b\otimes c) \overset{(iv)}{=} \Delta(a)(b\otimes c).$$

Applying $\epsilon\otimes \mathsf{id}$ to both sides and using the counitality axiom (iii) and (2.6),

$$(\epsilon\otimes \mathsf{id})((1\otimes a)E(b\otimes c)) = (\epsilon\otimes \mathsf{id})(\Delta(a)(b\otimes c)).$$

So the first axiom in (vi) holds true. The assumption about regularity — which has not yet been used so far — allows for a symmetric verification of the second axiom in (vi). $\square$

For *arbitrary* weak multiplier Hopf algebras in [13], however, the second axiom in Definition 2.1 (vi) does not seem to hold. Consequences of this will be discussed further in Section 6.

Theorem 2.10. *For a unital algebra A over a field, there is a bijective correspondence between*

- *weak bialgebra structures on A,*
- *and weak multiplier bialgebra structures on A.*

Proof. A unital algebra A is clearly idempotent with a non-degenerate multiplication, and its multiplier algebra $\mathbb{M}(A)$ coincides with A. So in this case the axioms in Definition 2.1 (i) become trivial identities, and any weak multiplier bialgebra structure on A is regular. By axioms (ii) and (iii), a weak multiplier bialgebra structure on A is given by a coassociative counital comultiplication $A \to A \otimes A$ that is a multiplicative map, and a compatible idempotent element of $A \otimes A$. By the uniqueness of the idempotent element E obeying axiom (iv), it follows that $E = \Delta(1)$. Then axiom (v) is the usual weak bialgebra axiom expressing the weak comultiplicativity of the unit. By Proposition 2.6 and Proposition 2.7, axiom (vi) is equivalent to the usual weak bialgebra axiom expressing the weak multiplicativity of the counit (cf. parts (4) of the quoted propositions). □

Among weak bialgebras A over a field, bialgebras are distinguished by the equivalent properties that $\Delta(1) = 1 \otimes 1$, or $\epsilon(ab) = \epsilon(a)\epsilon(b)$ for all $a, b \in A$, or $\overline{\sqcap}^L(a) = \epsilon(a)1$ for all $a \in A$, or $\overline{\sqcap}^R(a) = \epsilon(a)1$ for all $a \in A$. As shown in the next theorem, these properties (in appropriate forms) remain equivalent also for a weak multiplier bialgebra A.

Theorem 2.11. *Let A be a weak multiplier bialgebra over a field. The following assertions are equivalent.*

(1) $E = 1$ *as elements of* $\mathbb{M}(A \otimes A)$.
(2) $\epsilon(ab) = \epsilon(a)\epsilon(b)$ *for all* $a, b \in A$.
(3) $\overline{\sqcap}^L(a) = \epsilon(a)1$ *as elements of* $\mathbb{M}(A)$, *for all* $a \in A$.
(4) $\overline{\sqcap}^R(a) = \epsilon(a)1$ *as elements of* $\mathbb{M}(A)$, *for all* $a \in A$.

Proof. (1)⇒(2). For any $a, b \in A$, $\epsilon(a)\epsilon(b) \overset{(1)}{=} (\epsilon \otimes \epsilon)[(a \otimes 1)E(1 \otimes b)] = \epsilon(ab)$, where the last equality follows by Proposition 2.6 (3).

(2)⇒(1). Using Proposition 2.5 (1) in the first equality, it follows for any $a, b, c, d \in A$ that

$$\begin{aligned}
(ab \otimes 1)E(1 \otimes cd) &= ((\mathsf{id} \otimes \overline{\sqcap}^L)T_2(a \otimes b))(1 \otimes cd) \\
&\overset{(2.1)}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes T_2)(T_2 \otimes \mathsf{id})(a \otimes b \otimes c))(1 \otimes 1 \otimes d)] \\
&= (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})[(T_2(a \otimes b) \otimes 1)(1 \otimes T_1(c \otimes d))] \\
&\overset{(2)}{=} (\mathsf{id} \otimes \epsilon)T_2(a \otimes b) \otimes (\epsilon \otimes \mathsf{id})T_1(c \otimes d) \overset{(iii)}{=} ab \otimes cd,
\end{aligned}$$

from which we conclude by the density of $A \otimes A$ in $\mathbb{M}(A \otimes A)$.

(1)⇒(3). For any $a, b \in A$, $b\overline{\sqcap}^L(a) \overset{(2.1)}{=} (\epsilon \otimes \mathsf{id})[(a \otimes b)E] \overset{(1)}{=} b\epsilon(a)$, from which we conclude by the density of A in $\mathbb{M}(A)$.

(1)⇒(4) follows symmetrically making use of (2.2).

(3)⇒(1). Using Proposition 2.5 (1) in the first equality, it follows for any $a, b \in A$ that

$$(ab \otimes 1)E = (\mathsf{id} \otimes \overline{\sqcap}^L)T_2(a \otimes b) \overset{(3)}{=} (\mathsf{id} \otimes \epsilon)T_2(a \otimes b) \otimes 1 \overset{(iii)}{=} ab \otimes 1,$$

from which we conclude by the density of $A \otimes A$ in $\mathbb{M}(A \otimes A)$.

(4)⇒(1) follows symmetrically applying (2.3). □

Whenever the equivalent conditions in Theorem 2.11 hold for a weak multiplier bialgebra over a field, it would be most natural to term it a *multiplier bialgebra.*

Note, however, that this notion is different from both notions in [6] and [9], which were given the same name.

The next three examples do not belong to any of the previously discussed classes.

Example 2.12. Take a small category possibly with infinitely many objects and arrows. For a fixed field, let A be the vector space spanned by all of the arrows. It can be equipped with the following non-unital algebra structure. For any arrows a and b, let their product be the composite arrow ab if they are composable and zero otherwise. Since the identity arrows of the category give rise to local units, this is an idempotent algebra with a non-degenerate multiplication. It can be equipped with the structure of a regular weak multiplier bialgebra. The comultiplication takes an arrow a to $a \otimes a$ regarded as an element of the multiplier algebra $\mathbb{M}(A \otimes A)$. The counit takes each arrow to 1. The idempotent element E in $\mathbb{M}(A \otimes A)$ is given by $E(a \otimes b) = a \otimes b$ if the arrows a and b have equal targets and $E(a \otimes b) = 0$ otherwise, and $(a \otimes b)E = a \otimes b$ if the arrows a and b have equal sources and $(a \otimes b)E = 0$ otherwise. All these maps are then linearly extended.

Example 2.13. Take again a small category possibly with infinitely many objects and arrows. For a fixed field, let A be the vector space of functions of finite support on the arrow set $\mathcal{S}$. It is a non-unital algebra via pointwise multiplication (i.e. $(fg)(a) := f(a)g(a)$ for every $f, g \in A$ and $a \in \mathcal{S}$). The characteristic functions of the finite subsets of $\mathcal{S}$ serve as local units for A; hence it is an idempotent algebra with a non-degenerate multiplication.

For any arrows a and b of common source, assume that there are only finitely many arrows c such that $ca = b$. Symmetrically, for any arrows a and b of common target, assume that there are only finitely many arrows c such that $ac = b$. (These assumptions evidently hold for a groupoid.) Then A carries the structure of a regular weak multiplier bialgebra. In terms of the characteristic functions δ_p of the one element subsets $\{p\}$ of $\mathcal{S}$, the comultiplication Δ takes $f \in A$ to the multiplier $\Delta(f)$ described by

$$\Delta(f)(g \otimes h) = \sum_{p,q \in \mathcal{S}} g(p)h(q)f(pq)\delta_p \otimes \delta_q = (g \otimes h)\Delta(f)$$

for any $g, h \in A$. Note that in this sum there are only finitely many non-zero terms since g, h and f have finite supports. The maps T_j (for $j \in \{1, 2, 3, 4\}$) land in $A \otimes A$ by the assumption that we made about the set of arrows. The counit takes f to the sum of the values $f(i)$ for the *identity* arrows i (which contains finitely many non-zero terms by assumption). The idempotent element E in $\mathbb{M}(A \otimes A)$ is given by

$$E(g \otimes h) = \sum_{\{p,q \in \mathcal{S} \text{ composable}\}} g(p)h(q)\delta_p \otimes \delta_q = (g \otimes h)E, \quad \forall g, h \in A.$$

It was shown in [13] that whenever the categories in the above examples are groupoids, then both constructions yield regular weak multiplier Hopf algebras in the sense of [13].

Example 2.14. In this example we show that any direct sum of weak multiplier bialgebras over a field — so in particular any infinite direct sum of weak bialgebras over a field — is a weak multiplier bialgebra.

For any index set I, consider a family of idempotent algebras $\{A_j\}_{j \in I}$ over a field k with non-degenerate multiplications μ_j. Let $A := \bigoplus_{j \in I} A_j$ denote the direct sum vector space with the inclusions $i_j : A_j \to A$ and the projections $p_j : A \to A_j$. The elements of A are the I-tuples $\underline{a} = \{a_j \in A_j\}_{j\in I}$ such that $a_j := p_j(\underline{a})$ is non-zero only for finitely many indices $j \in I$. Clearly, A can be equipped with the structure of an idempotent algebra with a non-degenerate multiplication $\mu : \underline{a} \otimes \underline{b} \mapsto \underline{a}\,\underline{b}$, uniquely characterized by $p_j(\underline{a}\,\underline{b}) = a_j b_j$, for any $\underline{a}, \underline{b} \in A$ and $j \in I$ (so that i_j becomes multiplicative as well).

The multiplier algebra of A is isomorphic to the direct (in fact, Cartesian) product $\prod_{j\in I} \mathbb{M}(A_j)$, regarded as a unital algebra via the factorwise multiplication. (Its elements are I-tuples $\{\omega_j \in \mathbb{M}(A_j)\}_{j\in I}$ without any restriction on the number of the non-zero elements.) Indeed, $i_j(A_j)$ is an ideal in A for any $j \in I$. Hence for any $\omega \in \mathbb{M}(A)$, any $j \in I$, and any $a, b \in A_j$,

$$\omega i_j(ab) = \omega(i_j(a)i_j(b)) = (\omega i_j(a))i_j(b)$$

is an element of $i_j(A_j)$. So by the idempotency of A_j, $\omega i_j(a) \in i_j(A_j)$ and, symmetrically, $i_j(a)\omega \in i_j(A_j)$, for any $j \in I$ and $a \in A_j$. This proves the existence of multipliers $\omega_j \in \mathbb{M}(A_j)$ such that

$$i_j(\omega_j a) := \omega i_j(a) \qquad \text{and} \qquad i_j(a\omega_j) := i_j(a)\omega, \qquad \forall a \in A_j.$$

Hence there is a map

$$\varphi : \mathbb{M}(A) \to \prod_{j\in I} \mathbb{M}(A_j), \qquad \omega \mapsto \{\omega_j\}_{j\in I}. \tag{2.7}$$

It has the inverse $\{\omega_j\}_{j\in I} \mapsto \omega$ such that $p_j(\omega\underline{a}) = \omega_j a_j$ and $p_j(\underline{a}\omega) = a_j\omega_j$ for all $\underline{a} \in A$ and $j \in I$.

Let us now take two families $\{A_j\}_{j\in I}$ and $\{B_j\}_{j\in I}$ of idempotent algebras with non-degenerate multiplications, together with a family of multiplicative linear maps $\{\gamma_j : A_j \to \mathbb{M}(B_j)\}_{j\in I}$ and idempotent elements $\{e_j \in \mathbb{M}(B_j)\}_{j\in I}$ such that for all $j \in I$, $\gamma_j(A_j)B_j = e_jB_j$ and $B_j\gamma_j(A_j) = B_je_j$. Then it follows by Theorem 1.1 that there exist unique multiplicative linear maps $\{\overline{\gamma}_j : \mathbb{M}(A_j) \to \mathbb{M}(B_j)\}_{j\in I}$ extending γ_j such that $\overline{\gamma}_j(1_j) = e_j$. Put $A := \bigoplus_{j\in I} A_j$ and $B := \bigoplus_{j\in I} B_j$ as before and in terms of the map (2.7) define

$$e := \varphi^{-1}(\{e_j\}_{j\in I}) \in \mathbb{M}(B) \quad \text{and} \quad \gamma : A \to \mathbb{M}(B), \qquad \underline{a} \mapsto \varphi^{-1}(\{\gamma_j(a_j)\}_{j\in I}).$$

Then for all $j \in I$, $p_j(\gamma(A)B) = \gamma_j(A_j)B_j = e_jB_j = p_j(eB)$, so that $\gamma(A)B = eB$. Symmetrically, $B\gamma(A) = Be$. Thus by Theorem 1.1, γ extends to a unique multiplicative linear map $\overline{\gamma} : \mathbb{M}(A) \to \mathbb{M}(B)$ such that $\overline{\gamma}(1) = e$. Explicitly,

$$p_j(\overline{\gamma}(\omega)\underline{b}) = \overline{\gamma}_j(\omega_j)b_j \qquad \text{and} \qquad p_j(\underline{b}\overline{\gamma}(\omega)) = b_j\overline{\gamma}_j(\omega_j), \tag{2.8}$$

for all $j \in I$, $\underline{b} \in B$ and $\omega \in \mathbb{M}(A)$.

Assume next that for all $j \in I$, A_j carries a weak multiplier bialgebra structure with comultiplication $\Delta_j : A_j \to \mathbb{M}(A_j \otimes A_j)$, counit $\epsilon_j : A_j \to k$ and idempotent element $E_j \in \mathbb{M}(A_j \otimes A_j)$. Since $A \otimes A \cong \bigoplus_{j,l\in I} A_j \otimes A_l$, its multiplier algebra $\mathbb{M}(A \otimes A)$ is isomorphic to $\prod_{j,l\in I} \mathbb{M}(A_j \otimes A_l)$. Hence $\mathbb{M}(A \otimes A)$ has a non-unital subalgebra $\prod_{j\in I}\mathbb{M}(A_j\otimes A_j)$. In terms of the map (2.7), we define

$$\begin{aligned}
\Delta:&\quad A\to\prod_{j\in I}\mathbb{M}(A_j\otimes A_j)\subset\mathbb{M}(A\otimes A), &&\underline{a}\mapsto\varphi^{-1}(\{\Delta_j(a_j)\}_{j\in I})\\
\epsilon:&\quad A\to k, &&\underline{a}\mapsto\sum_{j\in I}\epsilon_j(a_j)\\
E\in&\quad \prod_{j\in I}\mathbb{M}(A_j\otimes A_j)\subset\mathbb{M}(A\otimes A), &&E:=\varphi^{-1}(\{E_j\}_{j\in I}).
\end{aligned}$$

Note that the counit ϵ is well-defined because the sum has only finitely many non-zero terms. This equips A with the structure of a weak multiplier bialgebra. Moreover, if A_j is a regular weak multiplier bialgebra for all $j\in I$, then so is the direct sum $A=\bigoplus_{j\in I}A_j$.

3. The base algebras

Let A be a weak multiplier bialgebra over a field. The aim of this section is to study the properties of the maps

$$\overline{\sqcap}^L: A\to\mathbb{M}(A),\qquad a\mapsto(\epsilon\otimes\mathsf{id})((a\otimes 1)E)$$

in (2.1) and

$$\overline{\sqcap}^R: A\to\mathbb{M}(A),\qquad a\mapsto(\mathsf{id}\otimes\epsilon)(E(1\otimes a))$$

in (2.2) in a remarkable analogy with the unital case. Their images in $\mathbb{M}(A)$ are termed as the *base algebras* (they are indeed subalgebras of $\mathbb{M}(A)$ by Lemma 3.4 below), and they will be investigated further in Section 4.

Lemma 3.1. *For any weak multiplier bialgebra A over a field and any $a,b\in A$,*

$$\epsilon(\overline{\sqcap}^L(a)b)=\epsilon(ab)\qquad\textit{and}\qquad\epsilon(a\overline{\sqcap}^R(b))=\epsilon(ab).$$

Proof. For any $a,b\in A$, $\epsilon(\overline{\sqcap}^L(a)b)\overset{(2.1)}{=}(\epsilon\otimes\epsilon)T_2(a\otimes b)\overset{(iii)}{=}\epsilon(ab)$. The other equality is proven symmetrically. □

Lemma 3.2. *For any weak multiplier bialgebra A over a field and any $a,b\in A$,*

$$\overline{\sqcap}^L(\overline{\sqcap}^L(a)b)=\overline{\sqcap}^L(ab)\qquad\textit{and}\qquad\overline{\sqcap}^R(a\overline{\sqcap}^R(b))=\overline{\sqcap}^R(ab).$$

Proof. Using Lemma 3.1 in the second equality,

$$\overline{\sqcap}^L(\overline{\sqcap}^L(a)b)=(\epsilon\otimes\mathsf{id})[(\overline{\sqcap}^L(a)b\otimes 1)E]=(\epsilon\otimes\mathsf{id})[(ab\otimes 1)E]=\overline{\sqcap}^L(ab),$$

for any $a,b\in A$. The other equality is proven symmetrically. □

Lemma 3.3. *For any weak multiplier bialgebra A over a field and any $a\in A$,*

$$\begin{aligned}
\overline{\Delta}\overline{\sqcap}^L(a)=&\quad(\overline{\sqcap}^L(a)\otimes 1)E&&=E(\overline{\sqcap}^L(a)\otimes 1)\qquad\textit{and}\\
\overline{\Delta}\overline{\sqcap}^R(a)=&\quad(1\otimes\overline{\sqcap}^R(a))E&&=E(1\otimes\overline{\sqcap}^R(a)).
\end{aligned}$$

Proof. For any $a\in A$, $(a\otimes 1)E\in A\otimes\mathbb{M}(A)$ by Proposition 2.5 (2). Hence

$$\begin{aligned}
\overline{\Delta}\overline{\sqcap}^L(a)&=\overline{\Delta}(\epsilon\otimes\mathsf{id})[(a\otimes 1)E]=(\epsilon\otimes\mathsf{id})(\mathsf{id}\otimes\overline{\Delta})[(a\otimes 1)E]\\
&\overset{(v)}{=}(\epsilon\otimes\mathsf{id})[(a\otimes 1\otimes 1)(E\otimes 1)(1\otimes E)]\\
&=((\epsilon\otimes\mathsf{id})[(a\otimes 1)E]\otimes 1)E=(\overline{\sqcap}^L(a)\otimes 1)E.
\end{aligned}$$

All of the remaining equalities are verified analogously. □

Lemma 3.4. *For any weak multiplier bialgebra A over a field, and any $a, b \in A$,*

$$\overline{\sqcap}^L(a\overline{\sqcap}^L(b)) = \overline{\sqcap}^L(a)\overline{\sqcap}^L(b) \qquad \textit{and} \qquad \overline{\sqcap}^R(\overline{\sqcap}^R(a)b) = \overline{\sqcap}^R(a)\overline{\sqcap}^R(b).$$

Proof. For any $\psi \in \mathbb{M}(A)$ such that $\overline{\Delta}(\psi) = (\psi \otimes 1)E$, and for any $a, b \in A$,

$$\begin{aligned}
\overline{\sqcap}^L(a\psi)b &\overset{(2.1)}{=} (\epsilon \otimes \mathsf{id})T_2(a\psi \otimes b) \overset{(iv)}{=} (\epsilon \otimes \mathsf{id})[(a\psi \otimes 1)E\Delta(b)] \\
&= (\epsilon \otimes \mathsf{id})[(a \otimes 1)\Delta(\psi b)] = (\epsilon \otimes \mathsf{id})T_2(a \otimes \psi b) \overset{(2.1)}{=} \overline{\sqcap}^L(a)\psi b,
\end{aligned}$$

where in the third equality we used the assumption about ψ and the multiplicativity of $\overline{\Delta}$. Since A is dense in $\mathbb{M}(A)$, this proves $\overline{\sqcap}^L(a\psi) = \overline{\sqcap}^L(a)\psi$. We conclude by Lemma 3.3 that $\overline{\sqcap}^L(a\overline{\sqcap}^L(b)) = \overline{\sqcap}^L(a)\overline{\sqcap}^L(b)$, for any $a, b \in A$. The other equality is proven symmetrically. □

Lemma 3.5. *For any weak multiplier bialgebra A over a field, and any $a, b \in A$,*

$$\overline{\sqcap}^R(a)\overline{\sqcap}^L(b) = \overline{\sqcap}^L(b)\overline{\sqcap}^R(a).$$

Proof. For any $a, b \in A$,

$$\begin{aligned}
\overline{\sqcap}^R(a)\overline{\sqcap}^L(b) &= (\mathsf{id} \otimes \epsilon)[E(1 \otimes a)](\epsilon \otimes \mathsf{id})[(b \otimes 1)E] \\
&= (\epsilon \otimes \mathsf{id} \otimes \epsilon)[(1 \otimes E)(b \otimes 1 \otimes a)(E \otimes 1)] \\
&= (\epsilon \otimes \mathsf{id} \otimes \epsilon)[(b \otimes 1 \otimes 1)(1 \otimes E)(E \otimes 1)(1 \otimes 1 \otimes a)] \\
&\overset{(v)}{=} (\epsilon \otimes \mathsf{id} \otimes \epsilon)[(b \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes E)(1 \otimes 1 \otimes a)] \\
&= (\epsilon \otimes \mathsf{id})[(b \otimes 1)E](\mathsf{id} \otimes \epsilon)[E(1 \otimes a)] = \overline{\sqcap}^L(b)\overline{\sqcap}^R(a).
\end{aligned}$$

□

Lemma 3.6. *For any weak multiplier bialgebra A over a field, and for any $a, b, c, d \in A$,*

$$(ab \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_1(c \otimes d)) = ((\mathsf{id} \otimes \overline{\sqcap}^L)T_2(a \otimes b))(1 \otimes cd).$$

Proof. Both expressions in the claim are equal to $(ab \otimes 1)E(1 \otimes cd)$; see Proposition 2.5 (1) and the proof of Proposition 2.6 (2)⇒(3). □

Whenever A is a *regular* weak multiplier bialgebra over a field, the above considerations can be repeated in the opposite algebra. That is, we can define the maps

$$\sqcap^L : A \to \mathbb{M}(A), \qquad a \mapsto (\epsilon \otimes \mathsf{id})(E(a \otimes 1)) \tag{3.1}$$

$$\sqcap^R : A \to \mathbb{M}(A), \qquad a \mapsto (\mathsf{id} \otimes \epsilon)((1 \otimes a)E). \tag{3.2}$$

They obey the following properties, for all $a, b, c, d \in A$.

(3.3) $\sqcap^R(a)b = (\mathsf{id} \otimes \epsilon)T_3(b \otimes a)$ and $b \sqcap^L(a) = (\epsilon \otimes \mathsf{id})T_4(a \otimes b)$,

(3.4) $(1 \otimes ab)E = (\sqcap^R \otimes \mathsf{id})T_3(b \otimes a)$ and $E(ab \otimes 1) = (\mathsf{id} \otimes \sqcap^L)T_4(b \otimes a)$,

(3.5) $\epsilon(a \sqcap^L(b)) = \epsilon(ab)$ and $\epsilon(\sqcap^R(a)b) = \epsilon(ab)$,

(3.6) $\sqcap^L(a \sqcap^L(b)) = \sqcap^L(ab)$ and $\sqcap^R(\sqcap^R(a)b) = \sqcap^R(ab)$,

(3.7) $\overline{\Delta} \sqcap^L(a) = (\sqcap^L(a) \otimes 1)E = E(\sqcap^L(a) \otimes 1)$ and
$\overline{\Delta} \sqcap^R(a) = (1 \otimes \sqcap^R(a))E = E(1 \otimes \sqcap^R(a))$,

(3.8) $\sqcap^L(\sqcap^L(a)b) = \sqcap^L(a) \sqcap^L(b)$ and $\sqcap^R(a \sqcap^R(b)) = \sqcap^R(a) \sqcap^R(b)$,

(3.9) $\sqcap^L(a) \sqcap^R(b) = \sqcap^R(b) \sqcap^L(a)$,

(3.10) $((\sqcap^R \otimes \mathsf{id})T_3(a \otimes b))(cd \otimes 1) = (1 \otimes ba)((\mathsf{id} \otimes \sqcap^L)T_4(d \otimes c))$.

In a weak bialgebra possessing a unit, the above maps behave as generalized counits: $\mu(\sqcap^L \otimes \mathsf{id})\Delta = \mathsf{id} = \mu(\mathsf{id} \otimes \sqcap^R)\Delta$ and $\mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})\Delta = \mathsf{id} = \mu^{\mathsf{op}}(\mathsf{id} \otimes \overline{\sqcap}^R)\Delta$. In the following lemma these identities are generalized to weak multiplier bialgebras.

Lemma 3.7. *For a regular weak multiplier bialgebra A over a field, the following equalities hold.*

(1) $\mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})T_3 = \mu^{\mathsf{op}}$.
(2) $\mu^{\mathsf{op}}(\mathsf{id} \otimes \overline{\sqcap}^R)T_4 = \mu^{\mathsf{op}}$.
(3) $\mu(\sqcap^L \otimes \mathsf{id})T_1 = \mu$.
(4) $\mu(\mathsf{id} \otimes \sqcap^R)T_2 = \mu$.

Proof. We spell out the proof only for (1); all other assertions are proven symmetrically. For any $a, b, c \in A$,

$$\begin{aligned}(\mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})T_3(a \otimes b))c &= \mu^{\mathsf{op}}[((\overline{\sqcap}^L \otimes \mathsf{id})T_3(a \otimes b))(c \otimes 1)] \\ &\overset{(2.1)}{=} (\epsilon \otimes \mathsf{id})[(1 \otimes b)\Delta(a)\Delta(c)] = (\epsilon \otimes \mathsf{id})T_3(ac \otimes b) \\ &\overset{(iii)}{=} bac,\end{aligned}$$

so we conclude by non-degeneracy of the multiplication. $\square$

Lemma 3.8. *For a regular weak multiplier bialgebra A over a field, and any $a, b \in A$,*

$$\epsilon(\sqcap^L(a)b) = \epsilon(\overline{\sqcap}^R(b)a) \quad \textit{and} \quad \epsilon(a \sqcap^R(b)) = \epsilon(b\overline{\sqcap}^L(a)).$$

Proof. In light of (3.1) and (2.2), respectively, both sides of the first equality are equal to $(\epsilon \otimes \epsilon)[E(a \otimes b)]$, hence also to each other. The second equality follows symmetrically. $\square$

Lemma 3.9. *For a regular weak multiplier bialgebra A over a field, and any $a \in A$, the following equalities hold.*

$$(\sqcap^R(a) \otimes 1)E = (1 \otimes \overline{\sqcap}^L(a))E \qquad \textit{and} \qquad E(\overline{\sqcap}^R(a) \otimes 1) = E(1 \otimes \sqcap^L(a)).$$

Proof. By axiom (iv) in Definition 2.1, the first equality in the claim is equivalent to

(3.11) $$(\sqcap^R(a) \otimes 1)T_1(b \otimes c) = (1 \otimes \overline{\sqcap}^L(a))T_1(b \otimes c), \quad \forall b, c \in A.$$

Using the identities $c \sqcap^R(a)b = (\mathsf{id} \otimes \epsilon)[(c \otimes a)\Delta(b)] = (\mathsf{id} \otimes \epsilon)[(1 \otimes a)T_2(c \otimes b)]$ and

$$(\epsilon \otimes \mathsf{id})[(a \otimes 1)T_1(b \otimes c)] = (\epsilon \otimes \mathsf{id})[T_2(a \otimes b)(1 \otimes c)] \overset{(2.1)}{=} \overline{\sqcap}^L(a)bc$$

(for any $a, b, c \in A$) in the first and the penultimate equalities, respectively, one computes

$$\begin{aligned}(d \sqcap^R (a) \otimes 1)T_1(b \otimes c) &= (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})[(1 \otimes a \otimes 1)((T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(d \otimes b \otimes c))]\\ &\overset{(ii)}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})[(1 \otimes a \otimes 1)((\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})(d \otimes b \otimes c))]\\ &= (1 \otimes \overline{\sqcap}^L(a))(d \otimes 1)\Delta(b)(1 \otimes c)\\ &= (d \otimes \overline{\sqcap}^L(a))T_1(b \otimes c),\end{aligned}$$

for all $a, b, c, d \in A$. So we conclude by the non-degeneracy of the multiplication that (3.11), and hence $(\sqcap^R(a) \otimes 1)E = (1 \otimes \overline{\sqcap}^L(a))E$, holds for all $a \in A$. The other equality is proven symmetrically. □

Lemma 3.10. *For any regular weak multiplier bialgebra A over a field, and for any $a, b, c, d \in A$,*

$$\begin{aligned}(1 \otimes ab)((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(c \otimes d)) &= ((\sqcap^R \otimes \mathsf{id})T_2^{\mathsf{op}}(a \otimes b))(dc \otimes 1) \quad \textit{and}\\ ((\overline{\sqcap}^L \otimes \mathsf{id})T_3(a \otimes b))(cd \otimes 1) &= (1 \otimes ba)((\mathsf{id} \otimes \sqcap^L)T_1^{\mathsf{op}}(c \otimes d)),\end{aligned}$$

where $T_1^{\mathsf{op}} = \mathsf{tw}\, T_1$ and $T_2^{\mathsf{op}} = \mathsf{tw}\, T_2$.

Proof. For any $a, b, c, d \in A$,

$$\begin{aligned}(1 \otimes 1 \otimes a)((\mathsf{id} \otimes T_1^{\mathsf{op}}\,\mathsf{tw})\ (T_4 \otimes \mathsf{id})(c \otimes d \otimes b))\\ &= (1 \otimes 1 \otimes a)(1 \otimes \Delta^{\mathsf{op}}(b))(\Delta(d) \otimes 1)(c \otimes 1 \otimes 1)\\ &= ((T_3 \otimes \mathsf{id})(\mathsf{id} \otimes T_2^{\mathsf{op}})(d \otimes a \otimes b))(c \otimes 1 \otimes 1).\end{aligned}$$

Applying $\mathsf{id} \otimes \epsilon \otimes \mathsf{id}$ to both sides and using the identities (2.2) and (3.3), we obtain the first equality in the claim. The second equality is proven symmetrically. □

Lemma 3.11. *Let A be a regular weak multiplier bialgebra over a field. For any $a, b \in A$,*

$$\begin{aligned}\sqcap^R(a\overline{\sqcap}^R(b)) = \sqcap^R(a)\overline{\sqcap}^R(b) = \overline{\sqcap}^R(\sqcap^R(a)b) \quad \textit{and}\\ \sqcap^L(\overline{\sqcap}^L(a)b) = \overline{\sqcap}^L(a) \sqcap^L (b) = \overline{\sqcap}^L(a \sqcap^L (b)).\end{aligned}$$

Proof. Applying the multiplicativity of $\overline{\Delta} : \mathbb{M}(A) \to \mathbb{M}(A \otimes A)$, Lemma 3.3 and axiom (iv) in the second equality,

$$T_3(\overline{\sqcap}^R(a)b \otimes c) = (1 \otimes c)\Delta(\overline{\sqcap}^R(a)b) = (1 \otimes c\overline{\sqcap}^R(a))\Delta(b) = T_3(b \otimes c\overline{\sqcap}^R(a)),$$

for any $a, b, c \in A$. Hence using (3.3),

$$\sqcap^R(a)\overline{\sqcap}^R(b)c = (\mathsf{id} \otimes \epsilon)T_3(\overline{\sqcap}^R(b)c \otimes a) = (\mathsf{id} \otimes \epsilon)T_3(c \otimes a\overline{\sqcap}^R(b)) = \sqcap^R(a\overline{\sqcap}^R(b))c.$$

This proves the first equality in the claim, and all other equalities are checked analogously. □

Lemma 3.12. *Let A be a regular weak multiplier bialgebra over a field. For any $a, b \in A$, the following hold:*

$$\begin{aligned}\sqcap^R(a\overline{\sqcap}^L(b)) = \sqcap^R(b) \sqcap^R (a) \qquad & \sqcap^L(\overline{\sqcap}^R(a)b) = \sqcap^L(b) \sqcap^L (a)\\ \overline{\sqcap}^R(\sqcap^L(b)a) = \overline{\sqcap}^R(a)\overline{\sqcap}^R(b) \qquad & \overline{\sqcap}^L(b \sqcap^R (a)) = \overline{\sqcap}^L(a)\overline{\sqcap}^L(b).\end{aligned}$$

Proof. We only prove the first assertion, all of the remaining ones are proven symmetrically. For any $a, b, c \in A$,

$$\sqcap^R(a\overline{\sqcap}^L(b))c \overset{(3.3)}{=} (\mathsf{id}\otimes\epsilon)T_3(c\otimes a\overline{\sqcap}^L(b)) = \sqcap^R(b)(\mathsf{id}\otimes\epsilon)T_3(c\otimes a) \overset{(3.3)}{=} \sqcap^R(b)\sqcap^R(a)c,$$

where the second equality follows by axiom (iv) in Definition 2.1 and Lemma 3.9. So we conclude by the density of A in $\mathbb{M}(A)$. □

In the following theorem, for any vector spaces V and W, $\mathsf{Lin}(V, W)$ denotes the vector space of linear maps $V \to W$.

Theorem 3.13. *For a regular weak multiplier bialgebra A over a field k, the following assertions are equivalent to each other.*

(1) *The comultiplication is* right full *in the sense that*

$$\langle(\mathsf{id}\otimes\omega)T_1(a\otimes b) \mid a, b \in A, \omega \in \mathsf{Lin}(A, k)\rangle = A.$$

(2) *The comultiplication is* right full *in the sense that*

$$\langle(\mathsf{id}\otimes\omega)T_3(a\otimes b) \mid a, b \in A, \omega \in \mathsf{Lin}(A, k)\rangle = A.$$

(3) $\langle(\mathsf{id}\otimes\epsilon)T_1(a\otimes b) \mid a, b \in A\rangle = A$.
(4) $\langle(\mathsf{id}\otimes\epsilon)T_3(a\otimes b) \mid a, b \in A\rangle = A$.
(5) $\{\sqcap^R(a) \mid a \in A\} = \{\overline{\sqcap}^R(a) \mid a \in A\}$.

The following assertions are equivalent to each other, too.

(1)' *The comultiplication is* left full *in the sense that*

$$\langle(\omega\otimes\mathsf{id})T_2(a\otimes b) \mid a, b \in A, \omega \in \mathsf{Lin}(A, k)\rangle = A.$$

(2)' *The comultiplication is* left full *in the sense that*

$$\langle(\omega\otimes\mathsf{id})T_4(a\otimes b) \mid a, b \in A, \omega \in \mathsf{Lin}(A, k)\rangle = A.$$

(3)' $\langle(\epsilon\otimes\mathsf{id})T_2(a\otimes b) \mid a, b \in A\rangle = A$.
(4)' $\langle(\epsilon\otimes\mathsf{id})T_4(a\otimes b) \mid a, b \in A\rangle = A$.
(5)' $\{\sqcap^L(a) \mid a \in A\} = \{\overline{\sqcap}^L(a) \mid a \in A\}$.

Proof. We only prove the equivalence of the first five assertions. The equivalence of the second quintuple is proven analogously.

(1)⇔(2) is proven in [12, Lemma 1.11].

(3)⇒(1) and (4)⇒(2) are trivial.

(1) and (2)⇒(5). For any $a, b \in A$, it follows by (3.4) and (2.3) that

$$(\sqcap^R\otimes\mathsf{id})T_1(a\otimes b) = (1\otimes a)E(1\otimes b) = (\overline{\sqcap}^R\otimes\mathsf{id})T_3(b\otimes a).$$

Then (1) implies

$$\{\sqcap^R(a) \mid a \in A\} = \langle(\mathsf{id}\otimes\omega)[(1\otimes a)E(1\otimes b)] \mid a, b \in A, \omega \in \mathsf{Lin}(A, k)\rangle,$$

and (2) implies

$$\{\overline{\sqcap}^R(a) \mid a \in A\} = \langle(\mathsf{id}\otimes\omega)[(1\otimes a)E(1\otimes b)] \mid a, b \in A, \omega \in \mathsf{Lin}(A, k)\rangle$$

proving the claim.

(5)⇒(3). By Lemma 3.7 (4) and the idempotency of the algebra A,

$$\langle a \sqcap^R (b) \mid a, b \in A\rangle = A.$$

Hence by (5),

$$A = \langle a\overline{\sqcap}^R(b) \mid a, b \in A\rangle \stackrel{(2.2)}{=} \langle(\mathsf{id} \otimes \epsilon)T_1(a \otimes b) \mid a, b \in A\rangle.$$

(5)⇒(4) is proven symmetrically. □

4. Firm separable Frobenius structure of the base algebras

In a weak bialgebra with a unit, the coinciding range of the maps $\overline{\sqcap}^L$ and $\sqcap^L$, and the coinciding range of $\overline{\sqcap}^R$ and $\sqcap^R$ in the previous section, carry the structures of anti-isomorphic *separable Frobenius algebras* (with units). The aim of this section is to see that in a regular weak multiplier bialgebra with a left and right full comultiplication, the base algebras still carry the structures of anti-isomorphic coseparable and co-Frobenius coalgebras. Consequently, they are firm Frobenius algebras in the sense of [1].

It follows immediately from Lemma 3.4 that for any weak multiplier bialgebra A, the ranges of $\overline{\sqcap}^L$ and of $\overline{\sqcap}^R$ are non-unital subalgebras of the multiplier algebra $\mathbb{M}(A)$. We turn to proving that — whenever A is regular with a left and right full comultiplication — they carry coalgebra structures as well. First we look for the candidate counit.

Proposition 4.1. *Let A be a regular weak multiplier bialgebra over a field k with a right full comultiplication. Then the counit ϵ determines a linear map*

$$\varepsilon : \{\sqcap^R(a) \mid a \in A\} \to k, \qquad \sqcap^R(a) \mapsto \epsilon(a).$$

Proof. In order to see that ε is a well-defined linear map, we need to show that $\sqcap^R(p) = 0$ implies $\epsilon(p) = 0$, for any $p \in A$. Since A is an idempotent algebra by assumption, we can write any element p of A as a linear combination $\sum_i a^i b^i$ in terms of finitely many elements $a^i, b^i \in A$. So omitting throughout the summation symbol for brevity, it is enough to prove that $\sqcap^R(a^i b^i) = 0$ implies $\epsilon(a^i b^i) = 0$, for any finite set of elements $a^i, b^i \in A$. If $\sqcap^R(a^i b^i) = 0$, then

$$\begin{aligned} 0 &\stackrel{(3.5)}{=} (\mathsf{id} \otimes \epsilon)[(1 \otimes a^i b^i)((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(h \otimes d))(1 \otimes c)] \\ &= (\epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes \sqcap^R)T_2(a^i \otimes b^i))(c \otimes dh)] \\ &= (\epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes \sqcap^R)T_2(a^i \otimes b^i))(\overline{\sqcap}^R(c) \otimes 1)]dh, \end{aligned}$$

for all $c, d, h \in A$. In the second equality above, we used the first statement in Lemma 3.10, and the third equality follows by Lemma 3.1. By Theorem 3.13, the vector subspaces $\sqcap^R(A) := \{\sqcap^R(a) \mid a \in A\}$ and $\overline{\sqcap}^R(A) := \{\overline{\sqcap}^R(a) \mid a \in A\}$ of $\mathbb{M}(A)$ coincide. So by the density of A in $\mathbb{M}(A)$, the map

$$\sqcap^R(A) \to \sqcap^R(A), \qquad \overline{\sqcap}^R(c) \mapsto (\epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes \sqcap^R)T_2(a^i \otimes b^i))(\overline{\sqcap}^R(c) \otimes 1)]$$

is the zero map. We introduce the notation $T_2(a^i \otimes b^i) =: a' \otimes b'$ (allowing for implicit summation on both sides of the equality). Using that the evident map $\mathsf{Lin}(\sqcap^R(A), k) \otimes \sqcap^R(A) \to \mathsf{Lin}(\sqcap^R(A), \sqcap^R(A))$, $\Phi \otimes x \mapsto \Phi(-)x$ is injective, we conclude that

$$\epsilon(a'-) \otimes \sqcap^R(b') \in \mathsf{Lin}(\sqcap^R(A), k) \otimes \sqcap^R(A)$$

is equal to zero. Applying to it the evaluation map $\mathsf{Lin}(\sqcap^R(A), k) \otimes \sqcap^R(A) \to k$, $\Phi \otimes x \mapsto \Phi(x)$, and using Lemma 3.7 (4), we prove that

$$\epsilon(a' \sqcap^R (b')) = \epsilon\mu(\mathsf{id} \otimes \sqcap^R)T_2(a^i \otimes b^i) = \epsilon(a^i b^i)$$

is equal to zero as needed. □

For the construction of the comultiplication on $\sqcap^R(A)$, the following technical lemma is needed.

Lemma 4.2. *Let A be a regular weak multiplier bialgebra over a field. For any $a, b \in A$, the following assertions hold.*

(1) *The element $(\mathsf{id}\otimes\overline{\sqcap}^R)T_4(a\otimes b)$ of $A\otimes\mathbb{M}(A)$ depends on a and b only through the product ba.*
(2) *The element $(\mathsf{id}\otimes\sqcap^R)T_2(a\otimes b)$ of $A\otimes\mathbb{M}(A)$ depends on a and b only through the product ab.*

Proof. We prove only part (1); part (2) follows analogously. Applying twice the first identity in Lemma 3.10, for all $a, b, c, d, f, g \in A$,

$$\begin{aligned}(1 \otimes cd)((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(a \otimes b))(f \otimes g) &= ((\sqcap^R \otimes \mathsf{id})T_2^{\mathsf{op}}(c \otimes d))(baf \otimes g) \\ &= (1 \otimes cd)((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(f \otimes ba))(1 \otimes g).\end{aligned}$$

So if $ba = b'a'$ for some $a, b, a', b' \in A$, then for all $f, g \in A$,

$$\begin{aligned}((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(a \otimes b))(f \otimes g) &= ((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(f \otimes ba))(1 \otimes g) \\ &= ((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(f \otimes b'a'))(1 \otimes g) \\ &= ((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(a' \otimes b'))(f \otimes g),\end{aligned}$$

proving $(\mathsf{id} \otimes \overline{\sqcap}^R)T_4(a \otimes b) = (\mathsf{id} \otimes \overline{\sqcap}^R)T_4(a' \otimes b')$. □

Proposition 4.3. *For a regular weak multiplier bialgebra A over a field, the following assertions hold.*

(1) *The maps $A \otimes A \to A \otimes A$,*

$$\begin{aligned}&a \otimes bc \mapsto ((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(c \otimes b))(a \otimes 1) \quad \textit{and} \\ &ab \otimes c \mapsto (1 \otimes c)((\mathsf{id} \otimes \sqcap^R)T_2(a \otimes b))\end{aligned}$$

(where $T_4^{\mathsf{op}} = \mathsf{tw}T_4$) determine an element of $\mathbb{M}(A \otimes A)$, to be denoted by F.
(2) *For any element $a \in A$ and $F \in \mathbb{M}(A \otimes A)$ as in (1), $(\sqcap^R(a) \otimes 1)F$ and $F(1\otimes\sqcap^R(a))$ are equal elements of $\sqcap^R(A)\otimes\sqcap^R(A)$, to be denoted by $\delta\sqcap^R(a)$.*
(3) *The map*

$$\delta : \sqcap^R(A) \to \sqcap^R(A) \otimes \sqcap^R(A), \qquad \sqcap^R(a) \mapsto (\sqcap^R(a) \otimes 1)F = F(1 \otimes \sqcap^R(a))$$

provides a $\sqcap^R(A)$-bimodule section of the multiplication in $\sqcap^R(A)$.
(4) *The map $\delta : \sqcap^R(A) \to \sqcap^R(A) \otimes \sqcap^R(A)$ in part (3) is a coassociative comultiplication.*

Proof. (1) Both maps in the claim are well-defined by Lemma 4.2, and they define a multiplier by the first statement in Lemma 3.10.

(2) Centrality of F in the $\sqcap^R(A)$-bimodule $\mathbb{M}(A \otimes A)$ follows by the following computation, for all $a, b, c, d \in A$.

$$\begin{aligned}(bc \otimes d)(\sqcap^R(a) \otimes 1)F &= (1 \otimes d)((\mathsf{id} \otimes \sqcap^R)T_2(b \otimes c \sqcap^R (a))) \\ &= (1 \otimes d)((\mathsf{id} \otimes \sqcap^R)(T_2(b \otimes c)\overline{\Delta} \sqcap^R (a))) \\ &= (1 \otimes d)((\mathsf{id} \otimes \sqcap^R)(T_2(b \otimes c)(1 \otimes \sqcap^R(a)))) \\ &= (1 \otimes d)((\mathsf{id} \otimes \sqcap^R)T_2(b \otimes c))(1 \otimes \sqcap^R(a)) \\ &= (bc \otimes d)F(1 \otimes \sqcap^R(a)).\end{aligned}$$

The second equality follows by the explicit form of T_2 and the multiplicativity of $\overline{\Delta}$. In the third equality we used that by (3.7), $\overline{\Delta} \sqcap^R (a) = E(1 \otimes \sqcap^R(a))$, and by axiom (iv) in Definition 2.1, $T_2(b \otimes c)E = T_2(b \otimes c)$. The fourth equality follows by (3.8). The stated elements belong to $\sqcap^R(A) \otimes \sqcap^R(A)$ by the following reasoning. For any $a, b, d, f \in A$, denote $a' \otimes b' := T_2(a \otimes b)$ and $f' \otimes d' := T_4(f \otimes d)$ (allowing for implicit summation). Then for all $a, b, c, d, f \in A$,

$$\begin{aligned}(\sqcap^R(ab) \otimes 1)F(c \otimes df) &= \sqcap^R(ab)\overline{\sqcap}^R(d')c \otimes f' = \sqcap^R(ab\overline{\sqcap}^R(d'))c \otimes f' \\ &= \sqcap^R(a')c \otimes \sqcap^R(b')df = (\sqcap^R \otimes \sqcap^R)T_2(a \otimes b)(c \otimes df).\end{aligned}$$

The second equality follows by Lemma 3.11, and the third one follows by the first assertion in Lemma 3.10. This proves

$$\delta \sqcap^R (ab) = (\sqcap^R \otimes \sqcap^R)T_2(a \otimes b), \quad \forall a, b \in A, \tag{4.1}$$

so by the idempotency of A, $\delta \sqcap^R (a) \in \sqcap^R(A) \otimes \sqcap^R(A)$, for all $a \in A$.

(3) In view of (4.1), for any $a, b \in A$,

$$\mu\delta \sqcap^R (ab) = \mu(\sqcap^R \otimes \sqcap^R)T_2(a \otimes b) = \sqcap^R\mu(\mathsf{id} \otimes \sqcap^R)T_2(a \otimes b) = \sqcap^R(ab).$$

The second identity follows by (3.8), and the last one does by Lemma 3.7 (4).

(4) Let us use a Heyneman-Sweedler type index notation $\delta(r) =: r_1 \otimes r_2$ for any $r \in \sqcap^R(A)$, where implicit summation is understood. It follows by part (3) that $\sqcap^R(A)$ is an idempotent algebra. So the coassociativity of δ follows by

$$(\delta \otimes \mathsf{id})\delta(sr) = \delta(sr_1) \otimes r_2 = s_1 \otimes s_2r_1 \otimes r_2 = s_1 \otimes \delta(s_2r) = (\mathsf{id} \otimes \delta)\delta(sr),$$

for all $s, r \in \sqcap^R(A)$. In the first and the penultimate equalities we used that δ is a morphism of left $\sqcap^R(A)$-modules, and in the second and the last equalities we used that it is a morphism of right $\sqcap^R(A)$-modules. ☐

Theorem 4.4. *Let A be a regular weak multiplier bialgebra over a field k with a right full comultiplication. Then $\sqcap^R(A)$ is a coalgebra via the counit $\varepsilon : \sqcap^R(A) \to k$ in Proposition* 4.1 *and the comultiplication $\delta : \sqcap^R(A) \to \sqcap^R(A) \otimes \sqcap^R(A)$ in Proposition* 4.3 (3).

Proof. The map δ is a coassociative comultiplication by Proposition 4.3 (4). It remains to prove its counitality. For any $a, b \in A$,

$$(\mathsf{id} \otimes \varepsilon)\delta \sqcap^R (ab) \overset{(4.1)}{=} (\mathsf{id} \otimes \varepsilon)(\sqcap^R \otimes \sqcap^R)T_2(a \otimes b) = \sqcap^R(\mathsf{id} \otimes \epsilon)T_2(a \otimes b) \overset{(iii)}{=} \sqcap^R(ab).$$

In order to prove counitality on the other side, we need an alternative expression of δ. To this end, note that for any $a, b, c \in A$,

$$(\mathsf{id} \otimes \epsilon)[(1 \otimes a)T_2(b \otimes c)] = (\mathsf{id} \otimes \epsilon)[(b \otimes 1)T_3(c \otimes a)] \overset{(3.3)}{=} b \sqcap^R (a)c. \tag{4.2}$$

On the other hand, for any $a, b, c \in A$,

$$\begin{aligned}(\mathsf{id} \otimes \epsilon)[T_3(b \otimes a)(1 \otimes c)] &= (\mathsf{id} \otimes \epsilon)[(1 \otimes a)T_1(b \otimes c)] \\ &= (\mathsf{id} \otimes \epsilon)[(1 \otimes \overline{\sqcap}^L(a))T_1(b \otimes c)] \\ &= (\mathsf{id} \otimes \epsilon)[(\sqcap^R(a) \otimes 1)T_1(b \otimes c)].\end{aligned}$$

The second equality follows by Lemma 3.1, and the third one follows by Lemma 3.9. Therefore,

$$\begin{aligned}(4.3) \quad (\sqcap^R \otimes \epsilon)\ [T_3(b \otimes a)(1 \otimes c)] &= (\sqcap^R \otimes \epsilon)[(\sqcap^R(a) \otimes 1)T_1(b \otimes c)] \\ = \quad (\sqcap^R \otimes \epsilon)[(a \otimes 1)T_1(b \otimes c)] &= (\sqcap^R \otimes \epsilon)[T_2(a \otimes b)(1 \otimes c)],\end{aligned}$$

where the second equality follows by (3.6). With these identities at hand, for any $a, b, c, d, f, g \in A$,

$$\begin{aligned}(4.4)\ (f \otimes g)\ &((\sqcap^R \otimes \sqcap^R)T_3(b \otimes a))(c \otimes d) \\ &\overset{(4.2)}{=} (f \otimes 1)(\sqcap^R \otimes \epsilon \otimes \mathsf{id})[(T_3(b \otimes a) \otimes 1)(1 \otimes T_2^{\mathsf{op}}(g \otimes d))](c \otimes 1) \\ &\overset{(4.3)}{=} (f \otimes 1)(\sqcap^R \otimes \epsilon \otimes \mathsf{id})[(T_2(a \otimes b) \otimes 1)(1 \otimes T_2^{\mathsf{op}}(g \otimes d))](c \otimes 1) \\ &\overset{(4.2)}{=} (f \otimes g)((\sqcap^R \otimes \sqcap^R)T_2(a \otimes b))(c \otimes d),\end{aligned}$$

so that by (4.1),

$$(4.5) \qquad \delta \sqcap^R (ab) = (\sqcap^R \otimes \sqcap^R)T_3(b \otimes a), \qquad \forall a, b \in A.$$

Using this expression of δ,

$$(\varepsilon \otimes \mathsf{id})\delta \sqcap^R (ab) \overset{(4.5)}{=} (\varepsilon \otimes \mathsf{id})(\sqcap^R \otimes \sqcap^R)T_3(b \otimes a) = \sqcap^R(\epsilon \otimes \mathsf{id})T_3(b \otimes a) \overset{(iii)}{=} \sqcap^R(ab).$$

□

Lemma 4.5. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Then the comultiplication δ and the counit ε in Theorem 4.4 satisfy the following identities, for all $a, b \in A$.*

(1) $\delta\overline{\sqcap}^R(ab) = (\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_4^{\mathsf{op}}(b \otimes a)$.
(2) $\delta\overline{\sqcap}^R(ab) = (\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_1^{\mathsf{op}}(a \otimes b)$.
(3) $\varepsilon\overline{\sqcap}^R(a) = \epsilon(a)$.

Proof. (1) Symmetrically to the derivation of (4.1), for any $a, b, c, d, f \in A$ denote $T_4(b \otimes a) =: b' \otimes a'$ and $T_2(c \otimes d) =: c' \otimes d'$, allowing for implicit summation. Then

$$\begin{aligned}(cd \otimes f)\delta\overline{\sqcap}^R(ab) &= (cd \otimes f)F(1 \otimes \overline{\sqcap}^R(ab)) = c' \otimes f \sqcap^R (d')\overline{\sqcap}^R(ab) \\ &= c' \otimes f\overline{\sqcap}^R(\sqcap^R(d')ab) = cd\overline{\sqcap}^R(a') \otimes f\overline{\sqcap}^R(b') \\ &= (cd \otimes f)((\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_4^{\mathsf{op}}(b \otimes a)).\end{aligned}$$

The third equality follows by Lemma 3.11, and the fourth equality follows by the first assertion in Lemma 3.10.

(2) Applying the equality (4.4) in the opposite of the algebra A, we obtain

$$(\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_4(b \otimes a) = (\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_1(a \otimes b).$$

Hence the claim follows by part (1).

(3) Applying Proposition 4.1 to the opposite of the algebra A, there is a linear map $\overline{\sqcap}^R(A) \to k$, $\overline{\sqcap}^R(a) \mapsto \epsilon(a)$. Using parts (1) and (2), it can be seen to be the counit for δ, proving that it is equal to ε. □

The following theorem describes the rich algebraic structure carried by the base algebras. Such a result was obtained for *regular* weak multiplier Hopf algebras in [11].

Theorem 4.6. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Then the following assertions hold.*

(1) *Via the coalgebra structure in Theorem* 4.4 *and the restriction of the multiplication in $\mathbb{M}(A)$, $\sqcap^R(A)$ is a coseparable coalgebra, hence a firm Frobenius algebra.*
(2) *The multiplication in the firm Frobenius algebra in part* (1) *is non-degenerate. Moreover, it has (idempotent) local units.*
(3) *The coalgebra $\sqcap^R(A)$ in part* (1) *is a co-Frobenius coalgebra. Hence there exists a unique isomorphism of non-unital algebras $\vartheta : \sqcap^R(A) \to \sqcap^R(A)$ — known as the* Nakayama automorphism *— such that $\varepsilon(sr) = \varepsilon(\vartheta(r)s)$, for all $s, r \in \sqcap^R(A)$.*

Proof. (1) By Theorem 4.4, $\sqcap^R(A)$ is a coalgebra. By Proposition 4.3 (3), the multiplication in $\sqcap^R(A)$ is a bicomodule retraction (i.e. left inverse) of the comultiplication. This precisely means a coseparable coalgebra structure. Then $\sqcap^R(A)$ is a firm Frobenius algebra by the considerations in [1, Section 6.4].

(2) For some $a \in A$, assume that $\sqcap^R(a)\overline{\sqcap}^R(b) = \sqcap^R(a\overline{\sqcap}^R(b)) = 0$, for all $b \in A$ (where the first equality follows by Lemma 3.11). Then also

$$0 = \varepsilon \sqcap^R (a\overline{\sqcap}^R(b)) = \epsilon(a\overline{\sqcap}^R(b)) = \epsilon(ab), \quad \forall b \in A,$$

where the last equality follows by Lemma 3.1. This implies that

$$0 = (\mathsf{id} \otimes \epsilon)[(1 \otimes a)T_2(b \otimes c)] = (\mathsf{id} \otimes \epsilon)[(b \otimes 1)T_3(c \otimes a)] \overset{(3.3)}{=} b \sqcap^R (a)c, \quad \forall b, c \in A,$$

proving $\sqcap^R(a) = 0$. Since $\sqcap^R(A) = \overline{\sqcap}^R(A)$ by Theorem 3.13, this proves the non-degeneracy of the multiplication on the right. Non-degeneracy on the left is proven symmetrically. The existence of local units follows by [1, Proposition 7].

(3) In light of part (2), it follows by [1, Proposition 7] that the coalgebra $\sqcap^R(A)$ in Theorem 4.4 is left and right co-Frobenius. So the existence of the Nakayama automorphism follows by [4, Section 6]. □

The following symmetric version is immediate.

Theorem 4.7. *Let A be a regular weak multiplier bialgebra over a field k with a left full comultiplication. Then the following assertions hold.*

(1) *The subalgebra $\sqcap^L(A)$ of $\mathbb{M}(A)$ is a coseparable coalgebra, hence a firm Frobenius algebra.*
(2) *The multiplication in the firm Frobenius algebra in part* (1) *is non-degenerate. Moreover, it has (idempotent) local units.*
(3) *The coalgebra $\sqcap^L(A)$ in part* (1) *is a co-Frobenius coalgebra (hence its counit has a Nakayama automorphism).*

Our next aim is to find a more explicit expression of the Nakayama automorphisms in Theorem 4.6 (3) and Theorem 4.7 (3).

Lemma 4.8. *For a regular weak multiplier bialgebra A over a field, the following assertions hold.*

(1) *If the comultiplication is left full, then there is a linear anti-multiplicative map*
$$\sigma : \sqcap^L(A) = \overline{\sqcap}^L(A) \to \sqcap^R(A), \qquad \overline{\sqcap}^L(a) \mapsto \sqcap^R(a).$$
(2) *If the comultiplication is left full, then there is a linear anti-multiplicative map*
$$\overline{\sigma} : \sqcap^L(A) = \overline{\sqcap}^L(A) \to \overline{\sqcap}^R(A), \qquad \sqcap^L(a) \mapsto \overline{\sqcap}^R(a).$$
(3) *If the comultiplication is right full, then there is a linear anti-multiplicative map*
$$\tau : \sqcap^R(A) = \overline{\sqcap}^R(A) \to \overline{\sqcap}^L(A), \qquad \sqcap^R(a) \mapsto \overline{\sqcap}^L(a).$$
(4) *If the comultiplication is right full, then there is a linear anti-multiplicative map*
$$\overline{\tau} : \sqcap^R(A) = \overline{\sqcap}^R(A) \to \sqcap^L(A), \qquad \overline{\sqcap}^R(a) \mapsto \sqcap^L(a).$$

If the comultiplication is both left and right full, then $\tau = \sigma^{-1}$ and $\overline{\tau} = \overline{\sigma}^{-1}$.

Proof. We prove part (1); all other parts follow symmetrically. Denote the counit of the coalgebra $\sqcap^L(A) = \overline{\sqcap}^L(A)$ by ε. If $\overline{\sqcap}^L(a) = 0$, then for all $b, c \in A$,

$$\begin{aligned} 0 &= (\mathsf{id} \otimes \varepsilon(\overline{\sqcap}^L(a)\overline{\sqcap}^L(-)))T_2(b \otimes c) = (\mathsf{id} \otimes \varepsilon\overline{\sqcap}^L(a\overline{\sqcap}^L(-)))T_2(b \otimes c) \\ &= (\mathsf{id} \otimes \epsilon(a\overline{\sqcap}^L(-)))T_2(b \otimes c) = (\mathsf{id} \otimes \epsilon(- \sqcap^R(a)))T_2(b \otimes c) \\ &= (\mathsf{id} \otimes \epsilon)[(b \otimes 1)\Delta(c)(1 \otimes \sqcap^R(a))] = (\mathsf{id} \otimes \epsilon)T_2(b \otimes c \sqcap^R(a)) \stackrel{(iii)}{=} bc \sqcap^R(a), \end{aligned}$$

proving that $\sqcap^R(a) = 0$. The second equality follows by Lemma 3.4, and the fourth one follows by Lemma 3.8. In the penultimate equality we applied axiom (iv) in Definition 2.1, (3.7) and the multiplicativity of $\overline{\Delta}$. This proves the existence of the stated linear map σ. Using Lemma 3.4 in the first equality and Lemma 3.12 in the penultimate equality,

$$\sigma(\overline{\sqcap}^L(a)\overline{\sqcap}^L(b)) = \sigma\overline{\sqcap}^L(a\overline{\sqcap}^L(b)) = \sqcap^R(a\overline{\sqcap}^L(b)) = \sqcap^R(b)\sqcap^R(a) = (\sigma\overline{\sqcap}^L(b))(\sigma\overline{\sqcap}^L(a)),$$

for any $a, b \in A$; that is, σ is anti-multiplicative. □

Proposition 4.9. *Let A be a regular weak multiplier bialgebra over a field with a left and right full comultiplication. Then the maps σ and $\overline{\sigma}$ in Lemma 4.8 are anti-coalgebra isomorphisms. Moreover, the Nakayama automorphism of $\sqcap^R(A)$ is equal to $\sigma\overline{\sigma}^{-1}$, and the Nakayama automorphism of $\sqcap^L(A)$ is equal to $\overline{\sigma}^{-1}\sigma$.*

Proof. By the left-right symmetric counterpart of Lemma 4.5 (1), for any $a, b \in A$, $\delta\overline{\sqcap}^L(ab) = (\overline{\sqcap}^L \otimes \overline{\sqcap}^L)T_3^{\mathsf{op}}(b \otimes a)$. Therefore,

$$(\sigma \otimes \sigma)\delta^{\mathsf{op}}\overline{\sqcap}^L(ab) = (\sigma\overline{\sqcap}^L \otimes \sigma\overline{\sqcap}^L)T_3(b \otimes a) = (\sqcap^R \otimes \sqcap^R)T_3(b \otimes a) \stackrel{(4.5)}{=} \delta \sqcap^R(ab),$$

so that σ is anti-comultiplicative. By the left-right symmetric counterpart of Lemma 4.5 (3), $\varepsilon\sigma\overline{\sqcap}^L(a) = \varepsilon \sqcap^R(a) = \epsilon(a) = \varepsilon\overline{\sqcap}^L(a)$ for any $a \in A$, proving that σ is an anti-coalgebra map. It is proven by symmetric steps that also $\overline{\sigma}$ is an anti-coalgebra homomorphism.

Applying Lemma 3.1 in the second equality, it follows for all $a, b \in A$ that

$$\epsilon(a\overline{\sigma} \sqcap^L (b)) = \epsilon(a\overline{\sqcap}^R(b)) = \epsilon(ab) \stackrel{(3.5)}{=} \epsilon(a \sqcap^L (b)).$$

Since $\sqcap^L(A) = \overline{\sqcap}^L(A)$ by Theorem 3.13, this implies that $\epsilon(a\overline{\sigma}\overline{\sqcap}^L(b)) = \epsilon(a\overline{\sqcap}^L(b))$, for all $a, b \in A$. Using this identity in the fourth equality and Lemma 3.8 in the fifth one,

$$\begin{aligned}\varepsilon(\sqcap^R(a)\overline{\sigma}\ \sigma^{-1} \sqcap^R (b)) &= \varepsilon(\sqcap^R(a)\overline{\sigma}\overline{\sqcap}^L(b)) \stackrel{(3.8)}{=} \varepsilon \sqcap^R (a\overline{\sigma}\overline{\sqcap}^L(b)) = \epsilon(a\overline{\sigma}\overline{\sqcap}^L(b)) \\ &= \epsilon(a\overline{\sqcap}^L(b)) = \epsilon(b \sqcap^R (a)) = \varepsilon \sqcap^R (b \sqcap^R (a)) \stackrel{(3.8)}{=} \varepsilon(\sqcap^R(b) \sqcap^R (a)),\end{aligned}$$

for all $a, b \in A$. This proves that $\sigma\overline{\sigma}^{-1}$ is the Nakayama automorphism of $\sqcap^R(A)$ and symmetric considerations prove that $\overline{\sigma}^{-1}\sigma$ is the Nakayama automorphism of $\sqcap^L(A)$. ☐

Finally, we want to find a relation between the multipliers E and F.

Lemma 4.10. *Let A be a regular weak multiplier bialgebra over a field. Then for all $a \in A$,*

$$(1 \otimes \overline{\sqcap}^L(a))E \in \sqcap^R(A) \otimes \overline{\sqcap}^L(A) \quad \textit{and} \quad E(1 \otimes \sqcap^L(a)) \in \overline{\sqcap}^R(A) \otimes \sqcap^L(A).$$

In particular, if the comultiplication is right and left full, then we can regard E as an element of $\mathbb{M}(\sqcap^R(A) \otimes \sqcap^L(A)^{\mathsf{op}})$.

Proof. For any $c, d, f \in A$, $(f \otimes 1)T_4(d \otimes c) = (f \otimes 1)\Delta(c)(d \otimes 1) = T_2(f \otimes c)(d \otimes 1)$. Hence multiplying on the left both sides of (3.10) by $f \otimes 1$ and simplifying on the right the resulting equality by $d \otimes 1$, we obtain the identity

$$(f \otimes 1)((\sqcap^R \otimes \mathsf{id})T_3(a \otimes b))(c \otimes 1) = (1 \otimes ba)((\mathsf{id} \otimes \sqcap^L)T_2(f \otimes c)),$$

for all $a, b, c, f \in A$. Using this identity in the fourth equality and Lemma 3.11 in the third one,

$$\begin{aligned}(a \otimes 1)(1 \otimes \overline{\sqcap}^L(bc))\, E\, (d \otimes f) &= (1 \otimes \overline{\sqcap}^L(bc))(a \otimes 1)E(d \otimes f) \\ &\stackrel{(3.4)}{=} (1 \otimes \overline{\sqcap}^L(bc))((\mathsf{id} \otimes \sqcap^L)T_2(a \otimes d))(1 \otimes f) \\ &= (1 \otimes \overline{\sqcap}^L)[(1 \otimes bc)((\mathsf{id} \otimes \sqcap^L)T_2(a \otimes d))](1 \otimes f) \\ &= (1 \otimes \overline{\sqcap}^L)[(a \otimes 1)((\sqcap^R \otimes \mathsf{id})T_3(c \otimes b))(d \otimes 1)](1 \otimes f) \\ &= (a \otimes 1)((\sqcap^R \otimes \overline{\sqcap}^L)T_3(c \otimes b))(d \otimes f),\end{aligned}$$

for all $a, b, c, d, f \in A$. This proves

$$(1 \otimes \overline{\sqcap}^L(bc))E = (\sqcap^R \otimes \overline{\sqcap}^L)T_3(c \otimes b), \qquad \forall b, c \in A, \tag{4.6}$$

hence by the idempotency of A also the first claim. The second claim is proven symmetrically. ☐

Let A be a regular weak multiplier bialgebra over a field with a right and left full comultiplication. The algebra $\sqcap^R(A) \otimes \sqcap^L(A)$ is idempotent by Proposition 4.3 (3) and its symmetric counterpart. Hence the multiplicative and bijective map $\mathsf{id} \otimes \sigma : \sqcap^R(A) \otimes \sqcap^L(A)^{\mathsf{op}} \to \sqcap^R(A) \otimes \sqcap^R(A)$ in Lemma 4.8 (1) is non-degenerate and thus extends to an algebra homomorphism $\overline{\mathsf{id} \otimes \sigma} : \mathbb{M}(\sqcap^R(A) \otimes \sqcap^L(A)^{\mathsf{op}}) \to \mathbb{M}(\sqcap^R(A) \otimes \sqcap^R(A))$.

Proposition 4.11. *Let A be a regular weak multiplier bialgebra over a field with a right and left full comultiplication. Then $(\overline{\mathsf{id}\otimes\sigma})(E) = F$ as elements of $\mathbb{M}(\sqcap^R(A)\otimes\sqcap^R(A))$.*

Proof. For any $a, b, c \in A$,

$$\begin{aligned}
((\overline{\mathsf{id}\otimes\sigma})(E))(\sqcap^R(a)\otimes\sqcap^R(bc)) &= (\overline{\mathsf{id}\otimes\sigma})[(1\otimes\overline{\sqcap}^L(bc))E(\sqcap^R(a)\otimes 1)]\\
&\overset{(4.6)}{=} (\overline{\mathsf{id}\otimes\sigma})[((\sqcap^R\otimes\overline{\sqcap}^L)T_3(c\otimes b))(\sqcap^R(a)\otimes 1)]\\
&= ((\sqcap^R\otimes\sqcap^R)T_3(c\otimes b))(\sqcap^R(a)\otimes 1)\\
&\overset{(4.5)}{=} F(\sqcap^R(a)\otimes\sqcap^R(bc)),
\end{aligned}$$

where in the first and the third equalities we used part (1) of Lemma 4.8 and that $\overline{\mathsf{id}\otimes\sigma}$ is multiplicative. $\square$

5. A monoidal category of modules

Bialgebras over a field can be characterized by the property that the category of their (left or right) modules is monoidal such that the forgetful functor to the category of vector spaces is strict monoidal. More generally, the category of (left or right) modules over a weak bialgebra is monoidal such that the forgetful functor to the category of bimodules over the (separable Frobenius) base algebra is strict monoidal (see e.g. [8]). The aim of this section is to prove a similar property of regular weak multiplier bialgebras with a (left or right) full comultiplication. The key point in doing so is to find the appropriate notion of module in the absence of an algebraic unit.

Definition 5.1. Let A be an idempotent algebra over a field with a non-degenerate multiplication. By a *non-unital* right A-module we mean a vector space V equipped with a linear map (called the A-*action*) $V\otimes A\to V$, $v\otimes a\mapsto va$ satisfying the associativity condition

$$(va)b = v(ab), \qquad \forall v\in V,\ a,b\in A.$$

A right A-module V is said to be *idempotent* (or *unital*) if the A-action $V\otimes A\to V$ is surjective. It is called *firm* if the quotient map $V\otimes_A A\to V$, $v\otimes_A a\mapsto va$ (to the A-module tensor product $V\otimes_A A$) is bijective. Finally, V is a *non-degenerate* A-module if for any $v\in V$, $(va = 0\ \forall a\in A)$ implies $v = 0$. Left A-modules are defined as right modules over the opposite algebra A^{op}, with action denoted by $V\otimes A^{\mathsf{op}}\to V$, $v\otimes a\mapsto av$, and A-bimodules are both left and right A-modules V with commuting actions (i.e. such that $a(vb) = (av)b$, for all $v\in V$ and $a, b\in A$).

The morphisms of non-unital modules over an idempotent and non-degenerate algebra A are the linear maps $f : V\to W$ such that $f(va) = f(v)a$, for all $v\in V$ and $a\in A$. Throughout, we denote by $M_{(A)}$ the category of idempotent and non-degenerate right A-modules. The category of firm A-bimodules (i.e. of bimodules which are firm both as left and right modules) will be denoted by ${}_AM_A$. Whenever A is a firm algebra — that is, the quotient map $A\otimes_A A\to A$, $a\otimes_A b\mapsto ab$ is bijective — ${}_AM_A$ is a monoidal category via the module tensor product $\otimes_A$ and the neutral object A.

Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. By Theorem 4.6 (1), $R := \sqcap^R(A)$ is a firm algebra so there is a monoidal category ${}_RM_R$.

Proposition 5.2. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Any object of $M_{(A)}$ can be regarded as a firm $R := \sqcap^R(A)$-bimodule. This gives rise to a functor $U : M_{(A)} \to {}_RM_R$, acting on the morphisms as the identity map.*

Proof. Using that any object V of $M_{(A)}$ is an idempotent A-module, define the R-actions on V with the help of the map τ in Lemma 4.8 (3) by

$$(va) \cdot \sqcap^R(b) := v(a \sqcap^R (b)) \quad \text{and} \quad \sqcap^R (b) \cdot (va) := v(a(\tau \sqcap^R (b))) = v(a\overline{\sqcap}^L(b)).$$

In order to see that these actions are well-defined, assume that $va = 0$. Then for all $b, c \in A$,

$$0 = (va)(\sqcap^R(b)c) = v(a(\sqcap^R(b)c)) = v((a \sqcap^R (b))c) = (v(a \sqcap^R (b)))c.$$

So by the non-degeneracy of V, $0 = v(a\sqcap^R(b))$, proving that the right R-action on V is well-defined. One checks symmetrically that also the left R-action is well-defined. Associativity of both actions is evident by the associativity of the multiplication in $\mathbb{M}(A)$ and the anti-multiplicativity of τ (cf. Lemma 4.8 (3)). The left and right R-actions commute by Lemma 3.5 (since by the right fullness of the comultiplication $\sqcap^R(A) = \overline{\sqcap}^R(A)$; see Theorem 3.13). Finally, R has local units by Theorem 4.6 (2). So in order to see that V is a firm R-bimodule, it is enough to see that it is idempotent as a left and as a right R-module. Since both the algebra A and the module V are idempotent, any element of V can be written as a linear combination of elements of the form $v(ab) = v(a' \sqcap^R (b')) = (va') \cdot \sqcap^R(b')$, in terms of $v \in V$ and $a, b \in A$, where $a' \otimes b' := T_2(a \otimes b)$ (allowing for implicit summation), and the first equality follows by Lemma 3.7 (4). Symmetrically, any element of V can be written as a linear combination of elements of the form $w(cd) = w(c'\overline{\sqcap}^L(d')) = \sqcap^R(d')\cdot(wc')$, in terms of $w \in V$ and $c, d \in A$, where $d' \otimes c' := T_3(d \otimes c)$ (allowing for implicit summation), and the first equality follows by Lemma 3.7 (1).

With respect to the stated R-actions, any A-module map is evidently a morphism of R-bimodules. This proves the existence of the stated functor U. □

Proposition 5.3. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Then $R := \sqcap^R(A)$ carries the structure of an idempotent and non-degenerate right A-module. The functor U in Proposition 5.2 takes this object R of $M_{(A)}$ to the R-bimodule R with the actions provided by the multiplication.*

Proof. For any $a, b \in A$, put

$$\sqcap^R(a) \triangleleft b := \sqcap^R(\sqcap^R(a)b) \overset{(3.6)}{=} \sqcap^R(ab).$$

It is clearly a well-defined associative action. Let us see that it is idempotent. For any $a, b \in A$, denote $b' \otimes a' := T_4(b \otimes a)$, allowing for implicit summation. By the right fullness of the comultiplication, $\sqcap^R(A) = \overline{\sqcap}^R(A)$; cf. Theorem 3.13. So by Lemma 3.7 (2),

$$\overline{\sqcap}^R(a') \triangleleft b' = \sqcap^R(\overline{\sqcap}^R(a')b') = \sqcap^R(ab).$$

By the idempotency of A, this proves that the A-action on R is surjective. In order to see its non-degeneracy, assume that for some $a \in A$, $\sqcap^R(ab) = 0$ for all $b \in A$.

Then for all $b, c, d \in A$,

$$\begin{aligned}
0 &= (\mu(\mathsf{id} \otimes \sqcap^R)[(1 \otimes a)T_2(b \otimes c)])d \\
&= (\mu(\mathsf{id} \otimes \sqcap^R)[(b \otimes 1)T_3(c \otimes a)])d \\
&\overset{(3.3)}{=} b(\mu(\mathsf{id} \otimes \mathsf{id} \otimes \epsilon)(\mathsf{id} \otimes T_3\mathsf{tw})(T_3 \otimes \mathsf{id})(c \otimes a \otimes d)) \\
&= b((\mathsf{id} \otimes \epsilon)(\mu \otimes \mathsf{id})(\mathsf{id} \otimes T_3\mathsf{tw})(T_3 \otimes \mathsf{id})(c \otimes a \otimes d)) \\
&= b((\mathsf{id} \otimes \epsilon)T_3(cd \otimes a)) \overset{(3.3)}{=} b \sqcap^R (a)cd.
\end{aligned}$$

By the density of A in $\mathbb{M}(A)$, this proves $\sqcap^R(a) = 0$, hence the non-degeneracy of the action. In the penultimate equality we used that

$$(\mu\otimes\mathsf{id})(\mathsf{id}\otimes T_3\mathsf{tw})(T_3\otimes\mathsf{id})(c\otimes a\otimes d) = (1\otimes a)\Delta(c)\Delta(d) = (1\otimes a)\Delta(cd) = T_3(cd\otimes a).$$

Applying the functor $U : M_{(A)} \to {}_RM_R$ in Proposition 5.2 to the object R of $M_{(A)}$ above, the right action in the resulting R-bimodule comes out as the right multiplication. Indeed,

$$\sqcap^R(ab) \cdot \sqcap^R(c) = \sqcap^R(a) \triangleleft (b \sqcap^R (c)) = \sqcap^R(ab \sqcap^R (c)) \overset{(3.8)}{=} \sqcap^R(ab) \sqcap^R (c),$$

for all $a, b, c \in A$. The left R-action is also given by the multiplication since

$$\sqcap^R(c) \cdot \sqcap^R(ab) = \sqcap^R(a) \triangleleft (b\overline{\sqcap}^L(c)) = \sqcap^R(ab\overline{\sqcap}^L(c)) = \sqcap^R(c) \sqcap^R (ab),$$

where the last equality follows by Lemma 3.12. □

Lemma 5.4. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Regard any objects V and W of $M_{(A)}$ as firm $R := \sqcap^R(A)$-bimodules as in Proposition* 5.2. *Then the R-module tensor product $V \otimes_R W$ is isomorphic to*

$$\langle (v \otimes w)((a \otimes b)E) \mid v \in V,\ w \in W,\ a, b \in A\rangle.$$

Proof. By Theorem 4.6 (1), R is a coseparable coalgebra. Then by [3, Proposition 2.17], $V\otimes_R W$ is isomorphic to the image of the idempotent map $\theta : V\otimes W \to V\otimes W$,

$$v \cdot \sqcap^R(a) \otimes w \mapsto (v \cdot (-) \otimes (-) \cdot w)\delta \sqcap^R (a),$$

where $\delta : R \to R \otimes R$ is the (R-bilinear) comultiplication in Proposition 4.3 (3). In order to obtain a more explicit expression of this map θ, note that by the idempotency of A and Lemma 3.7 (4), any element of A can be written as a linear combination of elements of the form $a\sqcap^R(bc)$ — so any element of V can be written as a linear combination of elements of the form $va \sqcap^R (bc)$ — in terms of $v \in V$, $a, b, c \in A$. Now

$$\begin{aligned}
\theta(va \sqcap^R (bc) \otimes wd) &\overset{(4.1)}{=} (va \otimes wd)((\sqcap^R \otimes \overline{\sqcap}^L)T_2(b \otimes c)) \\
&= (va \otimes wd)((\sqcap^R \otimes \mathsf{id})[(bc \otimes 1)E]) \\
&= (v \otimes w)((a \sqcap^R (bc) \otimes d)E),
\end{aligned}$$

where the second equality follows by Proposition 2.5 (1), and in the last equality we used that for all $a, b, c, d \in A$,

$$\begin{aligned}
(\sqcap^R \otimes \mathsf{id})[(ab \otimes cd)E] &\overset{(3.4)}{=} (\sqcap^R \otimes \mathsf{id})[(ab \otimes 1)((\sqcap^R \otimes \mathsf{id})T_3(d \otimes c))] \\
&\overset{(3.8)}{=} (\sqcap^R(ab) \otimes 1)((\sqcap^R \otimes \mathsf{id})T_3(d \otimes c)) \\
&\overset{(3.4)}{=} (\sqcap^R(ab) \otimes cd)E,
\end{aligned}$$

hence $(\sqcap^R \otimes \mathsf{id})[(ab \otimes 1)E] = (\sqcap^R(ab) \otimes 1)E$. This proves that the image of θ is spanned by the stated elements $(v \otimes w)((a \otimes b)E)$. $\square$

Proposition 5.5. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Regard any objects V and W of $M_{(A)}$ as firm $R := \sqcap^R(A)$-bimodules as in Proposition 5.2. Then the R-module tensor product $V \otimes_R W$ carries the structure of an idempotent and non-degenerate A-module too.*

Proof. Observe that for each $c \in A$, there is a well-defined linear map $V \otimes W \to V \otimes W$ given by

$$va \otimes wb \mapsto (va \otimes w)T_3(c \otimes b) = (v \otimes wb)T_2(a \otimes c) = (v \otimes w)((a \otimes b)\Delta(c))$$

for any $a, b \in A$, $v \in V$ and $w \in W$. Since this map is R-balanced by Lemma 3.9, it induces an action of A on $V \otimes_R W$ defined by

$$(va \otimes_R wb)c = \pi((v \otimes w)((a \otimes b)\Delta(c))),$$

where $\pi : V \otimes W \to V \otimes_R W$ denotes the canonical epimorphism. This action is associative by the multiplicativity of Δ. In order to see that it is idempotent and non-degenerate, let us apply the isomorphism in Lemma 5.4. It takes the above A-action on $V \otimes_R W$ to

$$(v \otimes w)((a \otimes b)E)c = (v \otimes w)((a \otimes b)\Delta(c)). \tag{5.1}$$

It is an idempotent action by axiom (iv) in Definition 2.1. In order to see that it is non-degenerate, assume that $(v \otimes w)((a \otimes b)\Delta(c)) = 0$ for all $c \in A$. Then

$$\begin{aligned} &0 = (v \otimes w)((a \otimes b)\Delta(c))(d \otimes f) = (v \otimes w)(a \otimes b)(\Delta(c)(d \otimes f)), \quad \forall c, d, f \in A \stackrel{(iv)}{\Longrightarrow} \\ &0 = (v \otimes w)(a \otimes b)(E(c \otimes d)) = (v \otimes w)((a \otimes b)E)(c \otimes d), \qquad \forall c, d \in A. \end{aligned}$$

By [6, Lemma 1.11], $V \otimes W$ is a non-degenerate $A \otimes A$-module. Hence $0 = (v \otimes w)((a \otimes b)E)$, proving the non-degeneracy of the A-module $V \otimes_R W$.

Applying the functor $U : M_{(A)} \to {}_R M_R$ in Proposition 5.2 to the object $V \otimes_R W$ of $M_{(A)}$ above, it follows by Lemma 3.3 and (3.7) that the resulting R-bimodule has the actions

$$\sqcap^R(a) \cdot (v \otimes_R w) \cdot \sqcap^R(b) = (\sqcap^R(a) \cdot v) \otimes_R (w \cdot \sqcap^R(b)).$$

$\square$

Theorem 5.6. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Then $M_{(A)}$ is a monoidal category and the functor $U : M_{(A)} \to {}_R M_R$ in Proposition 5.2 is strict monoidal.*

Proof. In view of Proposition 5.3 and Proposition 5.5, we only need to show that the associativity and unit constraints of ${}_R M_R$ — if evaluated on objects of $M_{(A)}$ — are morphisms of A-modules.

Take any objects V, W, Z in $M_{(A)}$. In view of Lemma 5.4, $(V \otimes_R W) \otimes_R Z$ is isomorphic to the vector subspace of $V \otimes W \otimes Z$ spanned by the elements of the form

$$\begin{aligned} &((v \otimes w)((a \otimes b)E) \otimes z)((c \otimes d)E) \\ &\quad \stackrel{(5.1)}{=} (v \otimes w \otimes z)((a \otimes b \otimes 1)(\Delta \otimes \mathsf{id})((c \otimes d)E)) \\ &\quad \stackrel{(v)(iv)}{=} (v \otimes w \otimes z)((a \otimes b \otimes 1)(\Delta(c) \otimes d)(1 \otimes E)) \end{aligned}$$

(for $a,b,c,d \in A$, $v \in V$, $w \in W$ and $z \in Z$), hence in light of axiom (iv) in Definition 2.1, by elements of the form

$$(v \otimes w \otimes z)((a \otimes b \otimes d)(E \otimes 1)(1 \otimes E)) \overset{(v)}{=} (v \otimes w \otimes z)((a \otimes b \otimes d)(1 \otimes E)(E \otimes 1))$$

(for $a,b,d \in A$, $v \in V$, $w \in W$ and $z \in Z$). A symmetric computation shows the isomorphism of the same vector subspace of $V \otimes W \otimes Z$ to $V \otimes_R (W \otimes_R Z)$, and the associator isomorphism $(V \otimes_R W) \otimes_R Z \to V \otimes_R (W \otimes_R Z)$ is given by the composite of these isomorphisms. Its A-module map property is thus equivalent to the equality of both induced actions

$$\begin{aligned} &((v \otimes w \otimes z)((a \otimes b \otimes c)(E \otimes 1)(1 \otimes E)))d \\ &\quad= ((v \otimes w)((a \otimes b)E) \otimes z)T_3(d \otimes c) \\ &\quad= (v \otimes wb \otimes z)((T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_3)(a \otimes d \otimes c)) \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} &((v \otimes w \otimes z)((a \otimes b \otimes c)(1 \otimes E)(E \otimes 1)))d \\ &\quad= (v \otimes (w \otimes z)((b \otimes c)E))T_2(a \otimes d) \\ &\quad= (v \otimes wb \otimes z)((\mathsf{id} \otimes T_3)(T_2 \otimes \mathsf{id})(a \otimes d \otimes c)), \end{aligned} \tag{5.3}$$

where we used the equivalent forms $(v \otimes w)((a \otimes b)E)c = (va \otimes w)T_3(c \otimes b) = (v \otimes wb)T_2(a \otimes c)$ of the action in (5.1). The actions (5.2) and (5.3) are equal by the alternative form $(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_3) = (\mathsf{id} \otimes T_3)(T_2 \otimes \mathsf{id})$ of the coassociativity axiom (ii) in Definition 2.1 (obtained by evaluating both sides of (ii) on any $a \otimes b \otimes c \in A \otimes A \otimes A$, multiplying on the left by $1 \otimes 1 \otimes d$ and simplifying on the right by $1 \otimes 1 \otimes c$).

In order to see that the left unit constraint $R \otimes_R V \to V$, $\sqcap^R(a) \otimes_R v \mapsto v\overline{\sqcap}^L(a)$ is a morphism of right A-modules, take any $\sqcap^R(a) \otimes_R vb \in R \otimes_R V$. Applying to it the left unit constraint and next the action by any $c \in A$ results in $vb\overline{\sqcap}^L(a)c$. On the other hand, acting first by $c \in A$ on $\sqcap^R(a) \otimes_R vb$ yields $\pi((\sqcap^R(a) \otimes v)T_3(c \otimes b))$, where we used the notation $\pi : R \otimes V \to R \otimes_R V$ for the canonical epimorphism. Applying now the left unit constraint, we obtain $v(\mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})[(a \otimes 1)T_3(c \otimes b)])$. Using Lemma 3.2, Lemma 3.3 and Lemma 3.7 (1) in the first, second and last equalities, respectively, we see that for any $a,b,c \in A$,

$$\begin{aligned} \mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})[(a \otimes 1)T_3(c \otimes b)] &= \mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})[(\overline{\sqcap}^L(a) \otimes 1)T_3(c \otimes b)] \\ &= \mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})T_3(\overline{\sqcap}^L(a)c \otimes b) = b\overline{\sqcap}^L(a)c. \end{aligned}$$

This proves that the left unit constraint in ${}_RM_R$ evaluated on an object V of $M_{(A)}$ is a morphism of A-modules. A symmetric reasoning applies for the right unit constraint. □

6. The antipode

The *antipode* of a Hopf algebra A is defined as the convolution inverse of the identity map $A \to A$. In a *weak* Hopf algebra, the antipode is no longer a strict inverse of the identity map in the convolution algebra of the linear maps $A \to A$. However, it is a 'weak' inverse in the following sense. The maps $\sqcap^L$ and $\sqcap^R$: $A \to A$ are idempotent elements in the convolution algebra, and they serve as left, respectively, right units for the identity map $A \to A$. The antipode is then a linear map $A \to A$ for whom $\sqcap^L$ and $\sqcap^R$ serve as a right, respectively, left unit and whose convolution products with the identity map in both possible orders yield $\sqcap^L$ and

$\sqcap^R$, respectively. In what follows, we equip a weak multiplier bialgebra with an antipode in the above spirit: as a generalized convolution inverse. The resulting structure is compared with a (regular or not) weak multiplier Hopf algebra in [13].

Let A be a regular weak multiplier bialgebra over a field. By Proposition 4.3 (1), for all $a, b \in A$ $(ab \otimes 1)F = (\mathsf{id} \otimes \sqcap^R)T_2(a \otimes b)$ is an element of $A \otimes \mathbb{M}(A)$. So by the idempotency of A, $(a \otimes 1)F \in A \otimes \mathbb{M}(A)$ for all $a \in A$, allowing for the definition of a linear map:

$$G_1 : A \otimes A \to A \otimes A, \qquad a \otimes b \mapsto (a \otimes 1)F(1 \otimes b). \tag{6.1}$$

The notation G_1 is motivated by the fact that it is the same map appearing under the same name in [13, Proposition 1.11]:

Proposition 6.1. *Let A be a regular weak multiplier bialgebra. The map* (6.1) *satisfies the equality*

$$(G_1 \otimes \mathsf{id})[\Delta_{13}(a)(1 \otimes b \otimes c)] = \Delta_{13}(a)(1 \otimes E)(1 \otimes b \otimes c), \qquad \forall a, b, c \in A.$$

Hence it is the same map denoted by G_1 in [13, Proposition 1.11].

Proof. For any $a, b, c, d \in A$,

$$\begin{aligned}(G_1 \otimes \mathsf{id})[\Delta_{13}(a)(1 \otimes bd \otimes c)] &= \Delta_{13}(a)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes b) \otimes c) \\ &= \Delta_{13}(a)(1 \otimes (\mathsf{id} \otimes \sqcap^L)T_4(d \otimes b))(1 \otimes 1 \otimes c) \\ &= \Delta_{13}(a)(1 \otimes E)(1 \otimes bd \otimes c).\end{aligned}$$

The first equality follows by Proposition 4.3 (1), the second one follows by Lemma 3.9 and the last equality follows by (3.4). So we conclude by the idempotency of A. $\square$

In [13], the form (6.1) of G_1 was proven for *regular* weak multiplier Hopf algebras, but it was left open if it has the above form for *arbitrary* weak multiplier Hopf algebras.

Proposition 6.2. *Let A be a regular weak multiplier bialgebra over a field. If the comultiplication is left and right full, then the following hold.*

1. *The image of the map $E_1 : A \otimes A \to A \otimes A$, $a \otimes b \mapsto E(a \otimes b)$ is isomorphic to the $\sqcap^L(A)$-module tensor square of A with respect to the actions*

$$\sqcap^L(a) \cdot b := \sqcap^L(a)b \qquad \text{and} \qquad b \cdot \sqcap^L(a) := \overline{\sqcap}^R(a)b, \qquad \text{for } a, b \in A.$$

2. *The image of the map $G_1 : A \otimes A \to A \otimes A$ in* (6.1) *is isomorphic to the $\sqcap^R(A)$-module tensor square of A with respect to the actions*

$$\sqcap^R(a) \cdot b := \sqcap^R(a)b \qquad \text{and} \qquad b \cdot \sqcap^R(a) := b \sqcap^R(a), \qquad \text{for } a, b \in A.$$

Proof. (1) By a symmetric version of Lemma 5.4, E_1 is equal to the map

$$a \otimes \sqcap^L(bc)d \mapsto ((\overline{\sqcap}^R \otimes \sqcap^L)T_1(b \otimes c))(a \otimes d),$$

whose image is equal to the stated module tensor product.

(2) Since $\sqcap^R(A)$ is a coseparable coalgebra by Theorem 4.6 (1), it follows by [3, Proposition 2.17] that the stated module tensor product is isomorphic to the image of the map

$$a \sqcap^R(b) \otimes c \mapsto (a \otimes 1)(\delta \sqcap^R(b))(1 \otimes c) = (a \sqcap^R(b) \otimes 1)F(1 \otimes c) = G_1(a \sqcap^R(b) \otimes c),$$

where $F \in \mathbb{M}(A \otimes A)$ appeared in Proposition 4.3 (1) and $\delta : \sqcap^R(A) \to \sqcap^R(A) \otimes \sqcap^R(A)$ is the comultiplication in Proposition 4.3 (3). Since by Lemma 3.7 (4) and by the idempotency of A any element of A is a linear combination of elements of the form $a\sqcap^R(b)$, for $a,b\in A$, we have the claim proven. □

For any weak multiplier bialgebra A over a field, consider the vector space

$$\mathcal{L} := \{L : A\otimes A\to A\otimes A \mid \quad L(a\otimes bc) = L(a\otimes b)(1\otimes c)\ \forall a,b,c\in A, \\ (T_2\otimes \mathsf{id})(\mathsf{id}\otimes L) = (\mathsf{id}\otimes L)(T_2\otimes\mathsf{id})\}.$$

Denoting the vector space of right A-module maps $A\to A$ by $\mathsf{End}_A(A)$, there is a linear map

$$\mathcal{L}\to \mathsf{Lin}(A,\mathsf{End}_A(A)),\qquad L\mapsto [\lambda_L : a\mapsto (\epsilon\otimes\mathsf{id})L(a\otimes -)]. \tag{6.2}$$

With its help, for any $a,b,c\in A$ and $L\in\mathcal{L}$,

$$\begin{aligned}((\mathsf{id}\otimes\lambda_L)T_2(a\otimes b))(1\otimes c) &= (\mathsf{id}\otimes\epsilon\otimes\mathsf{id})(\mathsf{id}\otimes L)(T_2\otimes\mathsf{id})(a\otimes b\otimes c)\\ &= (\mathsf{id}\otimes\epsilon\otimes\mathsf{id})(T_2\otimes\mathsf{id})(\mathsf{id}\otimes L)(a\otimes b\otimes c) \overset{(iii)}{=} (a\otimes 1)L(b\otimes c).\end{aligned} \tag{6.3}$$

Applying this together with the non-degeneracy of the multiplication in $A\otimes A$, we conclude that the map (6.2) is injective. Clearly, $\mathcal{L}$ is an algebra via the composition of maps. For $L,L'\in\mathcal{L}$ and $a\in A$,

$$\lambda_{L'L}(a) = (\epsilon\otimes\mathsf{id})L'L(a\otimes -) = \mu(\lambda_{L'}\otimes\mathsf{id})L(a\otimes -) \tag{6.4}$$

(where $\mu:\mathsf{End}_A(A)\otimes A\to A$ denotes the evaluation map $\Phi\otimes a\mapsto \Phi a\equiv\Phi(a)$), generalizing the convolution product $(\lambda_{L'}*\lambda_L)(a) = \mu(\lambda_{L'}\otimes\lambda_L)\Delta(a)$ of endomorphisms $\lambda_{L'}$ and λ_L on a (weak) bialgebra.

Proposition 6.3. *Let A be a regular weak multiplier bialgebra over a field.*

(1) *The maps T_1, $E_1 := E(-\otimes -)$ and G_1 in (6.1) from $A\otimes A$ to $A\otimes A$ are elements of $\mathcal{L}$.*
(2) *The map (6.2) takes the elements of $\mathcal{L}$ in part (1) to $[a\mapsto a(-)]$, $[a\mapsto \sqcap^L(a)(-)]$ and $[a\mapsto\sqcap^R(a)(-)]$, respectively.*
(3) *$E_1^2 = E_1$, $G_1^2 = G_1$ and $E_1T_1 = T_1 = T_1G_1$.*

Proof. (1) Evidently, all of T_1, E_1 and G_1 are right A-module maps. The compatibility of T_1 with T_2 is axiom (ii) in Definition 2.1. The compatibility of E_1 with T_2 follows in the same way as in [13, Proposition 2.2]: for all $a,b,c\in A$,

$$\begin{aligned}(T_2\otimes\mathsf{id})(\mathsf{id}\otimes E_1)(a\otimes b\otimes c) &= (a\otimes 1\otimes 1)(\Delta\otimes\mathsf{id})(E(b\otimes c))\\ &\overset{(v)}{=} (a\otimes 1\otimes 1)(1\otimes E)(E\otimes 1)(\Delta(b)\otimes c)\\ &\overset{(iv)}{=} (1\otimes E)(a\otimes 1\otimes 1)(\Delta(b)\otimes c)\\ &= (\mathsf{id}\otimes E_1)(T_2\otimes\mathsf{id})(a\otimes b\otimes c).\end{aligned}$$

It remains to prove the compatibility of G_1 with T_2. By Proposition 4.3 (1), for all $a,b,c\in A$,

$$G_1(a\otimes bc) = (a\otimes 1)((\overline{\sqcap}^R\otimes\mathsf{id})T_4^{\mathsf{op}}(c\otimes b)). \tag{6.5}$$

Using this form (6.5) of G_1 in the first and the last equalities and the multiplicativity of $\overline{\Delta}$ and Lemma 3.3 together with axiom (iv) in the second one, it follows for any

$a, b, c, d \in A$ that

$$\begin{aligned}
&(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1)(a \otimes b \otimes cd) \\
&\quad = (a \otimes 1 \otimes 1)(\Delta \otimes \mathsf{id})[(b \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c))] \\
&\quad = (a \otimes 1 \otimes 1)(\Delta(b) \otimes 1)(1 \otimes (\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c)) \\
&\quad = (\mathsf{id} \otimes G_1)(T_2 \otimes \mathsf{id})(a \otimes b \otimes cd),
\end{aligned}$$

from which we conclude by the idempotency of A.

(2) Take any $a, b \in A$. By axiom (iii), $(\epsilon \otimes \mathsf{id})T_1(a \otimes b) = ab$. By (3.1), $(\epsilon \otimes \mathsf{id})E_1(a \otimes b) = \sqcap^L(a)b$. Finally, using Lemma 3.1 in the second equality, it follows for all $a, b, c \in A$ that

$$\begin{aligned}
(\epsilon \otimes \mathsf{id})G_1(a \otimes bc) &\overset{(6.5)}{=} (\mathsf{id} \otimes \epsilon)[(1 \otimes a)(\mathsf{id} \otimes \overline{\sqcap}^R)T_4(c \otimes b)] \\
&= (\mathsf{id} \otimes \epsilon)[(1 \otimes a)T_4(c \otimes b)] \\
&= (\mathsf{id} \otimes \epsilon)[T_3(b \otimes a)(c \otimes 1)] \overset{(3.3)}{=} \sqcap^R(a)bc.
\end{aligned}$$

(3) $E_1^2 = E_1$ is evident by the fact that E is an idempotent element of $\mathbb{M}(A \otimes A)$. By Lemma 3.3, for any $a, b, c \in A$,

$$(1 \otimes \overline{\sqcap}^R(a))T_4(b \otimes c) = T_4(b \otimes \overline{\sqcap}^R(a)c). \tag{6.6}$$

For any $b, c \in A$, denote $T_4(c \otimes b) =: c' \otimes b'$ allowing for implicit summation. Then by Lemma 3.7 (2),

$$\overline{\sqcap}^R(b')c' = \mu^{\mathsf{op}}(\mathsf{id} \otimes \overline{\sqcap}^R)T_4(c \otimes b) = bc. \tag{6.7}$$

With these identities at hand and applying Lemma 3.4 in the second equality, it follows for $a, b, c, d \in A$ that

$$\begin{aligned}
G_1^2(a \otimes bcd) &\overset{(6.5)}{=} (a\overline{\sqcap}^R(b') \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c')) \\
&= (a \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})[(\overline{\sqcap}^R(b') \otimes 1)T_4^{\mathsf{op}}(d \otimes c')]) \\
&\overset{(6.6)}{=} (a \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes \overline{\sqcap}^R(b')c') \\
&\overset{(6.7)}{=} (a \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes bc)) \overset{(6.5)}{=} G_1(a \otimes bcd),
\end{aligned}$$

proving $G_1^2 = G_1$. The equality $E_1T_1 = T_1$ is immediate by axiom (iv). Finally, using again the notation $T_4(c \otimes b) =: c' \otimes b'$ (allowing for implicit summation), for all $a, b, c \in A$,

$$\begin{aligned}
T_1G_1(a \otimes bc) &\overset{(6.5)}{=} \Delta(a\overline{\sqcap}^R(b'))(1 \otimes c') = \Delta(a)(1 \otimes \overline{\sqcap}^R(b')c') \\
&\overset{(6.7)}{=} \Delta(a)(1 \otimes bc) = T_1(a \otimes bc).
\end{aligned}$$

The second equality follows by the multiplicativity of $\overline{\Delta}$, Lemma 3.3 and axiom (iv). ☐

Applying the same reasoning as in [13, Proposition 2.3], the following can be shown.

Proposition 6.4. *Let A be a regular weak multiplier bialgebra over a field. If there is a linear map $R_1 : A \otimes A \to A \otimes A$ such that $R_1T_1 = G_1$, $T_1R_1 = E_1$ and $R_1T_1R_1 = R_1$, then $R_1 \in \mathcal{L}$.*

Symmetrically to the above considerations, we can define

$$\mathcal{R} := \{K : A\otimes A \to A\otimes A \mid \quad K(ab\otimes c) = (a\otimes 1)K(b\otimes c)\ \forall a,b,c\in A \\ (\mathsf{id}\otimes T_1)(K\otimes \mathsf{id}) = (K\otimes \mathsf{id})(\mathsf{id}\otimes T_1)\}.$$

Denoting the vector space of left A-module maps $A\to A$ by ${}_A\mathsf{End}(A)$, there is an injective linear map

$$\mathcal{R}\to \mathsf{Lin}(A, {}_A\mathsf{End}(A)), \qquad K\mapsto [\rho_K : a\mapsto (\mathsf{id}\otimes\epsilon)K(-\otimes a)] \tag{6.8}$$

such that

$$(a\otimes 1)((\rho_K\otimes \mathsf{id})T_1(b\otimes c)) = K(a\otimes b)(1\otimes c), \qquad \forall a,b,c\in A. \tag{6.9}$$

For $K, K'\in\mathcal{R}$,

$$\rho_{K'K}(a) = \mu(\mathsf{id}\otimes\rho_{K'})K(-\otimes a), \quad \forall a\in A \tag{6.10}$$

(where $\mu : A\otimes {}_A\mathsf{End}(A)\to A$ denotes the evaluation map $a\otimes\Phi\mapsto a\Phi\equiv\Phi(a)$). The linear map (6.8) takes the elements

$$T_2, \quad E_2 : a\otimes b\mapsto (a\otimes b)E, \quad G_2 : ab\otimes c\mapsto ((\mathsf{id}\otimes \overline{\sqcap}^L)T_3^{\mathsf{op}}(b\otimes a))(1\otimes c) \tag{6.11}$$

of $\mathcal{R}$ to $[a\mapsto (-)a]$, $[a\mapsto (-)\sqcap^R(a)]$ and $[a\mapsto (-)\sqcap^L(a)]$, respectively. The equalities $E_2^2 = E_2$, $G_2^2 = G_2$ and $E_2T_2 = T_2 = T_2G_2$ hold. If there is a linear map $R_2 : A\otimes A\to A\otimes A$ such that $R_2T_2 = G_2$, $T_2R_2 = E_2$ and $R_2T_2R_2 = R_2$, then $R_2\in\mathcal{R}$.

Proposition 6.5. *Let A be a regular weak multiplier bialgebra over a field and for $i\in\{1,2\}$, let $E_i, G_i : A\otimes A\to A\otimes A$ be the same maps as before. Then the following hold.*

1. $(\mathsf{id}\otimes E_1)(E_2\otimes\mathsf{id}) = (E_2\otimes\mathsf{id})(\mathsf{id}\otimes E_1)$.
2. $(\mathsf{id}\otimes G_1)(G_2\otimes\mathsf{id}) = (G_2\otimes\mathsf{id})(\mathsf{id}\otimes G_1)$.
3. $(\mathsf{id}\otimes G_1)(E_2\otimes\mathsf{id}) = (E_2\otimes\mathsf{id})(\mathsf{id}\otimes G_1)$.
4. $(\mathsf{id}\otimes E_1)(G_2\otimes\mathsf{id}) = (G_2\otimes\mathsf{id})(\mathsf{id}\otimes E_1)$.

Proof. Assertion (1) is evident and (2) follows easily by the explicit forms of G_1 in (6.5) and G_2 in (6.11). Concerning (3), take any $a,b,c,d\in A$ and denote $T_4(d\otimes c) =: d'\otimes c'$, allowing for implicit summation. Then using Lemma 3.3 in the second equality,

$$\begin{aligned}(\mathsf{id}\otimes G_1)(E_2\otimes\mathsf{id})(a\otimes b\otimes cd) &\overset{(6.5)}{=} (a\otimes b)E(1\otimes\overline{\sqcap}^R(c'))\otimes d' \\ &= (a\otimes b\overline{\sqcap}^R(c'))E\otimes d' \overset{(6.5)}{=} (E_2\otimes\mathsf{id})(\mathsf{id}\otimes G_1)(a\otimes b\otimes cd).\end{aligned}$$

Part (4) is proven symmetrically. $\square$

Consider the vector subspace

$$\mathcal{M} := \{(L,K)\in\mathcal{L}\times\mathcal{R} \mid a((\epsilon\otimes\mathsf{id})L(b\otimes c)) = ((\mathsf{id}\otimes\epsilon)K(a\otimes b))c,\ \forall a,b,c\in A\}$$

of $\mathcal{L}\times\mathcal{R}$. The maps (6.2) and (6.8) induce a linear map

$$\mathcal{M}\to\mathsf{Lin}(A,\mathbb{M}(A)), \qquad (L,K)\mapsto [a\mapsto (\lambda_L(a),\rho_K(a))].$$

For $(L,K)\in\mathcal{M}$, assume that $L=0$. Then for any $a\in A$, in $(\lambda_L(a),\rho_K(a))\in\mathbb{M}(A)$ the component $\lambda_L(a)$ is zero. Hence also $\rho_K(a)=0$ for any $a\in A$ so $\rho_K=0$. Thus by the injectivity of (6.8), also $K=0$. Symmetrically, $K=0$ implies $L=0$.

By part (2) of Proposition 6.3 and its symmetric counterpart, (T_1,T_2), (E_1,G_2) and (G_1,E_2) are elements of $\mathcal{M}$. For $i\in\{1,2\}$, assume that there exist linear maps

$R_i : A \otimes A \to A \otimes A$ such that $R_i T_i = G_i$, $T_i R_i = E_i$ and $R_i T_i R_i = R_i$. Then $(R_1, R_2) \in \mathcal{L} \times \mathcal{R}$, and our next aim is to show that in fact $(R_1, R_2) \in \mathcal{M}$.

Proposition 6.6. *Let A be a regular weak multiplier bialgebra over a field, and for $i \in \{1,2\}$, let $E_i, G_i, R_i : A \otimes A \to A \otimes A$ be the same maps as before. Then the following hold.*

(1) $(\mathsf{id} \otimes E_1)(R_2 \otimes \mathsf{id}) = (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)$.
(2) $(\mathsf{id} \otimes R_1)(E_2 \otimes \mathsf{id}) = (E_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)$.
(3) $(\mathsf{id} \otimes G_1)(R_2 \otimes \mathsf{id}) = (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1)$.
(4) $(\mathsf{id} \otimes R_1)(G_2 \otimes \mathsf{id}) = (G_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)$.
(5) $(\mathsf{id} \otimes R_1)(R_2 \otimes \mathsf{id}) = (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)$.

Proof. (1) Applying part (4) of Proposition 6.5 in the second equality and its part (1) in the penultimate equality,

$$\begin{aligned}(\mathsf{id} \otimes E_1)(R_2 \otimes \mathsf{id}) &= (\mathsf{id} \otimes E_1)(G_2 R_2 \otimes \mathsf{id}) = (G_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(R_2 \otimes \mathsf{id})\\
&= (R_2 T_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(R_2 \otimes \mathsf{id}) \stackrel{E_1 \in \mathcal{L}}{=} (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(T_2 R_2 \otimes \mathsf{id})\\
&= (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(E_2 \otimes \mathsf{id}) = (R_2 E_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)\\
&= (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1).\end{aligned}$$

Parts (2)-(4) are proven analogously.

(5) Using part (1) of the current proposition in the second equality and part (3) in the penultimate equality,

$$\begin{aligned}(\mathsf{id} \otimes R_1)(R_2 \otimes \mathsf{id}) &= (\mathsf{id} \otimes R_1 E_1)(R_2 \otimes \mathsf{id}) = (\mathsf{id} \otimes R_1)(R_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)\\
&= (\mathsf{id} \otimes R_1)(R_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1 R_1) \stackrel{R_2 \in \mathcal{R}}{=} (\mathsf{id} \otimes R_1 T_1)(R_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)\\
&= (\mathsf{id} \otimes G_1)(R_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1) = (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1 R_1)\\
&= (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1).\end{aligned}$$ □

Corollary 6.7. *Let A be a regular weak multiplier bialgebra over a field and for $i \in \{1,2\}$, let $T_i, E_i, G_i : A \otimes A \to A \otimes A$ be the same maps as before. Assume that there exist linear maps $R_i : A \otimes A \to A \otimes A$ such that $R_i T_i = G_i$, $T_i R_i = E_i$ and $R_i T_i R_i = R_i$. Then $(R_1, R_2) \in \mathcal{M}$; hence there is a corresponding linear map $S := (\lambda_{R_1}, \rho_{R_2}) : A \to \mathbb{M}(A)$.*

Proof. Using in the second equality that $(G_1, E_2) \in \mathcal{M}$, it follows for any $a, b, c \in A$ that

$$\begin{aligned}a(\epsilon \otimes \mathsf{id}) R_1(b \otimes c) &= a(\epsilon \otimes \mathsf{id}) G_1 R_1 (b \otimes c)\\
&= \mu(\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(E_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)(a \otimes b \otimes c)\\
&= \mu(\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(T_2 R_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)(a \otimes b \otimes c)\\
&\stackrel{(iii)}{=} \mu(\mu \otimes \mathsf{id})(R_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)(a \otimes b \otimes c).\end{aligned}$$

Symmetrically, using in the second equality that $(E_1, G_2) \in \mathcal{M}$,

$$\begin{aligned}(\mathsf{id} \otimes \epsilon) R_2(a \otimes b) c &= (\mathsf{id} \otimes \epsilon) G_2 R_2 (a \otimes b) c\\
&= \mu(\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(R_2 \otimes \mathsf{id})(a \otimes b \otimes c)\\
&= \mu(\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(\mathsf{id} \otimes T_1 R_1)(R_2 \otimes \mathsf{id})(a \otimes b \otimes c)\\
&\stackrel{(iii)}{=} \mu(\mathsf{id} \otimes \mu)(\mathsf{id} \otimes R_1)(R_2 \otimes \mathsf{id})(a \otimes b \otimes c).\end{aligned}$$

They are equal by the associativity of μ and Proposition 6.6 (5). □

The map $S : A \to \mathbb{M}(A)$ in Corollary 6.7 — whenever it exists — will be termed the *antipode* for the following reason.

Theorem 6.8. *For any regular weak multiplier bialgebra A over a field, there is a bijective correspondence between the following data.*

(1) *For $i \in \{1,2\}$, a linear map $R_i : A \otimes A \to A \otimes A$ such that $R_iT_i = G_i$, $T_iR_i = E_i$ and $R_iT_iR_i = R_i$.*

(2) *A linear map $S : A \to \mathbb{M}(A)$ satisfying for all $a,b,c \in A$:*
   (vii) $T_1[((\mathsf{id} \otimes S)T_2(a \otimes b))(1 \otimes c)] = \Delta(a)(b \otimes c)$,
   (viii) $T_2[(a \otimes 1)((S \otimes \mathsf{id})T_1(b \otimes c))] = (a \otimes b)\Delta(c)$,
   (ix) $\mu(S \otimes \mathsf{id})[E(a \otimes 1)] = S(a)$ *(equivalently,* $\mu(\mathsf{id} \otimes S)[(1 \otimes a)E] = S(a)$*).*

Proof. (1)$\mapsto$(2). By Corollary 6.7, there is a linear map $(\lambda_{R_1}, \rho_{R_2}) =: S : A \to \mathbb{M}(A)$. Using in the penultimate equality that $T_1R_1 = E_1$,

$$\begin{aligned} T_1[((\mathsf{id} \otimes S)T_2(a \otimes b))(1 \otimes c)] &\overset{(6.3)}{=} T_1[(a \otimes 1)R_1(b \otimes c)] = \Delta(a)T_1R_1(b \otimes c) \\ &= \Delta(a)E(b \otimes c) \overset{(iv)}{=} \Delta(a)(b \otimes c) \end{aligned}$$

for all $a,b,c \in A$, so that (vii) holds. Symmetrically, (viii) follows by $T_2R_2 = E_2$. Using in the second equality $R_1E_1 = R_1$,

$$\mu(S \otimes \mathsf{id})[E(a \otimes b)] \overset{(6.4)}{=} \lambda_{R_1E_1}(a)b = \lambda_{R_1}(a)b = S(a)b,$$

for all $a,b \in A$, proving the first form of (ix). The second form follows symmetrically by $R_2E_2 = R_2$.

(2)$\mapsto$(1). By axiom (iv) in Definition 2.1,

$$\mathsf{Im}(T_1) \subseteq \langle E(a \otimes b) \mid a,b \in A\rangle = \langle \Delta(a)(b \otimes c) \mid a,b,c \in A\rangle.$$

Conversely, by (vii) $\langle \Delta(a)(b \otimes c) \mid a,b,c \in A\rangle \subseteq \mathsf{Im}(T_1)$, so that $\mathsf{Im}(T_1) = \mathsf{Im}(E_1)$. By Proposition 6.3 (3), $T_1G_1 = T_1$ so that $\mathsf{Ker}(G_1) \subseteq \mathsf{Ker}(T_1)$. In order to see the converse, note that applying $\mathsf{id} \otimes \epsilon$ to both sides of (viii) and making use of the counitality axiom (iii) and (3.3), we conclude, since A is non-degenerate, that

$$\mu(S \otimes \mathsf{id})T_1 = \mu(\sqcap^R \otimes \mathsf{id}). \tag{6.12}$$

Assume that for some $b,c \in A$, $T_1(b \otimes c) = 0$ (where implicit summation is allowed). Then for all $a \in A$,

$$\begin{aligned} 0 &= (\mathsf{id} \otimes \mu)(\mathsf{id} \otimes S \otimes \mathsf{id})(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(a \otimes b \otimes c) \\ &\overset{(ii)}{=} (\mathsf{id} \otimes \mu)(\mathsf{id} \otimes S \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})(a \otimes b \otimes c) \\ &\overset{(6.12)}{=} (\mathsf{id} \otimes \mu)(\mathsf{id} \otimes \sqcap^R \otimes \mathsf{id})(T_2 \otimes \mathsf{id})(a \otimes b \otimes c) \\ &= G_1(ab \otimes c) = (a \otimes 1)G_1(b \otimes c). \end{aligned}$$

In the penultimate equality we used an alternative form of the map G_1 in (6.1) derived from Proposition 4.3 (1), and in the last equality we used that G_1 in (6.1) is a morphism of left A-modules. By the non-degeneracy of the multiplication in $A \otimes A$, this proves $G_1(b \otimes c) = 0$, hence $\mathsf{Ker}(G_1) = \mathsf{Ker}(T_1)$. By the same reasoning applied in [13, Proposition 2.3], the above information about the image and the kernel of T_1 implies that there is a linear map $R_1 : A \otimes A \to A \otimes A$ with the desired properties. A bit more explicitly, for any $a,b \in A$,

$$R_1 : T_1(a \otimes b) \mapsto G_1(a \otimes b), \tag{6.13}$$

gives R_1 on $\mathsf{Im}(E_1) = \mathsf{Im}(T_1)$, while R_1 is defined as zero on $\mathsf{Ker}(E_1)$. The map R_2 is constructed symmetrically. Note that we have not made use of property (ix) so far.

It remains to see the bijectivity of the above correspondence. From the expression (6.13) of R_1, it is clear that it does not depend on the actual choice of the map S in part (2) (only on its existence). Hence starting with the data (R_1, R_2) as in part (1), we get from the relation $R_1 T_1 = G_1$ that R_1 must be defined by (6.13) on $\mathsf{Im}(T_1)$; and because $R_1 = R_1 E_1$, R_1 must be equal to zero on $\mathsf{Ker}(E_1)$. Similarly for R_2. Conversely, starting with a map S as in part (2) and iterating the above constructions $S \mapsto (R_1, R_2) \mapsto (\lambda_{R_1}, \rho_{R_2})$, we obtain the map $\lambda_{R_1} : A \to \mathsf{End}_A(A)$ taking $a \in A$ to

$$\begin{aligned} b \quad &\mapsto \quad (\epsilon \otimes \mathsf{id}) R_1(a \otimes b) = (\epsilon \otimes \mathsf{id}) G_1 R_1 (a \otimes b) = \mu(\sqcap^R \otimes \mathsf{id}) R_1(a \otimes b) \\ &\overset{(6.12)}{=} \quad \mu(S \otimes \mathsf{id}) T_1 R_1 (a \otimes b) = \mu(S \otimes \mathsf{id})[E(a \otimes b)]. \end{aligned}$$

In the second equality we used Proposition 6.3 (2). This element $\lambda_{R_1}(a)b$ is equal to $S(a)b$ for all $a, b \in A$ if and only if the first form of (ix) holds. Symmetrically, $a\rho_{R_2}(b)$ is equal to $aS(b)$ for all $a, b \in A$ if and only if the second form of (ix) holds, which proves in particular the equivalence of both stated forms of (ix). □

Theorem 6.8 implies in particular that if the antipode exists, then it is unique.

Let us stress that the antipode axioms in part (2) of Theorem 6.8 imply the identities

$$\begin{aligned} &\mu(S \otimes \mathsf{id}) T_1 = \mu(\sqcap^R \otimes \mathsf{id}), && \mu(\mathsf{id} \otimes S) T_2 = \mu(\mathsf{id} \otimes \sqcap^L), \\ &\mu(S \otimes \mathsf{id}) E_1 = \mu(S \otimes \mathsf{id}) \quad \Leftrightarrow && \mu(\mathsf{id} \otimes S) E_2 = \mu(\mathsf{id} \otimes S), \end{aligned} \tag{6.14}$$

expressing the requirement that S is the (widely generalized) convolution inverse of the map $A \to \mathbb{M}(A)$, $a \mapsto (a(-), (-)a)$. However, the identities in (6.14) do not seem to be equivalent to the axioms (vii)-(ix).

Combining Theorem 2.9 and Theorem 6.8, we conclude that any regular weak multiplier Hopf algebra in the sense of [13] is a regular weak multiplier bialgebra in the sense of the current paper possessing an antipode. On the other hand, if a regular weak multiplier bialgebra in the sense of the current paper admits an antipode, then it is also a weak multiplier Hopf algebra — though not necessarily a regular one — in the sense of [13]. That is to say, our regular weak multiplier bialgebras possessing an antipode are *between regular and arbitrary* weak multiplier Hopf algebras in [13].

In view of Theorem 2.10, a unital algebra possesses a weak Hopf algebra structure as in [2] if and only if via the same structure maps, it is a regular weak multiplier bialgebra with an antipode.

From Theorem 6.8 and Example 2.14, we obtain the following example.

Example 6.9. For a family $\{A_j\}_{j \in I}$ of regular weak multiplier bialgebras over a field, labelled by any index set I, the direct sum regular weak multiplier bialgebra $\bigoplus_{j \in I} A_j$ in Example 2.14 possesses an antipode if and only if A_j does, for all $j \in I$. In this case, for any $\underline{a} \in A$, $S(\underline{a}) = \varphi^{-1}(\{S_j(a_j)\}_{j \in I})$ in terms of the map (2.7) and the antipode S_j of A_j.

Our final task is to investigate the properties of the antipode.

Lemma 6.10. *Let A be a regular weak multiplier bialgebra over a field. Assume that A possesses an antipode $S : A \to \mathbb{M}(A)$. For $i \in \{1,2\}$, denote by $R_i : A \otimes A \to A \otimes A$ the corresponding maps in Theorem* 6.8 (1). *Then the following hold:*

$$\mu R_1 = \mu(\sqcap^L \otimes \mathsf{id}) \qquad \textit{and} \qquad \mu R_2 = \mu(\mathsf{id} \otimes \sqcap^R).$$

Proof. For any $a, b, c \in A$,

$$a(\mu R_1(b \otimes c)) \overset{(6.3)}{=} (\mu(\mathsf{id} \otimes S)T_2(a \otimes b))c \overset{(6.14)}{=} a \sqcap^L (b)c.$$

This proves the first assertion, and the second one is proven symmetrically. □

Lemma 6.11. *Let A be a regular weak multiplier bialgebra over a field. Assume that A possesses an antipode $S : A \to \mathbb{M}(A)$. For $i \in \{1,2\}$, denote by $R_i : A \otimes A \to A \otimes A$ the corresponding maps in Theorem* 6.8 (1). *Then the following hold:*

$$\mu(\sqcap^R \otimes \mathsf{id})R_1 = \mu(S \otimes \mathsf{id}) \qquad \textit{and} \qquad \mu(\mathsf{id} \otimes \sqcap^L)R_2 = \mu(\mathsf{id} \otimes S).$$

Proof. Using part (2) of Proposition 6.3 in the first equality, $G_1 R_1 = R_1$ in the third one, and the relation between S and λ_{R_1} in Theorem 6.8 in the last one, it follows for all $a, b \in A$ that

$$\mu(\sqcap^R \otimes \mathsf{id})R_1(a \otimes b) = \mu(\lambda_{G_1} \otimes \mathsf{id})R_1(a \otimes b) \overset{(6.4)}{=} \lambda_{G_1 R_1}(a)b = \lambda_{R_1}(a)b = S(a)b.$$

This proves the first assertion, and the second one is proven symmetrically. □

Although the following theorem is contained in [13, Proposition 3.5], we prefer to give an alternative proof not referring to Heyneman-Sweedler type indices.

Theorem 6.12. *Let A be a regular weak multiplier bialgebra over a field. If A possesses an antipode $S : A \to \mathbb{M}(A)$, then it is anti-multiplicative.*

Proof. For $i \in \{1,2\}$, denote by $R_i : A \otimes A \to A \otimes A$ the maps in Theorem 6.8 (1). Consider the composite map

$$W := \mu^2(R_2 \otimes \mathsf{id})(\mathsf{id} \otimes \mu \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{id} \otimes R_1)(\mathsf{id} \otimes \mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{id} \otimes R_1)(\mathsf{id} \otimes \mathsf{tw} \otimes \mathsf{id})$$

from $A \otimes A \otimes A \otimes A$ to A. We shall evaluate it on an arbitrary element $a \otimes b \otimes c \otimes d$ in two different ways. In one case, we will get $aS(bc)d$, and in the other case it will yield $aS(c)S(b)d$.

To begin with, compute

$$\begin{aligned}
(\mu(\mathsf{id} \otimes S) \otimes \mathsf{id})(\mathsf{id} \otimes T_1) &\overset{(6.8)}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(R_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) \\
&\overset{R_2 \in \mathcal{R}}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(R_2 \otimes \mathsf{id}) \\
&\overset{(iii)}{=} (\mathsf{id} \otimes \mu)(R_2 \otimes \mathsf{id}).
\end{aligned}$$

With its help,

$$W(a \otimes b \otimes c \otimes d) = a(\mu(S \otimes \mathsf{id})T_1(\mu \otimes \mathsf{id})(\mathsf{id} \otimes R_1)(\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes R_1)(c \otimes b \otimes d)).$$

Next, for all $b, c, d \in A$,

$$T_1(\mu \otimes \mathsf{id})(\mathsf{id} \otimes R_1)(b \otimes c \otimes d) = \Delta(b)(T_1 R_1(c \otimes d)) \overset{(iv)}{=} T_1(b \otimes d)(c \otimes 1).$$

Using this computation,

$$\begin{aligned} W(a\otimes b\otimes c\otimes d) &= a(\mu(S\otimes \mathsf{id})[(T_1R_1(b\otimes d))(c\otimes 1)]) \\ &= a(\mu(S\otimes \mathsf{id})E_1(bc\otimes d)) \overset{(6.14)}{=} aS(bc)d \end{aligned}$$

for all $a, b, c, d \in A$. On the other hand, by Lemma 6.10 and (3.6),

$$\mu R_2(\mathsf{id}\otimes \mu) = \mu(\mathsf{id}\otimes \sqcap^R\mu) = \mu(\mathsf{id}\otimes \sqcap^R)(\mathsf{id}\otimes \mu(\sqcap^R\otimes \mathsf{id})) = \mu R_2(\mathsf{id}\otimes \mu(\sqcap^R\otimes \mathsf{id})),$$

hence

$$\begin{aligned} W = \mu^2\, &(R_2\otimes \mathsf{id})(\mathsf{id}\otimes \mu(\sqcap^R\otimes \mathsf{id})\otimes \mathsf{id}) \\ &(\mathsf{id}\otimes \mathsf{id}\otimes R_1)(\mathsf{id}\otimes \mathsf{tw}\otimes \mathsf{id})(\mathsf{id}\otimes \mathsf{id}\otimes R_1)(\mathsf{id}\otimes \mathsf{tw}\otimes \mathsf{id}). \end{aligned}$$

Moreover, for any $a, b, c, d \in A$,

$$\begin{aligned} R_2(a\otimes \sqcap^R(b)c)(1\otimes d) &\overset{(6.9)}{=} (a\otimes 1)((S\otimes \mathsf{id})T_1(\sqcap^R(b)c\otimes d)) \\ &\overset{(3.7)}{=} (a\otimes 1)((S\otimes \mathsf{id})((1\otimes \sqcap^R(b))T_1(c\otimes d))) \\ &= (a\otimes \sqcap^R(b))((S\otimes \mathsf{id})T_1(c\otimes d)). \end{aligned}$$

Therefore,

$$\begin{aligned} &\mu^2(R_2\otimes \mathsf{id})(\mathsf{id}\otimes \mu(\sqcap^R\otimes \mathsf{id})\otimes \mathsf{id})(\mathsf{id}\otimes \mathsf{id}\otimes R_1)(a\otimes b\otimes c\otimes d) \\ &\quad= \mu[(a\otimes \sqcap^R(b))((S\otimes \mathsf{id})T_1R_1(c\otimes d))] \\ &\quad= \mu[(a\otimes \sqcap^R(b))((S\otimes \mathsf{id})(E(c\otimes d)))] \\ &\quad\overset{(3.7)}{=} a(\mu(S\otimes \mathsf{id})E_1(c\otimes \sqcap^R(b)d)) \overset{(6.14)}{=} aS(c)\sqcap^R(b)d. \end{aligned}$$

Substituting this identity and applying Lemma 6.11, we obtain

$$W(a\otimes b\otimes c\otimes d) = \mu^3(\mathsf{id}\otimes S\otimes \sqcap^R\otimes \mathsf{id})(\mathsf{id}\otimes \mathsf{id}\otimes R_1)(a\otimes c\otimes b\otimes d) = aS(c)S(b)d$$

for any $a, b, c, d \in A$. By the density of A in $\mathbb{M}(A)$, this proves $S(bc) = S(c)S(b)$, for all $b, c \in A$. □

The following proposition is contained in [13, Proposition 3.6]. However, in our setting a much shorter proof can be given.

Proposition 6.13. *Let A be a regular weak multiplier bialgebra over a field which possesses an antipode $S : A \to \mathbb{M}(A)$. Whenever the comultiplication is left and right full, S is a non-degenerate map.*

Proof. Using the idempotency of the algebra A, Lemma 3.7 (1), the fact that $\sqcap^L(A) = \overline{\sqcap}^L(A)$ (cf. Theorem 3.13) and (6.14),

$$A = A^2 \subseteq A\overline{\sqcap}^L(A) = A\sqcap^L(A) \subseteq AS(A) \subseteq A$$

so that $A = AS(A)$. A symmetrical reasoning shows that also $A = S(A)A$. □

We conclude by Theorem 1.1 that in the situation in Proposition 6.13 the antipode extends to algebra homomorphisms $\overline{S} : \mathbb{M}(A)^{\mathsf{op}} \to \mathbb{M}(A)$, $\overline{\mathsf{id}\otimes S} : \mathbb{M}(A\otimes A^{\mathsf{op}}) \cong \mathbb{M}(A^{\mathsf{op}}\otimes A)^{\mathsf{op}} \to \mathbb{M}(A\otimes A)$, $\overline{S\otimes \mathsf{id}} : \mathbb{M}(A\otimes A) \to \mathbb{M}(A^{\mathsf{op}}\otimes A) \cong \mathbb{M}(A\otimes A^{\mathsf{op}})^{\mathsf{op}}$ and $\overline{S\otimes S} : \mathbb{M}(A\otimes A)^{\mathsf{op}} \to \mathbb{M}(A\otimes A)$.

Lemma 6.14. *Let A be a regular weak multiplier bialgebra over a field. Assume that A possesses an antipode $S : A \to \mathbb{M}(A)$. Then for any $a, b \in A$, the following hold:*

$$S(a\overline{\sqcap}^L(b)) = \sqcap^R(b)S(a) \qquad S(\overline{\sqcap}^L(b)a) = S(a)\sqcap^R(b)$$
$$S(a\overline{\sqcap}^R(b)) = \sqcap^L(b)S(a) \qquad S(\overline{\sqcap}^R(b)a) = S(a)\sqcap^L(b).$$

Proof. Using Lemma 3.9 in the second equality, it follows for any $a, b, c \in A$ that

$$\begin{aligned} aS(b\overline{\sqcap}^L(c)) &\overset{(6.14)}{=} \mu(\mathsf{id}\otimes S)[(a\otimes b\,\overline{\sqcap}^L(c))E] \\ &= \mu(\mathsf{id}\otimes S)[(a\sqcap^R(c)\otimes b)E] \overset{(6.14)}{=} a\sqcap^R(c)S(b). \end{aligned}$$

By the density of A in $\mathbb{M}(A)$, this proves the first claim. It also implies that

$$S(a)\sqcap^R(c)S(b)d = S(a)S(b\overline{\sqcap}^L(c))d = S(\overline{\sqcap}^L(c)a)S(b)d$$

for all $a, b, c, d \in A$, where in the second equality we used the anti-multiplicativity of S. Using the non-degeneracy of S and the density of A in $\mathbb{M}(A)$, we have the second claim proven. The remaining assertions follow symmetrically. $\square$

In view of Proposition 2.5 and (3.4), in any regular weak multiplier bialgebra A over a field, we may regard E as an element of $\mathbb{M}(A \otimes A^{\mathsf{op}})$. The following proposition — and thus its Corollary 6.16 — was proven in [13] only for regular weak multiplier Hopf algebras.

Proposition 6.15. *Let A be a regular weak multiplier bialgebra over a field with a left and right full comultiplication. Assume that A possesses an antipode $S : A \to \mathbb{M}(A)$. Then the elements $E \in \mathbb{M}(A \otimes A^{\mathsf{op}})$ and $F \in \mathbb{M}(A \otimes A)$ in Proposition 4.3 (1) are related via the extensions of S as*

$$(\overline{\mathsf{id}\otimes S})(E) = F \qquad \text{and} \qquad (\overline{S\otimes \mathsf{id}})(F) = E^{\mathsf{op}},$$

where $(a \otimes 1)E^{\mathsf{op}}(1 \otimes b) := \mathsf{tw}[(1 \otimes a)E(b \otimes 1)]$ and $(1 \otimes b)E^{\mathsf{op}}(a \otimes 1) := \mathsf{tw}[(b \otimes 1)E(1 \otimes a)]$ define $E^{\mathsf{op}} \in \mathbb{M}(A^{\mathsf{op}} \otimes A)$.

Proof. Since A is idempotent and S is non-degenerate (by Proposition 6.13), any element of $A \otimes A$ can be written as a linear combination of elements of the form $ab \otimes cS(d)$, in terms of $a, b, c, d \in A$. Moreover, using the anti-multiplicativity of S in the first equality, applying Proposition 2.5 (1) in the second equality, Lemma 6.14 in the third one and Proposition 4.3 (1) in the last one, it follows for any $a, b, c, d \in A$ that

$$\begin{aligned} (ab\otimes cS(d))(\overline{\mathsf{id}\otimes S})(E) &= (1\otimes c)((\mathsf{id}\otimes S)[(ab\otimes 1)E(1\otimes d)]) \\ &= (1\otimes c)((\mathsf{id}\otimes S)[((\mathsf{id}\otimes\overline{\sqcap}^L)T_2(a\otimes b))(1\otimes d)]) \\ &= (1\otimes cS(d))((\mathsf{id}\otimes\sqcap^R)T_2(a\otimes b)) = (ab\otimes cS(d))F. \end{aligned}$$

This proves the first assertion. Symmetrically, in order to prove the second one, write any element of $A\otimes A$ as a linear combination of elements of the form $aS(b)\otimes cd$. Using again the anti-multiplicativity of S in the first equality, Proposition 4.3 (1) in the second equality, Lemma 6.14 in the third one and (3.4) in the fourth one, it follows for any $a, b, c, d \in A$ that

$$\begin{aligned}
&(aS(b)\otimes 1)(\overline{S\otimes \mathsf{id}})(F)(1\otimes cd)\\
&\quad = (a\otimes 1)((S\otimes \mathsf{id})[F(b\otimes cd)])\\
&\quad = (a\otimes 1)((S\otimes \mathsf{id})[((\overline{\sqcap}^R\otimes \mathsf{id})T_4^{\mathsf{op}}(d\otimes c))(b\otimes 1)])\\
&\quad = (aS(b)\otimes 1)((\sqcap^L\otimes \mathsf{id})T_4^{\mathsf{op}}(d\otimes c))\\
&\quad = (aS(b)\otimes 1)\mathsf{tw}(E(cd\otimes 1))\\
&\quad = \mathsf{tw}((1\otimes aS(b))E(cd\otimes 1))\\
&\quad = (aS(b)\otimes 1)E^{\mathsf{op}}(1\otimes cd). \qquad \square
\end{aligned}$$

Corollary 6.16. *Let A be a regular weak multiplier bialgebra over a field with a left and right full comultiplication. If A possesses an antipode $S : A \to \mathbb{M}(A)$, then S is anti-comultiplicative in the sense of the commutative diagram*

$$\begin{tikzcd}
A^{\mathsf{op}} \arrow[r, "S"] \arrow[d, "\Delta^{\mathsf{op}}"'] & \mathbb{M}(A) \arrow[d, "\overline{\Delta}"] \\
\mathbb{M}(A\otimes A)^{\mathsf{op}} \arrow[r, "\overline{S\otimes S}"'] & \mathbb{M}(A\otimes A).
\end{tikzcd}$$

Proof. By [13, Proposition 3.7], $\overline{\Delta}S(a) = ((\overline{S\otimes S})\Delta^{\mathsf{op}}(a))E$, for all $a \in A$. By Proposition 6.15, $E = (\overline{S\otimes S})(E^{\mathsf{op}})$, so that by the anti-multiplicativity of $\overline{S\otimes S}$,

$$\begin{aligned}
\overline{\Delta}S(a) &= ((\overline{S\otimes S})\Delta^{\mathsf{op}}(a))E = ((\overline{S\otimes S})\Delta^{\mathsf{op}}(a))(\overline{S\otimes S})(E^{\mathsf{op}})\\
&= (\overline{S\otimes S})(E^{\mathsf{op}}\Delta^{\mathsf{op}}(a)) \overset{(iv)}{=} (\overline{S\otimes S})\Delta^{\mathsf{op}}(a). \qquad \square
\end{aligned}$$

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