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SOME EQUATIONS FOR THE UNIVERSAL KUMMER VARIETY

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ABSTRACT. We give a method to find quartic equations for Kummer varieties and we give some explicit examples. From these equations for g-dimensional Kummer varieties one obtains equations for the moduli space of $g+1$-dimensional Kummer varieties. These again define modular forms which vanish on the period matrices of Riemann surfaces. The modular forms that we find for $g=5$ appear to be new and of lower weight than known before.

INTRODUCTION

The classical Kummer surface is a quartic surface in $\mathbf{P}^3$ with 16 nodes. It is the quotient of an abelian surface by the involution $x \mapsto -x$. The translations by points of order two on the abelian surface induce projective transformations that map the Kummer surface into itself. This group action lifts to an action of a finite (non-abelian) Heisenberg group on $\mathbf{C}^4$.

Using classical theta function theory, the equation of this surface can be written as a Heisenberg invariant quartic polynomial whose coefficients are polynomials of degree 12 in the second order theta constants. Thus this polynomial is best seen as a polynomial of (bi)degree $(12, 4)$ in $\mathbf{C}[\mathbf{u}, \mathbf{x}]$ which, upon substituting the second order theta constants for the variables $\mathbf{u}$, gives the equation of the Kummer surface. Such a polynomial is called an equation for the universal Kummer variety.

More generally, the Kummer variety of a (principally polarized) abelian variety of dimension g admits a map to a projective space of dimension $2^g - 1$. It is defined by second order theta functions. For a general Kummer variety, the ideal of the image is generated by homogeneous polynomials of degree at most four. Equations of degree four were given in [K, Theorem 3.6].

A classical result asserts that Kummer surfaces are tangent hyperplane sections of the Igusa quartic threefold in $\mathbf{P}^4$. Somewhat remarkably, generalizations of the Igusa quartic are quite useful for finding equations for Kummer varieties and their moduli space. In Proposition 3.4 we show that a generalized Igusa equation (see Definition 3.2) provides a Heisenberg invariant quartic equation for the universal Kummer variety. In the case of dimension $g = 3, 4$ we showed (with computer computations) that there exist generalized Igusa equations of degree four. This implies that there are equations of degree $(12, 4)$ for the universal Kummer variety in these cases (see Sections 3.7, 3.8).

After having read a first version of this paper, R. Salvati Manni suggested that our method would also apply to the non-Heisenberg invariant quartic equations. This is indeed the case. A generalized Igusa equation of degree d, for g-dimensional abelian varieties, will provide a universal non-Heisenberg invariant quartic equation

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of bidegree $(4(d-1),4)$ for the Kummer varieties of dimension $g+1$ (see Proposition 4.6). Thus we find explicit quartic, non-Heisenberg invariant, equations for the universal Kummer variety in genus $3,4,5$ of degree $(12,4)$; see Section 4.7.

The equations for the moduli space of Kummer varieties are the universal Kummer equations of degree $(d,0)$ for some d. In Section 5 we show that a generalized Igusa equation of degree d, for g-dimensional abelian varieties, provides an equation of degree $4d$ for the moduli space of $g+1$-dimensional abelian varieties. Finally we recall that, using the Schottky-Jung relations, such an equation leads to an equation for the locus of Jacobians inside the moduli space of abelian varieties of dimension $g+2$.

These new equations for the moduli space and the Jacobi locus present a major improvement. For example, for $g=4$ we find equations of degree 16 for the moduli space in $\mathbf{P}^{15}$, whereas the known equations, obtained by 'rationalizing' equations between classical theta constants, are of degree 32. The resulting equations for the Jacobi locus for $g=5$, as well as those which we find for $g=6$, were not known before either. These new equations should be helpful in the study of the Schottky problem, which asks for equations defining the Jacobi locus in the moduli space of abelian varieties. In fact, the degree of the known equations for the Jacobi locus, which are polynomials in the classical theta constants obtained from the Schottky-Jung relations, have a degree which increases rapidly with g. If generalized Igusa equations of degree 4 exist for all $g\geq 3$, then our results give degree 16 equations for the Jacobi locus in any $g\geq 4$. This would be an important new insight in the classical Schottky problem and might contribute towards a solution.

This paper was motivated by the papers [RSSS] and [GSM] where an explicit equation of the Coble quartic was given. The singular locus of this quartic hypersurface in $\mathbf{P}^7$ is the Kummer variety of the Jacobian of a non-hyperelliptic curve. The Coble quartic is also the moduli space of rank two bundles with trivial determinant on the curve. The equation for the Coble quartic is in fact an equation for the universal Kummer variety of degree $(28,4)$. It is not (yet) clear how it is related to the 27 equations of bidegree $(12,4)$ given in Section 3.7 of this paper.

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1. Basics

1.1. **Theta functions.** For $\tau\in\mathcal{H}_g$, the Siegel upper halfspace of $g\times g$ symmetric complex matrices with positive definite imaginary part, one defines a (principally polarized) abelian variety A_τ by $A_\tau:=\mathbf{C}^g/(\mathbf{Z}^g+\mathbf{Z}^g\tau)$, were the vectors are row vectors. We denote by L_τ a symmetric line bundle on A_τ defining the principal polarization.

The classical theta functions with characteristics $\epsilon,\epsilon'\in\{0,1\}^g$ are the holomorphic functions on $\mathcal{H}_g\times\mathbf{C}^g$ defined by

$$\theta[{}^{\epsilon}_{\epsilon'}](\tau,z)\;:=\;\sum_{n\in\mathbf{Z}^g}\exp\left[\pi i(n+\epsilon/2)\tau^t(n+\epsilon/2)+2\pi i(n+\epsilon/2)^t(z+\epsilon'/2)\right]$$

(cf. [RSSS, (2.3)]). We will also write $\theta_m=\theta[{}^{\epsilon}_{\epsilon'}]$ with $m=[{}^{\epsilon}_{\epsilon'}]$. The function θ_m defines a global section of a translate of L_τ by a point of order two.

The second order theta functions are defined as

$$\Theta[\sigma](\tau,z)\;:=\;\theta[{}^{\sigma}_{0}](2\tau,2z)\qquad(\sigma\in\{0,1\}^g).$$

They provide a basis of $H^0(A_\tau, L_\tau^{\otimes 2})$. The (totally symmetric) line bundle $L_\tau^{\otimes 2}$ is intrinsically defined by the principal polarization on A_τ.

1.2. **The Kummer variety.** The second order theta functions are all even functions, $\Theta[\sigma](\tau, z) = \Theta[\sigma](\tau, -z)$. The map

$$\Theta_\tau \,:\, A_\tau \longrightarrow \mathbf{P}^{2^g-1}, \qquad z \longmapsto (\ldots : \Theta[\sigma](\tau, z) : \ldots)$$

which they define thus factors over the quotient variety $A_\tau/\{\pm 1\}$ which is known as the Kummer variety of the abelian variety A_τ. The singular locus of the Kummer variety consists of 2^{2g} points which are the images of the fixed points of the map $x \mapsto -x$ on A_τ; these are the 2-torsion points of A_τ. In case the ppav A_τ is not a product of ppav's of lower dimension, the image of Θ_τ is isomorphic to its Kummer variety.

1.3. **The universal Kummer variety.** The coefficients of the equations for the Kummer variety which we will give in this paper depend only on the coordinates of the point $\Theta_\tau(0)$. This leads us (similar to [RSSS, Section 3]) to consider the map

$$\Theta \,:\, \mathcal{H}_g \times \mathbf{C}^g \longrightarrow \mathbf{P}^{2^g-1} \times \mathbf{P}^{2^g-1}\ , \qquad (\tau, z) \longmapsto (u, x) = \big(\Theta_\tau(0),\, \Theta_\tau(z)\big).$$

The image of this map is a quasi-projective variety of dimension $g + g(g+1)/2$. The closure of the image is called the universal Kummer variety $\mathcal{K}_g(2,4)$, and we will exhibit polynomials in $\mathbf{C}[\mathbf{u}, \mathbf{x}]$ (actually in $\mathbf{Q}[\mathbf{u}, \mathbf{x}]$) which are in the ideal $\mathcal{I}_g$ of $\mathcal{K}_g(2,4)$. In particular, for genus $2, 3, 4$ we show that there are such polynomials of bidegree $(12, 4)$ (these are new for $g = 3, 4$).

As $\Theta_\tau(0)$ lies in the image of the Kummer variety, any universal Kummer equation $F(\mathbf{u}, \mathbf{x})$ gives an equation $F(\mathbf{u}, \mathbf{u})$ for the image of the Siegel upper halfspace $\mathcal{H}_g$ in $\mathbf{P}^{2^g-1}$ under the map

$$\mathcal{H}_g \longrightarrow \mathbf{P}^{2^g-1}, \qquad \tau \longmapsto \Theta_\tau(0).$$

The new equations of bidegree $(12, 4)$ for $\mathcal{K}_g(2,4)$ have the extra property that $F(\mathbf{u}, \mathbf{u})$ is identically zero as a polynomial in $\mathbf{u}$ if the equation is Heisenberg invariant (as a polynomial in $\mathbf{x}$). The non-Heisenberg invariant equations do give non-trivial equations for the image, and we find equations for $\mathcal{K}_g(2,4)$ of bidegree $(16, 0)$ for $g = 3, 4, 5$ (which are new for $g = 4, 5$).

1.4. **The Heisenberg group.** Any $a \in A_\tau[2]$, the subgroup of two torsion points of A_τ, defines a biholomorphic map $t_a : A_\tau \to A_\tau$, $x \mapsto x + a$. The line bundles L_τ and $t_a^* L_\tau$ are isomorphic, and such an isomorphism is unique up to scalar multiple. However, one cannot choose the isomorphisms for the various a in such a way as to obtain an action of $A_\tau[2]$ on $H^0(A_\tau, L_\tau^{\otimes 2})$. Instead one obtains a representation of a non-commutative Heisenberg group H_g, an extension of $A_\tau[2] \cong (\mathbf{Z}/2\mathbf{Z})^{2g}$ by $\mathbf{C}^\times$, on $H^0(A_\tau, L_\tau^{\otimes 2})$.

The map Θ_τ is equivariant for the action of H_g, where the action on $\mathbf{P}^{2^g-1}$ is generated by the following projective transformations (which we refer to as sign changes and translations):

$$x_\sigma \longmapsto (-1)^{\sigma^t\alpha} x_\sigma\,; \qquad x_\sigma \longmapsto x_{\sigma+\beta} \qquad\qquad (\sigma \in (\mathbf{Z}/2\mathbf{Z})^g)\ ,$$

where the x_σ are the homogeneous coordinates on $\mathbf{P}^{2^g-1}$ and $\alpha, \beta \in (\mathbf{Z}/2\mathbf{Z})^g$ correspond to the elements $\alpha/2, \beta\tau/2 \in \frac{1}{2}\Lambda_\tau/\Lambda_\tau = A_\tau[2]$ respectively.

2. Equations for the Kummer variety

2.1. **The general theory.** We will write a (homogeneous) polynomial of degree d in the 2^g variables x_σ as

$$F := \sum_\alpha c_\alpha x^\alpha \qquad (\in \mathbf{C}[\mathbf{x}] := \mathbf{C}[\ldots, x_\sigma, \ldots]),$$

where

$$\alpha = (\ldots, \alpha_\sigma, \ldots) \in \mathbf{Z}^{2^g}_{\geq 0}, \qquad |\alpha| := \sum_\sigma \alpha_\sigma = d, \qquad x^\alpha := \prod_\sigma x_\sigma^{\alpha_\sigma}.$$

The polynomial F is an equation for the Kummer variety of A_τ if the theta function

$$(\Theta_\tau^* F)(z) := \sum_\alpha c_\alpha \prod_\sigma \Theta[\sigma](\tau, z)^{\alpha_\sigma}$$

is zero as a function of $z \in \mathbf{C}^g$. So Θ_τ^* substitutes $x_\sigma := \Theta[\sigma](\tau, z)$.

Since F is homogeneous of degree d, the function $(\Theta_\tau^* F)(z))$ is a theta function of order $2d$; equivalently, it corresponds to a global section of $L_\tau^{\otimes 2d}$. Let $\vartheta_1, \ldots, \vartheta_N$, where $N = N_d$, be theta functions which provide a basis of $H^0(A_\tau, L_\tau^{\otimes 2d})$. Then $(\Theta_\tau^* F)(z)) = \sum_{i=1}^N a_i \vartheta_i(z)$ for certain complex numbers a_i. Moreover, F is an equation for the Kummer variety if and only if $a_i = 0$ for all i. Thus the $\mathbf{C}$-vector space $I_d = I_d(\Theta_\tau(A_\tau))$ of equations of degree d of the Kummer variety is

$$I_d := \ker(\, \Theta_\tau^* : \mathbf{C}[\mathbf{x}]_d \longrightarrow H^0(A_\tau, L_\tau^{\otimes 2d})\,),$$

where $\mathbf{C}[\mathbf{x}]_d$ denotes the vector space of homogeneous polynomials of degree d. (Equivalently, I_d is the kernel of the natural map $S^d H^0(A_\tau, L_\tau^{\otimes 2}) \to H^0(A_\tau, L_\tau^{\otimes 2d})$.)

Finding the equations of degree d is thus a problem in linear algebra once the matrix of the $\mathbf{C}$-linear map Θ_τ^* w.r.t. bases of its domain and image are known. For degree $d = 2$ this matrix is the diagonal matrix with entries $\theta[{\epsilon \atop \epsilon'}](\tau, 0)$. Unfortunately, for $d = 3$ this matrix is not known explicitly. For $d = 4$ the matrix was given in [vG, Proposition 4], and we recall part of the result in Section 2.4.

2.2. **The map Θ_τ^* in degree two.** We recall that for degree $d = 2$ the matrix of Θ_τ^*, w.r.t. suitable bases, is a diagonal matrix. For $\epsilon, \epsilon' \in (\mathbf{Z}/2\mathbf{Z})^g$ we define a polynomial in $\mathbf{C}[\mathbf{x}]_2$ by

$$Q[{\epsilon \atop \epsilon'}] := \sum_\sigma (-1)^{\sigma^t \epsilon'} x_\sigma x_{\sigma+\epsilon}$$

with summation over $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$. In case $\epsilon^t \epsilon' = 0$ this polynomial is not identically zero. The $2^{g-1}(2^g + 1)$ polynomials one finds in this way are a basis of $\mathbf{C}[\mathbf{x}]_2$. (They are also eigenvectors for the action of the Heisenberg group.) A classical theta function formula shows that ([RSSS, (2.8)])

$$\Theta_\tau^*(Q[{\epsilon \atop \epsilon'}]) = Q[{\epsilon \atop \epsilon'}](\ldots, \Theta[\sigma](\tau, z), \ldots) = \theta[{\epsilon \atop \epsilon'}](\tau, 0)\theta[{\epsilon \atop \epsilon'}](\tau, 2z).$$

The functions $\theta[{\epsilon \atop \epsilon'}](\tau, 2z)$, with $\epsilon^t \epsilon' = 0$, are a basis of the subspace of even theta functions in $H^0(A_\tau, L_\tau^{\otimes 4})$. Therefore the matrix of Θ_τ^* is now a diagonal matrix with entries the even theta constants $\theta[{\epsilon \atop \epsilon'}](\tau, 0)$. In particular, there are quadratic equations for the Kummer variety of A_τ if and only if at least one of these theta constants is zero.

2.3. **The Heisenberg invariant quartics.** The vector space $\mathbf{C}[\mathbf{x}]_4$ has the obvious basis given by the monomials $\prod_\sigma x_\sigma^{\alpha_\sigma}$ with $|\alpha| = 4$. As the map Θ_τ^* is equivariant w.r.t. the action of the Heisenberg group, the matrix will be in block form if we choose bases adapted to this action.

In [vG, Proposition 1(iii)] it is shown that

$$\mathbf{C}[\mathbf{x}]_4 = \bigoplus_\chi \mathbf{C}[\mathbf{x}]_{4,\chi},$$

where $\chi : H_g \to \{\pm 1\}$ runs over the characters of H_g; these maps factor over $(\mathbf{Z}/2\mathbf{Z})^{2g}$. The dimension of the eigenspace with character χ is $(2^g+1)(2^{g-1}+1)/3$ if $\chi = 0$ (i.e. $\chi(x) = 1$ for all x) and it is $(2^{g-1}+1)(2^{g-2}+1)/3$ for the $2^{2g}-1$ non-trivial characters; see below.

A basis for $\mathbf{C}[\mathbf{x}]_{4,0}$, the quartic Heisenberg invariants, can be found as follows. Let $T \subset (\mathbf{Z}/2\mathbf{Z})^g$ be a subgroup of order at most 4. So we have the following possibilities for T:

$$T = \{0\}, \quad \{0,\alpha\}, \quad \{0,\alpha,\beta,\alpha+\beta\} \qquad (\alpha,\beta \in (\mathbf{Z}/2\mathbf{Z})^g) \qquad (\alpha \neq \beta).$$

In all cases we can in fact write $T = \{0,\alpha,\beta,\alpha+\beta\}$, possibly with $\alpha = \beta = 0$ or $\beta = 0$. There are $1 + (2^g-1) + (2^g-1)(2^g-2)/6$ such subgroups. For each such subgroup T the monomial $x_0 x_\alpha x_\beta x_{\alpha+\beta}$ is invariant under the sign changes in the Heisenberg group. Now we simply take the sum of the monomials in the orbit of this monomial under the translations in the Heisenberg group. Then we obtain the polynomial

$$P_T := \sum_{\rho \in (\mathbf{Z}/2\mathbf{Z})^g} x_\rho x_{\rho+\alpha} x_{\rho+\beta} x_{\rho+\alpha+\beta}\ , \qquad T = \{0,\alpha,\beta,\alpha+\beta\}.$$

These P_T's are a basis of $\mathbf{C}[\mathbf{x}]_{4,0}$.

We will need to know the partial derivative of P_T w.r.t. x_σ. In case $T = \{0\}$, we have $P_T = \sum_\rho x_\rho^4$ and $\partial P_T/\partial x_\sigma = 4x_\sigma^3$. In case $T = \{0,\alpha\}$, we have to consider the summands of P_T with $\rho = \sigma$, which is $x_\sigma^2 x_{\sigma+\alpha}^2$, and $\rho = \sigma+\alpha$, which is $x_{\sigma+\alpha}^2 x_\sigma^2$, so $\partial P_T/\partial x_\sigma = 4x_\sigma x_{\sigma+\alpha}^2$. In case $\sharp T = 4$, there are four monomials which contribute to the partial derivative, and each gives $x_{\sigma+\alpha}x_{\sigma+\beta}x_{\sigma+\alpha+\beta}$. So we find the following result, which holds for all subgroups T:

$$\frac{\partial P_T}{\partial x_\sigma} = 4x_{\sigma+\alpha}x_{\sigma+\beta}x_{\sigma+\alpha+\beta}\ , \qquad T = \{0,\alpha,\beta,\alpha+\beta\}.$$

2.4. **The map Θ_τ^* in degree four.** The decomposition of the image of Θ_τ^* into eigenspaces for the Heisenberg group is as follows:

$$H^0(A_\tau, L_\tau^{\otimes 8}) = \bigoplus_\chi H^0(A_\tau, L_\tau^{\otimes 8})_\chi\ ,$$

where $\dim H^0(A_\tau, L_\tau^{\otimes 8})_\chi$ is 2^g (but if $\chi \neq 0$, the subspace of even theta functions has dimension 2^{g-1}). With these decompositions, we have, for all characters χ of H_g,

$$\Theta_\tau^*(\mathbf{C}[\ldots,x_\sigma,\ldots]_{4,\chi}) \ \subset\ H^0(A_\tau, L_\tau^{\otimes 8})_\chi.$$

The 'multiplication by two' map [2] on the abelian variety A_τ (so $[2](x) := 2x$ for $x \in A_\tau$) gives the following isomorphism:

$$H^0(A_\tau, L_\tau^{\otimes 8})_0 = [2]^* H^0(A_\tau, L_\tau^{\otimes 2}), \qquad \text{so the} \quad \Theta[\sigma](\tau, 2z) \quad (\sigma \in (\mathbf{Z}/2\mathbf{Z})^g)$$

are a basis of $H^0(A_\tau, L_\tau^{\otimes 8})_0$.

We now consider the multiplication map for the case $\chi = 0$:

$$\Theta_\tau^* : \ \mathbf{C}[\ldots, x_\sigma, \ldots]_{4,0} = \bigoplus_T \mathbf{C}P_T \ \longrightarrow \ H^0(A_\tau, L_\tau^{\otimes 8})_0 = \bigoplus_\sigma \mathbf{C}\Theta[\sigma](\tau, 2z).$$

So, given a subgroup $T = \{0, \alpha, \beta, \alpha+\beta\}$, we want to find the complex numbers $a_{\sigma,T}$ such that

$$\Theta_\tau^*(P_T) = \sum_\sigma a_{\sigma,T}\, \Theta[\sigma](\tau, 2z). \tag{2.4.a}$$

Riemann's theta formula implies that (see [vG, Proposition 4]):

$$a_{\sigma,T} = \Theta[\sigma+\alpha](\tau,0)\Theta[\sigma+\beta](\tau,0)\Theta[\sigma+\alpha+\beta](\tau,0). \tag{2.4.b}$$

Using the derivatives of the P_T, there is an attractive way to write this result:

$$4a_{\sigma,T} = \frac{\partial P_T}{\partial x_\sigma}(\Theta_\tau(0)) \ ,$$

that is, the entries of the matrix, up to a factor 4 which does not affect the kernel, are just the partial derivatives of the P_T's evaluated in the point $\Theta_\tau(0)$ which has coordinates $x_\sigma = \Theta[\sigma](\tau, 0)$. In other words, the matrix $(a_{\sigma,T})$ is the transpose of the Jacobi matrix of the polynomials P_T, evaluated in $\Theta_\tau(0)$.

2.5. **Quartic equations.** Using Cramer's rule, applied to a $2^g \times (2^g+1)$ submatrix of rank 2^g of $(a_{\sigma,T})$, one obtains quartic Heisenberg equations for the Kummer variety whose coefficients are of degree $3 \cdot 2^g$ in the $\Theta[\sigma](\tau, 0)$. The quartic equations given by Khaled ([K, Theorem 3.6(b)]) appear to be of degree $(2^{8g}, 4)$ (see the definition of the $\tilde{q}$'s on p. 208 of [K]), but he also considers more general embeddings of Kummer varieties.

In Section 2.7 we recall that the classical equation for the Kummer surface in $\mathbf{P}^3$ is obtained in this way. In Section 2.8 we recall a result from [RSSS] which shows that there are equations of lower degree, $(16, 4)$ rather than $(24, 4)$, in case $g = 3$. However, for $g = 3, 4$ there also exist equations of degree $(12, 4)$ as we will show in Sections 3.7, 3.8.

There is a similar result for the matrix of $\Theta_\tau^* : \mathbf{C}[\ldots, x_\sigma, \ldots]_{4,\chi} \to H^0(A_\tau, L_\tau^{\otimes 8})_\chi$ for non-trivial χ. However, the entries of this matrix are no longer polynomials in the $\Theta[\sigma](\tau, 0)$, but they involve more general theta constants; see Section 4.2.

Before discussing the cases $g = 2, 3$, we give a characterization of the quartic, Heisenberg invariant, equations of the Kummer variety which will be exploited in the next section.

2.6. **Proposition.** *A quartic, Heisenberg invariant, polynomial P is an equation for the Kummer variety of A_τ if and only if the point $\Theta_\tau(0) := (\ldots : \Theta[\sigma](\tau, 0) : \ldots)$ is a singular point of the algebraic variety defined by $P = 0$ in $\mathbf{P}^{2^g-1}$.*

Proof. Since $P \in \mathbf{C}[\mathbf{x}]_{4,0}$, there are $c_T \in \mathbf{C}$ such that $P = \sum_T c_T P_T$. Using equation (2.4.a), the image of P in $H^0(A_\tau, L_\tau^{\otimes 8})$ is

$$\begin{aligned}\Theta_\tau^*(P) &= \sum_T c_T \Theta_\tau^*(P_T) = \sum_T c_T \left(\sum_\sigma a_{\sigma,T} \Theta[\sigma](\tau, 2z) \right) \\ &= \sum_\sigma \left(\sum_T a_{\sigma,T} c_T \right) \Theta[\sigma](\tau, 2z).\end{aligned}$$

As P is an equation for the Kummer variety if and only if $\Theta_\tau^*(P)=0$ on $\mathbf{C}^g$, we find, using the determination of the $a_{\sigma,T}$ above, that this is the case exactly when

$$0=\sum_T a_{\sigma,T}c_T=\sum_T c_T\frac{\partial P_T}{\partial x_\sigma}(\Theta_\tau(0))=\frac{\partial P}{\partial x_\sigma}(\Theta_\tau(0))\qquad\text{for all }\sigma\in(\mathbf{Z}/2\mathbf{Z})^g.$$

This again is equivalent to $\Theta_\tau(0)$ being a singular point on the variety (better: subscheme) defined by $P=0$. □

2.7. **The case** $g=2$. In this case the multiplication map Θ_τ^* is given by the 4×5 matrix of derivatives of the polynomials $P_0=x_{00}^4+\ldots+x_{11}^4$,

$$P_1=2(x_{00}^2x_{01}^2+x_{10}^2x_{11}^2),\quad P_2=2(x_{00}^2x_{10}^2+x_{01}^2x_{11}^2),\quad P_3=2(x_{00}^2x_{11}^2+x_{01}^2x_{10}^2)\ ,$$

and $P_{12}=4x_{00}x_{01}x_{10}x_{11}$, evaluated in $x_\sigma=u_\sigma$ with $u_\sigma:=\Theta[\sigma](\tau,0)$. Using Cramer's rule, the equation of the Kummer surface $a_0P_0+\ldots+a_{12}P_{12}=0$ is then a determinant (cf. [RSSS, (1.1)]):

$$F(\mathbf{u},\mathbf{x})\ :=\ \det\begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_{12}\\ u_{00}^3 & u_{00}u_{01}^2 & u_{00}u_{10}^2 & u_{00}u_{11}^2 & u_{01}u_{10}u_{11}\\ u_{01}^3 & u_{00}^2u_{01} & u_{01}u_{11}^2 & u_{01}u_{10}^2 & u_{00}u_{10}u_{11}\\ u_{10}^3 & u_{10}u_{11}^2 & u_{00}^2u_{10} & u_{01}^2u_{10} & u_{00}u_{01}u_{11}\\ u_{11}^3 & u_{10}^2u_{11} & u_{01}^2u_{11} & u_{00}^2u_{11} & u_{00}u_{01}u_{10}\end{pmatrix}\ =\ 0\ ,$$

where we took out a factor 4 from each of the last four rows. The polynomial $F\in\mathbf{C}[\mathbf{u},\mathbf{x}]$ is of bidegree $(12,4)$, and it is the equation defining the universal Kummer variety $\mathcal{K}_2(2,4)$ in $\mathbf{P}^3\times\mathbf{P}^3$.

2.8. **The case** $g=3$. The basis of Heisenberg invariant quartics is given by the following $15=1+7+7$ polynomials: $P_0=\sum x_\sigma^4$, $P_i=2(x_0^2x_\sigma^2+\ldots)$ where i and σ correspond through $i=4\sigma_1+2\sigma_2+\sigma_3$, and finally $P_{ij}=4(x_0x_\alpha x_\beta x_{\alpha+\beta}+\ldots)$, where $P_{ij}=P_T$ with T the subgroup of order 4 of $(\mathbf{Z}/2\mathbf{Z})^3$ generated by α,β corresponding to $i,j\in\{1,\ldots,7\}$ (and we assume that $\alpha+\beta$ corresponds to k with $i<j<k$).

To be explicit, here are some of these P_T's (up to 2-power factors, they coincide with the polynomials in the x_σ in [RSSS, (2.11)]; the others are quite similar and some can be recovered from the matrix of partial derivatives below):

$$\begin{aligned} P_2 &= 2(x_{000}^2x_{010}^2+x_{001}^2x_{011}^2+x_{100}^2x_{110}^2+x_{101}^2x_{111}^2),\\ P_{24} &= 4(x_{000}x_{010}x_{100}x_{110}+x_{001}x_{011}x_{101}x_{111}),\\ P_{12} &= 4(x_{000}x_{001}x_{010}x_{011}+x_{100}x_{101}x_{110}x_{111}).\end{aligned}$$

In this case the multiplication matrix has size 8×15, but it has submatrices of rank 6. This was suggested by [RSSS, Lemma 8.2]. We checked that any 7×7 minor of the 8×7 submatrix formed by the Jacobian matrix $(\partial P_T/\partial x_\sigma)$ of the 7 quartics $P_2,P_3,P_{24},P_{34},P_{25},P_{35},P_{12}$ is identically zero. This leads to an equation for the universal Kummer threefold which is a linear combination of these seven P_T's.

Now we consider the following 7×7 matrix; the last six rows are the partial derivatives (divided by 4) of the polynomials in the first row, w.r.t. the six variables

$x_{001}, \dots, x_{110}$, and the derivatives are evaluated in the point with coordinates $x_\sigma = u_\sigma$:

$$\begin{pmatrix}
P_2 & P_3 & P_{24} & P_{34} & P_{25} & P_{35} & P_{12} \\
u_{001}u_{011}^2 & u_{001}u_{010}^2 & u_{011}u_{101}u_{111} & u_{010}u_{101}u_{110} & u_{011}u_{100}u_{110} & u_{010}u_{100}u_{111} & u_{000}u_{010}u_{011} \\
u_{000}^2u_{010} & u_{001}^2u_{010} & u_{000}u_{100}u_{110} & u_{001}u_{101}u_{110} & u_{000}u_{101}u_{111} & u_{001}u_{100}u_{111} & u_{000}u_{001}u_{011} \\
u_{001}^2u_{011} & u_{000}^2u_{011} & u_{001}u_{101}u_{111} & u_{000}u_{100}u_{111} & u_{001}u_{100}u_{110} & u_{000}u_{101}u_{110} & u_{000}u_{001}u_{010} \\
u_{100}u_{110}^2 & u_{100}u_{111}^2 & u_{000}u_{010}u_{110} & u_{000}u_{011}u_{111} & u_{001}u_{011}u_{110} & u_{001}u_{010}u_{111} & u_{101}u_{110}u_{111} \\
u_{101}u_{111}^2 & u_{101}u_{110}^2 & u_{001}u_{011}u_{111} & u_{001}u_{010}u_{110} & u_{000}u_{010}u_{111} & u_{000}u_{011}u_{110} & u_{100}u_{110}u_{111} \\
u_{100}^2u_{110} & u_{101}^2u_{110} & u_{000}u_{010}u_{100} & u_{001}u_{010}u_{101} & u_{001}u_{011}u_{100} & u_{000}u_{011}u_{101} & u_{100}u_{101}u_{111}
\end{pmatrix}.$$

The determinant of this matrix is a linear combination $\sum_T a_T P_T$ where each a_T is a degree $18 = 3 \cdot 6$ polynomial in the u_σ. Each coefficient a_T turns out to be a multiple of $u_{000}u_{111}$. After dividing the determinant by this monomial, we obtain a polynomial of bidegree $(16, 4)$ which is an equation for the universal Kummer threefold. Writing the polynomial as

$$s_{010}P_2 + s_{011}P_3 + t_{001}P_{24} + t_{011}P_{34} + t_{101}P_{25} + t_{111}P_{35} + t_{100}P_{12}$$

one obtains the coefficients $s_\sigma, t_\sigma \in \mathbf{C}[\mathbf{u}]$ of the polynomial in [RSSS, Lemma 8.2].

3. Generalized Igusa quartics

3.1. **Singular Heisenberg invariant quartics.** In view of Proposition 2.6, we want to find quartic Heisenberg invariant polynomials F such that all partial derivatives of F vanish in the point $\Theta_\tau(0)$ with $\tau \in \mathcal{H}_g$. We consider the following variation: find quartic Heisenberg invariant polynomials, whose coefficients are polynomials in the coordinates of a point $u = (\dots : u_\sigma : \dots) \in \mathbf{P}^{2^g-1}$, such that all their partial derivatives vanish in u. So we do not require that $u = \Theta_\tau(0)$ for some τ. Such a polynomial $F \in \mathbf{C}[\mathbf{u}, \mathbf{x}]$ will be an equation for the universal Kummer variety.

The following proposition provides a method for finding such polynomials. It uses polynomials in variables p_T,

$$\sum_\beta d_\beta p^\beta, \qquad p^\beta := \prod_T p_T^{\beta_T}$$

with $d_\beta \in \mathbf{C}$ and $\beta = (\dots, \beta_T, \dots)$, where T runs over the set of subgroups of $(\mathbf{Z}/2\mathbf{Z})^g$ of order at most four. The basis $P_T = P_T(\mathbf{x})$ of quartic Heisenberg invariants is indexed by the same set, and we have an evaluation map

$$x^* : \mathbf{C}[\mathbf{p}] := \mathbf{C}[\dots, p_T, \dots] \longrightarrow \mathbf{C}[\mathbf{x}], \qquad R \longmapsto x^*(R) := R(\dots, P_T(\mathbf{x}), \dots),$$

which substitutes the quartic polynomial P_T in the x_σ for the variable p_T. Similarly, we have an evaluation map

$$u^* : \mathbf{C}[\mathbf{p}] \longrightarrow \mathbf{C}[\mathbf{u}], \qquad R \longmapsto u^*(R) := R(\ldots, P_T(\mathbf{u}), \ldots) ,$$

so $u^*(p_T) = P_T(\mathbf{u})$.

3.2. **Definition.** A homogeneous polynomial $R \in \mathbf{C}[\mathbf{p}]$ of degree d such that $x^*(R) = 0$ is called a generalized Igusa equation of degree d (in dimension g). In case $d = 4$, such an R will be called a generalized Igusa quartic (cf. Section 3.6).

3.3. **Definition.** Let $R \in \mathbf{C}[\mathbf{p}]_d$, so R is a homogeneous polynomial of degree d in the variables p_T. Then we define

$$F_R := F_R(\mathbf{u}, \mathbf{x}) = \sum_T u^* \left(\frac{\partial R}{\partial p_T} \right) P_T(\mathbf{x}) \qquad (\in \mathbf{C}[\mathbf{u}, \mathbf{x}]_{4(d-1),4}).$$

For each $u \in \mathbf{C}^{2^g}$, the polynomial $F_R(u, \mathbf{x})$ is a linear combination of the $P_T(\mathbf{x})$, hence it is a Heisenberg invariant quartic in $\mathbf{C}[\mathbf{x}]$. The coefficient of P_T is the partial derivative of R w.r.t. p_T, in which one substitutes $p_S := P_S(\mathbf{u})$ for all indices S. As $\partial R / \partial p_T$ is homogeneous of degree $d-1$, this coefficient is a homogeneous polynomial in the u_σ of degree $4(d-1)$.

Upon substituting $\mathbf{x} := \mathbf{u}$ in $F_R(\mathbf{x}, \mathbf{u})$ one finds $R_F(\mathbf{u}, \mathbf{u}) = du^* R$; in fact:

$$F_R(\mathbf{u}, \mathbf{u}) = \sum_T u^* \left(\frac{\partial R}{\partial p_T} \right) P_T(\mathbf{u}) = u^* \left(\sum_T \frac{\partial R}{\partial p_T} p_T \right) = u^*(dR) ,$$

where we used Euler's relation.

3.4. **Proposition.** *Let $R \in \mathbf{C}[\mathbf{p}]_d$ be a generalized Igusa equation, so $x^* R = 0$. Then $F_R \in \mathbf{C}[\mathbf{u}, \mathbf{x}]$ is an equation for the universal Kummer variety which moreover satisfies $F_R(\mathbf{u}, \mathbf{u}) = 0$.*

Proof. By assumption $x^* R = 0$, hence also $u^* R = 0$. The partial derivatives $\partial F_R / \partial x_\sigma$ of F_R, evaluated in $\mathbf{x} = \mathbf{u}$, are then

$$\left(\frac{\partial F_R}{\partial x_\sigma} \right)_{|\mathbf{x}=\mathbf{u}} = \sum_T u^* \left(\frac{\partial R}{\partial p_T} \right) \left(\frac{\partial P_T}{\partial x_\sigma} \right)_{|\mathbf{x}=\mathbf{u}} = \frac{\partial (u^* R)}{\partial u_\sigma} = 0 ,$$

where we used the chain rule for differentiation. Thus, for any $u = (\ldots, u_\sigma, \ldots) \in \mathbf{C}^{2^g}$, the Heisenberg invariant quartic $F(u, \mathbf{x})$ defines a variety in $\mathbf{P}^{2^g - 1}$ which is singular in the point $x = (\ldots : u_\sigma : \ldots)$. In particular, considering the case $u = \Theta_\tau(0)$, we find that $F_R(u, \mathbf{x})$ is an equation for the Kummer variety of A_τ by Proposition 2.6. Thus F_R is an equation for the universal Kummer variety. Finally, $F_R(\mathbf{u}, \mathbf{u}) = du^* R = 0$. $\square$

3.5. *Remark.* Proposition 3.4 shows that the kernel of the ring homomorphism x^* provides elements of the ideal $\mathcal{I}_g$ of the universal Kummer variety:

$$\ker(x^* : \mathbf{C}[\mathbf{p}] \longrightarrow \mathbf{C}[\mathbf{x}]) \longrightarrow \mathcal{I}_g , \qquad R \longmapsto F_R.$$

The kernel of the map x^* has the following geometrical interpretation. Consider the map given by the Heisenberg invariant quartics:

$$\mathbf{P}^{2^g - 1} \longrightarrow \mathbf{P}^N, \qquad x = (\ldots : x_\sigma : \ldots) \longmapsto (\ldots : P_T(x) : \ldots) ,$$

where $N+1=(2^g+1)(2^{g-1}+1)/3$. For any g, this map factors over the Heisenberg quotient $\mathbf{P}^{2^g-1}/H_g$. In case $g=2$, one can show that it embeds $\mathbf{P}^3/H_2$ into $\mathbf{P}^4$ as the Igusa quartic (see also Section 3.6). (For $g\geq 3$ however, the Heisenberg invariant quartics do not generate the ring of all Heisenberg invariant polynomials in $\mathbf{C}[\mathbf{x}]$, since the dimension of the space of degree eight Heisenberg invariant polynomials, computed in [DG, section 1.8], is then larger than the dimension of the space of quadratic polynomials in the Heisenberg invariant quartics.) In any case, the homogeneous elements R in the kernel of x^*, that is, the generalized Igusa equations, are the equations for the image of this map. As $2^g-1<N$ for $g\geq 2$, the kernel is non-trivial.

A final observation is that for all g this map has no base points: in fact, the polynomials $Q[{\epsilon \atop \epsilon'}]$ from Section 2.2 are a basis of the quadratic polynomials, hence they have no base points, and the $Q[{\epsilon \atop \epsilon'}]^2$ are Heisenberg invariant quartics.

3.6. **The case $g=2$.** In this case there is a unique quartic R_2 in the p_T's which is identically zero as a polynomial in the x_σ, i.e. $x^*(R_2)=0$. It is the well-known Igusa quartic:

$$R_2 := p_{12}^4+(p_0^2-p_1^2-p_2^2-p_3^2)p_{12}^2+p_1^2p_2^2+p_1^2p_3^2+p_2^2p_3^2-2p_0p_1p_2p_3.$$

From R_2 one finds the equation of the universal Kummer surface:

$$F_{R_2}=u^*\Big(2p_0p_{12}^2-2p_1p_2p_3\Big)P_0\,+\,\ldots\,+\,u^*\Big(4p_{12}^3+2(p_0^2-p_1^2-p_2^2-p_3^2)p_{12}\Big)P_{12}.$$

In fact, $\partial R_2/\partial p_0=2(p_0p_{12}^2-p_1p_2p_3)$ etc. As $u^*(p_T)=P_T(\mathbf{u})$, the polynomial F_{R_2} has bidegree $(12,4)$ in $\mathbf{C}[\mathbf{u},\mathbf{x}]$. An explicit computation verifies that F_{R_2} coincides with the determinant from Section 2.7, up to a scalar multiple.

3.7. **The case $g=3$.** The restriction of x^* to polynomials of degree ≤ 3 is injective, but the kernel of $x^*:\mathbf{C}[\mathbf{p}]_4\to\mathbf{C}[\mathbf{x}]_{16}$ has dimension 27. An example of a quartic polynomial in the kernel is:

$$\begin{aligned}R_3 &:= (p_{14}p_{16}-p_1p_{12})(-p_{24}^2-p_{25}^2+p_{34}^2+p_{35}^2)+p_{14}p_{16}(p_2^2-p_3^2)\\ &+p_{34}p_{35}(p_0p_2+p_1p_3-p_4p_6-p_5p_7)-p_{24}p_{25}(p_0p_3+p_1p_2-p_4p_7-p_5p_6)\\ &-p_{12}(p_2p_4p_7+p_2p_5p_6-p_3p_4p_6-p_3p_5p_7).\end{aligned}$$

The corresponding basis of Heisenberg invariant quartics is given Section 2.8. The polynomial $F_{R_3}(\mathbf{u},\mathbf{x})$ has 728 terms.

Using the action of $Sp(6,\mathbf{Z})$ (or the normalizer of the Heisenberg group) one can find a basis of the 27-dimensional vector space. As a consequence, for each R in this 27-dimensional vector space we find an equation of bidegree $(12,4)$ of the Kummer threefold. Therefore [RSSS, Conjecture 8.6] cannot be correct as the proposed generators for the ideal of the universal Kummer threefold have bidegree (a,b) with $a>12$.

3.8. **The case $g=4$.** In this case we found that the kernel of x^* restricted to $\mathbf{C}[\mathbf{p}]_4$ has dimension 510. This implies by Proposition 3.4 that there are equations of degree $(12,4)$ for the Kummer fourfold in $\mathbf{P}^{15}$.

The space of Heisenberg invariant quartics in $2^4=16$ variables has dimension 51, and they define a map $\mathbf{P}^{15}\to\mathbf{P}^{50}$. The quartics in the kernel of x^* are equations for the image of this map. We verified that the variety Z (better: subscheme) in $\mathbf{P}^{50}$ defined by these equations has the image as an irreducible component. For this we only had to find a point $x \in \mathbf{P}^{15}$ such that tangent space T_yZ has dimension 15, where y is the image of x. We do not know whether Z is irreducible.

3.9. **The general case.** In Sections 3.7, 3.8 we showed that there exist generalized Igusa quartics for $g = 3, 4$. We were not able to show that such Igusa quartics exist for any $g > 2$. Using representation theory, one can make some guesses as to what to expect in general.

The action of $\Gamma_g := Sp(2g, \mathbf{Z})$ on the Siegel space $\mathcal{H}_g$ induces a projective representation of Γ_g on the 2^g-dimensional vector space of second order theta functions. Identifying this vector space with $\mathbf{C}[\mathbf{x}]_1$, the linear forms in the 2^g variables x_σ, we obtain a projective representation of Γ_g on $\mathbf{C}[\mathbf{x}]$. On the subspace of Heisenberg invariant polynomials, one obtains a linear representation of $Sp(2g, \mathbf{F}_2) = \Gamma_g/\Gamma_g(2)$, where $\Gamma_g(2)$ is the subgroup of matrices congruent to I mod 2.

Similarly, identifying $\mathbf{C}[\mathbf{p}]_1 = \mathbf{C}[\mathbf{x}]_{4,0}$ by $p_T \mapsto P_T$, we obtain a representation of $Sp(2g, \mathbf{F}_2)$ on $\mathbf{C}[\mathbf{p}]$. The representation on $\mathbf{C}[\mathbf{p}]_1$ is irreducible, and it is the representation denoted by $\overline{V}_g$ in [F]. The kernel of the map $x^* : \mathbf{C}[\mathbf{p}]_d \longrightarrow \mathbf{C}[\mathbf{x}]_{4d}$ is a subrepresentation. This should be helpful in understanding the kernel.

The results on the kernel of x^* found above fit into the following pattern. The group $Sp(2g, \mathbf{F}_2)$ has an irreducible representation of dimension $(2^g-1)(2^{g-1}-1)/3$ (on the Heisenberg invariants in $\bigwedge^4(\mathbf{C}^{2^g})$), denoted by $\overline{U}_g$ in [F], for $g \geq 2$. The representation $Sym^2(\overline{U}_g)$ is reducible for $g > 2$ (in case $g = 2$ it is the trivial representation). In case $g = 3$ there are two irreducible components; one is the trivial representation, the other has dimension 27 (and this is the representation on the 27-dimensional kernel of x^*). For $g > 3$, $Sym^2(\overline{U}_g)$ has a subrepresentation of dimension $2^{g-1}(2^g - 1)$ (this subrepresentation is the image of the map $Sym^2(\overline{U}_g) \to (\bigwedge^8 \mathbf{C}^{2^g}))$. A computation shows that the complementary representation in $Sym^2(\overline{U}_g)$ has dimension

$$\frac{2}{9}(2^g+1)(2^g-1)(2^{g-1}+1)(2^{g-3}-1)$$

for $g \geq 4$. The dimension is thus 510, 11594, 210210 for $g = 4, 5, 6$. We checked, using Magma, that this representation is irreducible for $g = 4, 5, 6$. We also found that this representation occurs, with multiplicity one, in the representation of $Sp(2g, \mathbf{F}_2)$ on the subspace $Sym^4(\mathbf{C}[\mathbf{x}]_{4,0}) = \mathbf{C}[\mathbf{p}]_4$ for $g = 3, \dots, 6$.

This suggests that $\ker(x^* : \mathbf{C}[\mathbf{p}]_4 \longrightarrow \mathbf{C}[\mathbf{x}]_{16})$ might have this dimension also for $g > 4$.

In any case, if this kernel is non-trivial, then we would have equations of bidegree $(12, 4)$ for the universal Kummer variety for all $g \geq 2$. One still has to check that not all coefficients, which are degree 12 polynomials in the u_σ, are zero on the image of $\mathcal{H}_g$ in $\mathbf{P}^{2^g-1}$.

4. Non-Heisenberg invariant quartic equations

4.1. **The map Θ_τ^* in degree four (bis).** In Section 2.4 we considered the map $\Theta_\tau^* : \mathbf{C}[\mathbf{x}]_{4,0} \to H^0(A_\tau, L_\tau^{\otimes 8})_0$, whose kernels are the quartic Heisenberg invariant equations for the Kummer variety of A_τ. We now consider the case of a non-trivial character χ of the Heisenberg group H_g. The action of $Sp(2g, \mathbf{Z})$ on the non-trivial characters of H_g is transitive. Thus we concentrate on a fixed character.

It will be convenient to consider $g+1$-dimensional abelian varieties and to introduce the polynomial rings $\mathbf{C}[\mathbf{y}]$ and $\mathbf{C}[\mathbf{v}]$, in 2^{g+1} variables y_σ and v_σ with

$\sigma \in (\mathbf{Z}/2\mathbf{Z})^{g+1}$ respectively, which are the analogues of the rings $\mathbf{C}[\mathbf{x}]$ and $\mathbf{C}[\mathbf{u}]$ in the 2^g variables x_σ and u_σ for $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$ respectively.

We consider the character $\chi : H_{g+1} \to \{\pm 1\}$ which is trivial on the sign changes and which is $(-1)^{\beta_{g+1}}$ on the translation $y_\sigma \mapsto y_{\sigma+\beta}$ with $\beta \in (\mathbf{Z}/2\mathbf{Z})^{g+1}$.

In Proposition 4.6 we will find quartic equations for the Kummer variety in the χ-eigenspace of $\mathbf{C}[\mathbf{v}]$, that is, elements of the kernel of

$$\Theta_\tau^* : \mathbf{C}[\mathbf{y}]_{4,\chi} \longrightarrow H^0(A_\tau, L_\tau^{\otimes 8})_\chi^+ ,$$

where the codomain is the subspace of even theta functions in $H^0(A_\tau, L_\tau^{\otimes 8})_\chi$.

4.2. **A matrix for Θ_τ^*.** Notice that $\dim \mathbf{C}[\mathbf{y}]_{4,\chi} = (2^g+1)(2^{g-1}+1)/3$ (cf. Section 2.3), which is also the dimension of $\mathbf{C}[\mathbf{x}]_{4,0}$. The subspace $\mathbf{C}[\mathbf{x}]_{4,0}$ has basis P_T, which leads us to define a ring homomorphism

$$\mathbf{C}[\mathbf{p}] \longrightarrow \mathbf{C}[\mathbf{y}], \qquad R \longmapsto \tilde{R}_\chi ,$$

which is defined by the assignment

$$p_T \longmapsto \tilde{p}_{T,\chi} := P_T(\ldots, y_{\sigma 0}, \ldots) - P_T(\ldots, y_{\sigma 1}, \ldots) ,$$

where the P_T are a basis of the Heisenberg invariant quartics (in the 2^g variables x_σ). These are thus evaluated first in $x_\sigma := y_{\sigma 0}$ and next in $x_\sigma := y_{\sigma 1}$.

As the P_T are H_g-invariants, it is easy to see that $\tilde{p}_{T,\chi} \in \mathbf{C}[\mathbf{y}]_{4,\chi}$. Moreover, this homomorphism induces an isomorphism $\mathbf{C}[\mathbf{p}]_1 \cong \mathbf{C}[\mathbf{y}]_{4,\chi}$. Thus the polynomials $\tilde{p}_{T,\chi}$, where T runs over the $(2^g+1)(2^{g-1}+1)/3$ subgroups of order at most four of $(\mathbf{Z}/2\mathbf{Z})^g$, form a basis of $\mathbf{C}[\mathbf{y}]_{4,\chi}$.

A basis of $H^0(A_\tau, L_\tau^{\otimes 8})_\chi^+$, the even theta functions in $H^0(A_\tau, L_\tau^{\otimes 8})_\chi$, is given by the $\theta[\begin{smallmatrix}\sigma 0\\ 0\,1\end{smallmatrix}](2\tau, 4z)$, where σ runs over $(\mathbf{Z}/2\mathbf{Z})^g$ (here $\tau \in \mathcal{H}_{g+1}$, $z \in \mathbf{C}^{g+1}$).

W.r.t. these bases, the map

$$\Theta_\tau^* : \mathbf{C}[\mathbf{y}]_{4,\chi} = \bigoplus_T \mathbf{C}\tilde{p}_{T,\chi} \longrightarrow H^0(A_\tau, L_\tau^{\otimes 8})_\chi^+ = \bigoplus_\sigma \mathbf{C}\theta[\begin{smallmatrix}\sigma 0\\ 0\,1\end{smallmatrix}](2\tau, 4z)$$

is determined by the complex numbers $b_{\sigma,\tau}$ such that

$$\Theta_\tau^*(\tilde{p}_{T,\chi}) = \sum_\sigma b_{\sigma,T}\theta[\begin{smallmatrix}\sigma 0\\ 0\,1\end{smallmatrix}](2\tau, 4z). \tag{4.1.a}$$

Using Riemann's theta formula one finds (see [vG, Proposition 4]):

$$b_{\sigma,T} = \theta[\begin{smallmatrix}\sigma+\alpha & 0\\ 0 & 1\end{smallmatrix}](2\tau, 0)\theta[\begin{smallmatrix}\sigma+\beta & 0\\ 0 & 1\end{smallmatrix}](2\tau, 0)\theta[\begin{smallmatrix}\sigma+\alpha+\beta & 0\\ 0 & 1\end{smallmatrix}](2\tau, 0). \tag{4.1.b}$$

4.3. **The matrix coefficients.** The coefficients $b_{\sigma,T}$ of the matrix for Θ_τ^* on the χ-eigenspaces are cubic monomials in the $\theta[\begin{smallmatrix}\sigma 0\\ 0\,1\end{smallmatrix}](2\tau, 0)$, quite similar to the coefficients of Θ_τ^* on the Heisenberg invariants. To understand these coefficients better, we observe that they are the coordinates of a point of order four on the $g+1$-dimensional abelian variety A_τ.

In fact, let $b := (1/4)e_{g+1} \in \mathbf{C}^{g+1}$, where the e_j are the standard basis of $\mathbf{C}^{g+1}$. One easily computes that $\Theta_\tau(b) = (\ldots : y_\sigma : \ldots)$ with

$$y_{\sigma_1\ldots\sigma_g 1} = \Theta[\sigma_1 \ldots \sigma_g 1](\tau, b) = \theta[\begin{smallmatrix}\sigma_1 \ldots \sigma_g & 1\\ 0 \ldots 0 & 0\end{smallmatrix}](2\tau, (1/2)e_{g+1}) = \theta[\begin{smallmatrix}\sigma_1 \ldots \sigma_g & 1\\ 0 \ldots 0 & 1\end{smallmatrix}](2\tau, 0) = 0 ,$$

since the last theta function is odd. The other coordinates of $\Theta_\tau(b)$ are indeed:

$$y_{\sigma_1\ldots\sigma_g 0} = \Theta[\sigma 0](\tau, b) = \theta[\begin{smallmatrix}\sigma 0\\ 0\,0\end{smallmatrix}](2\tau, (1/2)e_g) = \theta[\begin{smallmatrix}\sigma 0\\ 0\,1\end{smallmatrix}](2\tau, 0).$$

Since we are interested in universal equations for the Kummer varieties, we will need to know that certain polynomials in the 2^g theta constants $\theta[\begin{smallmatrix}\sigma 0\\ 0\,1\end{smallmatrix}](2\tau, 0)$ are also polynomials in the 2^{g+1} coordinates $\Theta[\sigma_1\dots\sigma_{g+1}](\tau,0)$ of $\Theta_\tau(0)$. This is achieved by the following lemma.

4.4. **Lemma.** *Let $T\subset(\mathbf{Z}/2\mathbf{Z})^g$ be a subgroup with at most four elements. Let $P_T\in\mathbf{C}[\mathbf{x}]$ be the corresponding Heisenberg invariant quartic in 2^g variables x_σ and let $\tilde p_{T,\chi}\in\mathbf{C}[\mathbf{y}]$ be the quartic polynomial in the 2^{g+1} variables y_σ as in Section 4.2.*

Then for any $\tau\in\mathbf{H}_{g+1}$ we have:

$$(*)\qquad P_T(\dots,\Theta[\sigma_1\dots\sigma_g0](\tau,b),\dots)=\tilde p_{T,\chi}(\Theta_\tau(0)).$$

Proof. We consider the image $\tilde T\subset(\mathbf{Z}/2\mathbf{Z})^{g+1}$ of T under the map $(\mathbf{Z}/2\mathbf{Z})^g\hookrightarrow(\mathbf{Z}/2\mathbf{Z})^{g+1}$, $(\sigma_1,\dots,\sigma_g)\mapsto(\sigma_1,\dots,\sigma_g,0)$. The corresponding Heisenberg invariant quartic polynomial $P_{\tilde T}\in\mathbf{C}[\mathbf{y}]_{4,0}$ can be written as

$$P_{\tilde T}=\sum_\sigma y_\sigma y_{\sigma+\alpha}y_{\sigma+\beta}y_{\sigma+\alpha+\beta}=P_T(\dots,y_{\sigma_1\dots\sigma_g0},\dots)\,+\,P_T(\dots,y_{\sigma_1\dots\sigma_g1},\dots),$$

where σ runs over $(\mathbf{Z}/2\mathbf{Z})^{g+1}$, $\tilde T=\{0,\alpha,\beta,\alpha+\beta\}$, and we split the summation over the σ with $\sigma_{g+1}=0$ and $\sigma_{g+1}=1$ respectively. Now we compute the value of $P_{\tilde T}(\Theta_\tau(b))$ in two different ways.

First of all, we substitute the coordinates of $\Theta_\tau(b)$ in $P_{\tilde T}$. As the coordinates with $\sigma_{g+1}=1$ are all zero, the result is simply

$$P_{\tilde T}(\Theta_\tau(b))=P_T(\dots,\Theta[\sigma_1,\dots,\sigma_g0](\tau,b),\dots).$$

Next we use the formulae (2.4a,b) for $\Theta_\tau^*(P_{\tilde T})$:

$$\begin{aligned}P_{\tilde T}(\Theta_\tau(b))&=\Theta_\tau^*(P_{\tilde T})_{|z=b}\\&=\sum_\sigma\Theta[\sigma+\alpha](\tau,0)\Theta[\sigma+\beta](\tau,0)\Theta[\sigma+\alpha+\beta](\tau,0)\Theta[\sigma](\tau,2b).\end{aligned}$$

The $\Theta[\sigma](\tau,2b)$ are the coordinates of the point $\Theta_\tau(2b)$. Now $2b$ is a point of order two in A_τ and the point $\Theta_\tau(2b)$ is obtained from the point $\Theta_\tau(0)$ by the action of the following sign change in H_{g+1}:

$$y_{\sigma_1\dots\sigma_{g+1}}\;\mapsto\;(-1)^{\sigma_{g+1}}y_{\sigma_1\dots\sigma_{g+1}}.$$

In fact, one computes that

$$\begin{aligned}\Theta[\sigma](\tau,2b)&=\theta[{\textstyle{\sigma\atop 0}}](2\tau,4b)=\theta[{\textstyle{\sigma\atop 0}}](2\tau,e_{g+1})\\&=(-1)^{\sigma_{g+1}}\theta[{\textstyle{\sigma\atop 0}}](2\tau,0)=(-1)^{\sigma_{g+1}}\Theta[\sigma](\tau,0).\end{aligned}$$

As $\alpha_{g+1}=\beta_{g+1}=0$ we find:

$$\begin{aligned}P_{\tilde T}(\Theta_\tau(b))=&\sum_{\sigma,\sigma_{g+1}=0}\Theta[\sigma+\alpha](\tau,0)\Theta[\sigma+\beta](\tau,0)\Theta[\sigma+\alpha+\beta](\tau,0)\Theta[\sigma](\tau,0)\\&-\sum_{\sigma,\sigma_{g+1}=1}\Theta[\sigma+\alpha](\tau,0)\Theta[\sigma+\beta](\tau,0)\Theta[\sigma+\alpha+\beta](\tau,0)\Theta[\sigma](\tau,0)\\&=P_T(\dots,\Theta[\sigma_1\dots\sigma_g0](2\tau,0),\dots)\,-\,P_T(\dots,\Theta[\sigma_1\dots\sigma_g1](2\tau,0),\dots)\\&=\tilde p_{T,\chi}(\Theta_\tau(0)).\end{aligned}$$

This concludes the proof of (*). □

4.5. The following proposition shows that generalized Igusa equations also provide non-Heisenberg invariant equations for the universal Kummer varieties. Note the similarity between this proposition and Proposition 3.4.

4.6. **Proposition.** *Let $R \in \mathbf{C}[\mathbf{p}]_d$ be a generalized Igusa equation of degree d in dimension g, so $R \in \ker(x^* : \mathbf{C}[\mathbf{p}]_d \to \mathbf{C}[\mathbf{x}]_{4d})$. Then the polynomial*

$$\tilde{F}_{R,\chi} := \sum_T v^* \widetilde{\left(\frac{\partial R}{\partial p_T}\right)}_\chi \tilde{p}_{T,\chi}(\mathbf{y}) \in \mathbf{C}[\mathbf{v}, \mathbf{y}]_{4(d-1),4}$$

is a quartic equation for the universal Kummer variety $\mathcal{K}_{g+1}$ which, as a polynomial in $\mathbf{y}$, lies in $(\mathbf{C}[\mathbf{v}])[\mathbf{y}]_{4,\chi}$, where $v^* : \mathbf{C}[\mathbf{y}] \to \mathbf{C}[\mathbf{v}]$ is the 'evaluation in $\mathbf{v}$' map $v^*F := F(\dots, v_\sigma, \dots)$.

Proof. Given $R \in \ker(x^* : \mathbf{C}[\mathbf{p}]_d \to \mathbf{C}[\mathbf{x}]_{4d})$, we write the polynomial F_R (cf. Definition 3.3) as

$$F_R = \sum_T c_T(\mathbf{u}) P_T(\mathbf{x}) \qquad \text{so} \quad c_T(\mathbf{u}) := u^* \left(\frac{\partial R}{\partial p_T}\right).$$

From the proof of Proposition 3.4 and Section 2.3 we have

$$0 = \left(\frac{\partial F_R}{\partial x_\sigma}\right)_{|\mathbf{x}=\mathbf{u}} = 4 \sum_T u_{\sigma+\alpha} u_{\sigma+\beta} u_{\sigma+\alpha+\beta} c_T(\mathbf{u}) \qquad (\in \mathbf{C}[\mathbf{u}]) ,$$

for all $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$. The monomial in front of $c_T(\mathbf{u})$ is similar to $b_{\sigma,T}$ and we exploit this fact.

Let $\tau \in \mathcal{H}_{g+1}$. Then we substitute

$$u_\sigma := y_{\sigma_1 \dots \sigma_g 0} = \Theta[\sigma 0](\tau, b) \quad \text{with} \quad \tau \in \mathcal{H}_{g+1}, \quad b = e_{g+1}/4.$$

Then $u_{\sigma+\alpha} u_{\sigma+\beta} u_{\sigma+\alpha+\beta}$ becomes $\theta[{\sigma+\alpha\, 0 \atop 0 \quad 1}](2\tau, 0)\theta[{\sigma+\beta\, 0 \atop 0 \quad 1}](2\tau, 0)\theta[{\sigma+\alpha+\beta\, 0 \atop 0 \qquad 1}](2\tau, 0) = b_{\sigma,T}$, and thus

$$0 = \sum_T b_{\sigma,T} c_T(\dots, \Theta[\sigma 0](\tau, b), \dots) = 0$$

for all $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$. This again implies that the polynomial

$$\sum_T c_T(\dots, \Theta[\sigma 0](\tau, b), \dots) \tilde{p}_{T,\chi}(\mathbf{y}) \qquad (\in \mathbf{C}[\mathbf{y}]_{4,\chi})$$

is an equation for the Kummer variety of the $g+1$-dimensional abelian variety A_τ.

To see that this is a universal Kummer equation, we need to show that the coefficients are actually polynomials in the $\Theta[\sigma](\tau, 0)$. Each partial derivative $\partial F/\partial p_T$ of F is a polynomial in the p_T, hence formula (*) from Lemma 4.4 implies that the coefficients are polynomials in the $\tilde{p}_{T,\chi}(\Theta_\tau(0))$. To be precise, if

$$\frac{\partial R}{\partial p_T} = \sum_\beta d_\beta \prod_S p_S^{\beta_S} \qquad (\in \mathbf{C}[\mathbf{p}]_{d-1}) ,$$

where $\beta = (\dots, \beta_S, \dots)$, and S runs over the subgroups of $(\mathbf{Z}/2\mathbf{Z})^g$ of order at most four, then

$$\begin{aligned} c_T(\dots, \Theta[\sigma 0](\tau, b), \dots) &= \sum_\beta d_\beta \prod P_S^{\beta_S}(\dots, \Theta[\sigma 0](\tau, b), \dots) \\ &= \sum_\beta d_\beta \prod \tilde{p}_{S,\chi}^{\beta_S}(\Theta_\tau(0)) \\ &= \widetilde{(\partial R/\partial p_T)}_\chi(\Theta_\tau(0)). \end{aligned}$$

Thus the coefficients of the equation of the Kummer variety are the polynomials $v^*(\widetilde{\partial R/\partial p_T})_\chi \in \mathbf{C}[\mathbf{v}]$, which are evaluated in $\mathbf{v} = \Theta_\tau(0)$. Hence we found a universal Kummer relation, which is exactly $\tilde{F}_{R,\chi}$. $\square$

4.7. **Example.** In case $g = 2$, the Igusa quartic R_2 generates the kernel of x^*. Using the formula for F_{R_2} from Section 3.6, one easily finds the corresponding universal Kummer equation for $g = 3$:

$$\begin{aligned}\tilde{F}_{R_2,\chi} &= v^*\Big(2\tilde{p}_{0,\chi}\tilde{p}_{12,\chi}^2 - 2\tilde{p}_{1,\chi}\tilde{p}_{2,\chi}\tilde{p}_{3,\chi}\Big)\tilde{p}_{0,\chi}(\mathbf{y}) \\ &\quad + \ldots + v^*\Big(4\tilde{p}_{12,\chi}^3 + 2(\tilde{p}_{0,\chi}^2 - \tilde{p}_{1,\chi}^2 - \tilde{p}_{2,\chi}^2 - \tilde{p}_{3,\chi}^2)\tilde{p}_{12,\chi}\Big)\tilde{p}_{12,\chi}(\mathbf{y}).\end{aligned}$$

Here the basis of $\mathbf{C}[\mathbf{v}]_{4,\chi}$ is (cf. Section 2.7):

$$\begin{array}{rcl} v^*(\tilde{p}_{0,\chi}) &=& v_{000}^4 + v_{010}^4 + v_{100}^4 + v_{110}^4 - v_{001}^4 - v_{011}^4 - v_{101}^4 - v_{111}^4, \\ v^*(\tilde{p}_{1,\chi}) &=& 2(v_{000}^2 v_{010}^2 + v_{100}^2 v_{110}^2 - v_{001}^2 v_{011}^2 - v_{101}^2 v_{111}^2), \\ v^*(\tilde{p}_{2,\chi}) &=& 2(v_{000}^2 v_{100}^2 + v_{010}^2 v_{110}^2 - v_{001}^2 v_{101}^2 - v_{011}^2 v_{111}^2), \\ v^*(\tilde{p}_{3,\chi}) &=& 2(v_{000}^2 v_{110}^2 + v_{010}^2 v_{100}^2 - v_{001}^2 v_{111}^2 - v_{011}^2 v_{101}^2), \\ v^*(\tilde{p}_{12,\chi}) &=& 4(v_{000}v_{010}v_{100}v_{110} - v_{001}v_{011}v_{101}v_{111}). \end{array}$$

As far as we know, this equation for the universal Kummer threefold was not known before. For each non-trivial character of H_3 one finds such a quartic relation in the corresponding eigenspace, thus we get 63 such equations.

Similarly, using the results from Sections 3.7, 3.8, one finds non-Heisenberg invariant quartics for $g = 4, 5$.

4.8. *Remark***.** The quartic polynomials $Q[{\epsilon \atop \epsilon'}]^2$ are Heisenberg invariant. It is not hard to find their image in $\mathbf{C}[\mathbf{v}]$:

$$v^*\left((\widetilde{Q[{\epsilon \atop \epsilon'}]^2})_\chi\right) = Q[{\epsilon\,0 \atop \epsilon'0}](\mathbf{v})Q[{\epsilon\,0 \atop \epsilon'1}](\mathbf{v}).$$

5. Equations for the moduli space

5.1. **Equations for the moduli space.** In this section we show that the generalized Igusa equations defined in Section 3.2 also produce equations for the moduli space $\Theta(\mathcal{H}_{g+1}) \subset \mathbf{P}^{2^{g+1}-1}$ (note that g increases by 1 in this process, as in the non-Heisenberg invariant equations for the universal Kummer variety in Section 4).

5.2. **Proposition.** *Let $R \in \mathbf{C}[\mathbf{p}]_d$ be a generalized Igusa equation of degree d, so $R \in \ker(x^* : \mathbf{C}[\mathbf{p}]_d \to \mathbf{C}[\mathbf{x}]_{4d})$. Then the polynomial $\widetilde{R}_\chi \in \mathbf{C}[\mathbf{v}]_d$, defined in Section 4.2, is an equation for (the closure of) $\Theta(\mathcal{H}_{g+1})$ in $\mathbf{P}^{2^{g+1}-1}$.*

Proof. Let $R \in \ker(x^* : \mathbf{C}[\mathbf{p}]_d \to \mathbf{C}[\mathbf{x}]_{4d})$. By Proposition 4.6 it defines the equation $\tilde{F}_{R,\chi} \in \mathbf{C}[\mathbf{v}, \mathbf{y}]$ for the universal Kummer variety. Thus $\tilde{F}_{R,\chi}(\Theta_\tau(0), \Theta_\tau(z)) = 0$ for all $\tau \in \mathcal{H}_g$ and all $z \in \mathbf{C}^{g+1}$. Putting $z = 0$, we see that $\tilde{F}_{R,\chi}(\mathbf{v}, \mathbf{v})$ is an equation for the moduli space. Euler's relation shows that

$$\tilde{F}_{R,\chi}(\mathbf{v},\mathbf{v}) = \sum_T \widetilde{\left(\frac{\partial R}{\partial p_T}\right)}_a \tilde{p}_{T,\chi}(\mathbf{v}) = \left(\sum_T \left(\frac{\partial R}{\partial p_T}\right) p_T\right)_{|p_T = \tilde{p}_{T,\chi}(\mathbf{v})} = d\widetilde{R}_\chi(\mathbf{v}) \ ,$$

hence also $\widetilde{R}_\chi(\mathbf{v})$ is an equation for the moduli space. $\square$

5.3. **Example** $g = 2$. In case $g = 2$, the kernel of x^* is generated by the Igusa quartic R_2; see Section 3.6. It is also well-known that the image of the map $\Theta : \mathcal{H}_3 \to \mathbf{P}^7$, known as the Satake hypersurface, is defined by an irreducible homogeneous polynomial f_{16} of degree 16 (cf. [RSSS, Proposition 3.1]).

The space $\mathbf{C}[\mathbf{x}]_{4,0}$ is five dimensional; its basis P_T was given explicitly in Section 3.6. The $\tilde{p}_{T,\chi}$ are given in Section 4.7. The polynomial $\tilde{R}_{2,\chi}$ in the eight variables v_σ can then be computed by substituting $p_T := \tilde{p}_{T,\chi}$ in the polynomial R_2 from Section 3.6. One finds that $\tilde{R}_{2,\chi}$ is, up to multiplication by a non-zero constant, the polynomial f_{16}. Thus Proposition 5.2 produces non-trivial equations.

5.4. **Example** $g = 3$. We checked that the polynomial $\tilde{R}_{3,\chi}$ in the 16 variables obtained from the generalized Igusa equation R_3 from Section 3.7 is not identically zero. It is thus an equation of degree 16 for the image of $\mathcal{H}_4$ in $\mathbf{P}^{15}$.

Classically, equations of degree 32 for the moduli space were known. These are obtained by 'rationalizing' relations between the theta constants $\theta[{}^{\epsilon}_{\epsilon'}](\tau, 0)$ in order to obtain relations in which only the squares $\theta[{}^{\epsilon}_{\epsilon'}](\tau, 0)^2$ appear, and then express these squares in terms of the $\Theta[\sigma](\tau, 0)$ as in Section 2.2 with $z = 0$.

In [FO], Freitag and Oura show that there exists an equation of degree 24 for $\Theta(\mathcal{H}_4)$ in $\mathbf{P}^{15}$ which is invariant under $Sp(8, \mathbf{Z})$.

6. Schottky-Jung relations

6.1. **The classical Schottky-Jung relations.** The Schottky-Jung relations relate the theta constants $\theta[{}^{\eta}_{\eta'}](\tau, 0)$, where $\tau \in \mathcal{H}_g$ is a period matrix of the Jacobian of a genus g curve C, to the theta constants $\theta[{}^{\epsilon}_{\epsilon'}](\pi, 0)$, where $\pi \in \mathcal{H}_{g-1}$ is a period matrix of the Prym variety of an unramified double cover $\tilde{C} \to C$. The classical form of these relations is as follows:

$$\text{(SJ)} \qquad \theta[{}^{\epsilon}_{\epsilon'}]^2(\pi, 0) = c\, \theta[{}^{\epsilon\, 0}_{\epsilon' 0}](\tau, 0)\theta[{}^{\epsilon\, 0}_{\epsilon' 1}](\tau, 0)\ ,$$

for all even characteristics $[{}^{\epsilon}_{\epsilon'}]$, where c is a non-zero complex number.

Given a homogeneous polynomial in the $\theta[{}^{\epsilon}_{\epsilon'}]^2(\pi, 0)$ which is identically zero as a function of $\pi \in \mathcal{H}_{g-1}$, one obtains from the Schottky-Jung relations a polynomial in certain $\theta[{}^{\eta}_{\eta'}](\tau, 0)$ and thus a holomorphic function (actually a modular form) on $\mathcal{H}_g$. This function is thus zero in $\tau \in \mathcal{H}_g$ if τ is the period matrix of a Riemann surface.

A nice overview of the approach to the Schottky problem which uses Schottky-Jung relations and modular forms is given in [G, Section 3]; a quick derivation of the Schottky-Jung relations can be found in [C, Section 6.6].

6.2. **Example** $g = 4$. Let $f_{16} \in \mathbf{C}[\mathbf{v}]$ be the degree 16 polynomial from Section 5.3 which defines the image of $\Theta(\mathcal{H}_3) \subset \mathbf{P}^7$. As the quadratic polynomials $Q[{}^{\epsilon}_{\epsilon'}] \in \mathbf{C}[\mathbf{v}]$ are a basis of $\mathbf{C}[\mathbf{v}]_2$, one can write f_{16} as a degree 8 polynomial $\bar{f}_8$ in the $Q[{}^{\epsilon}_{\epsilon'}]$:

$$f_{16}(v_{000}, \ldots, v_{111}) = \bar{f}_8(\ldots, Q[{}^{\epsilon}_{\epsilon'}](\mathbf{v}), \ldots).$$

For $\pi \in \mathcal{H}_3$ one has $Q[{}^{\epsilon}_{\epsilon'}](\Theta_\pi(0)) = \theta[{}^{\epsilon}_{\epsilon'}]^2(\pi, 0)$ (see Section 2.2). Thus if we define

$$F(\tau) := \bar{f}_8(\ldots, \theta[{}^{\epsilon\, 0}_{\epsilon' 0}](\tau, 0)\theta[{}^{\epsilon\, 0}_{\epsilon' 1}](\tau, 0), \ldots),$$

then for a period matrix of a Riemann surface $\tau \in \mathcal{H}_4$ we have

$$F(\tau) = c^{-8} f_{16}(\Theta_\pi(0)) = 0.$$

Schottky verified that F is not identically zero on $\mathcal{H}_4$ and Igusa [I] (and independently Freitag [Fr]) showed that $F(\tau) = 0$ implies that τ is in the Jacobi locus $\mathcal{J}_4 \subset \mathcal{H}_4$, which is the closure of the locus of period matrices of Riemann surfaces in $\mathcal{H}_4$.

6.3. **Example $g = 5$.** Proceeding as in Example 6.2, the degree 16 equations for $\Theta(\mathcal{H}_4) \subset \mathbf{P}^{15}$ which we found in Section 5.4 lead to polynomials of degree 16 in the $\theta[{}^{\eta}_{\eta'}](\tau, 0)$'s with $\tau \in \mathcal{H}_5$. These are thus modular forms of weight 8 on $\Gamma_5(4, 8)$ which are zero on the Jacobi locus $\mathcal{J}_5$.

It might be interesting to see if the results from [C-SB, Theorem 1.3], which generalize a result of Grushevsky and Salvati Manni, can be used to show that these modular forms are actually cusp forms. See also [FO] for cusp forms of low weight on $\mathcal{H}_5$.

For recent progress on the characterization of $\mathcal{J}_5$ in $\mathcal{H}_5$ using Schottky-Jung relations and geometrical methods, we refer to [S].

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