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AMENABILITY AND COVARIANT INJECTIVITY OF LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. As is well known, the equivalence between amenability of a locally compact group G and injectivity of its von Neumann algebra $\mathcal{L}(G)$ does not hold in general beyond inner amenable groups. In this paper, we show that the equivalence persists for all locally compact groups if $\mathcal{L}(G)$ is considered as a $\mathcal{T}(L_2(G))$-module with respect to a natural action. In fact, we prove an appropriate version of this result for every locally compact quantum group.

1. INTRODUCTION

The connection between amenability of a locally compact group G and injectivity of the von Neumann algebra $\mathcal{L}(G)$ associated with the left regular representation has been a topic of interest in abstract harmonic analysis for decades. Amenability of G entails injectivity of $\mathcal{L}(G)$, however, the converse is not true, e.g., if $G = SL_n(\mathbb{R})$ for $n \geq 2$; indeed, a result of Connes' [5, Corollary 7], attributed to Dixmier, states that $\mathcal{L}(G)$ is injective for any separable connected locally compact group. In order to find a strengthening of injectivity which would be equivalent to amenability, there have been two main approaches: in terms of additional properties of the underlying group, or of the associated conditional expectations. In the spirit of the first approach, Lau and Paterson showed that G is amenable if and only if $\mathcal{L}(G)$ is injective and G is inner amenable [20, Corollary 3.2]. Following the second approach, Sołtan and Viselter recently proved, in the more general setting of locally compact quantum groups $\mathbb{G}$, that amenability is equivalent to the existence of a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ which maps $L_\infty(\mathbb{G})$ into the center of $L_\infty(\hat{\mathbb{G}})$ [30, Theorem 3]. In the present paper, we provide a new perspective on this connection, even at the level of locally compact quantum groups, by presenting new characterizations of amenability using the $\mathcal{T}(L_2(\mathbb{G}))$-module structure of $\mathcal{B}(L_2(\mathbb{G}))$. We note that the use of an action by $\mathcal{T}(L_2(\mathbb{G}))$, rather than one by $L_1(\mathbb{G})$, is crucial.

We begin in section 2 by recalling the relevant definitions and results from the theory of locally compact quantum groups, as introduced by Kustermans and Vaes [18, 19, 35].

Section 3 is devoted to an overview of the $\mathcal{T}(L_2(\mathbb{G}))$-bimodule structures on $\mathcal{B}(L_2(\mathbb{G}))$ and its relation to the spaces LUC($\mathbb{G}$) and RUC($\mathbb{G}$) of left and right uniformly continuous functionals on a locally compact quantum group $\mathbb{G}$, as introduced in [13, 29]. For any locally compact quantum group $\mathbb{G}$, there are two canonical completely contractive Banach algebra structures on $\mathcal{T}(L_2(\mathbb{G}))$, denoted

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by $(\mathcal{T}(L_2(\mathbb{G})), \lhd)$ and $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$, induced by the left and right fundamental unitaries of $\mathbb{G}$, respectively. This in turn yields two interesting bimodule structures on $\mathcal{B}(L_2(\mathbb{G}))$, which have been a recent topic of interest in the development of harmonic analysis on locally compact quantum groups [14, 15], and are closely related to $\mathrm{LUC}(\mathbb{G})$ and $\mathrm{RUC}(\mathbb{G})$.

The dual space of $\mathrm{LUC}(\mathbb{G})$ carries a natural Banach algebra structure. In [14], Hu, Neufang and Ruan studied various properties of this algebra, in particular through a weak*-weak* continuous, injective, completely contractive representation

$$\Theta^r : \mathrm{LUC}(\mathbb{G})^* \to \mathcal{CB}_{\mathcal{T}_\rhd}(\mathcal{B}(L_2(\mathbb{G})))$$

in the algebra of completely bounded right $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$-module maps on $\mathcal{B}(L_2(\mathbb{G}))$. This representation is the fundamental tool in our work, and is used in section 4 to show that a locally compact quantum group $\mathbb{G}$ is amenable if and only if the dual quantum group $\hat{\mathbb{G}}$ is what we shall call *covariantly injective*, meaning the corresponding projection of norm one commutes with the module action of $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$ on $\mathcal{B}(L_2(\mathbb{G}))$. As an application, we obtain a new proof of the recently answered question of Bédos and Tuset concerning the topological amenability of $\mathbb{G}$ [38]. By examining the remaining three $\mathcal{T}(L_2(\mathbb{G}))$-module structures on $\mathcal{B}(L_2(\mathbb{G}))$, we obtain new characterizations of amenability, co-commutativity, as well as injectivity of $\hat{\mathbb{G}}$. Moreover, compactness of $\mathbb{G}$ can be characterized in terms of normal conditional expectations respecting the $\mathcal{T}(L_2(\mathbb{G}))$-module structure.

We finish in section 5 by showing that a locally compact quantum group $\mathbb{G}$ is amenable if and only if $L_\infty(\hat{\mathbb{G}})$ is an injective operator $\mathcal{T}(L_2(\mathbb{G}))$-module. Even in the commutative case, this provides a new identification of classical amenability of a locally compact group G in terms of the injectivity of $\mathcal{L}(G)$ as a $\mathcal{T}(L_2(G))$-module. We also show that both amenability of $\mathbb{G}$ *and* of $\hat{\mathbb{G}}$ may be characterized through the injectivity of $\mathcal{B}(L_2(\mathbb{G}))$ as a left, respectively, right $\mathcal{T}(L_2(\mathbb{G}))$-module. This, along with other results in the paper suggests that these homological methods may provide a new approach to the duality problem of amenability and co-amenability for arbitrary locally compact quantum groups.

2. Preliminaries

A *locally compact quantum group* is a quadruple $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \varphi, \psi)$, where $L_\infty(\mathbb{G})$ is a Hopf-von Neumann algebra with a co-associative co-multiplication $\Gamma : L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G})\bar{\otimes}L_\infty(\mathbb{G})$, and φ and ψ are fixed (normal faithful semifinite) left and right Haar weights on $L_\infty(\mathbb{G})$, respectively [19, 35]. For every locally compact quantum group $\mathbb{G}$, there exists a *left fundamental unitary operator* W on $L_2(\mathbb{G}, \varphi) \otimes L_2(\mathbb{G}, \varphi)$ and a *right fundamental unitary operator* V on $L_2(\mathbb{G}, \psi) \otimes L_2(\mathbb{G}, \psi)$ implementing the co-multiplication Γ via

$$\Gamma(x) = W^*(1 \otimes x)W = V(x \otimes 1)V^* \quad (x \in L_\infty(\mathbb{G})).$$

Both unitaries satisfy the *pentagonal relation*; that is,

$$W_{12}W_{13}W_{23} = W_{23}W_{12} \quad \text{and} \quad V_{12}V_{13}V_{23} = V_{23}V_{12}. \tag{1}$$

By [19, Proposition 2.11], we may identify $L_2(\mathbb{G}, \varphi)$ and $L_2(\mathbb{G}, \psi)$, so we will simply use $L_2(\mathbb{G})$ for this Hilbert space throughout the paper.

Let $L_1(\mathbb{G})$ denote the predual of $L_\infty(\mathbb{G})$. Then the pre-adjoint of Γ induces an associative completely contractive multiplication on $L_1(\mathbb{G})$, defined by

$$\star : L_1(\mathbb{G})\widehat{\otimes}L_1(\mathbb{G}) \ni f \otimes g \mapsto f \star g = \Gamma_*(f \otimes g) \in L_1(\mathbb{G}).$$

The multiplication $\star$ is a complete quotient map from $L_1(\mathbb{G})\widehat{\otimes}L_1(\mathbb{G})$ onto $L_1(\mathbb{G})$, implying

$$\langle L_1(\mathbb{G}) \star L_1(\mathbb{G})\rangle = L_1(\mathbb{G}),$$

where $\langle L_1(\mathbb{G}) \star L_1(\mathbb{G})\rangle$ denotes the closed linear span of $f \star g$, with $f, g \in L_1(\mathbb{G})$. There is a canonical $L_1(\mathbb{G})$-bimodule structure on $L_\infty(\mathbb{G})$, defined by

$$\langle f \star x, g\rangle = \langle x, g \star f\rangle \quad \text{and} \quad \langle x \star f, g\rangle = \langle x, f \star g\rangle,$$

for $x \in L_\infty(\mathbb{G})$, and $f, g \in L_1(\mathbb{G})$. Using the co-multiplication Γ we may write

$$f \star x = (\iota \otimes f)\Gamma(x) \quad \text{and} \quad x \star f = (f \otimes \iota)\Gamma(x) \quad (x \in L_\infty(\mathbb{G}), f \in L_1(\mathbb{G})).$$

If X is an operator system in $L_\infty(\mathbb{G})$ that is also a left $L_1(\mathbb{G})$-submodule, then a *left invariant mean on* X, is a state $m \in X^*$ satisfying

$$\langle m, f \star x\rangle = \langle f, 1\rangle\langle m, x\rangle \quad (x \in X, f \in L_1(\mathbb{G})). \tag{2}$$

Right and two-sided invariant means are defined similarly. A locally compact quantum group $\mathbb{G}$ is said to be *amenable* if there exists a left invariant mean on $L_\infty(\mathbb{G})$. It is known that $\mathbb{G}$ is amenable if and only if there exists a right (equivalently, two-sided) invariant mean (cf. [8, Proposition 3]). $\mathbb{G}$ is said to be *co-amenable* if $L_1(\mathbb{G})$ has a bounded left (equivalently, right or two-sided) approximate identity (cf. [2, Theorem 3.1]).

Given a locally compact quantum group $\mathbb{G}$, the *left regular representation* $\lambda : L_1(\mathbb{G}) \to \mathcal{B}(L_2(\mathbb{G}))$ is defined by

$$\lambda(f) = (f \otimes \iota)(W) \quad (f \in L_1(\mathbb{G})),$$

and is an injective, completely contractive homomorphism from $L_1(\mathbb{G})$ into $\mathcal{B}(L_2(\mathbb{G}))$. Then $L_\infty(\hat{\mathbb{G}}) := \{\lambda(f) : f \in L_1(\mathbb{G})\}''$ is the von Neumann algebra associated with the dual quantum group $\hat{\mathbb{G}}$ of $\mathbb{G}$. Analogously, we have the *right regular representation* $\rho : L_1(\mathbb{G}) \to \mathcal{B}(L_2(\mathbb{G}))$ defined by

$$\rho(f) = (\iota \otimes f)(V) \quad (f \in L_1(\mathbb{G})),$$

which is also an injective, completely contractive homomorphism from $L_1(\mathbb{G})$ into $\mathcal{B}(L_2(\mathbb{G}))$. Then $L_\infty(\hat{\mathbb{G}}') := \{\rho(f) : f \in L_1(\mathbb{G})\}''$ is the von Neumann algebra associated to the quantum group $\hat{\mathbb{G}}'$. It follows that $L_\infty(\hat{\mathbb{G}}') = L_\infty(\hat{\mathbb{G}})'$, and the fundamental unitaries satisfy $W \in L_\infty(\mathbb{G})\bar{\otimes}L_\infty(\hat{\mathbb{G}})$ and $V \in L_\infty(\hat{\mathbb{G}}')\bar{\otimes}L_\infty(\mathbb{G})$ [19, Proposition 2.15]. Moreover, dual quantum groups always satisfy $L_\infty(\mathbb{G})\cap L_\infty(\hat{\mathbb{G}}) = L_\infty(\mathbb{G}) \cap L_\infty(\hat{\mathbb{G}}') = \mathbb{C}1$ [37].

If G is a locally compact group, we let $\mathbb{G}_a = (L_\infty(G), \Gamma_a, \varphi_a, \psi_a)$ denote the *commutative* quantum group associated with the commutative von Neumann algebra $L_\infty(G)$, where the co-multiplication is given by $\Gamma_a(f)(s,t) = f(st)$, and φ_a and ψ_a are integration with respect to a left and right Haar measure, respectively. The dual quantum group $\hat{\mathbb{G}}_a$ of $\mathbb{G}_a$ is the *co-commutative* quantum group $\mathbb{G}_s = (\mathcal{L}(G), \Gamma_s, \varphi_s, \psi_s)$, where $\mathcal{L}(G)$ is the left group von Neumann algebra with co-multiplication $\Gamma_s(\lambda(t)) = \lambda(t) \otimes \lambda(t)$, and $\varphi_s = \psi_s$ is Haagerup's Plancherel weight (cf. [32, §VII.3]). Here, $\mathbb{G}_s$ is called co-commutative since its co-multiplication is symmetric. We also consider the quantum group $\hat{\mathbb{G}}'_a = \mathbb{G}'_s$ associated to the right group von Neumann algebra $\mathcal{R}(G)$ with the co-multiplication $\Gamma'_s(\rho(t)) = \rho(t) \otimes \rho(t)$. Then $L_1(\mathbb{G}_a)$ is the usual group convolution algebra $L_1(G)$, and $L_1(\mathbb{G}_s) = L_1(\mathbb{G}'_s)$ is the Fourier algebra $A(G)$. It is known that every commutative locally compact quantum group is of the form $\mathbb{G}_a$ [31, Theorem 2; §2; 36]. Therefore, every commutative locally compact quantum group is co-amenable, and is amenable if and only if the underlying locally compact group is amenable. By duality, every co-commutative locally compact quantum group is of the form $\mathbb{G}_s$, which is always amenable [27, Theorem 4], and is co-amenable if and only if the underlying locally compact group is amenable, by Leptin's classical theorem.

By using the regular representations of the quantum groups $\hat{\mathbb{G}}$ and $\hat{\mathbb{G}}'$, we arrive at the *reduced quantum group C^*-algebra* of $L_\infty(\mathbb{G})$, defined as

$$C_0(\mathbb{G}) = \overline{\hat{\lambda}(L_1(\hat{\mathbb{G}}))}^{\|\cdot\|} = \overline{\hat{\rho}(L_1(\hat{\mathbb{G}}'))}^{\|\cdot\|}.$$

$\mathbb{G}$ is said to be *compact* if $C_0(\mathbb{G})$ is a unital C^*-algebra. For quantum groups arising from locally compact groups G, it follows that $C_0(\mathbb{G}_a)$ is $C_0(G)$, the algebra of continuous functions vanishing at infinity, and $C_0(\mathbb{G}_s)$ is the left group C^*-algebra $C^*_\lambda(G)$. The multiplier algebra of $C_0(\mathbb{G})$ will be denoted $M(C_0(\mathbb{G}))$.

3. LUC($\mathbb{G}$) and LUC($\mathbb{G}$)*

Let $\mathbb{G}$ be a locally compact quantum group. The right fundamental unitary V of $\mathbb{G}$ induces a co-associative co-multiplication

$$\Gamma^r : \mathcal{B}(L_2(\mathbb{G})) \ni x \mapsto V(x \otimes 1)V^* \in \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G})),$$

and the restriction of Γ^r to $L_\infty(\mathbb{G})$ yields the original co-multiplication Γ on $L_\infty(\mathbb{G})$. The pre-adjoint of Γ^r induces an associative completely contractive multiplication on $\mathcal{T}(L_2(\mathbb{G}))$, defined by

$$\rhd : \mathcal{T}(L_2(\mathbb{G}))\widehat{\otimes}\mathcal{T}(L_2(\mathbb{G})) \ni \omega \otimes \tau \mapsto \omega \rhd \tau = \Gamma^r_*(\omega \otimes \tau) \in \mathcal{T}(L_2(\mathbb{G})),$$

where $\widehat{\otimes}$ denotes the operator space projective tensor product. Analogously, the left fundamental unitary W of $\mathbb{G}$ induces a co-associative co-multiplication

$$\Gamma^l : \mathcal{B}(L_2(\mathbb{G})) \ni x \mapsto W^*(1 \otimes x)W \in \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G})),$$

and the restriction of Γ^l to $L_\infty(\mathbb{G})$ is also equal to Γ. The pre-adjoint of Γ^l induces another associative completely contractive multiplication

$$\lhd : \mathcal{T}(L_2(\mathbb{G}))\widehat{\otimes}\mathcal{T}(L_2(\mathbb{G})) \ni \omega \otimes \tau \mapsto \omega \lhd \tau = \Gamma^l_*(\omega \otimes \tau) \in \mathcal{T}(L_2(\mathbb{G})).$$

These two products on $\mathcal{T}(L_2(\mathbb{G}))$ are quite different in general. It is known that $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$ is always left faithful, and right faithful if and only if $\mathbb{G}$ is trivial. Similarly, $(\mathcal{T}(L_2(\mathbb{G})), \lhd)$ is always right faithful, and is left faithful if and only if $\mathbb{G}$ is trivial (cf. [14]).

For commutative and co-commutative quantum groups, this type of multiplicative structure on $\mathcal{T}(L_2(\mathbb{G}))$ has been studied in [1, 22–24, 26], and the general case has been investigated in [14, 15, 17]. In particular, it was shown in [14, Lemma 5.2] that the pre-annihilator $L_\infty(\mathbb{G})_\perp$ of $L_\infty(\mathbb{G})$ in $\mathcal{T}(L_2(\mathbb{G}))$ is a norm closed two-sided ideal in $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$ and $(\mathcal{T}(L_2(\mathbb{G})), \lhd)$, respectively, and the complete quotient map

$$\pi : \mathcal{T}(L_2(\mathbb{G})) \ni \omega \mapsto f = \omega|_{L_\infty(\mathbb{G})} \in L_1(\mathbb{G}) \tag{3}$$

is a completely contractive algebra homomorphism from $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$ and $(\mathcal{T}(L_2(\mathbb{G})), \lhd)$, respectively, onto $L_1(\mathbb{G})$. Therefore, we have the completely isometric Banach algebra identifications

$$(L_1(\mathbb{G}), \star) \cong (\mathcal{T}(L_2(\mathbb{G})), \rhd)/L_\infty(\mathbb{G})_\perp \quad \text{and} \quad (L_1(\mathbb{G}), \star) \cong (\mathcal{T}(L_2(\mathbb{G})), \lhd)/L_\infty(\mathbb{G})_\perp.$$

This allows us to view each of $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$ and $(\mathcal{T}(L_2(\mathbb{G})), \lhd)$ as a lifting of $(L_1(\mathbb{G}), \star)$.

The multiplication $\rhd$ defines a completely contractive $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$-bimodule structure on $\mathcal{B}(L_2(\mathbb{G}))$ via

$$\mathcal{B}(L_2(\mathbb{G}))\widehat{\otimes}\mathcal{T}(L_2(\mathbb{G})) \ni (x, \omega) \mapsto x \rhd \omega = (\omega \otimes \iota)V(x \otimes 1)V^* \in L_\infty(\mathbb{G}) \subseteq \mathcal{B}(L_2(\mathbb{G}));$$
$$\mathcal{T}(L_2(\mathbb{G}))\widehat{\otimes}\mathcal{B}(L_2(\mathbb{G})) \ni (\omega, x) \mapsto \omega \rhd x = (\iota \otimes \omega)V(x \otimes 1)V^* \in \mathcal{B}(L_2(\mathbb{G})).$$

Note that since $V \in L_\infty(\hat{\mathbb{G}}')\bar{\otimes}L_\infty(\mathbb{G})$, the bimodule action on $L_\infty(\hat{\mathbb{G}})$ becomes rather trivial. Indeed, for $\hat{x} \in L_\infty(\hat{\mathbb{G}})$ and $\omega \in \mathcal{T}(L_2(\mathbb{G}))$

(4) $\hat{x} \rhd \omega = (\omega \otimes \iota)V(\hat{x} \otimes 1)V^* = \langle \omega, \hat{x}\rangle 1$ and $\omega \rhd \hat{x} = (\iota \otimes \omega)V(\hat{x} \otimes 1)V^* = \langle \omega, 1\rangle \hat{x}$.

Remark 3.1. Observe that the left action of $\mathcal{T}(L_2(\mathbb{G}))$ on $L_\infty(\mathbb{G})$ satisfies $\omega \rhd x = \pi(\omega) \star x$, i.e., it is implemented by a left $L_1(\mathbb{G})$ action. However, the homological properties of the resulting right action on $L_1(\mathbb{G})$ are not equivalent to those corresponding to the canonical right action of $L_1(\mathbb{G})$ on itself. For instance, $L_1(G)$ is always right projective over itself for any locally compact group G, while it is projective as a right $\mathcal{T}(L_2(G))$-module if and only if G is discrete. See [6, Theorem 3.3.32] and [26, Theorem 3.4] for details.

The multiplication $\lhd$ defines analogously a completely contractive $(\mathcal{T}(L_2(\mathbb{G})), \lhd)$-bimodule structure on $\mathcal{B}(L_2(\mathbb{G}))$ via

$$\mathcal{T}(L_2(\mathbb{G}))\widehat{\otimes}\mathcal{B}(L_2(\mathbb{G})) \ni (\omega, x) \mapsto \omega \lhd x = (\iota \otimes \omega)W^*(1 \otimes x)W \in L_\infty(\mathbb{G}) \subseteq \mathcal{B}(L_2(\mathbb{G}));$$
$$\mathcal{B}(L_2(\mathbb{G}))\widehat{\otimes}\mathcal{T}(L_2(\mathbb{G})) \ni (x, \omega) \mapsto x \lhd \omega = (\omega \otimes \iota)W^*(1 \otimes x)W \in \mathcal{B}(L_2(\mathbb{G})).$$

In particular, for $x \in L_\infty(\mathbb{G})$ and $f = \omega|_{L_\infty(\mathbb{G})}$ with $\omega \in \mathcal{T}(L_2(\mathbb{G}))$, we have

(5) $x \rhd \omega = x \lhd \omega = (\omega \otimes \iota)\Gamma(x) = x \star f$ and $\omega \lhd x = \omega \rhd x = (\iota \otimes \omega)\Gamma(x) = f \star x$.

As above, we see that the bimodule actions of $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$ and $(\mathcal{T}(L_2(\mathbb{G})), \lhd)$ on $\mathcal{B}(L_2(\mathbb{G}))$ are liftings of the usual bimodule action of $L_1(\mathbb{G})$ on $L_\infty(\mathbb{G})$.

If $\mathbb{G}$ is a locally compact quantum group, the subspaces $\mathrm{LUC}(\mathbb{G})$ and $\mathrm{RUC}(\mathbb{G})$ of $L_\infty(\mathbb{G})$ are defined by [13, 29]

$$\mathrm{LUC}(\mathbb{G}) = \langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G})\rangle \quad \text{and} \quad \mathrm{RUC}(\mathbb{G}) = \langle L_1(\mathbb{G}) \star L_\infty(\mathbb{G})\rangle.$$

It was shown by Runde in [29, Theorem 2.4] that $\mathrm{LUC}(\mathbb{G})$ and $\mathrm{RUC}(\mathbb{G})$ are operator systems in $L_\infty(\mathbb{G})$ such that

$$\mathbb{C}_0(\mathbb{G}) \subseteq \mathrm{LUC}(\mathbb{G}), \mathrm{RUC}(\mathbb{G}) \subseteq M(C_0(\mathbb{G})). \tag{6}$$

In the classical setting of locally compact groups G, $\mathrm{LUC}(\mathbb{G}_a)$ (respectively, $\mathrm{RUC}(\mathbb{G}_a)$) is the usual space $\mathrm{LUC}(G)$ (respectively, $\mathrm{RUC}(G)$) of bounded left (respectively, right) uniformly continuous functions on G, and $\mathrm{LUC}(\mathbb{G}_s) = \mathrm{RUC}(\mathbb{G}_s)$ is the space $\mathrm{UCB}(\hat{G})$ of uniformly continuous linear functionals on $A(G)$ introduced by Granirer [10]. Using the extended module actions of $\mathcal{T}(L_2(\mathbb{G}))$ on $\mathcal{B}(L_2(\mathbb{G}))$, it was shown in [14, Proposition 5.3] that

$$\begin{aligned}\mathrm{LUC}(\mathbb{G}) &= \langle \mathrm{LUC}(\mathbb{G}) \star L_1(\mathbb{G})\rangle = \langle \mathcal{B}(L_2(\mathbb{G})) \rhd \mathcal{T}(L_2(\mathbb{G}))\rangle;\\ \mathrm{RUC}(\mathbb{G}) &= \langle L_1(\mathbb{G}) \star \mathrm{RUC}(\mathbb{G})\rangle = \langle \mathcal{T}(L_2(\mathbb{G})) \lhd \mathcal{B}(L_2(\mathbb{G}))\rangle.\end{aligned}$$

For every locally compact quantum group $\mathbb{G}$, we have the left and right Arens products $\Box$ and $\Diamond$ on $L_\infty(\mathbb{G})^* = L_1(\mathbb{G})^{**}$, which are defined by

$$\langle m\Box n, x\rangle = \langle m, n\Box x\rangle \quad \text{and} \quad \langle m\Diamond n, x\rangle = \langle n, x\Diamond m\rangle \quad (m, n \in L_\infty(\mathbb{G})^*, x \in L_\infty(\mathbb{G})),$$

where $n\Box x$ and $x\Diamond m$ are elements of $L_\infty(\mathbb{G})$ given by

$$\langle n\Box x, f\rangle = \langle n, x\star f\rangle \quad \text{and} \quad \langle x\Diamond m, f\rangle = \langle m, f\star x\rangle \quad (f \in L_1(\mathbb{G})).$$

Then $(L_\infty(\mathbb{G})^*, \Box)$ and $(L_\infty(\mathbb{G})^*, \Diamond)$ are completely contractive Banach algebras.

Given $m \in \mathrm{LUC}(\mathbb{G})^*$, we define a bounded linear map m_L on $L_\infty(\mathbb{G})$ by

$$m_L : L_\infty(\mathbb{G}) \ni x \mapsto m\Box x \in L_\infty(\mathbb{G}),$$

where the product $m\Box x \in L_\infty(\mathbb{G})$ is given as above, noticing that $L_\infty(\mathbb{G})\star L_1(\mathbb{G}) \subseteq \mathrm{LUC}(\mathbb{G})$. This map is completely bounded, with $\|m_L\|_{cb} \le \|m\|$, and a right $L_1(\mathbb{G})$-module map, since

$$\begin{aligned}\langle m\Box(x\star f), g\rangle &= \langle m, x\star(f\star g)\rangle = \langle m\Box x, f\star g\rangle\\ &= \langle (m\Box x)\star f, g\rangle \quad (x \in L_\infty(\mathbb{G}), f, g \in L_1(\mathbb{G})).\end{aligned}$$

Therefore, m_L maps $\mathrm{LUC}(\mathbb{G})$ into $\mathrm{LUC}(\mathbb{G})$, and so the left Arens product $\Box$ on $L_\infty(\mathbb{G})^*$ induces a completely contractive multiplication on $\mathrm{LUC}(\mathbb{G})^*$, also denoted $\Box$, so that the restriction

$$L_\infty(\mathbb{G})^* \ni m \mapsto m|_{\mathrm{LUC}(\mathbb{G})} \in \mathrm{LUC}(\mathbb{G})^*$$

is a completely contractive, multiplicative quotient map from $(L_\infty(\mathbb{G})^*, \Box)$ onto $(\mathrm{LUC}(\mathbb{G})^*, \Box)$.

Let $m \in \mathrm{LUC}(\mathbb{G})^*$. Then, as $\mathrm{LUC}(\mathbb{G}) = \langle \mathcal{B}(L_2(\mathbb{G})) \rhd \mathcal{T}(L_2(\mathbb{G}))\rangle$, the module map m_L may be extended to a right $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$-module map on $\mathcal{B}(L_2(\mathbb{G}))$ via

$$\langle m_L(x), \omega\rangle = \langle m, x\rhd\omega\rangle = \langle m, (\omega\otimes\iota)V(x\otimes 1)V^*\rangle \quad (x \in \mathcal{B}(L_2(\mathbb{G})), \omega \in \mathcal{T}(L_2(\mathbb{G}))).$$

In this case, we also have $\|m_L\|_{cb} \le \|m\|$, and if we let $\mathcal{CB}_{\mathcal{T}_\rhd}(\mathcal{B}(L_2(\mathbb{G})))$ denote the algebra of completely bounded right $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$-module maps on $\mathcal{B}(L_2(\mathbb{G}))$, it follows that

$$\Theta^r : \mathrm{LUC}(\mathbb{G})^* \ni m \mapsto m_L \in \mathcal{CB}_{\mathcal{T}_\rhd}(\mathcal{B}(L_2(\mathbb{G}))) \tag{7}$$

is a weak*-weak* continuous, injective, completely contractive algebra homomorphism [14, Proposition 6.5]. Moreover, [14, Theorem 7.1] entails that

$$\Theta^r(\mathrm{LUC}(\mathbb{G})^*) \subseteq \mathcal{CB}_{\mathcal{T}_\rhd}(\mathcal{B}(L_2(\mathbb{G}))) \cap \mathcal{CB}^{L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(\mathcal{B}(L_2(\mathbb{G}))), \tag{8}$$

where $\mathcal{CB}^{L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(\mathcal{B}(L_2(\mathbb{G})))$ is the algebra of completely bounded $L_\infty(\hat{\mathbb{G}})$-bimodule maps on $\mathcal{B}(L_2(\mathbb{G}))$ that leave $L_\infty(\mathbb{G})$ invariant. Analogously, the right Arens product $\Diamond$ induces a completely contractive Banach algebra structure on $\mathrm{RUC}(\mathbb{G})^*$, and there exists a weak*-weak* continuous, injective, completely contractive anti-homomorphism

$$\Theta^l : \mathrm{RUC}(\mathbb{G})^* \to {}_{\mathcal{T}_\lhd}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G}))), \tag{9}$$

where ${}_{\mathcal{T}_{\lhd}}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ is the algebra of completely bounded left $(\mathcal{T}(L_2(\mathbb{G})),\lhd)$-module maps on $\mathcal{B}(L_2(\mathbb{G}))$.

4. Covariant injectivity

In this section we introduce and study versions of injectivity of $L_\infty(\hat{\mathbb{G}})$ that capture fundamental properties of $\mathbb{G}$, such as amenability, compactness, and co-commutativity. The underlying idea is to refine injectivity through a covariance condition, by which we mean the existence of a conditional expectation respecting the natural $\mathcal{T}(L_2(\mathbb{G}))$-module structure of $\mathcal{B}(L_2(\mathbb{G}))$ and $L_\infty(\hat{\mathbb{G}})$.

Definition 4.1. For a locally compact quantum group $\mathbb{G}$, we say that a mapping $\Phi\in\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ is *covariant* if $\Phi(x\rhd\rho)=\Phi(x)\rhd\rho$ for all $x\in\mathcal{B}(L_2(\mathbb{G}))$ and $\rho\in\mathcal{T}(L_2(\mathbb{G}))$.

We begin by using the representation (7) of $\mathrm{LUC}(\mathbb{G})^*$ to establish a one-to-one correspondence between right invariant means on $\mathrm{LUC}(\mathbb{G})$ and covariant conditional expectations onto $L_\infty(\hat{\mathbb{G}})$.

Theorem 4.2. *Let $\mathbb{G}$ be a locally compact quantum group. The following statements are equivalent:*

(1) *$\mathbb{G}$ is amenable;*
(2) *there is a right invariant mean on* $\mathrm{LUC}(\mathbb{G})$*;*
(3) *there is a covariant conditional expectation $E:\mathcal{B}(L_2(\mathbb{G}))\to L_\infty(\hat{\mathbb{G}})$.*

Proof. (1) $\Rightarrow$ (2) Restriction of a right invariant mean on $L_\infty(\mathbb{G})$ yields (2).

(2) $\Rightarrow$ (3) Let $m\in\mathrm{LUC}(\mathbb{G})^*$ be a right invariant mean. Then $m\Box y=\langle m,y\rangle 1$ for all $y\in\mathrm{LUC}(\mathbb{G})$ by right invariance, which gives

$$\langle m\Box m,y\rangle=\langle m,m\Box y\rangle=\langle m,y\rangle\langle m,1\rangle=\langle m,y\rangle.$$

Hence, m is a norm one idempotent in $\mathrm{LUC}(\mathbb{G})^*$, making $\Theta^r(m)$ a projection of norm one in $\mathcal{CB}_{\mathcal{T}_{\rhd}}(\mathcal{B}(L_2(\mathbb{G})))$. As such, its image is equal to its fixed points, denoted $\mathcal{H}_{\Theta(m)}$. First observe that $L_\infty(\hat{\mathbb{G}})\subseteq\mathcal{H}_{\Theta(m)}$ as $\Theta^r(m)(\hat{x})=(\iota\otimes m)V(\hat{x}\otimes 1)V^*=\hat{x}$. On the other hand, as $\Theta^r(m)$ is a $\mathcal{T}(L_2(\mathbb{G}))$-module map, its fixed points form a $\mathcal{T}(L_2(\mathbb{G}))$-submodule of $\mathcal{B}(L_2(\mathbb{G}))$. Thus, $x\rhd\omega\in\mathcal{H}_{\Theta(m)}$ for every $x\in\mathcal{H}_{\Theta(m)}$ and $\omega\in\mathcal{T}(L_2(\mathbb{G}))$. But if $y\in\mathcal{H}_{\Theta(m)}\cap\mathrm{LUC}(\mathbb{G})$, then $y=\Theta^r(m)(y)=m\Box y=\langle m,y\rangle 1$. Hence, if $x\in\mathcal{H}_{\Theta(m)}$ and $\omega\in\mathcal{T}(L_2(\mathbb{G}))$, then $x\rhd\omega=\langle m,x\rhd\omega\rangle 1$, so that for any $\tau\in\mathcal{T}(L_2(\mathbb{G}))$

$$\begin{aligned}\langle\Gamma(x),\omega\otimes\tau\rangle&=\langle x,\omega\rhd\tau\rangle=\langle x\rhd\omega,\tau\rangle=\langle m,x\rhd\omega\rangle\langle 1,\tau\rangle\\&=\langle\Theta^r(m)(x),\omega\rangle\langle 1,\tau\rangle=\langle x\otimes 1,\omega\otimes\tau\rangle.\end{aligned}$$

As $\omega,\tau\in\mathcal{T}(L_2(\mathbb{G}))$ were arbitrary, it follows that $\Gamma(x)=V(x\otimes 1)V^*=x\otimes 1$, that is, $V(x\otimes 1)=(x\otimes 1)V$. Applying the slice map $(\iota\otimes f)$ to both sides of this equation yields $\rho(f)x=x\rho(f)$, for all $f\in L_1(\mathbb{G})$. Therefore $x\in\rho(L_1(\mathbb{G}))'=L_\infty(\hat{\mathbb{G}})$, making $E:=\Theta^r(m)$ the required projection.

(3) $\Rightarrow$ (1) If $E:\mathcal{B}(L_2(\mathbb{G}))\to L_\infty(\hat{\mathbb{G}})$ is a conditional expectation in $\mathcal{CB}_{\mathcal{T}_{\rhd}}(\mathcal{B}(L_2(\mathbb{G})))$, then $E(\mathrm{LUC}(\mathbb{G}))\subseteq\mathrm{LUC}(\mathbb{G})\cap L_\infty(\hat{\mathbb{G}})=\mathbb{C}1$. Thus, by restriction we obtain a bounded linear functional $n\in\mathrm{LUC}(\mathbb{G})^*$ satisfying $\langle n,y\rangle 1=E(y)$ for all $y\in\mathrm{LUC}(\mathbb{G})$. Moreover, considering the associated map $\Theta^r(n)\in\mathcal{CB}_{\mathcal{T}_{\rhd}}(\mathcal{B}(L_2(\mathbb{G})))$, we see that

$$\langle E(x),\omega\rangle=E(x)\rhd\omega=E(x\rhd\omega)=\langle n,x\rhd\omega\rangle=\langle\Theta^r(n)(x),\omega\rangle$$

for all $x \in \mathcal{B}(L_2(\mathbb{G}))$ and $\omega \in \mathcal{T}(L_2(\mathbb{G}))$. This ensures that $E = \Theta^r(n)$, so in particular we have $E(L_\infty(\mathbb{G})) \subseteq L_\infty(\mathbb{G}) \cap L_\infty(\hat{\mathbb{G}}) = \mathbb{C}1$ by (8). Put $m := E|_{L_\infty(\mathbb{G})}$. Then $m \in L_\infty(\mathbb{G})^*$ is a state satisfying

$$\langle m, x \star f\rangle = E(x \star f) = E(x) \star f = \langle m, x\rangle\langle 1, f\rangle$$

for every $x \in L_\infty(\mathbb{G})$ and $f \in L_1(\mathbb{G})$. Hence, m is a right invariant mean on $L_\infty(\mathbb{G})$. $\square$

Corollary 4.3. *A locally compact group G is amenable if and only if there is a covariant conditional expectation $E : \mathcal{B}(L_2(G)) \to \mathcal{L}(G)$.*

Remark 4.4. In [2], a notion of *topological amenability* for locally compact quantum groups $\mathbb{G}$ was defined by the existence of an invariant mean on $M(C_0(\mathbb{G}))$. The authors then asked if this notion of amenability is equivalent to the original one. The answer was recently provided, in the affirmative, by Zobeidi [38], generalizing the partial result of Runde in the co-amenable setting [29, Theorem 3.6]. As we always have $\mathrm{LUC}(\mathbb{G}) \subseteq M(C_0(\mathbb{G}))$, Theorem 4.2 provides an alternative proof (which had been found independently from [38]) for arbitrary locally compact quantum groups.

There is a corresponding result involving left invariant means on $\mathrm{RUC}(\mathbb{G})$ and conditional expectations in ${}_{\mathcal{T}_\triangleleft}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$. We state the result for completeness and for later use, but omit the details of the proof as the argument can easily be adapted from above using the left representation (9).

Theorem 4.5. *Let $\mathbb{G}$ be a locally compact quantum group. The following statements are equivalent:*

1. $\mathbb{G}$ *is amenable;*
2. *there is a left invariant mean on* $\mathrm{RUC}(\mathbb{G})$*;*
3. *there is a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}}')$ in ${}_{\mathcal{T}_\triangleleft}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$.*

As $\mathbb{G}$ is compact if and only if it admits a left invariant mean in $L_1(\mathbb{G})$ [2, Proposition 3.1], and the maps $\Theta^r(f), \Theta^l(f) \in \mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ are normal for all $f \in L_1(\mathbb{G})$ [16, §4], the following corollary is immediate.

Corollary 4.6. *Let $\mathbb{G}$ be a locally compact quantum group. The following statements are equivalent:*

1. $\mathbb{G}$ *is compact;*
2. *there is a normal covariant conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$;*
3. *there is a normal conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}}')$ in ${}_{\mathcal{T}_\triangleleft}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$.*

In Theorem 4.2, we characterized the amenability of $\mathbb{G}$ by means of a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ commuting with the right $(\mathcal{T}(L_2(\mathbb{G})), \triangleright)$-module action on $\mathcal{B}(L_2(\mathbb{G}))$. As there are three other $\mathcal{T}(L_2(\mathbb{G}))$-module structures on $\mathcal{B}(L_2(\mathbb{G}))$, a natural problem is to study the existence of module projections $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ in each of the remaining cases. To this end, we denote by ${}_{\mathcal{T}_\triangleright}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ (respectively, $\mathcal{CB}_{\mathcal{T}_\triangleleft}(\mathcal{B}(L_2(\mathbb{G})))$) the algebra of completely bounded left $(\mathcal{T}(L_2(\mathbb{G})), \triangleright)$-module (respectively, right $(\mathcal{T}(L_2(\mathbb{G})), \triangleleft)$-module) maps on $\mathcal{B}(L_2(\mathbb{G}))$, and for any subset $\mathcal{S}$ of $\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$, we denote its commutant in $\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ by $\mathcal{S}^c$.

Theorem 4.7. *Let $\mathbb{G}$ be a locally compact quantum group. There exists a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ in ${}_{\mathcal{T}_\rhd}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ if and only if $\mathbb{G}$ is amenable.*

Proof. By restriction, we may view any $f \in L_1(\mathbb{G}) \subseteq L_\infty(\mathbb{G})^*$ as an element of $\mathrm{LUC}(\mathbb{G})^*$. Moreover, if $\pi : (\mathcal{T}(L_2(\mathbb{G})), \rhd) \to (L_1(\mathbb{G}), \star)$ denotes the restriction map (3), for $\omega, \rho \in \mathcal{T}(L_2(\mathbb{G}))$ and $x \in \mathcal{B}(L_2(\mathbb{G}))$ we have

$$\langle \rho \rhd x, \omega\rangle = \langle x, \omega \rhd \rho\rangle = \langle x \rhd \omega, \rho\rangle = \langle x \rhd \omega, \pi(\rho)\rangle = \langle \Theta^r(\pi(\rho))(x), \omega\rangle.$$

Thus, $\rho \rhd x = \Theta^r(\pi(\rho))(x)$, so that a map $\Phi \in \mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ is a left $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$-module homomorphism if and only if $\Phi \in \Theta^r(L_1(\mathbb{G}))^c$.

If $\mathbb{G}$ is amenable, then there exists a two-sided invariant mean m on $L_\infty(\mathbb{G})$. Denoting again by m its restriction to $\mathrm{LUC}(\mathbb{G})$, it follows that

$$m \Box f = f \Box m = \langle f, 1\rangle m \tag{10}$$

for every $f \in L_1(\mathbb{G})$. Hence, $\Theta^r(m) \in \Theta^r(L_1(\mathbb{G}))^c$ by (7). As m is also a right invariant mean on $\mathrm{LUC}(\mathbb{G})$, it follows from the proof of Theorem 4.2 that $\Theta^r(m)$ is a conditional expectation onto $L_\infty(\hat{\mathbb{G}})$.

Conversely, suppose that there exists a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ in ${}_{\mathcal{T}_\rhd}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$, and let $\hat{f} \in L_1(\hat{\mathbb{G}})$ be a state. For $\omega \in \mathcal{T}(L_2(\mathbb{G}))$ with $f = \pi(\omega) \in L_1(\mathbb{G})$ and $x \in L_\infty(\mathbb{G})$, the relations (4) imply

$$\langle \hat{f} \circ E, f \star x\rangle = \langle \hat{f} \circ E, \omega \rhd x\rangle = \langle \hat{f}, \omega \rhd E(x)\rangle = \langle \omega, 1\rangle \langle \hat{f} \circ E, x\rangle = \langle f, 1\rangle\langle \hat{f} \circ E, x\rangle.$$

Thus, $\hat{f} \circ E$ is a left invariant mean on $L_\infty(\mathbb{G})$. $\square$

Remark 4.8. We note that the existence of a conditional expectation with the module property as in Theorem 4.7 is equivalent to the amenability of the right fundamental unitary V, as defined by Bédos and Tuset in [2, §4].

Theorem 4.9. *Let $\mathbb{G}$ be a locally compact quantum group. There exists a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ in $\mathcal{CB}_{\mathcal{T}_\lhd}(\mathcal{B}(L_2(\mathbb{G})))$ if and only if $L_\infty(\hat{\mathbb{G}})$ is injective.*

Proof. Suppose that $\hat{\mathbb{G}}$ is injective. Then there exists a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$. By [33], E is an $L_\infty(\hat{\mathbb{G}})$-bimodule map on $\mathcal{B}(L_2(\mathbb{G}))$. We will show that it also lies in $\mathcal{CB}_{\mathcal{T}_\lhd}(\mathcal{B}(L_2(\mathbb{G})))$. To this end, observe that a map $\Phi \in \mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ is a right $(\mathcal{T}(L_2(\mathbb{G})), \lhd)$-module map if and only if $\Phi \in \Theta^l(L_1(\mathbb{G}))^c$ (see the proof of Theorem 4.7). If $f \in L_1(\mathbb{G})$, then from [16, Theorem 4.10], $\Theta^l(f)$ is a normal completely bounded $L_\infty(\hat{\mathbb{G}}')$-bimodule map on $\mathcal{B}(L_2(\mathbb{G}))$, which by an unpublished result of Haagerup [11] implies the existence of two nets $(\hat{a}_i)_{i\in I}$ and $(\hat{b}_i)_{i\in I}$ in $L_\infty(\hat{\mathbb{G}})$ such that

$$\Theta^l(f)(x) = \sum_{i\in I} \hat{a}_i x \hat{b}_i,$$

where the sum converges in the weak* topology of $\mathcal{B}(L_2(\mathbb{G}))$ for all $x \in \mathcal{B}(L_2(\mathbb{G}))$. Now, it follows from [21, Lemma 2.3] that we may approximate E in the weak* topology of $\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ by a net of normal completely bounded $L_\infty(\hat{\mathbb{G}})$-bimodule maps $(\Phi_j)_{j\in J}$. Consequently, for $x \in \mathcal{B}(L_2(\mathbb{G}))$ and $\rho \in \mathcal{T}(L_2(\mathbb{G}))$,

$$\begin{aligned}\langle E(\Theta^l(f)(x)), \rho\rangle &= \lim_{j\in J}\langle \Phi_j(\Theta^l(f)(x)), \rho\rangle = \lim_{j\in J}\sum_{i\in I}\langle \Phi_j(\hat{a}_i x \hat{b}_i), \rho\rangle \\ &= \lim_{j\in J}\sum_{i\in I}\langle \hat{a}_i \Phi_j(x)\hat{b}_i, \rho\rangle = \lim_{j\in J}\langle \Theta^l(f)(\Phi_j(x)), \rho\rangle \\ &= \langle \Theta^l(f)(E(x)), \rho\rangle.\end{aligned}$$

Since $f \in L_1(\mathbb{G})$ was arbitrary, we have $E \in \Theta^l(L_1(\mathbb{G}))^c$. As the converse is trivial, we are done. □

Corollary 4.10. *Let $\mathbb{G}$ be a locally compact quantum group for which there exists a state $m \in \mathcal{B}(L_2(\mathbb{G}))^*$ satisfying $m(\rho \rhd x) = m(x \lhd \rho)$ for all $x \in \mathcal{B}(L_2(\mathbb{G}))$ and $\rho \in \mathcal{T}(L_2(\mathbb{G}))$. Then $\mathbb{G}$ is amenable if and only if $L_\infty(\hat{\mathbb{G}})$ is injective.*

Proof. Suppose $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ is a conditional expectation. Then by Theorem 4.9, E is a right $(\mathcal{T}(L_2(\mathbb{G})), \lhd)$-module map. Thus, $n := m \circ E$ is a state on $\mathcal{B}(L_2(\mathbb{G}))$ satisfying $n(x \lhd \rho) = m(E(x) \lhd \rho) = m(\rho \rhd E(x)) = \langle \rho, 1\rangle n(x)$ for all $x \in \mathcal{B}(L_2(\mathbb{G}))$ and $\rho \in \mathcal{T}(L_2(\mathbb{G}))$. It follows that $n|_{L_\infty(\mathbb{G})}$ is a right invariant mean on $L_\infty(\mathbb{G})$. □

Remark 4.11. The above condition, i.e., the existence of a state $m \in \mathcal{B}(L_2(\mathbb{G}))^*$ such that $m(\rho \rhd x) = m(x \lhd \rho)$ for all $x \in \mathcal{B}(L_2(\mathbb{G}))$ and $\rho \in \mathcal{T}(L_2(\mathbb{G}))$, may be seen as a form of inner amenability for locally compact quantum groups. Indeed, a commutative quantum group $\mathbb{G}_a$ satisfies this property if and only if its underlying group G is inner amenable [3], i.e., there exists a state $n \in L_\infty(G)^*$ satisfying $n(\delta_s * f * \delta_{s^{-1}}) = n(f)$ for all $f \in L_\infty(G)$ and $s \in G$. Moreover, one can show that discrete Kac algebras are inner amenable, thus Corollary 4.10 entails the equivalence of injectivity and amenability for discrete Kac algebras – a concrete application of our techniques. The latter important result, due to Ruan [28], has also been derived in [30]. We hope to be able to use our approach to extend this equivalence to arbitrary discrete quantum groups, an outstanding open problem in the area.

Theorem 4.12. *Let $\mathbb{G}$ be a locally compact quantum group. There exists a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ in ${}_{\mathcal{T}_\lhd}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$ if and only if $\mathbb{G}$ is co-commutative, i.e., $L_\infty(\mathbb{G}) = \mathcal{L}(G)$ for some locally compact group G.*

Proof. If $\mathbb{G}$ is co-commutative, then $L_\infty(\mathbb{G}) = \mathcal{L}(G)$ for some locally compact group G, and by [27, Theorem 4] there exists a left invariant mean m on $\mathcal{L}(G)$. In this case, its restriction to $\mathrm{UCB}(\hat{G}) = \mathrm{RUC}(\mathbb{G})$ is also a left invariant mean, and Theorem 4.5 provides a conditional expectation $\Theta^l(m) : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}}')$ in ${}_{\mathcal{T}_\lhd}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$. By duality, $L_\infty(\hat{\mathbb{G}}) = L_\infty(G) = L_\infty(\hat{\mathbb{G}}')$, making $\Theta^l(m)$ the desired projection.

If $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ exists in ${}_{\mathcal{T}_\lhd}\mathcal{CB}(\mathcal{B}(L_2(\mathbb{G})))$, then a simple calculation implies that $(E \otimes \iota) \circ \Gamma^l = \Gamma^l \circ E$. As $\Gamma^l(\cdot) = W^*(1 \otimes (\cdot))W$, with $W \in L_\infty(\mathbb{G})\bar{\otimes}L_\infty(\hat{\mathbb{G}})$, and $E(\mathcal{B}(L_2(\mathbb{G}))) = L_\infty(\hat{\mathbb{G}})$, we must have $(E \otimes \iota) \circ \Gamma^l(x) = \Gamma^l \circ E(x) \in L_\infty(\mathbb{G})\bar{\otimes}L_\infty(\hat{\mathbb{G}})$ for every $x \in \mathcal{B}(L_2(\mathbb{G}))$. In particular, for $\hat{x}' \in L_\infty(\hat{\mathbb{G}}')$, we have

$$(E \otimes \iota) \circ \Gamma^l(\hat{x}') = (E \otimes \iota)(W^*(1 \otimes \hat{x}')W) = 1 \otimes \hat{x}' \in L_\infty(\mathbb{G})\bar{\otimes}L_\infty(\hat{\mathbb{G}}),$$

implying that $L_\infty(\hat{\mathbb{G}}') \subseteq L_\infty(\hat{\mathbb{G}})$. As $L_\infty(\hat{\mathbb{G}})$ is in standard form on $\mathcal{B}(L_2(\mathbb{G}))$, there exists a conjugate linear isometric involution $\hat{J}$ on $L_2(\mathbb{G})$ satisfying $\hat{J}L_\infty(\hat{\mathbb{G}})\hat{J} = L_\infty(\hat{\mathbb{G}}')$. We therefore obtain $L_\infty(\hat{\mathbb{G}}) \subseteq L_\infty(\hat{\mathbb{G}}')$, that is, $L_\infty(\hat{\mathbb{G}})$ is commutative. By [31, Theorem 2; §2; 36], $L_\infty(\hat{\mathbb{G}}) = L_\infty(G)$ for some locally compact group G, making $L_\infty(\mathbb{G})$ co-commutative. □

Remark 4.13. By the proof of [28, Theorem 2.1], it follows that a locally compact quantum group $\mathbb{G}$ is amenable if and only if there exists a *non-zero* left (respectively, right, two-sided) invariant functional $m \in L_\infty(\mathbb{G})^*$. Hence, the existence of a *completely bounded* covariant projection $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ is equivalent to the amenability of $\mathbb{G}$, and it follows that we may replace "conditional expectation" by "completely bounded projection" in the statement of every theorem and corollary in this section. For Theorem 4.10, recall that a von Neumann algebra $M \subseteq \mathcal{B}(H)$ is injective if and only if there exists a completely bounded projection $E : \mathcal{B}(H) \to M$ [4, 25].

5. Injective modules

Continuing in the spirit of the previous sections, here we establish a perfect duality between quantum group amenability and injectivity in the category of $\mathcal{T}(L_2(\mathbb{G}))$-modules. We also show that both amenability of $\mathbb{G}$ and of $\hat{\mathbb{G}}$ may be characterized through the injectivity of $\mathcal{B}(L_2(\mathbb{G}))$ as a left, respectively, right $\mathcal{T}(L_2(\mathbb{G}))$-module. This marks the starting point for subsequent work on homological properties of $\mathcal{T}(L_2(\mathbb{G}))$-modules and their connections to amenability.

Let $\mathcal{A}$ be a completely contractive Banach algebra and X be an operator space. We say that X is a right *operator $\mathcal{A}$-module* if it is a right Banach $\mathcal{A}$-module for which the module map $m : X\widehat{\otimes}\mathcal{A} \to X$ is completely contractive. We denote by $\mathbf{mod} - \mathcal{A}$ the category of right operator $\mathcal{A}$-modules with morphisms given by completely contractive module homomorphisms. If $X, Y \in \mathbf{mod} - \mathcal{A}$, an injective morphism $\Phi : X \to Y$ is called *admissible* if there exists a completely contractive map (not necessarily a morphism) $\Psi : Y \to X$ such that $\Psi \circ \Phi = \iota_X$. An operator module $X \in \mathbf{mod} - \mathcal{A}$ is *faithful* if for every non-zero $x \in X$, there is a non-zero $a \in \mathcal{A}$ such that $x \cdot a \neq 0$, and X is said to be *injective* if for every $Y, Z \in \mathbf{mod} - \mathcal{A}$, every injective admissible morphism $\Phi : Y \to Z$, and every morphism $\Psi : Y \to X$, there exists a morphism $\tilde{\Psi} : Z \to X$ such that $\tilde{\Psi} \circ \Phi = \Psi$. Left operator $\mathcal{A}$-modules are defined similarly, and there are analogous notions of admissibility, faithfulness and injectivity in this category, denoted by $\mathcal{A} - \mathbf{mod}$.

Let $X \in \mathbf{mod} - \mathcal{A}$. The unitization of $\mathcal{A}$, denoted $\mathcal{A}^+$, carries a natural operator space structure turning it into a completely contractive Banach algebra (cf. [34, §3.2]), and it follows that X becomes a right operator $\mathcal{A}^+$-module via the extended action

$$x \cdot (a + \lambda e) = x \cdot a + \lambda x \quad (a \in \mathcal{A}^+, \lambda \in \mathbb{C}, x \in X).$$

Then there is a canonical morphism $\Delta^+ : X \to \mathcal{CB}(\mathcal{A}^+, X)$ given by

$$\Delta^+(x)(a) = x \cdot a \quad (x \in X, a \in \mathcal{A}^+),$$

where the $\mathcal{A}$-bimodule structure on $\mathcal{CB}(\mathcal{A}^+, X)$ is defined by

$$(a \cdot \Psi)(b) = \Psi(ba) \quad \text{and} \quad (\Psi \cdot a)(b) = \Psi(ab) \quad (a \in \mathcal{A}, \Psi \in \mathcal{CB}(\mathcal{A}^+, X), b \in \mathcal{A}^+).$$

By the standard argument, it follows that X is injective if and only if there exists a morphism $\Phi : \mathcal{CB}(\mathcal{A}^+, X) \to X$ that is a left inverse to Δ^+. Moreover, if X is faithful, by the operator space version of [7, Proposition 1.7] (which can be proved using the operator space structure of $\mathcal{A}^+$, cf. [34, Proposition 3.2.7]), X is injective if and only if there exists a morphism $\Phi : \mathcal{CB}(\mathcal{A}, X) \to X$ that is a left inverse to $\Delta : X \to \mathcal{CB}(\mathcal{A}, X)$, where $\Delta(x)(a) := \Delta^+(x)(a)$ for all $x \in X$ and $a \in \mathcal{A}$.

Theorem 5.1. *Let $\mathbb{G}$ be a locally compact quantum group. The following statements are equivalent:*

(1) *$\mathbb{G}$ is amenable;*
(2) *$L_\infty(\hat{\mathbb{G}})$ is injective in $\mathbf{mod} - (\mathcal{T}(L_2(\mathbb{G})), \rhd)$;*
(3) *$L_\infty(\hat{\mathbb{G}})$ is injective in $(\mathcal{T}(L_2(\mathbb{G})), \rhd) - \mathbf{mod}$.*

Proof. (1) $\Rightarrow$ (2) Observe that if $\hat{x} \in L_\infty(\hat{\mathbb{G}})$ such that $0 = \hat{x} \rhd \rho = \langle \hat{x}, \rho \rangle 1$ for all $\rho \in \mathcal{T}(L_2(\mathbb{G}))$, then $\langle \hat{x}, \hat{f} \rangle = 0$ for all $\hat{f} \in L_1(\hat{\mathbb{G}})$, making $\hat{x} = 0$. Thus, $L_\infty(\hat{\mathbb{G}})$ is faithful in $\mathbf{mod} - (\mathcal{T}(L_2(\mathbb{G})), \rhd)$. It therefore suffices to provide a morphism which is a left inverse to the map $\Delta^r : L_\infty(\hat{\mathbb{G}}) \to \mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})), L_\infty(\hat{\mathbb{G}}))$ given by

$$\Delta^r(\hat{x})(\rho) = \hat{x} \rhd \rho \quad (\hat{x} \in L_\infty(\hat{\mathbb{G}}), \rho \in \mathcal{T}(L_2(\mathbb{G}))).$$

Identifying $\mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})), L_\infty(\hat{\mathbb{G}})) \cong \mathcal{B}(L_2(\mathbb{G})) \bar{\otimes} L_\infty(\hat{\mathbb{G}})$ (cf. [9]) via

$$\langle \Psi, \rho \otimes \hat{f} \rangle = \langle \Psi(\rho), \hat{f} \rangle \quad (\Psi \in \mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})), L_\infty(\hat{\mathbb{G}})), \rho \in \mathcal{T}(L_2(\mathbb{G})), \hat{f} \in L_1(\hat{\mathbb{G}})),$$

one easily sees that $\Delta^r(\hat{x}) = \hat{x} \otimes 1$ for all $\hat{x} \in L_\infty(\hat{\mathbb{G}})$, and the corresponding $\mathcal{T}(L_2(\mathbb{G}))$-module structure on $\mathcal{B}(L_2(\mathbb{G})) \bar{\otimes} L_\infty(\hat{\mathbb{G}})$ is given by

$$T \unrhd \rho = (\rho \otimes \iota \otimes \iota)(\Gamma^r \otimes \iota)(T)$$

for $T \in \mathcal{B}(L_2(\mathbb{G})) \bar{\otimes} L_\infty(\hat{\mathbb{G}})$ and $\rho \in \mathcal{T}(L_2(\mathbb{G}))$. Since $\mathbb{G}$ is amenable, there exists a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ that is a morphism in $\mathbf{mod} - (\mathcal{T}(L_2(\mathbb{G})), \rhd)$ (cf. Theorem 4.2). Fix a state $\hat{f} \in L_1(\hat{\mathbb{G}})$, and define $\Phi^r : \mathcal{B}(L_2(\mathbb{G})) \bar{\otimes} L_\infty(\hat{\mathbb{G}}) \to L_\infty(\hat{\mathbb{G}})$ by

$$\Phi^r(T) = E((\iota \otimes \hat{f})T) \quad (T \in \mathcal{B}(L_2(\mathbb{G})) \bar{\otimes} L_\infty(\hat{\mathbb{G}})).$$

Then Φ^r is a complete contraction, and for $\hat{x} \in L_\infty(\hat{\mathbb{G}})$ we have

$$\Phi^r(\Delta^r(\hat{x})) = \Phi^r(\hat{x} \otimes 1) = E(\hat{x}) = \hat{x},$$

so that Φ^r is a left inverse to Δ^r. Moreover, for $T \in \mathcal{B}(L_2(\mathbb{G})) \bar{\otimes} L_\infty(\hat{\mathbb{G}})$ and $\rho \in \mathcal{T}(L_2(\mathbb{G}))$, we have

$$\begin{aligned}\Phi^r(T \unrhd \rho) &= \Phi^r((\rho \otimes \iota \otimes \iota)(\Gamma^r \otimes \iota)(T)) = E((\rho \otimes \iota)\Gamma^r((\iota \otimes \hat{f})T)) \\ &= E(((\iota \otimes \hat{f})T) \rhd \rho) = E((\iota \otimes \hat{f})T) \rhd \rho = \Phi^r(T) \rhd \rho.\end{aligned}$$

(2) $\Rightarrow$ (1) If $L_\infty(\hat{\mathbb{G}})$ is injective in $\mathbf{mod} - (\mathcal{T}(L_2(\mathbb{G})), \rhd)$, there is a morphism $\Phi^r : \mathcal{B}(L_2(\mathbb{G})) \bar{\otimes} L_\infty(\hat{\mathbb{G}}) \to L_\infty(\hat{\mathbb{G}})$ that is a left inverse to Δ^r. Define $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ by $E(x) = \Phi^r(x \otimes 1)$ for all $x \in \mathcal{B}(L_2(\mathbb{G}))$. Then E is a morphism, and for $\hat{x} \in L_\infty(\hat{\mathbb{G}})$ we get

$$E(\hat{x}) = \Phi^r(\hat{x} \otimes 1) = \Phi^r(\Delta^r(\hat{x})) = \hat{x},$$

making E a projection of norm one onto $L_\infty(\hat{\mathbb{G}})$. Theorem 4.2 then entails the amenability of $\mathbb{G}$.

(1) ⇒ (3) As above, it follows that $L_\infty(\hat{\mathbb{G}})$ is faithful in $(\mathcal{T}(L_2(\mathbb{G})), \rhd) - \mathbf{mod}$. We therefore have to provide a morphism which is a left inverse to $\Delta^l : L_\infty(\hat{\mathbb{G}}) \to \mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})), L_\infty(\hat{\mathbb{G}}))$ given by

$$\Delta^l(\hat{x})(\rho) = \rho \rhd \hat{x} \quad (\hat{x} \in L_\infty(\hat{\mathbb{G}}), \rho \in \mathcal{T}(L_2(\mathbb{G}))).$$

With the identification $\mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})), L_\infty(\hat{\mathbb{G}})) \cong \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}L_\infty(\hat{\mathbb{G}})$, it follows that $\Delta^l(\hat{x}) = 1 \otimes \hat{x}$ for all $\hat{x} \in L_\infty(\hat{\mathbb{G}})$ and that the corresponding $\mathcal{T}(L_2(\mathbb{G}))$-module structure on $\mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}L_\infty(\hat{\mathbb{G}})$ is given by $\rho \trianglerighteq T = (\iota \otimes \rho \otimes \iota)(\Gamma^r \otimes \iota)(T)$ for $T \in \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}L_\infty(\hat{\mathbb{G}})$ and $\rho \in \mathcal{T}(L_2(\mathbb{G}))$. By amenability of $\mathbb{G}$, there exists a conditional expectation $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ that is a morphism in $(\mathcal{T}(L_2(\mathbb{G})), \rhd) - \mathbf{mod}$ (cf. Theorem 4.7). Fix a state $\hat{f} \in L_1(\hat{\mathbb{G}})$, put $m := \hat{f} \circ E \in \mathcal{B}(L_2(\mathbb{G}))^*$, and define $\Phi^l : \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}L_\infty(\hat{\mathbb{G}}) \to L_\infty(\hat{\mathbb{G}})$ by

$$\Phi^l(T) = (m \otimes \iota)(T) \quad (T \in \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}L_\infty(\hat{\mathbb{G}})).$$

Clearly Φ^l is a completely contractive left inverse to Δ^l. Furthermore, for $T \in \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}L_\infty(\hat{\mathbb{G}})$, $\rho \in \mathcal{T}(L_2(\mathbb{G}))$, and $\hat{g} \in L_1(\hat{\mathbb{G}})$, we have

$$\begin{aligned}\langle \Phi^l(\rho \trianglerighteq T), \hat{g}\rangle &= \langle (m \otimes \iota)(\rho \trianglerighteq T), \hat{g}\rangle = \langle m, (\iota \otimes \rho)\Gamma^r((\iota \otimes \hat{g})T)\rangle \\ &= \langle \hat{f}, E(\rho \rhd ((\iota \otimes \hat{g})T))\rangle = \langle \hat{f}, \rho \rhd E((\iota \otimes \hat{g})T)\rangle \\ &= \langle \rho, 1\rangle\langle m, (\iota \otimes \hat{g})(T)\rangle = \langle \rho, 1\rangle\langle \Phi^l(T), \hat{g}\rangle = \langle \rho \rhd \Phi^l(T), \hat{g}\rangle.\end{aligned}$$

(3) ⇒ (1) If $L_\infty(\hat{\mathbb{G}})$ is injective in $(\mathcal{T}(L_2(\mathbb{G})), \rhd) - \mathbf{mod}$, there is a morphism $\Phi^l : \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}L_\infty(\hat{\mathbb{G}}) \to L_\infty(\hat{\mathbb{G}})$ that is a left inverse to Δ^l. Define $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ by $E(x) = \Phi^l(x \otimes 1)$ for all $x \in \mathcal{B}(L_2(\mathbb{G}))$. Since $E(1) = \Phi^l(1 \otimes 1) = \Phi^l(\Delta^l(1)) = 1$, E is a unital morphism, and for any state $\hat{f} \in L_1(\hat{\mathbb{G}})$, we have $1 = \hat{f} \circ E(1) \leq \|\hat{f} \circ E\| \leq 1$, making $\hat{f} \circ E$ a state in $\mathcal{B}(L_2(\mathbb{G}))^*$. By the proof of Theorem 4.7, it then follows that the restriction of $\hat{f} \circ E$ to $L_\infty(\mathbb{G})$ is a left invariant mean. ☐

Remark 5.2. By the observations in Remark 4.13, it follows that $\mathbb{G}$ is amenable if and only if $L_\infty(\hat{\mathbb{G}})$ is injective as a right (respectively, left) module in the category of right (respectively, left) operator $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$-modules with *completely bounded* module homomorphisms. Note, however, that injectivity in this category is formally weaker than injectivity in $\mathbf{mod} - (\mathcal{T}(L_2(\mathbb{G})), \rhd)$, (respectively, $(\mathcal{T}(L_2(\mathbb{G})), \rhd) - \mathbf{mod}$).

By restricting to the commutative setting, we immediately obtain a new characterization of classical amenability, while, on the other hand, restricting to the co-commutative case, we see that $L_\infty(G)$ is an injective $\mathcal{T}(L_2(G))$-module for any locally compact group G.

Corollary 5.3. *Let G be a locally compact group. The following statements are equivalent:*

(1) *G is amenable;*
(2) *$\mathcal{L}(G)$ is injective in* $\mathbf{mod} - (\mathcal{T}(L_2(G)), \rhd)$;
(3) *$\mathcal{L}(G)$ is injective in* $(\mathcal{T}(L_2(G)), \rhd) - \mathbf{mod}$.

Corollary 5.4. *Let G be a locally compact group. Then $L_\infty(G)$ is injective in both* $\mathbf{mod} - (\mathcal{T}(L_2(G)), \rhd)$ *and* $(\mathcal{T}(L_2(G)), \rhd) - \mathbf{mod}$.

Recall that the multiplication in $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$ is a complete quotient map for any locally compact quantum group $\mathbb{G}$. Consequently,

$$\mathcal{T}(L_2(\mathbb{G})) = \langle \mathcal{T}(L_2(\mathbb{G})) \rhd \mathcal{T}(L_2(\mathbb{G})) \rangle,$$

and so if $x \in \mathcal{B}(L_2(\mathbb{G}))$ satisfies $\rho \rhd x = 0$ for all $\rho \in \mathcal{T}(L_2(\mathbb{G}))$, then $\langle \rho \rhd x, \omega \rangle = \langle x, \omega \rhd \rho \rangle = 0$ for all $\rho, \omega \in \mathcal{T}(L_2(\mathbb{G}))$, making $x = 0$. Thus, $\mathcal{B}(L_2(\mathbb{G}))$ is faithful in $(\mathcal{T}(L_2(\mathbb{G})), \rhd) - \mathbf{mod}$. By a similar argument it follows that $\mathcal{B}(L_2(\mathbb{G}))$ is also faithful in $\mathbf{mod} - (\mathcal{T}(L_2(\mathbb{G})), \rhd)$.

Theorem 5.5. *Let $\mathbb{G}$ be a locally compact quantum group. Then $\mathbb{G}$ is amenable if and only if $\mathcal{B}(L_2(\mathbb{G}))$ is injective in $(\mathcal{T}(L_2(\mathbb{G})), \rhd) - \mathbf{mod}$.*

Proof. Suppose $\mathbb{G}$ is amenable, and let $m \in L_\infty(\mathbb{G})^*$ be a two-sided invariant mean. Since $\mathcal{B}(L_2(\mathbb{G}))$ is faithful, it suffices to provide a morphism that is a left inverse for the map $\Delta : \mathcal{B}(L_2(\mathbb{G})) \to \mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})), \mathcal{B}(L_2(\mathbb{G})))$ given by

$$\Delta(x)(\rho) = \rho \rhd x \quad (x \in \mathcal{B}(L_2(\mathbb{G})), \rho \in \mathcal{T}(L_2(\mathbb{G}))). \tag{11}$$

Identifying $\mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})), \mathcal{B}(L_2(\mathbb{G}))) \cong \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))$ (cf. [9]) via

$$\langle \Phi, \rho \otimes \omega \rangle = \langle \Phi(\omega), \rho \rangle \quad (\Phi \in \mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})), \mathcal{B}(L_2(\mathbb{G}))), \rho, \omega \in \mathcal{T}(L_2(\mathbb{G}))),$$

one easily sees that $\Delta = \Gamma^r$, and that the left $(\mathcal{T}(L_2(\mathbb{G})), \rhd)$-module structure on $\mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))$ is given by $\rho \unrhd T = (\iota \otimes \iota \otimes \rho)(\iota \otimes \Gamma^r)(T)$, for $T \in \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))$ and $\rho \in \mathcal{T}(L_2(\mathbb{G}))$. We are therefore reduced to finding a morphism $\Phi : \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G})) \to \mathcal{B}(L_2(\mathbb{G}))$ satisfying $\Phi \circ \Gamma^r = \iota_{\mathcal{B}(L_2(\mathbb{G}))}$.

Let $n \in \mathcal{T}(L_2(\mathbb{G}))$ be a state. Then $m_n := n \circ \Theta^r(m)$ is a state on $\mathcal{B}(L_2(\mathbb{G}))$, and we define $\Phi : \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G})) \to \mathcal{B}(L_2(\mathbb{G}))$ by

$$\Phi(T) = (\iota \otimes m_n)(V^*TV) \quad (T \in \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))). \tag{12}$$

Clearly, Φ is a complete contraction, and for $x \in \mathcal{B}(L_2(\mathbb{G}))$, we have

$$\Phi(\Gamma^r(x)) = \Phi(V(x \otimes 1)V^*) = (\iota \otimes m_n(x \otimes 1)) = x,$$

so Φ is a left inverse for Γ^r. To show the module property, fix

$$T \in \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G})) \quad \text{and} \quad \rho \in \mathcal{T}(L_2(\mathbb{G})).$$

Then, using the standard leg notation, we obtain

$$\begin{aligned}\Phi(\rho \rhd T) &= \Phi((\iota \otimes \iota \otimes \rho)(V_{23}T_{12}V_{23}^*)) = (\iota \otimes m_n \otimes \rho)(V_{12}^*V_{23}T_{12}V_{23}^*V_{12}) \\ &= (\iota \otimes m_n \otimes \rho)(V_{13}V_{23}V_{12}^*T_{12}V_{12}V_{23}^*V_{13}^*) \\ &= (\iota \otimes \rho)(V(\iota \otimes m_n \otimes \iota)(V_{23}V_{12}^*T_{12}V_{12}V_{23}^*)V^*).\end{aligned}$$

Now, for any $\tau, \omega \in \mathcal{T}(L_2(\mathbb{G}))$, recalling that $\pi : \mathcal{T}(L_2(\mathbb{G})) \to L_1(\mathbb{G})$ denotes the canonical quotient map, we have

$$\begin{aligned}&\langle (\iota \otimes m_n \otimes \iota)(V_{23}V_{12}^*T_{12}V_{12}V_{23}^*), \tau \otimes \omega \rangle \\ &\quad = \langle (m_n \otimes \iota)V((\tau \otimes \iota)(V^*TV) \otimes 1)V^*, \omega \rangle \\ &\quad = \langle m_n, \Theta^r(\pi(\omega))((\tau \otimes \iota)V^*TV) \rangle \\ &\quad = \langle n, \langle \omega, 1 \rangle \Theta^r(m)((\tau \otimes \iota)V^*TV) \rangle \quad \text{(by equation (10))} \\ &\quad = \langle m_n \otimes \omega, (\tau \otimes \iota)(V^*TV) \otimes 1 \rangle \\ &\quad = \langle (\iota \otimes m_n \otimes \iota)(V^*TV \otimes 1), \tau \otimes \omega \rangle.\end{aligned}$$

Since τ and ω in $\mathcal{T}(L_2(\mathbb{G}))$ were arbitrary, it follows that

$$\begin{aligned}\Phi(\rho \triangleright T) &= (\iota \otimes \rho)(V(\iota \otimes m_n \otimes \iota)(V_{23}V_{12}^*T_{12}V_{12}V_{23}^*)V^*)\\ &= (\iota \otimes \rho)(V(\iota \otimes m_n \otimes \iota)(V^*TV \otimes 1)V^*)\\ &= (\iota \otimes \rho)(V(\Phi(T) \otimes 1)V^*) = \rho \triangleright \Phi(T).\end{aligned}$$

Conversely, suppose that there exists a morphism $\Phi : \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G})) \to \mathcal{B}(L_2(\mathbb{G}))$ that is a left inverse to Γ^r. Then $\Gamma^r \circ \Phi$ is a conditional expectation onto the image of Γ^r, and $\Gamma^r \circ \Phi = (\Phi \otimes \iota)(\iota \otimes \Gamma^r)$ as Φ is a module map. Define a map $E : \mathcal{B}(L_2(\mathbb{G})) \to \mathcal{B}(L_2(\mathbb{G}))$ by

$$E(x) = \Phi(x \otimes 1) \quad (x \in \mathcal{B}(L_2(\mathbb{G}))).$$

Then E is a complete contraction, and for $x \in \mathcal{B}(L_2(\mathbb{G}))$ we have

$$\begin{aligned}\Gamma^r(E(x)) = \Gamma^r(\Phi(x \otimes 1)) &= (\Phi \otimes \iota)(\iota \otimes \Gamma^r)(x \otimes 1) = (\Phi \otimes \iota)(x \otimes 1 \otimes 1)\\ &= \Phi(x \otimes 1) \otimes 1 = E(x) \otimes 1,\end{aligned}$$

which by the standard argument shows that $E(x) \in L_\infty(\hat{\mathbb{G}})$. Moreover, $E(\hat{x}) = \Phi(\hat{x} \otimes 1) = \Phi(\Gamma^r(\hat{x})) = \hat{x}$ for all $\hat{x} \in L_\infty(\hat{\mathbb{G}})$ making E a projection of norm one onto $L_\infty(\hat{\mathbb{G}})$.

Since $\Gamma^r \circ \Phi$ is a conditional expectation onto $\Gamma^r(\mathcal{B}(L_2(\mathbb{G})))$, it follows from [33] that

$$(\Gamma^r \circ \Phi)(\Gamma^r(x)T\Gamma^r(y)) = \Gamma^r(x)(\Gamma^r \circ \Phi(T))\Gamma^r(y) = \Gamma^r(x\Phi(T)y),$$

which, by the injectivity of Γ^r, implies $\Phi(\Gamma^r(x)T\Gamma^r(y)) = x\Phi(T)y$, for all $x, y \in \mathcal{B}(L_2(\mathbb{G}))$ and $T \in \mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))$. Taking $T = x' \otimes 1 \in L_\infty(\mathbb{G})'\bar{\otimes}L_\infty(\mathbb{G})'$ and $x \in L_\infty(\mathbb{G})$, we therefore have $\Phi(x' \otimes 1)x = x\Phi(x' \otimes 1)$. Consequently, $E(x') = \Phi(x' \otimes 1) \in L_\infty(\mathbb{G})'$ for every $x' \in L_\infty(\mathbb{G})'$. Since $L_\infty(\mathbb{G})$ is standard in $\mathcal{B}(L_2(\mathbb{G}))$, there is a conjugate linear involution J on $L_2(\mathbb{G})$ satisfying $JL_\infty(\mathbb{G})J = L_\infty(\mathbb{G})'$. Moreover, $JL_\infty(\hat{\mathbb{G}})J \subseteq L_\infty(\hat{\mathbb{G}})$ [19, Proposition 2.1], so that $E_J : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ given by

$$E_J(x) = JE(JxJ)J \quad (x \in \mathcal{B}(L_2(\mathbb{G})))$$

also defines a conditional expectation onto $L_\infty(\hat{\mathbb{G}})$. Clearly, $E_J(L_\infty(\mathbb{G})) \subseteq L_\infty(\mathbb{G}) \cap L_\infty(\hat{\mathbb{G}}) = \mathbb{C}1$, so [30, Theorem 3] entails the amenability of $\mathbb{G}$. □

Remark 5.6. Contrary to Theorem 5.1, it is not immediately obvious if one can weaken the hypothesis of Theorem 5.5 to injectivity in the category of left $(\mathcal{T}(L_2(\mathbb{G})), \triangleright)$-modules with *completely bounded* morphisms (cf. Remark 5.2).

By considering the category of left operator $\mathcal{T}(L_2(\mathbb{G}))$-modules with *normal* completely contractive morphisms, denoted $(\mathcal{T}(L_2(\mathbb{G})), \triangleright) - \mathbf{nmod}$, we obtain the following characterization of compactness.

Corollary 5.7. *Let $\mathbb{G}$ be a locally compact quantum group. Then $\mathbb{G}$ is compact if and only if $\mathcal{B}(L_2(\mathbb{G}))$ is injective in $(\mathcal{T}(L_2(\mathbb{G})), \triangleright) - \mathbf{nmod}$.*

Proof. If $\mathbb{G}$ is compact, then there is a two-sided invariant mean $m \in L_1(\mathbb{G})$, and one may define a normal morphism as in equation (12) to produce a left inverse to Δ, as defined in (11). Conversely, one may repeat the second half of the proof of Theorem 5.5 to obtain a normal conditional expectation from $\mathcal{B}(L_2(\mathbb{G}))$ onto $L_\infty(\hat{\mathbb{G}})$ mapping $L_\infty(\mathbb{G})$ into $\mathbb{C}1$. Then [17, Theorem 4.2] implies that $\hat{\mathbb{G}}$ is discrete whence $\mathbb{G}$ is compact. □

Proposition 5.8. *Let $\mathbb{G}$ be a locally compact quantum group. If $\hat{\mathbb{G}}$ is amenable, then $\mathcal{B}(L_2(\mathbb{G}))$ is injective in* $\mathbf{mod}-(\mathcal{T}(L_2(\mathbb{G})),\rhd)$.

Proof. By [19, Proposition 2.15], the unitary operator $U\otimes U := \hat{J}J\otimes\hat{J}J$ on $L_2(\mathbb{G})\otimes L_2(\mathbb{G})$ intertwines the right fundamental unitaries of $\hat{\mathbb{G}}$ and $\hat{\mathbb{G}}'$, denoted $\hat{V}$ and $\hat{V}'$, respectively. One then obtains a one-to-one correspondence between invariant means on $L_\infty(\hat{\mathbb{G}})$ and $L_\infty(\hat{\mathbb{G}}')$ via conjugation with U, making $\hat{\mathbb{G}}$ amenable if and only if $\hat{\mathbb{G}}'$ is. Thus, assuming amenability of $\hat{\mathbb{G}}$, we let $\hat{m}'$ be a two-sided invariant mean on $L_\infty(\hat{\mathbb{G}}')$. Similar to the previous theorem, we must provide a morphism which is a left inverse to the map $\Delta:\mathcal{B}(L_2(\mathbb{G}))\to\mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})),\mathcal{B}(L_2(\mathbb{G})))$ given by

$$\Delta(x)(\rho)=x\rhd\rho\quad(x\in\mathcal{B}(L_2(\mathbb{G})),\rho\in\mathcal{T}(L_2(\mathbb{G}))).\tag{13}$$

In this case, we identify $\mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})),\mathcal{B}(L_2(\mathbb{G})))\cong\mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))$ (cf. [9]) via

$$\langle\Phi,\rho\otimes\omega\rangle=\langle\Phi(\rho),\omega\rangle\quad(\Phi\in\mathcal{CB}(\mathcal{T}(L_2(\mathbb{G})),\mathcal{B}(L_2(\mathbb{G}))),\rho,\omega\in\mathcal{T}(L_2(\mathbb{G}))).$$

This ensures $\Delta=\Gamma^r$, and that the corresponding $\mathcal{T}(L_2(\mathbb{G}))$-module structure on $\mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))$ is defined by $T\trianglerighteq\rho=(\rho\otimes\iota\otimes\iota)(\Gamma^r\otimes\iota)(T)$ for $T\in\mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))$ and $\rho\in\mathcal{T}(L_2(\mathbb{G}))$.

We take a normal state $n\in\mathcal{T}(L_2(\mathbb{G}))$, and define $\Phi:\mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))\to\mathcal{B}(L_2(\mathbb{G}))$ by

$$\Phi(T)=(\iota\otimes m_n)(V^*TV)\quad(T\in\mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))),$$

where $m_n:=n\circ\Theta^r(\hat{m}')$ is a state on $\mathcal{B}(L_2(\mathbb{G}))$, and Θ^r denotes the representation of $\mathrm{LUC}(\hat{\mathbb{G}}')$. Clearly, Φ is a completely contractive left inverse to Γ^r. To show that Φ is also a module map we follow along similar lines as in Theorem 5.5. Fix $T\in\mathcal{B}(L_2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{G}))$ and $\rho\in\mathcal{T}(L_2(\mathbb{G}))$. Then

$$\begin{aligned}\Phi(T\trianglerighteq\rho)&=\Phi((\rho\otimes\iota\otimes\iota)(V_{12}T_{13}V_{12}^*))=(\rho\otimes\iota\otimes m_n)(V_{23}^*V_{12}T_{13}V_{12}^*V_{23})\\&=(\rho\otimes\iota\otimes m_n)(V_{12}V_{23}^*V_{13}^*T_{13}V_{13}V_{23}V_{12}^*)\\&=(\rho\otimes\iota)(V(\iota\otimes\iota\otimes m_n)(V_{23}^*V_{13}^*T_{13}V_{13}V_{23})V^*).\end{aligned}$$

Now, denoting π by the canonical quotient map $\mathcal{T}(L_2(\mathbb{G}))\to L_1(\hat{\mathbb{G}}')$, and using the fact that $\hat{V}'=\sigma V^*\sigma$, where σ is the flip map on $L_2(\mathbb{G})\otimes L_2(\mathbb{G})$, for any $\tau,\omega\in\mathcal{T}(L_2(\mathbb{G}))$, we have

$$\begin{aligned}&\langle(\iota\otimes\iota\otimes\hat{m'}_n)(V_{23}^*V_{13}^*T_{13}V_{13}V_{23}),\tau\otimes\omega\rangle\\&=\langle(\iota\otimes\iota\otimes m_n)(V_{23}^*(\sigma\otimes 1)V_{23}^*T_{23}V_{23}(\sigma\otimes 1)V_{23}),\tau\otimes\omega\rangle\\&=\langle(\iota\otimes\iota\otimes m_n)(V_{13}^*V_{23}^*T_{23}V_{23}V_{13}),\omega\otimes\tau\rangle\\&=\langle(\iota\otimes m_n)(V^*(1\otimes(\tau\otimes\iota)(V^*TV))V),\omega\rangle\\&=\langle(m_n\otimes\iota)(\hat{V}'((\tau\otimes\iota)(V^*TV)\otimes 1)\hat{V}'^*),\omega\rangle\\&=\langle m_n,\Theta^r(\pi(\omega))((\tau\otimes\iota)(V^*TV))\rangle\\&=\langle n,\langle\omega,1\rangle\Theta^r(\hat{m}')((\tau\otimes\iota)(V^*TV))\rangle\\&=\langle m_n\otimes\omega,(\tau\otimes\iota)(V^*TV)\otimes 1\rangle\\&=\langle(\iota\otimes m_n\otimes\iota)(V^*TV\otimes 1),\tau\otimes\omega\rangle.\end{aligned}$$

As τ and ω were arbitrary, we have

$$\begin{aligned}\Phi(T \trianglerighteq \rho) &= (\rho \otimes \iota)(V(\iota \otimes \iota \otimes m_n)(V_{23}^* V_{13}^* T_{13} V_{13} V_{23})V^*)\\ &= (\rho \otimes \iota)(V(\iota \otimes m_n \otimes \iota)(V^* T V \otimes 1)V^*)\\ &= (\rho \otimes \iota)(V(\Phi(T) \otimes 1)V^*)\\ &= \Phi(T) \triangleright \rho.\end{aligned}$$

□

Remark 5.9. We remark that the converse of Proposition 5.8 holds in the setting of Kac algebras. The proof involves machinery from representations of completely bounded multipliers over quantum groups [16], and will therefore appear in subsequent work.

Remark 5.10. Let $\mathcal{A}$ be a completely contractive Banach algebra. We say that an element $X \in \mathcal{A} - \mathbf{mod}$ is *flat* if and only if X^* is injective in $\mathbf{mod} - \mathcal{A}$. In this case, we obtain a stronger version of operator flatness than the usual definition involving completely bounded morphisms [1] (see [12] for the classical setting). If $\mathbb{G}$ is a locally compact quantum group such that $\hat{\mathbb{G}}$ is amenable, then Proposition 5.8 implies that $(\mathcal{T}(L_2(\mathbb{G})), \triangleright)$ is flat in $(\mathcal{T}(L_2(\mathbb{G})), \triangleright) - \mathbf{mod}$, a property solely in terms of the Banach algebra structure of $(\mathcal{T}(L_2(\mathbb{G})), \triangleright)$, without involving the dual quantum group. If one can deduce the existence of a bounded right approximate identity in $(\mathcal{T}(L_2(\mathbb{G})), \triangleright)$, which is true in the case of $\mathbb{G}_s$, then $\mathbb{G}$ is co-amenable by [14, Proposition 5.4]. Since, on the other hand, co-amenability of $\mathbb{G}$ implies amenability of $\hat{\mathbb{G}}$ by [2, Theorem 3.2], one would thus obtain a solution to the long-standing conjecture on the duality of amenability and co-amenability for arbitrary locally compact quantum groups. Moreover, the results in this paper suggest further approaches to this open problem.

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