

FINITE GROUP ACTIONS ON CERTAIN STABLY PROJECTIONLESS C^* -ALGEBRAS WITH THE ROHLIN PROPERTY

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Dedicated to Professor Yasuo Watatani on the occasion of his 60th birthday.

ABSTRACT. We introduce the Rohlin property and approximate representability for finite group actions on stably projectionless C^* -algebras and study their basic properties. We give some examples of finite group actions on the Razak-Jacelon algebra \mathcal{W} and show some classification results of these actions. This study is based on the work of Izumi, Robert's classification theorem and Kirchberg's central sequence C^* -algebras.

1. INTRODUCTION

A C^* -algebra A is said to be *stably projectionless* if $A \otimes \mathbb{K}$ has no non-zero projections, where \mathbb{K} is the C^* -algebra of compact operators on an infinite-dimensional separable Hilbert space. Note that every stably projectionless C^* -algebra does not have a unit. In this paper we study finite group actions on stably projectionless C^* -algebras.

In the theory of operator algebras, the study of group actions on operator algebras is one of the most fundamental subjects and has a long history. We refer the reader to [14] and the references given there for this subject. Connes considered the Rohlin property for automorphisms of von Neumann algebras and classified automorphisms of the AFD factor of type II_1 in [5] and [6]. It is important to consider the Rohlin property for classifying group actions on operator algebras. Using a suitable formulation of the Rohlin property of automorphisms of unital C^* -algebras, Kishimoto classified a large class of automorphisms on UHF algebras and certain AT algebras in [23] and [24]. Nakamura showed that every aperiodic automorphism of a Kirchberg algebra has the Rohlin property and classified them in [32]. Izumi, Katsura and Matui showed classification results of large classes of \mathbb{Z}^N -actions (see [15], [18], [28] and [29]). Izumi defined the Rohlin property for finite group actions on unital C^* -algebras (see also [21] and [11]) and classified a large class of finite group actions on unital C^* -algebras in [12] and [13].

In the above formulations of the Rohlin property, a partition of unity consisting of projections (in the central sequence C^* -algebra) is used. Hence it is not clear how to define the Rohlin property for group actions on stably projectionless C^* -algebras.

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Although Evans and Kishimoto defined the Rohlin property for automorphisms of non-unital C^* -algebras and classified trace scaling automorphisms of certain AF algebras in [10], their definition requires that C^* -algebras have non-zero projections. Sato defined the weak Rohlin property for automorphisms of unital projectionless C^* -algebras and classified a large class of automorphisms of the Jiang-Su algebra \mathcal{Z} in [44]. Matui and Sato defined the weak Rohlin property for countable amenable group actions and classified large classes of actions of \mathbb{Z}^2 and the Klein bottle group on \mathcal{Z} in [30] and [31]. We do not need a partition of unity consisting of projections in the definition of the weak Rohlin property. But the weak Rohlin property is too weak from the viewpoint of the classification of finite group actions. Moreover their arguments are based on analyses of the actions on UHF algebras with the usual Rohlin property. Therefore we need to consider a suitable formulation of the Rohlin property for actions on stably projectionless C^* -algebras.

We shall define the Rohlin property for finite group actions on σ -unital C^* -algebras by using Kirchberg's central sequence C^* -algebras defined in [19]. Kirchberg's central sequence algebra $F(A)$ of a C^* -algebra A is defined as the quotient C^* -algebra of the central sequence C^* -algebra A_∞ by the annihilator of A . Kirchberg's central sequence C^* -algebras are useful for proving \mathcal{Z} -stability results for non-unital C^* -algebras (see [33] and [46]). One of the purposes of this paper is to show that Kirchberg's central sequence C^* -algebras are also useful for the classification of group actions on certain stably projectionless C^* -algebras.

Let \mathcal{W} be the Razak-Jacelon algebra studied in [16], which is a certain simple nuclear stably projectionless C^* -algebra having trivial K -groups and a unique tracial state and no unbounded traces. The Razak-Jacelon algebra \mathcal{W} is regarded as a stably finite analogue of the Cuntz algebra \mathcal{O}_2 . Moreover Robert showed that $\mathcal{W} \otimes \mathbb{K}$ is isomorphic to $\mathcal{O}_2 \rtimes_\alpha \mathbb{R}$ for some one parameter automorphism group α in [37] (see also [26], [27] and [8]). Hence it is natural and interesting to consider whether \mathcal{W} has similar properties of \mathcal{O}_2 .

In this paper we study mainly finite group actions on \mathcal{W} . This is based on Izumi's study of finite group actions on \mathcal{O}_2 in [12]. This paper is organized as follows: In Section 2, we collect notation and some results. In Section 3, we define the Rohlin property for finite group actions on σ -unital C^* -algebras and study their basic properties. In particular, we show that any two actions of a finite group on \mathcal{W} with the Rohlin property are conjugate (Corollary 3.7). We also show that there exist strong K -theoretical constraints for the Rohlin property as in the unital case. In Section 4, we introduce approximate representability for finite group actions on σ -unital C^* -algebras, which is the "dual" notion of the Rohlin property. We show a classification result for certain approximately representable actions (Corollary 4.5). Using this classification result, we give interesting examples of finite group actions on \mathcal{W} with the Rohlin property (Corollary 4.11). In Section 5, we classify certain "locally representable" actions of \mathbb{Z}_2 on \mathcal{W} up to conjugacy (Theorem 5.2) and up to cocycle conjugacy (Theorem 5.5). We also construct a locally representable action α of \mathbb{Z}_2 on \mathcal{W} such that $K_0(\mathcal{W} \rtimes_\alpha \mathbb{Z}_2) = \mathbb{Z}[\frac{1}{2}]$ and $K_1(\mathcal{W} \rtimes_\alpha \mathbb{Z}_2) = 0$.

2. PRELIMINARIES

In this section we shall collect notation and some results. We refer the reader to [3] and [34] for basic facts of operator algebras.

2.1. Notation. We say that a C*-algebra A is σ -unital if A has a countable approximate unit. In particular, if A is σ -unital, then there exists a positive element $s \in A$ such that $\{s^{\frac{1}{n}}\}_{n \in \mathbb{N}}$ is an approximate unit. Such a positive element s is called *strictly positive* in A . If A is separable, then A is σ -unital. We denote by \tilde{A} the unitization algebra of A . The *multiplier algebra*, denoted by $M(A)$, of A is the largest unital C*-algebra that contains A as an essential ideal. If α is an automorphism of A , then α extends uniquely to an automorphism $\bar{\alpha}$ of $M(A)$. A homomorphism ϕ from A to B is said to be *non-degenerate* if $\phi(A)B$ is dense in B . A non-degenerate homomorphism from A to B can be uniquely extended to a homomorphism $\bar{\phi}$ from $M(A)$ to $M(B)$. Note that if $A \subset B$ is a non-degenerate inclusion, then every approximate unit for A is an approximate unit for B .

For a unitary element u in $M(A)$, define an automorphism $\text{Ad}(u)$ of A by $\text{Ad}(u)(a) = uau^*$ for $a \in A$. Such an automorphism is called an *inner automorphism*. Let $\text{Aut}(A)$ denote the automorphism group of A , which is equipped with the topology of pointwise norm convergence. An automorphism α is said to be *approximately inner* if α is in the closure of the inner automorphism group. We say that two automorphisms α and β are *approximately unitarily equivalent* if $\alpha \circ \beta^{-1}$ is approximately inner.

For a locally compact group G , an *action* α of G on A is a continuous homomorphism from G to $\text{Aut}(A)$. We say that α is *outer* if α_g is not an inner automorphism for any $g \in G \setminus \{\iota\}$ where ι is the identity of G . An α -*cocycle* is a continuous map u from G to the unitary group $U(M(A))$ of $M(A)$ such that $u(gh) = u(g)\alpha_g(u(h))$ for any $g, h \in G$. If u is an α -cocycle, we define the action α^u of G on A by $\alpha_g^u(a) = \text{Ad}(u(g)) \circ \alpha_g(a)$ for any $a \in A$ and $g \in G$. For two G -actions α on A and β on B , we say that α and β are *conjugate*, written $(A, \alpha) \cong (B, \beta)$, if there exists an isomorphism θ from A onto B such that $\theta \circ \alpha_g = \beta_g \circ \theta$ for any $g \in G$. They are said to be *cocycle conjugate* if there exists an α -cocycle u such that α^u is conjugate to β .

Let $T(A)$ be the set of densely defined lower semicontinuous traces on A and $T_1(A)$ the set of tracial states on A , and put $T_0(A) := \{\tau \in T(A) \mid \|\tau\| \leq 1\}$. If A is unital, then $T_1(A)$ is a Choquet simplex (see [43, Theorem 3.1.18]). There exists a natural one-to-one correspondence between $T_0(A)$ and $T_1(\tilde{A})$, and hence $T_0(A)$ is a Choquet simplex. Every tracial state τ on A extends uniquely to a tracial state $\bar{\tau}$ on $M(A)$ (see, for example, [46, Proposition 2.9]). For $\tau \in T_1(A)$, consider the Gelfand-Naimark-Segal (GNS) construction (π_τ, H_τ, ξ) . Then τ extends uniquely to a normal tracial state $\tilde{\tau}$ on $\pi_\tau(A)''$. Note that if τ is an extremal tracial state, then $\pi_\tau(A)''$ is a finite factor. Let α be an automorphism of A such that $\tau \circ \alpha = \tau$. Then α extends uniquely to an automorphism $\tilde{\alpha}$ of $\pi_\tau(A)''$. Moreover if α is an action of G on A such that $\tau \circ \alpha_g = \tau$ for any $g \in G$, then α extends uniquely to a von Neumann algebraic action $\tilde{\alpha}$ on $\pi_\tau(A)''$. We say that an action α of G on a C*-algebra A with a unique tracial state τ is *strongly outer* if $\tilde{\alpha}_g$ is not inner in $\pi_\tau(A)''$ for any $g \in G \setminus \{\iota\}$.

We denote by $K(H)$, \mathbb{K} and M_{n^∞} for $n \in \mathbb{N}$ the C*-algebra of compact operators on a Hilbert space H , the C*-algebra of compact operators on an infinite-dimensional separable Hilbert space and the uniformly hyperfinite (UHF) algebra of type n^∞ , respectively. Let \mathbb{Z}_n denote the cyclic group of order n . For a \mathbb{Z}_n -action α , α_1 is denoted by α for simplicity. If G is a finite group, we denote by $|G|$ the order of G .

2.2. Crossed products and fixed point algebras. For an action α of G on A , we denote by $A \rtimes_\alpha G$ and A^α the reduced crossed product C^* -algebra and the fixed point algebra, respectively. If G is a compact group, then A^α is isomorphic to a corner algebra of $A \rtimes_\alpha G$ (see [41]). Indeed, define a projection e_α in $M(A \rtimes_\alpha G)$ by

$$e_\alpha := \int_G \lambda_g dg,$$

where λ_g is the implementing unitary of α_g in $M(A \rtimes_\alpha G)$ and dg stands for the normalized Haar measure. Then A^α is isomorphic to $e_\alpha(A \rtimes_\alpha G)e_\alpha$. Note that we have $M(A^\alpha) = M(A)^{\hat{\alpha}}$.

When G is a locally compact abelian group, let $\hat{\alpha}$ denote the dual action of α on $A \rtimes_\alpha G$. The Takesaki-Takai duality theorem [45] shows that there exists an isomorphism Φ from $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}$ onto $A \otimes K(\ell^2(G))$ such that $\Phi \circ \hat{\alpha} = (\alpha \otimes \text{Ad}(\rho)) \circ \Phi$ where ρ is the right regular representation of G . If G is a finite abelian group, then we see that $\bar{\Phi}(e_\alpha) = 1_{M(A)} \otimes e$ where $e \in K(\ell^2(G))$ is the projection onto the constant functions. We may regard $\Phi(a)$ as a diagonal matrix

$$\begin{pmatrix} a & & & \\ & \alpha_{g_1}(a) & & \\ & & \ddots & \\ & & & \alpha_{g_{|G|-1}}(a) \end{pmatrix},$$

where $a \in A$ and $G = \{1, g_1, \dots, g_{|G|-1}\}$.

Assume that τ is a tracial state on A such that $\tau \circ \alpha_g = \tau$ for any $g \in G$. Then we can construct a von Neumann algebraic crossed product $W^*(\pi_\tau(A)'', \tilde{\alpha}, G)$. If A is simple, then we may consider the inclusions

$$A \rtimes_\alpha G \subset M(A \rtimes_\alpha G) \subset W^*(\pi_\tau(A)'', \tilde{\alpha}, G).$$

Notice that $A \rtimes_\alpha G$ is weakly dense in $W^*(\pi_\tau(A)'', \tilde{\alpha}, G)$ because π_τ is a faithful representation of A . The following proposition is well known (see, for example, [1]).

Proposition 2.1. *Let A be a simple C^* -algebra with a unique tracial state τ , and let α be a strongly outer action of a discrete group G on A . Then $A \rtimes_\alpha G$ has a unique tracial state $E \circ \tau$ where E is the canonical conditional expectation from $A \rtimes_\alpha G$ onto A .*

Hiroki Matui told us the following lemmas.

Lemma 2.2. *Let $A \subset B$ be a non-degenerate inclusion of separable C^* -algebras. If $T_1(A)$ is compact, then $T_1(B)$ is compact.*

Proof. On the contrary, suppose that $T_1(B)$ were not compact. Then there exist a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ in $T_1(B)$ and an element τ in $T_0(B)$ such that τ_n converges to τ (in the weak- $*$ topology) and $\|\tau\| < 1$. Since $T_1(A)$ is compact, the restriction of τ on A is a tracial state on A . This is a contradiction because $A \subset B$ is a non-degenerate inclusion. □

Lemma 2.3. *Let B be a C^* -algebra, and let β be an action of a finite group G on B . Assume that $T_1(B)$ is compact and there exists a unique tracial state τ on B such that $\tau \circ \beta_g = \tau$ for any $g \in G$. Then B has at most $|G|$ extremal tracial states.*

Proof. By the Krein-Milman theorem, $T_1(B)$ is the closed convex hull of its extreme points $\text{ex}(T_1(B))$. Let τ_1 be an extremal tracial state on B . Then $\tau_1 \circ \beta_g$ is an extremal tracial state for any $g \in G$. Since τ is the unique tracial state on B such that $\tau \circ \beta_g = \tau$ for any $g \in G$, we have $\tau = \frac{1}{|G|} \sum_{g \in G} \tau_1 \circ \beta_g$. On the contrary, suppose that $\text{ex}(T_1(B))$ contained at least $|G| + 1$ points. Then there exists an extremal tracial τ_2 such that $\tau_2 \neq \tau_1 \circ \beta_g$ for any $g \in G$, and we have $\tau = \frac{1}{|G|} \sum_{g \in G} \tau_2 \circ \beta_g$. This is a contradiction because $T_0(B)$ is a Choquet simplex. \square

Proposition 2.4. *Let A be a separable C*-algebra with a unique tracial state τ , and let α be an action of a finite abelian group on A . Then $A \rtimes_{\alpha} G$ has at most $|G|$ extremal tracial states.*

Proof. Lemma 2.2 shows that $T_1(A \rtimes_{\alpha} G)$ is compact. Since $A \rtimes_{\alpha} G \rtimes_{\hat{\alpha}} \hat{G}$ is isomorphic to $A \otimes M_{|G|}(\mathbb{C})$, $A \rtimes_{\alpha} G \rtimes_{\hat{\alpha}} \hat{G}$ has a unique tracial state. Hence we see that there exists a single point in $T_1(A \rtimes_{\alpha} G)$ that is fixed by $\hat{\alpha}$. Therefore $A \rtimes_{\alpha} G$ has at most $|G|$ extremal tracial states by Lemma 2.3. \square

Proposition 2.5. *Let A be a simple separable C*-algebra with a unique tracial state τ , and let α be an action of \mathbb{Z}_2 on A . Then $A \rtimes_{\alpha} \mathbb{Z}_2$ has exactly two extremal tracial states if and only if there exists a unitary element U_{α} in $\pi_{\tau}(A)''$ such that $\tilde{\alpha} = \text{Ad}(U_{\alpha})$ and $U_{\alpha}^2 = 1$.*

Proof. Assume that there exists a unitary element U_{α} in $\pi_{\tau}(A)''$ such that $\tilde{\alpha} = \text{Ad}(U_{\alpha})$ and $U_{\alpha}^2 = 1$. Then $W^*(\pi_{\tau}(A)'', \tilde{\alpha}, \mathbb{Z}_2)$ is isomorphic to $\pi_{\tau}(A)'' \oplus \pi_{\tau}(A)''$. Hence we see that $W^*(\pi_{\tau}(A)'', \tilde{\alpha}, \mathbb{Z}_2)$ has two extremal normal tracial states σ_1 and σ_2 such that $\sigma_1(m_0 + m_1\lambda) = \tilde{\tau}(m_0) + \tilde{\tau}(m_1U_{\alpha})$ and $\sigma_2(m_0 + m_1\lambda) = \tilde{\tau}(m_0) - \tilde{\tau}(m_1U_{\alpha})$, where λ is the implementing unitary of α and $m_0, m_1 \in \pi_{\tau}(A)''$. Since $A \rtimes_{\alpha} \mathbb{Z}_2$ is weakly dense in $W^*(\pi_{\tau}(A)'', \tilde{\alpha}, \mathbb{Z}_2)$ and $\pi_{\tau}(A)''$ is a factor, we see that the restrictions of σ_1 and σ_2 on $A \rtimes_{\alpha} \mathbb{Z}_2$ are distinct extremal tracial states on $A \rtimes_{\alpha} \mathbb{Z}_2$. By the proposition above, $A \rtimes_{\alpha} \mathbb{Z}_2$ has exactly two extremal tracial states.

Conversely, assume that $A \rtimes_{\alpha} \mathbb{Z}_2$ has exactly two extremal tracial states. Then there exists a unitary element V in $\pi_{\tau}(A)''$ such that $\tilde{\alpha} = \text{Ad}(V)$ by Proposition 2.1. Since $\pi_{\tau}(A)''$ is a factor, there exists a real number t such that $V^2 = e^{2\pi it}1$. Put $U_{\alpha} := e^{-\pi it}V$. Then U_{α} has the desired property. \square

Remark 2.6. Choose a unitary element U_{α} in $\pi_{\tau}(A)''$ such that $\tilde{\alpha} = \text{Ad}(U_{\alpha})$ and $U_{\alpha}^2 = 1$.

(i) Let V be a unitary element in $\pi_{\tau}(A)''$ such that $\tilde{\alpha} = \text{Ad}(V)$. Then there exists a real number t such that $V = e^{2\pi it}U_{\alpha}$ because $\pi_{\tau}(A)''$ is a factor. Hence the value $|\tilde{\tau}(V)|$ does not depend on the choice of such a unitary element V and is determined by α . We have $|\tilde{\tau}(V)| \in [0, 1]$ and $|\tilde{\tau}(V)| = 1$ if and only if $\tilde{\alpha} = \text{id}$. Hence if α is an outer action, then $|\tilde{\tau}(V)| \in [0, 1)$.

(ii) The proof above shows that two extremal tracial states ω_1 and ω_2 on $A \rtimes_{\alpha} \mathbb{Z}_2$ are given by the restriction of σ_1 and σ_2 . Therefore we have

$$\bar{\omega}_1(x_0 + x_1\lambda) = \bar{\tau}(x_0) + \tilde{\tau}(x_1U_{\alpha}) \quad \text{and} \quad \bar{\omega}_2(x_0 + x_1\lambda) = \bar{\tau}(x_0) - \tilde{\tau}(x_1U_{\alpha}),$$

where λ is the implementing unitary of α and $x_0, x_1 \in M(A)$. Note that if W is a unitary element in $\pi_{\tau}(A)''$ such that $\tilde{\alpha} = \text{Ad}(W)$ and $W^2 = 1$, then $W = U_{\alpha}$ or $W = -U_{\alpha}$.

We shall define a conjugacy invariant for outer actions of \mathbb{Z}_2 on a simple C^* -algebra with a unique tracial state. By Proposition 2.5 and Remark 2.6, we see that the following definition is well defined.

Definition 2.7. Let α be an outer action of \mathbb{Z}_2 on a simple C^* -algebra A with a unique tracial state τ . Define an invariant $\epsilon(\alpha) \in [0, 1]$ of α by $\epsilon(\alpha) = 1$ if $A \rtimes_\alpha \mathbb{Z}_2$ has a unique tracial state and $\epsilon(\alpha) = |\tilde{\tau}(V)|$ if $A \rtimes_\alpha \mathbb{Z}_2$ has exactly two extremal tracial states, where V is a unitary element in $\pi_\tau(A)''$ such that $\tilde{\alpha} = \text{Ad}(V)$.

2.3. Kirchberg’s central sequence C^* -algebras. We shall recall some properties of Kirchberg’s central sequence C^* -algebras in [19]. See also [33, Section 5]. For a σ -unital C^* -algebra A , set

$$c_0(A) := \{(a_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n \rightarrow \infty} \|a_n\| = 0\}, \quad A^\infty := \ell^\infty(\mathbb{N}, A)/c_0(A).$$

Let B be a C^* -subalgebra of A . We identify A and B with the C^* -subalgebras of A^∞ consisting of equivalence classes of constant sequences. Put

$$A_\infty := A^\infty \cap A', \quad \text{Ann}(B, A^\infty) := \{(a_n)_n \in A^\infty \cap B' \mid (a_n)_n b = 0 \text{ for any } b \in B\}.$$

Then $\text{Ann}(B, A^\infty)$ is a closed two-sided ideal of $A^\infty \cap B'$, and define

$$F(A) := A_\infty / \text{Ann}(A, A^\infty).$$

We call $F(A)$ the *central sequence C^* -algebra* of A . A sequence $(a_n)_n$ is said to be *central* if $\lim_{n \rightarrow \infty} \|a_n a - a a_n\| = 0$ for all $a \in A$. A central sequence is a representative of an element in A_∞ . Since A is σ -unital, A has a countable approximate unit $\{h_n\}_{n \in \mathbb{N}}$. It is easy to see that $[(h_n)_n]$ is a unit in $F(A)$. If A is unital, then $F(A) = A_\infty$. Note that $F(A)$ is isomorphic to $M(A)^\infty \cap A' / \text{Ann}(A, M(A)^\infty)$ and $F(A \otimes \mathbb{K})$. If α is an automorphism of A , α induces natural automorphisms of A^∞ , A_∞ and $F(A)$. We denote them by the same symbol α for simplicity. Note that if α is an inner automorphism of A , then the induced automorphism of α on $F(A)$ is the identity map.

2.4. Strict comparison and almost stable rank one. For positive elements $a, b \in A$, we say that a is *Cuntz smaller than* b , written $a \precsim b$, if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of A such that $\|x_n^* b x_n - a\| \rightarrow 0$. Positive elements a and b are said to be *Cuntz equivalent*, written $a \sim b$, if $a \precsim b$ and $b \precsim a$. For $\tau \in T(A)$, put $d_\tau(h) = \lim_{n \rightarrow \infty} \tau \otimes \text{Tr}(h^{\frac{1}{n}})$ for $h \in (A \otimes \mathbb{K})_+$. Then d_τ is a dimension function. In this paper we say that A has *strict comparison* if $a, b \in (A \otimes \mathbb{K})_+$ with $d_\tau(a) < d_\tau(b) < \infty$ for any $\tau \in T(A)$, then $a \precsim b$. A C^* -algebra A is said to have *almost stable rank one* if for every σ -unital hereditary subalgebra $B \subseteq A \otimes \mathbb{K}$ we have $B \subseteq \overline{\text{GL}(\widetilde{B})}$. If A is unital, then A has stable rank one if and only if A has almost stable rank one. We have the following proposition. See, for example, [33, Corollary 3.3 and Proposition 3.4].

Proposition 2.8. *Let A be a simple exact σ -unital stably projectionless C^* -algebra, and let a and b be positive elements in A . Assume that A has strict comparison and almost stable rank one. If $d_\tau(a) = d_\tau(b) < \infty$ for any $\tau \in T(A)$, then \overline{aA} is isomorphic to \overline{bA} as right Hilbert A -modules.*

We shall consider the comparison theory for projections in multiplier algebras of certain stably projectionless C^* -algebras.

Proposition 2.9. *Let A be a simple exact σ -unital stably projectionless C*-algebra, and let p and q be projections in $M(A)$. Assume that A has strict comparison and almost stable rank one. If $\bar{\tau}(p) = \bar{\tau}(q) < \infty$ for any $\tau \in T(A)$, then p is Murray-von Neumann equivalent to q in $M(A)$.*

Proof. Consider right Hilbert A -modules pA and qA . Since pAp and qAq are σ -unital, there exist positive elements a and b in A such that $\overline{aA} = pA$ and $\overline{bA} = qA$. Note that $\{a^{\frac{1}{n}}\}_{n \in \mathbb{N}}$ and $\{b^{\frac{1}{n}}\}_{n \in \mathbb{N}}$ are approximate units for pAp and qAq , respectively. Hence we have $\bar{\tau}(p) = d_\tau(a)$ and $\bar{\tau}(q) = d_\tau(b)$ for any $\tau \in T(A)$. By Proposition 2.8, we see that pA is isomorphic to qA as right Hilbert A -modules. Therefore p is Murray-von Neumann equivalent to q in $M(A)$. □

We denote by \mathcal{Z} the Jiang-Su algebra constructed in [17]. The Jiang-Su algebra \mathcal{Z} is a unital separable simple infinite-dimensional nuclear C*-algebra whose K -theoretic invariant is isomorphic to that of complex numbers. A C*-algebra A is said to be \mathcal{Z} -stable if A is isomorphic to $A \otimes \mathcal{Z}$. If A is a simple exact \mathcal{Z} -stable C*-algebra with traces, then A has strict comparison (see [40, Theorem 4.5]). Robert showed that if A is a simple σ -unital \mathcal{Z} -stable stably projectionless C*-algebra, then A has almost stable rank one (see [38]). Therefore we have the following corollary.

Corollary 2.10. *Let A be a simple exact σ -unital stably projectionless \mathcal{Z} -stable C*-algebra, and let p and q be projections in $M(A)$. If $\bar{\tau}(p) = \bar{\tau}(q) < \infty$ for any $\tau \in T(A)$, then p is Murray-von Neumann equivalent to q in $M(A)$.*

2.5. Razak-Jacelon algebra and Robert’s classification theorem. Let \mathcal{W} be the Razak-Jacelon algebra studied in [16], which has trivial K -groups and a unique tracial state and no unbounded traces. The Razak-Jacelon algebra \mathcal{W} is constructed as an inductive limit C*-algebra of Razak’s building block in [36], that is,

$$A(n, m) = \left\{ f \in C([0, 1]) \otimes M_m(\mathbb{C}) \mid f(0) = \text{diag}(\overbrace{c, \dots, c}^k, 0_n), f(1) = \text{diag}(\overbrace{c, \dots, c}^{k+1}), \right. \\ \left. c \in M_n(\mathbb{C}) \right\},$$

where n and m are natural numbers with $n|m$ and $k := \frac{m}{n} - 1$. Let \mathcal{O}_2 denote the Cuntz algebra generated by 2 isometries S_1 and S_2 . For every $\lambda_1, \lambda_2 \in \mathbb{R}$ there exists by universality a one-parameter automorphism group α of \mathcal{O}_2 given by $\alpha_t(S_j) = e^{it\lambda_j} S_j$. Kishimoto and Kumjian in [26] and [27] showed that if λ_1 and λ_2 are all non-zero, of the same sign and λ_1 and λ_2 generate \mathbb{R} as a closed subgroup, then $\mathcal{O}_2 \rtimes_\alpha \mathbb{R}$ is a simple stably projectionless C*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace. Moreover Robert (see [37]) showed that $\mathcal{W} \otimes \mathbb{K}$ is isomorphic to $\mathcal{O}_2 \rtimes_\alpha \mathbb{R}$ for some λ_1 and λ_2 . (See also [8].)

We say that A is a *1-dimensional non-commutative CW (NCCW) complex* if A is a pullback C*-algebra of the form

$$\begin{array}{ccc} A & \xrightarrow{\pi_2} & E \\ \downarrow \pi_1 & & \downarrow \rho \\ C([0, 1]) \otimes F & \xrightarrow{\delta_0 \oplus \delta_1} & F \oplus F \end{array},$$

where E and F are finite-dimensional C*-algebras and δ_i is the evaluation map at i . Razak’s building block $A(n, m)$ is a 1-dimensional NCCW complex. Robert classified inductive limit C*-algebras of 1-dimensional NCCW complexes with trivial

K_1 -groups (and C^* -algebras that are stably isomorphic to such inductive limits) in [37]. In particular, if A is a simple stably projectionless C^* -algebra, then the classification invariant of A is the 3-tuple $(K_0(A), (T(A), T_1(A)), r_A)$ where r_A is the pairing between $K_0(A)$ and $T(A)$ (see [37, Proposition 6.2.3 and Corollary 6.2.4]). For a homomorphism ϕ from A to B , we denote by $K_0(\phi)$ and $T(\phi)$ the induced homomorphism from $K_0(A)$ to $K_0(B)$ and the induced map from $T(B)$ to $T(A)$, respectively. The following theorem is an immediate consequence of Robert’s classification theorem (see [37, Theorem 1.0.1, Proposition 3.1.7, Proposition 6.2.3 and Corollary 6.2.4]).

Theorem 2.11 (Robert). *Let A be a simple stably projectionless C^* -algebra that is stably isomorphic to an inductive limit C^* -algebra of 1-dimensional NCCW complexes with trivial K_1 -groups.*

(i) *If α and β are automorphisms of A , then α is approximately unitarily equivalent to β if and only if $K_0(\alpha) = K_0(\beta)$ and $T(\alpha) = T(\beta)$.*

(ii) *Let φ be an automorphism of the countable abelian group $K_0(A)$ and γ an automorphism of the topological cone $T(A)$ with $\gamma(T_1(A)) = T_1(A)$. If φ and γ are compatible with the pairing r_A , then there exists an automorphism α of A such that $K_0(\alpha) = \varphi$ and $T(\alpha) = \gamma$.*

Robert’s classification theorem shows that if A is a simple approximately finite-dimensional (AF) algebra with a unique tracial state and no unbounded traces, then $A \otimes \mathcal{W}$ is isomorphic to \mathcal{W} . In general, Robert conjectured that if A and B are separable nuclear simple C^* -algebras, then $T(A)$ is isomorphic to $T(B)$ as non-cancellative topological cones if and only if $A \otimes \mathcal{W} \otimes \mathbb{K}$ is isomorphic to $B \otimes \mathcal{W} \otimes \mathbb{K}$ (see [42]).

Note that every simple inductive limit C^* -algebra of 1-dimensional NCCW complexes has strict comparison and stable rank one. Moreover these C^* -algebras are \mathcal{Z} -stable by [46, Corollary 9.2].

3. THE ROHLIN PROPERTY

In this section we introduce the Rohlin property for finite group actions on σ -unital C^* -algebras and study some basic properties of finite group actions with the Rohlin property.

Definition 3.1. An action α of a finite group G on a σ -unital C^* -algebra A is said to have the *Rohlin property* if there exists a partition of unity $\{e_g\}_{g \in G} \subset F(A)$ consisting of projections satisfying

$$\alpha_g(e_h) = e_{gh},$$

for any $g, h \in G$. A family of projections $\{e_g\}_{g \in G}$ is called *Rohlin projections* of α .

If A is unital, then the definition above coincides with the definition in [12].

Example 3.2. Let G be a finite group, and let μ^G be the action of G on $M_{|G| \infty}$ given in [12, Example 3.2], that is,

$$\mu_g^G = \bigotimes_{n=1}^{\infty} \text{Ad}(\lambda_g)$$

for any $g \in G$, where λ_g is the left regular representation of G and we identify $B(\ell^2(G))$ with $M_{|G|}(\mathbb{C})$. Define an action ν^G of G on $M_{|G|^\infty} \otimes \mathcal{W} \cong \mathcal{W}$ by

$$\nu^G := \mu^G \otimes \text{id}.$$

Then ν^G has the Rohlin property. Indeed, let $\{h_n\}_{n \in \mathbb{N}}$ be an approximate unit for \mathcal{W} . Since μ^G has the Rohlin property (see [12]), there exist Rohlin projections $\{(p_{g,n})_n\}_{g \in G}$ of μ^G in $(M_{|G|^\infty})_\infty$. Put $e_g := [(p_{g,n} \otimes h_n)_n] \in F(M_{|G|^\infty} \otimes \mathcal{W})$ for any $g \in G$. Then we see that $\{e_g\}_{g \in G}$ are Rohlin projections of ν^G .

The example above shows that for any finite group G , there exists an action of G on \mathcal{W} with the Rohlin property. Moreover we see that $\mathcal{W} \rtimes_{\nu^G} G$ is isomorphic to \mathcal{W} because $M_{|G|^\infty} \rtimes_{\mu^G} G$ is a simple unital AF algebra with a unique tracial state.

If G is a finite abelian group, then we have the following characterization of the Rohlin property.

Proposition 3.3. *Let α be an action of a finite abelian group G on a σ -unital C*-algebra A . Then α has the Rohlin property if and only if there exists a unitary representation u of \hat{G} on $F(A)$ such that*

$$\alpha_g(u(\gamma)) = \gamma(g)u(\gamma),$$

for any $g \in G$ and $\gamma \in \hat{G}$.

Proof. Assume that α has the Rohlin property, and choose Rohlin projections $\{e_g\}_{g \in G}$ of α . Define a map u from \hat{G} to $F(A)$ by

$$u(\gamma) := \sum_{g \in G} \overline{\gamma(g)} e_g$$

for any $\gamma \in \hat{G}$. Then u is a unitary representation such that $\alpha_g(u(\gamma)) = \gamma(g)u(\gamma)$ for any $g \in G$ and $\gamma \in \hat{G}$.

Conversely, assume that there exists a unitary representation u of \hat{G} on $F(A)$ such that $\alpha_g(u(\gamma)) = \gamma(g)u(\gamma)$ for any $g \in G$ and $\gamma \in \hat{G}$. For any $g \in G$, put

$$e_g := \frac{1}{|G|} \sum_{\gamma \in \hat{G}} \gamma(g)u(\gamma).$$

Then $\{e_g\}_{g \in G}$ are Rohlin projections of α . □

The following lemma is an analogous result of [12, Lemma 3.3] in our case.

Lemma 3.4. *Let A be a σ -unital C*-algebra with $A \subset \overline{\text{GL}(\tilde{A})}$, and let α and β be actions of a finite group G on A with the Rohlin property. Assume that α_g and β_g are approximately unitarily equivalent for any $g \in G$. Then for every finite set $F \subset A$ and positive number ϵ , there exists a unitary element v in \tilde{A} such that the following hold for any $x \in F$:*

$$\begin{aligned} \|\beta_g(x) - \text{Ad}(v^*) \circ \alpha_g \circ \text{Ad}(v)(x)\| &< \epsilon, \quad g \in G, \\ \|[v, x]\| &< \epsilon + \sup_{g \in G} \|\beta_g(x) - \alpha_g(x)\|. \end{aligned}$$

Proof. Put

$$F_1 := \bigcup_{g \in G} \beta_g(F).$$

Then there exist unitary elements $\{v_g\}_{g \in G}$ in $M(A)$ such that

$$\|v_g \beta_g(y) v_g^* - \alpha_g(y)\| < \frac{\epsilon}{2}$$

for any $y \in F_1$. We choose Rohlin projections $\{e_g\}_{g \in G}$ of α . Then there exist positive contractions $\{f_g\}_{g \in G}$ in A_∞ such that $[f_g] = e_g$ for any $g \in G$. Note that we have $f_g f_h a = 0$ if $g \neq h$, $f_g^2 a = f_g a$ and $\alpha_g(f_h) a = f_{gh} a$ for any $a \in A$. Set

$$w := \sum_{g \in G} v_g f_g \in A^\infty.$$

For any $a \in A$, we have

$$\begin{aligned} w^* w a &= \sum_{g \in G} \sum_{h \in G} f_g v_g^* v_h f_h a = \sum_{g \in G} \sum_{h \in G} f_g v_g^* v_h a f_h \\ &= \sum_{g \in G} \sum_{h \in G} v_g^* v_h a f_g f_h = \sum_{g \in G} a f_g = a. \end{aligned}$$

Since $A \subset \overline{\text{GL}(\tilde{A})}$, there exists a bounded sequence $(z_n)_n$ consisting of invertible elements in \tilde{A} such that $w = (z_n)_n$ in $(\tilde{A})^\infty$. Put $u_n := z_n (z_n^* z_n)^{-\frac{1}{2}}$ and $u := (u_n)_n \in (\tilde{A})^\infty$. Then u_n is a unitary element in \tilde{A} and we have

$$\|(z_n - u_n) a\| = \|u_n ((z_n^* z_n)^{\frac{1}{2}} a - a)\| = \|(z_n^* z_n)^{\frac{1}{2}} a - a\| \rightarrow 0,$$

for any $a \in A$. Therefore we have $wa = ua$ in A^∞ . Note that we may not have $aw = au$. For $x \in F$, we have

$$\begin{aligned} \|\beta_g(x) - \text{Ad}(u^*) \circ \alpha_g \circ \text{Ad}(u)(x)\| &= \|u \beta_g(x) u^* - \alpha_g(uxu^*)\| \\ &= \|w \beta_g(x) w^* - \alpha_g(wxw^*)\| \\ &= \left\| \sum_{h \in G} v_h \beta_g(x) f_h v_h^* - \alpha_g \left(\sum_{h \in G} v_h x f_h v_h^* \right) \right\| \\ &= \left\| \sum_{h \in G} f_h v_h \beta_g(x) v_h^* - \alpha_g \left(\sum_{h \in G} f_h v_h x v_h^* \right) \right\| \\ &= \left\| \sum_{h \in G} f_{gh} v_{gh} \beta_g(x) v_{gh}^* - \sum_{h \in G} f_{gh} \alpha_g(v_h x v_h^*) \right\| \\ &= \left\| \sum_{h \in G} f_{gh} (v_{gh} \beta_g(x) v_{gh}^* - \alpha_g(v_h x v_h^*)) \right\|. \end{aligned}$$

Since $f_h f_k a = 0$ for any $a \in A$ if $h \neq k$, for $x \in F$, we have

$$\begin{aligned} &\|\beta_g(x) - \text{Ad}(u^*) \circ \alpha_g \circ \text{Ad}(u)(x)\| \\ &\leq \sup_{h \in G} \{ \|v_{gh} \beta_g(x) v_{gh}^* - \alpha_g(v_h x v_h^*)\| \} \\ &\leq \sup_{h \in G} \{ \|v_{gh} \beta_{gh}(\beta_{h^{-1}}(x)) v_{gh}^* - \alpha_{gh}(\beta_{h^{-1}}(x))\| + \|\alpha_{gh}(\beta_{h^{-1}}(x)) - \alpha_g(v_h x v_h^*)\| \} \\ &\leq \frac{\epsilon}{2} + \sup_{h \in G} \{ \|\alpha_h(\beta_{h^{-1}}(x)) - v_h \beta_h(\beta_{h^{-1}}(x)) v_h^*\| \} \\ &< \epsilon. \end{aligned}$$

For $x \in F$, we have

$$\begin{aligned} \| [u, x] \| &= \| u x u^* - x \| = \| w x w^* - x \| \\ &= \left\| \sum_{g \in G} f_g (v_g x v_g^* - x) \right\| \\ &\leq \sup_{g \in G} \{ \| v_g \beta_g (\beta_{g^{-1}}(x)) v_g^* - \alpha_g (\beta_{g^{-1}}(x)) \| + \| \alpha_g (\beta_{g^{-1}}(x)) - x \| \} \\ &< \epsilon + \sup_{g \in G} \| \beta_g(x) - \alpha_g(x) \|. \end{aligned}$$

Set $v := u_n$ for sufficiently large n . Then we obtain the conclusion of the lemma. \square

We can show the following theorem by using Lemma 3.4 and the Bratteli-Elliott-Evans-Kishimoto intertwining argument (see [10]). Indeed, the same proof as [12, Theorem 3.5] works by using Lemma 3.4 instead of [12, Lemma 3.3].

Theorem 3.5. *Let A be a separable C*-algebra with $A \subset \overline{\text{GL}(\tilde{A})}$, and let α and β be actions of a finite group G on A with the Rohlin property. Assume that α_g and β_g are approximately unitarily equivalent for any $g \in G$. Then there exists an approximately inner automorphism θ of A such that*

$$\theta \circ \alpha_g \circ \theta^{-1} = \beta_g, \quad g \in G.$$

Remark 3.6. Izumi showed such a statement for unital separable C*-algebras A without assuming that A has stable rank one. We do not know if the theorem above holds without assuming that $A \subset \overline{\text{GL}(\tilde{A})}$. Note that every simple separable \mathcal{Z} -stable stably projectionless C*-algebra has almost stable rank one (see Section 2.4).

Corollary 3.7. *Let α be an action of a finite group G on \mathcal{W} with the Rohlin property. Then α is conjugate to ν^G in Example 3.2.*

Proof. By Razak’s classification theorem [36], every automorphism of \mathcal{W} is approximately inner. Therefore Theorem 3.5 implies the conclusion of the corollary. \square

We shall show that there exists an action α of \mathbb{Z}_2 on \mathcal{W} , which does not have the Rohlin property. This action is locally representable (see Definition 4.7) and will be classified in Section 5.

Example 3.8. Let β be an action of \mathbb{Z}_2 on M_{2^∞} defined by

$$\beta = \bigotimes_{n=1}^\infty \text{Ad}(1_{2^{n-1}} \oplus (-1)) \quad \text{on} \quad \bigotimes_{n=1}^\infty M_{2^n}(\mathbb{C}).$$

Define a \mathbb{Z}_2 -action α on $M_{2^\infty} \otimes \mathcal{W} \cong \mathcal{W}$ by

$$\alpha := \beta \otimes \text{id}.$$

Then it is easily seen that β is an outer action on M_{2^∞} but there exists a unitary element U in the II_1 factor $\pi_\tau(M_{2^\infty})''$ such that $\tilde{\beta} = \text{Ad}(U)$ and $\tilde{\tau}(U) = 0$, where τ is the unique tracial state on M_{2^∞} . Hence we have $\tilde{\alpha} = \text{Ad}(U \otimes 1)$, and $\mathcal{W} \rtimes_\alpha \mathbb{Z}_2$ has exactly two extremal tracial states by Proposition 2.5. (See also [35, Example 2.9].) Therefore α is not cocycle conjugate to $\nu^{\mathbb{Z}_2}$ because $\mathcal{W} \rtimes_\alpha \mathbb{Z}_2$ is not isomorphic to \mathcal{W} . Corollary 3.7 implies that α does not have the Rohlin property. Note that the invariant $\epsilon(\alpha)$ defined in Definition 2.7 is equal to 0.

Let α be an action of a finite group G on a C^* -algebra A . Then the K -groups of A have G -module structures induced by α . We denote by $(K_0(A), \alpha)$ and $(K_1(A), \alpha)$ such G -modules. Izumi showed that there exist strong K -theoretical constraints for the Rohlin property in the unital case (see [12, Theorem 3.13] and [13, Theorem 3.3]). We do not use projections and unitary elements in \tilde{A} for the definition of the Rohlin property. Nevertheless, we can show that there exist the same K -theoretical constraints by the following proposition.

Proposition 3.9. *Let A be a separable simple C^* -algebra, and let α be an action of a finite group G on A with the Rohlin property. Then there exists a unital separable C^* -algebra B , and an action β of G on B with the Rohlin property such that $(K_i(A), \alpha)$ is isomorphic to $(K_i(B), \beta)$ as G -modules for $i = 0, 1$.*

Proof. Let \mathcal{O}_∞ be the Cuntz algebra generated by an infinite sequence of isometries. Then $A \otimes \mathcal{O}_\infty$ is purely infinite (see, for example, [39, Theorem 4.1.10]). Since $\alpha \otimes \text{id}$ is outer, $(A \otimes \mathcal{O}_\infty) \rtimes_{\alpha \otimes \text{id}} G$ is simple and purely infinite by [27, Lemma 10]. Therefore we see that $(A \otimes \mathcal{O}_\infty)^{\alpha \otimes \text{id}}$ is purely infinite, and hence $(A \otimes \mathcal{O}_\infty)^{\alpha \otimes \text{id}}$ has a non-zero projection p . Put $B := p(A \otimes \mathcal{O}_\infty)p$ and $\beta := (\alpha \otimes \text{id})|_{p(A \otimes \mathcal{O}_\infty)p}$. Choose Rohlin projections $\{e_g = [(f_{n,g})_n]\}_{g \in G}$ of α in $F(A)$ where $(f_{n,g})_n \in A_\infty$. Then $\{((f_{n,g} \otimes 1)p)_n\}_{g \in G}$ are Rohlin projections of β , and hence β has the Rohlin property. Brown’s theorem (see [4]) shows that the inclusion map ι from $p(A \otimes \mathcal{O}_\infty)p$ into $A \otimes \mathcal{O}_\infty$ induces isomorphisms of K -groups. Since we have $\iota \circ \beta = (\alpha \otimes \text{id}) \circ \iota$ and \mathcal{O}_∞ is KK -equivalent to \mathbb{C} , we see that $(K_i(B), \beta)$ is isomorphic to $(K_i(A), \alpha)$ as G -modules for $i = 0, 1$. □

We refer the reader to [13] for details of K -theoretical constraints. The following corollary is an immediate consequence of Proposition 3.9 and [13, Theorem 3.6].

Corollary 3.10. *Let A be a simple separable C^* -algebra such that either $K_0(A)$ or $K_1(A)$ is isomorphic to \mathbb{Z} . Then there exist no non-trivial finite group actions on A with the Rohlin property.*

For given K -theoretical invariants, Izumi constructed model actions with the Rohlin property on Kirchberg algebras [13, Theorem 5.3]. As an application of this construction, we obtain the following corollary by a similar argument as in the proof of Proposition 3.9 and [13, Corollary 5.5].

Corollary 3.11. *Let α be an action of a finite group G with the Rohlin property on a simple separable nuclear C^* -algebra A in the UCT class. Then $A \rtimes_\alpha G$ is in the UCT class.*

4. APPROXIMATELY REPRESENTABLE ACTIONS

In this section we introduce approximate representability for finite group actions on σ -unital C^* -algebras and study some basic properties of approximately representable actions. Moreover we give some examples of Rohlin actions on \mathcal{W} by using a classification result of some approximately representable actions.

Definition 4.1. An action α of a finite abelian group G on a σ -unital C^* -algebra A is said to be *approximately representable* if there exist elements $\{w(g)\}_{g \in G}$ in $(A^\alpha)_\infty$ such that the map u from G to $F(A^\alpha)$ given by $u(g) = [w(g)]$ is a unitary

representation of G , and

$$\alpha_g(a) = w(g)aw(g)^*$$

in A^∞ for any $a \in A$ and $g \in G$.

If A is unital, then the definition above coincides with the definition in [12].

Remark 4.2. For a general finite group G , we can define approximate representability as in [12, Remark 3.7]. Put $F(A^\alpha, A) := A^\infty \cap (A^\alpha)' / \text{Ann}(A^\alpha, A^\infty)$ (see Section 2.3). An action α of a finite group G on a σ -unital C*-algebra A is said to be approximately representable if there exist elements $\{w(g)\}_{g \in G}$ in $A^\infty \cap (A^\alpha)'$ such that the map from G to $F(A^\alpha, A)$ given by $u(g) = [w(g)]$ is a unitary representation of G and

$$\begin{aligned} \alpha_g(a) &= w(g)aw(g)^*, & \text{in } A^\infty, \quad a \in A, \quad g \in G, \\ \alpha_g(u(h)) &= u(ghg^{-1}), & \text{in } F(A^\alpha, A), \quad g, h \in G. \end{aligned}$$

We shall show that approximate representability is the “dual” notion of the Rohlin property. Note that if α is an action of a finite group G on A , then we have $(A^\alpha)^\infty = (A^\infty)^\alpha$. In a similar way, we see the following lemma.

Lemma 4.3. *Let α be an action of a finite group G on A , and let B be a subalgebra of A^α . If w is an element in $A^\infty \cap B'$ such that $wa = \alpha_g(w)a$ for any $a \in A$ and $g \in G$, then there exists an element $y \in (A^\alpha)^\infty \cap B'$ such that $wa = ya$ in A^∞ for any $a \in A$.*

Proof. Let $(w_n)_n$ be a representative of w . Define $y_n := \frac{1}{|G|} \sum_{g \in G} \alpha_g(w_n) \in A^\alpha$, and put $y = (y_n)_n$. Then y has the desired property. □

See [12, Lemma 3.8] for the unital case of the following proposition.

Proposition 4.4. *Let α be an action of a finite abelian group G on a σ -unital C*-algebra A . Then*

- (i) α has the Rohlin property if and only if $\hat{\alpha}$ is approximately representable;
- (ii) α is approximately representable if and only if $\hat{\alpha}$ has the Rohlin property.

Proof. (i) Assume that α has the Rohlin property. Then there exists a unitary representation u of \hat{G} on $F(A)$ such that $\alpha_g(u(\gamma)) = \gamma(g)u(\gamma)$ for any $g \in G$ and $\gamma \in \hat{G}$ by Proposition 3.3. Choose elements $\{w(\gamma)\}_{\gamma \in \hat{G}}$ in A_∞ such that $[w(\gamma)] = u(\gamma)$, and let λ_g be the implementing unitary of α_g in $M(A \rtimes_\alpha G)$. Then we have

$$a\lambda_g w(\gamma)\lambda_g^* = a\gamma(g)w(\gamma),$$

for any $a \in A$, $g \in G$ and $\gamma \in \hat{G}$. This equation implies

$$w(\gamma)^* a\lambda_g w(\gamma) = \gamma(g)a\lambda_g,$$

and hence

$$\hat{\alpha}_\gamma(x) = w(\gamma)^* x w(\gamma),$$

for any $x \in A \rtimes_\alpha G$ and $\gamma \in \hat{G}$. Since we have $(A \rtimes_\alpha G)^{\hat{\alpha}} = A$, $\hat{\alpha}$ is approximately representable. The converse follows from a similar computation.

(ii) Assume that α is approximately representable and take elements $\{w(g)\}_{g \in G}$ in $(A^\alpha)^\infty$ as in Definition 4.1. Then we have

$$\lambda_g a \lambda_g^* = w(g)aw(g)^*, \quad a \in A, g \in G,$$

and

$$\lambda_g w(h) \lambda_g^* = w(h), \quad g, h \in G,$$

where λ_g is the implementing unitary of α_g in $M(A \rtimes_\alpha G)$. Put $z(g) := w(g) \lambda_g \in (A \rtimes_\alpha G)^\infty$ for any $g \in G$. By the equations above, we have

$$a \lambda_h z(g) = a \lambda_h w(g) \lambda_g = a w(g) \lambda_h \lambda_g = a w(g) \lambda_g \lambda_h = w(g) \lambda_g a \lambda_h = z(g) a \lambda_h,$$

for any $a \in A$ and $g, h \in G$. Therefore $\{z_g\}_{g \in G}$ are elements in $(A \rtimes_\alpha G)_\infty$. Define a map v from G to $F(A \rtimes_\alpha G)$ by $v(g) = [z(g)]$. Then it can be easily checked that v is a unitary representation and $\hat{\alpha}_\gamma(v(g)) = \gamma(g)v(g)$ for any $\gamma \in \hat{G}$ and $g \in G$. Proposition 3.3 implies that $\hat{\alpha}$ has the Rohlin property.

Conversely, assume that $\hat{\alpha}$ has the Rohlin property. Then there exists a unitary representation u of G on $F(A \rtimes_\alpha G)$ such that $\hat{\alpha}_\gamma(u(g)) = \gamma(g)u(g)$ for any $\gamma \in \hat{G}$ and $g \in G$ by Proposition 3.3. Choose elements $\{z(g)\}_{g \in G}$ in $(A \rtimes_\alpha G)_\infty$ such that $[z(g)] = u(g)$. Then we have

$$\hat{\alpha}_\gamma(\lambda_g z(g)^*) x = \gamma(g) \lambda_g \overline{\gamma(g)} z(g)^* x = \lambda_g z(g)^* x,$$

for any $x \in A \rtimes_\alpha G$, $g \in G$ and $\gamma \in \hat{G}$, where λ_g is the implementing unitary of α_g in $M(A \rtimes_\alpha G)$. Since we have $\lambda_g z(g)^* b = b \lambda_g z(g)^*$ for any $b \in A^\alpha$ and $(A \rtimes_\alpha G)^{\hat{\alpha}} = A$, Lemma 4.3 implies that there exist elements $\{y(g)\}_{g \in G}$ in $A^\infty \cap (A^\alpha)'$ such that $y(g)x = \lambda_g z(g)^* x$ for any $x \in A \rtimes_\alpha G$ and $g \in G$. For any $a \in A$ and $g, h \in G$, we have

$$\alpha_h(y(g))a = \lambda_h \lambda_g z(g)^* \lambda_h^* a = \lambda_h \lambda_g \lambda_h^* a z(g)^* = \lambda_g z(g)^* a = y(g)a.$$

Hence there exist elements $\{w(g)\}_{g \in G}$ in $(A^\alpha)_\infty$ such that $w(g)a = \lambda_g z(g)^* a$ for any $a \in A$ and $g \in G$ by Lemma 4.3. It can be easily checked that the map v from G to $F(A^\alpha)$ given by $v(g) = [w(g)]$ is a unitary representation of G and $\alpha_g(a) = w(g) a w(g)^*$ for any $a \in A$ and $g \in G$. Consequently α is approximately representable. \square

By Theorem 3.5, Proposition 4.4 and the Takesaki-Takai duality, we can show the following classification result for some approximately representable actions. See [12, Corollary 3.9] for the unital case.

Corollary 4.5. *Let α and β be actions of a finite abelian group G on a separable C^* -algebra A , and let e_α and e_β be the projections in $M(A \rtimes_\alpha G)$ and $M(A \rtimes_\beta G)$, respectively, defined in Section 2.2. Assume that α and β are approximately representable and $A \rtimes_\alpha G$ and $A \rtimes_\beta G$ have almost stable rank one. Then α and β are conjugate if and only if there exists an isomorphism θ from $A \rtimes_\alpha G$ onto $A \rtimes_\beta G$ such that $\theta \circ \hat{\alpha}_\gamma \circ \theta^{-1}$ is approximately unitarily equivalent to $\hat{\beta}_\gamma$ for any $\gamma \in \hat{G}$ and $\bar{\theta}(e_\alpha)$ is Murray-von Neumann equivalent to e_β (in $M(A \rtimes_\beta G)$).*

Proof. The only if part is obvious. We will show the if part. By Theorem 3.5 and Proposition 4.4, there exists an approximately inner automorphism θ_1 of $A \rtimes_\beta G$ such that $\theta_1 \circ \theta \circ \hat{\alpha}_\gamma \circ \theta^{-1} \circ \theta_1^{-1} = \hat{\beta}_\gamma$ for any $\gamma \in \hat{G}$. Replacing θ with $\theta_1 \circ \theta$, we may assume that θ is an isomorphism from $A \rtimes_\alpha G$ onto $A \rtimes_\beta G$ such that $\theta \circ \hat{\alpha}_\gamma = \hat{\beta}_\gamma \circ \theta$ for any γ , and $\bar{\theta}(e_\alpha)$ is Murray-von Neumann equivalent to e_β because θ_1 is approximately inner. Then there exists an isomorphism Ψ from $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}$ onto $A \rtimes_\beta G \rtimes_{\hat{\beta}} \hat{G}$ such that $\Psi \circ \hat{\alpha}_g = \hat{\beta}_g \circ \Psi$ for any $g \in G$ and $\bar{\Psi}(e_\alpha)$ is Murray-von Neumann equivalent to e_β in $M(A \rtimes_\beta G \rtimes_{\hat{\beta}} \hat{G})^{\bar{\beta}}$. Hence the Takesaki-Takai duality

theorem implies that there exists an automorphism Θ of $A \otimes K(\ell^2(G))$ such that Θ intertwines $\alpha \otimes \text{Ad}(\rho)$ and $\beta \otimes \text{Ad}(\rho)$. Let $e \in K(\ell^2(G))$ be the projection onto the constant functions. Then we have

$$\begin{aligned} (A, \alpha) &\cong ((1 \otimes e)(A \otimes K(\ell^2(G)))(1 \otimes e), \alpha \otimes \text{Ad}(\rho)|_{(1 \otimes e)(A \otimes K(\ell^2(G)))(1 \otimes e)}) \\ &\cong (\bar{\Theta}(1 \otimes e)(A \otimes K(\ell^2(G)))\bar{\Theta}(1 \otimes e), \beta \otimes \text{Ad}(\rho)|_{\bar{\Theta}(1 \otimes e)(A \otimes K(\ell^2(G)))\bar{\Theta}(1 \otimes e)}). \end{aligned}$$

Since $\bar{\Psi}(e_\alpha)$ is Murray-von Neumann equivalent to e_β in $M(A \rtimes_\beta G \rtimes_\beta \hat{G})^{\bar{\beta}}$, we see that $\bar{\Theta}(1 \otimes e)$ is Murray-von Neumann equivalent to $1 \otimes e$ in $M(A \otimes K(\ell^2(G)))^{\bar{\beta} \otimes \text{Ad}(\rho)}$ (see Section 2.2). Therefore we have

$$\begin{aligned} (A, \alpha) &\cong (\bar{\Theta}(1 \otimes e)(A \otimes K(\ell^2(G)))\bar{\Theta}(1 \otimes e), \beta \otimes \text{Ad}(\rho)|_{\bar{\Theta}(1 \otimes e)(A \otimes K(\ell^2(G)))\bar{\Theta}(1 \otimes e)}) \\ &\cong ((1 \otimes e)(A \otimes K(\ell^2(G)))(1 \otimes e), \beta \otimes \text{Ad}(\rho)|_{(1 \otimes e)(A \otimes K(\ell^2(G)))(1 \otimes e)}) \\ &\cong (A, \beta). \end{aligned}$$

□

Remark 4.6. Let s be a strictly positive element in A^α . Define a positive element s_α in $A \rtimes_\alpha G$ by

$$s_\alpha := \frac{1}{|G|} \sum_{g \in G} s \lambda_g,$$

where λ_g is the implementing unitary of α_g in $M(A \rtimes_\alpha G)$. It can be checked that $\overline{s_\alpha(A \rtimes_\alpha G)} = e_\alpha(A \rtimes_\alpha G)$ and the Cuntz class $[s_\alpha]$ does not depend on the choice of s . Since $A \rtimes_\alpha G$ and $A \rtimes_\beta G$ have almost stable rank one, we see that the condition that $\bar{\theta}(e_\alpha)$ is Murray-von Neumann equivalent to e_β is equivalent to the condition that $\theta(s_\alpha)$ is Cuntz equivalent to s_β by [7, Theorem 3].

We shall consider examples of approximately representable actions.

Definition 4.7. Let \mathcal{C} be a class of C*-algebras. An action α of a finite group G on a C*-algebra A is said to be *locally representable* if there exists an α -invariant increasing sequence of C*-subalgebras $\{A_n\}_{n \in \mathbb{N}}$ whose union is dense in A , and a unitary representation $u^{(n)}$ of G in $M(A_n)$ such that

$$\alpha_g(a) = \text{Ad}(u^{(n)}(g))(a),$$

for any $a \in A_n$ and $g \in G$. Moreover if A_n is contained in \mathcal{C} for any $n \in \mathbb{N}$, we say that α is *locally representable in \mathcal{C}* .

Note that we do not assume that $A_n \subset A_{n+1}$ is a non-degenerate inclusion in the definition above. Hence $M(A_n)$ may not be contained in $M(A_{n+1})$. We denote by \mathcal{C}_F the class of all finite-dimensional C*-algebras. We shall show that every locally representable action on a separable C*-algebra is approximately representable.

Lemma 4.8. *Let α be an action of a finite group G on a separable C*-algebra A . Assume that there exists an α -invariant increasing sequence of C*-subalgebras $\{A_n\}_{n \in \mathbb{N}}$ whose union is dense in A . Then there exists an approximate unit $\{h_n\}_{n \in \mathbb{N}}$ for A such that $h_n \in (A_n)^\alpha$ for any $n \in \mathbb{N}$.*

Proof. Let $\{F_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets in A such that $F_n \subset A_n$ and $\bigcup_{n=1}^\infty F_n$ is dense in A . Since $(A_n)^\alpha \subset A_n$ is a non-degenerate

inclusion, there exists a positive contraction h_n in $(A_n)^\alpha$ such that

$$\|h_n x - x\| < \frac{1}{n},$$

for any $x \in F_n$. Then $\{h_n\}_{n \in \mathbb{N}}$ is an approximate unit for A . □

Proposition 4.9. *If α is a locally representable action of a finite group G on a separable C^* -algebra A , then α is approximately representable.*

Proof. Let $\{A_n\}_{n \in \mathbb{N}}$ be an increasing sequence of C^* -subalgebras of A and $\{u^{(n)}\}_{n \in \mathbb{N}}$ unitary representations as in Definition 4.7. There exists an approximate unit $\{h_n\}_{n \in \mathbb{N}}$ for A such that $h_n \in (A_n)^\alpha$ for any $n \in \mathbb{N}$ by Lemma 4.8. For any $g \in G$, put $w(g) := (u^{(n)}(g)h_n)_n \in A^\infty$. Then we have

$$\alpha_g(a) = w(g)aw(g)^*$$

in A^∞ for any $a \in A$ and $g \in G$, and hence $w(g) \in A^\infty \cap (A^\alpha)'$. Since we have $u^{(n)}(g)h_n = h_n u^{(n)}(g)$ for any $n \in \mathbb{N}$ and $g \in G$, it can be easily checked that a map v from G to $F(A^\alpha, A)$ given by $v(g) = [w(g)]$ is a unitary representation of G and

$$\alpha_g(v(h)) = v(ghg^{-1}),$$

for any $g, h \in G$. Therefore α is approximately representable. □

Izumi showed that if β is an outer action of a finite group G on a separable simple unital nuclear C^* -algebra A , then the action $\beta \otimes \text{id}$ of G on $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ has the Rohlin property (see [12, Corollary 4.3]). This result can be regarded as an equivariant version of Kirchberg-Phillips's theorem (see [20, Theorem 3.8]). We cannot show such a statement for \mathcal{W} . Indeed, Example 3.8 shows that there exists an outer action β on M_{2^∞} such that the action $\beta \otimes \text{id}$ on $M_{2^\infty} \otimes \mathcal{W} \cong \mathcal{W}$ does not have the Rohlin property. Note that this action is not strongly outer and $(M_{2^\infty} \otimes \mathcal{W}) \rtimes_{\beta \otimes \text{id}} \mathbb{Z}_2$ has two extremal tracial states. Hence it is natural to consider the following question.

Question 4.10. Let A be a separable simple nuclear C^* -algebra with a unique tracial state and no unbounded traces, and let β be a strongly outer action of a finite group G on A . Does the action $\beta \otimes \text{id}$ of G on $A \otimes \mathcal{W}$ have the Rohlin property?

We shall show the following (non-trivial) partial result of the question above as an application of Corollary 4.5.

Corollary 4.11. *Let A be a simple AF algebra with a unique tracial state and no unbounded traces, and let β be a strongly outer action of a finite abelian group G on A . Assume that β is approximately representable and $A \rtimes_\beta G$ is an AF algebra. Then*

$$(A \otimes \mathcal{W}, \beta \otimes \text{id}) \cong (\mathcal{W}, \nu^G)$$

where ν^G is the action in Example 3.2, and hence $\beta \otimes \text{id}$ has the Rohlin property.

Proof. Proposition 2.1 and [22, Theorem 3.1] imply that $A \rtimes_\beta G$ is a simple C^* -algebra with a unique tracial state. It is easy to see that $A \rtimes_\beta G$ has no unbounded traces. Since $(A \otimes \mathcal{W}) \rtimes_{\beta \otimes \text{id}} G$ is isomorphic to $(A \rtimes_\beta G) \otimes \mathcal{W}$, $(A \otimes \mathcal{W}) \rtimes_{\beta \otimes \text{id}} G$ is isomorphic to \mathcal{W} . On the other hand, $\mathcal{W} \rtimes_{\nu^G} G$ is isomorphic to \mathcal{W} and Proposition 4.9 implies that ν^G is approximately representable. Let θ be an isomorphism from $\mathcal{W} \rtimes_{\nu^G} G$ onto $(A \otimes \mathcal{W}) \rtimes_{\beta \otimes \text{id}} G$. We denote by $\omega_{\beta \otimes \text{id}}$ and ω_ν the unique tracial

state on $(A \otimes \mathcal{W}) \rtimes_{\beta \otimes \text{id}} G$ and the unique tracial state on $\mathcal{W} \rtimes_{\nu, G} G$, respectively. Then we have

$$\bar{\omega}_{\beta \otimes \text{id}}(e_{\alpha \otimes \text{id}}) = \bar{\omega}_{\nu}(e_{\nu, G}) = \frac{1}{|G|}.$$

Hence we see that $\bar{\theta}(e_{\nu, G})$ is Murray-von Neumann equivalent to $e_{\alpha \otimes \text{id}}$ by Proposition 2.9. Since every automorphism of \mathcal{W} is approximately inner, we obtain the conclusion by Corollary 4.5. \square

Remark 4.12. (i) If an action β on a C*-algebra A is locally representable in \mathcal{C}_F , then $A \rtimes_{\beta} G$ is an AF algebra.

(ii) There exists a strongly outer action β of a finite abelian group on M_{2^∞} such that β is locally representable in \mathcal{C}_F and β does not have the Rohlin property. Indeed, define an action β of \mathbb{Z}_2 on M_{2^∞} by

$$\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(1_{2^{n-1+1}} \oplus -1_{2^{n-1-1}}) \quad \text{on} \quad \bigotimes_{n=1}^{\infty} M_{2^n}(\mathbb{C}).$$

Then β is such an action (see [35, Example 3.7] for the proof). But the corollary above shows that the action $\beta \otimes \text{id}$ on $M_{2^\infty} \otimes \mathcal{W} \cong \mathcal{W}$ has the Rohlin property.

5. SYMMETRIES OF \mathcal{W}

Let \mathcal{C}_R be the class of C*-algebras A in Robert’s classification theorem [37, Theorem 1.0.1], that is, A is either a 1-dimensional NCCW complex with trivial K_1 -group, an inductive limit C*-algebra of such C*-algebras or a C*-algebra stably isomorphic to one such inductive limit. It is obvious that the outer actions on \mathcal{W} given in Example 3.2 and Example 3.8 are locally representable in \mathcal{C}_R . In this section we shall classify the locally representable outer \mathbb{Z}_2 -actions (\mathcal{W}, α) in \mathcal{C}_R up to conjugacy and cocycle conjugacy. We denote by τ the unique tracial state on \mathcal{W} in this section.

Lemma 5.1. *Let A be a simple stably projectionless C*-algebra with no unbounded traces, and let α be an outer action of a finite group G on A . Assume that α is locally representable in \mathcal{C}_R . Then $A \rtimes_{\alpha} G$ is a simple stably projectionless C*-algebra with no unbounded traces and is contained in \mathcal{C}_R .*

Proof. It is easy to see that $A \rtimes_{\alpha} G$ has no unbounded traces and A^{α} is stably projectionless. Since α is outer, $A \rtimes_{\alpha} G$ is simple by [22, Theorem 3.1]. Hence we see that $A \rtimes_{\alpha} G$ is a simple stably projectionless C*-algebra with no unbounded traces. Let $\{A_n\}_{n \in \mathbb{N}}$ be an increasing sequence of C*-subalgebras of A and $\{u^{(n)}\}_{n \in \mathbb{N}}$ unitary representations as in Definition 4.7. Then we have

$$A \rtimes_{\alpha} G = \overline{\bigcup_{n=1}^{\infty} A_n \rtimes_{\text{Ad}(u^{(n)})} G}.$$

Since $A_n \rtimes_{\text{Ad}(u^{(n)})} G$ is isomorphic to the tensor product of A_n and a finite-dimensional C*-algebra, we see that $A \rtimes_{\alpha} G$ is contained in \mathcal{C}_R . \square

Theorem 5.2. *Let α and β be outer actions of \mathbb{Z}_2 on \mathcal{W} . Assume that α and β are locally representable in \mathcal{C}_R . Then α is conjugate to β if and only if $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ is isomorphic to $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$ and $\epsilon(\alpha) = \epsilon(\beta)$ where $\epsilon(\alpha)$ is the invariant defined in Definition 2.7.*

Proof. It is easy to see that if α is conjugate to β , then $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ is isomorphic to $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$ and $\epsilon(\alpha) = \epsilon(\beta)$. We will show the converse. Let θ be an isomorphism from $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ onto $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$.

First, we shall show $K_0(\hat{\alpha}) = -\text{id}$. Put $A := \mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$, and let i_1 and i_2 denote the inclusion from A into $A \rtimes_{\hat{\alpha}} \mathbb{Z}_2$ and the inclusion from $A \rtimes_{\hat{\alpha}} \mathbb{Z}_2$ into $A \rtimes_{\hat{\alpha}} \mathbb{Z}_2 \rtimes_{\hat{\alpha}} \mathbb{Z}_2 \cong A \otimes M_2(\mathbb{C})$, respectively. Since $A \rtimes_{\hat{\alpha}} \mathbb{Z}_2$ is isomorphic to $\mathcal{W} \otimes M_2(\mathbb{C})$ and $K_0(\mathcal{W}) = 0$, we have $K_0(i_2 \circ i_1) = 0$. The Takesaki-Takai duality implies

$$i_2 \circ i_1(a) = \begin{pmatrix} a & 0 \\ 0 & \hat{\alpha}(a) \end{pmatrix}$$

for any $a \in A$ (see Section 2.2). Hence we have $K_0(\hat{\alpha}) = -\text{id}$. Note that the same argument shows $K_0(\hat{\beta}) = -\text{id}$.

(i) The case where $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ and $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$ have a unique tracial state. We denote by ω_{α} and ω_{β} the unique tracial state on $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ and the unique tracial state on $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$, respectively. We see that $\hat{\beta}$ is approximately unitarily equivalent to $\theta \circ \hat{\alpha} \circ \theta^{-1}$ by Lemma 5.1 and Robert’s classification theorem (Theorem 2.11). Moreover we have

$$\bar{\omega}_{\alpha}(e_{\alpha}) = \bar{\omega}_{\beta}(e_{\beta}) = \frac{1}{2},$$

and hence $\bar{\theta}(e_{\alpha})$ is Murray-von Neumann equivalent to e_{β} by Proposition 2.9. Therefore Corollary 4.5 shows that α is conjugate to β .

(ii) The case where $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ and $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$ have two extremal tracial states. Let $\omega_{\alpha,0}$ and $\omega_{\alpha,1}$ be extremal tracial states on $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ and $\omega_{\beta,0}$ and $\omega_{\beta,1}$ extremal tracial states on $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$. Since $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2 \rtimes_{\hat{\alpha}} \mathbb{Z}_2$ and $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2 \rtimes_{\hat{\beta}} \mathbb{Z}_2$ have a unique tracial state, we have $\omega_{\alpha,0} \circ \hat{\alpha} = \omega_{\alpha,1}$ and $\omega_{\beta,0} \circ \hat{\beta} = \omega_{\beta,1}$. Hence we see that $\hat{\beta}$ is approximately unitarily equivalent to $\theta \circ \hat{\alpha} \circ \theta^{-1}$ by Lemma 5.1 and Robert’s classification theorem (Theorem 2.11). Choose a unitary element U_{α} in $\pi_{\tau}(\mathcal{W})''$ such that $\tilde{\alpha} = \text{Ad}(U_{\alpha})$ and $U_{\alpha}^2 = 1$ (see Proposition 2.5 and Remark 2.6). By Remark 2.6, we have either

$$(\bar{\omega}_{\alpha,0}(e_{\alpha}), \bar{\omega}_{\alpha,1}(e_{\alpha})) = \left(\frac{1}{2}(1 + \tilde{\tau}(U_{\alpha})), \frac{1}{2}(1 - \tilde{\tau}(U_{\alpha}))\right)$$

or

$$(\bar{\omega}_{\alpha,0}(e_{\alpha}), \bar{\omega}_{\alpha,1}(e_{\alpha})) = \left(\frac{1}{2}(1 - \tilde{\tau}(U_{\alpha})), \frac{1}{2}(1 + \tilde{\tau}(U_{\alpha}))\right).$$

In the same way, we can compute $\bar{\omega}_{\beta,0}(e_{\beta})$ and $\bar{\omega}_{\beta,1}(e_{\beta})$. Since $\epsilon(\alpha) = \epsilon(\beta)$, we have either

$$\bar{\omega}_{\alpha,0}(e_{\alpha}) = \bar{\omega}_{\beta,0}(e_{\beta}) \quad \text{and} \quad \bar{\omega}_{\alpha,1}(e_{\alpha}) = \bar{\omega}_{\beta,1}(e_{\beta})$$

or

$$\bar{\omega}_{\alpha,0}(e_{\alpha}) = \bar{\omega}_{\beta,1}(e_{\beta}) \quad \text{and} \quad \bar{\omega}_{\alpha,1}(e_{\alpha}) = \bar{\omega}_{\beta,0}(e_{\beta}).$$

Hence replacing θ with $\theta \circ \hat{\alpha}$ if necessary, we have $\bar{\omega}(\bar{\theta}(e_{\alpha})) = \bar{\omega}(e_{\beta})$ for any $\omega \in T_1(\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2)$. Therefore α is conjugate to β by Proposition 2.9 and Corollary 4.5. \square

Remark 5.3. (i) With notation as above, $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ is isomorphic to $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$ if and only if $T(\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2) \cong T(\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2)$ as topological cones and $K_0(\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2) \cong K_0(\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2)$ as abelian groups, and that these isomorphisms are compatible with the pairing between T and K_0 by Lemma 5.1 and Robert’s classification theorem.

(ii) If $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ and $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$ have a unique tracial state, then α is conjugate to β if and only if $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ is isomorphic to $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$.

We shall consider the classification up to cocycle conjugacy.

Proposition 5.4. *Let A be a simple exact σ -unital stably projectionless C*-algebra with a unique tracial state and no unbounded traces, and let α and β be actions of a finite abelian group G on A . Assume that A has strict comparison and almost stable rank one. Then α is cocycle conjugate to β if and only if $\hat{\alpha}$ is conjugate to $\hat{\beta}$.*

Proof. In general, it is easy to see that if α is cocycle conjugate to β , then $\hat{\alpha}$ is conjugate to $\hat{\beta}$. We will show the if part. Since we see that $\hat{\alpha}$ is conjugate to $\hat{\beta}$, there exists an automorphism θ of $A \otimes K(\ell^2(G))$ such that θ intertwines $\alpha \otimes \text{Ad}(\rho)$ and $\beta \otimes \text{Ad}(\rho)$ by the Takesaki-Takai duality theorem. Note that $A \otimes K(\ell^2(G))$ is a simple exact σ -unital stably projectionless C*-algebra with a unique tracial state ω and no unbounded traces, which has strict comparison and almost stable rank one. Let $e \in K(\ell^2(G))$ be the projection onto the constant functions. Then we have

$$(A, \alpha) \cong (\bar{\theta}(1 \otimes e)(A \otimes K(\ell^2(G)))\bar{\theta}(1 \otimes e), \beta \otimes \text{Ad}(\rho)|_{\bar{\theta}(1 \otimes e)(A \otimes K(\ell^2(G)))\bar{\theta}(1 \otimes e)})$$

by the proof of Corollary 4.5. Since ω is the unique tracial state on $A \otimes K(\ell^2(G))$, we have $\bar{\omega}(\bar{\theta}(1 \otimes e)) = \bar{\omega}(1 \otimes e)$. By Proposition 2.9, there exists a partial isometry v in $M(A \otimes K(\ell^2(G)))$ such that

$$v^*v = \bar{\theta}(1 \otimes e) \quad \text{and} \quad vv^* = 1 \otimes e.$$

For any $g \in G$, define

$$u(g) := \overline{v\beta_g \otimes \text{Ad}\rho_g(v^*)}.$$

Then we have $(1 \otimes e)u(g)(1 \otimes e) = u(g)$ for any $g \in G$, and we may regard u as a β -cocycle in $M(A)$. It can be easily checked that

$$(\bar{\theta}(1 \otimes e)(A \otimes K(\ell^2(G)))\bar{\theta}(1 \otimes e), \beta \otimes \text{Ad}(\rho)|_{\bar{\theta}(1 \otimes e)(A \otimes K(\ell^2(G)))\bar{\theta}(1 \otimes e)}) \cong (A, \beta^u),$$

and hence we obtain the conclusion. □

Theorem 5.5. *Let α and β be outer actions of \mathbb{Z}_2 on \mathcal{W} . Assume that α and β are locally representable in \mathcal{C}_R . Then α is cocycle conjugate to β if and only if $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ is isomorphic to $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$.*

Proof. The only if part is obvious. We will show the if part. Let θ be an isomorphism from $\mathcal{W} \rtimes_{\alpha} \mathbb{Z}_2$ onto $\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2$. The proof of Theorem 5.2 implies that $\hat{\beta}$ is approximately unitarily equivalent to $\theta \circ \hat{\alpha} \circ \theta^{-1}$. Hence $\hat{\alpha}$ is conjugate to $\hat{\beta}$ by Theorem 3.5 and Proposition 4.4. Proposition 5.4 shows that α is cocycle conjugate to β . □

The following example is based on Blackadar’s construction of symmetries on M_{2^∞} in [2] and Robert’s classification theorem.

Example 5.6. [9, Theorem 5.2.1 and Theorem 5.2.2] show that there exists a simple stably projectionless C*-algebra A such that A is expressible as an inductive limit C*-algebra of 1-dimensional NCCW complexes with trivial K_1 -groups and $(K_0(A), (T(A), T_1(A)), r_A) = (\mathbb{Z}, (\mathbb{R}_+, \emptyset), 0)$. (Note that in the case $K_0^+ = 0$, we do not need the assumption that every additive map on G arises through the pairing from some element of C in [9, Theorem 5.2.2].) By [33, Proposition 5.2], there exists a hereditary subalgebra B of A such that B has a unique tracial state ω and no unbounded traces. Since B is stably isomorphic to A by Brown’s theorem (see [4]), B is contained in \mathcal{C}_R and $(K_0(B), (T(B), T_1(B)), r_B) = (\mathbb{Z}, (\mathbb{R}_+, \{\omega\}), 0)$.

By Robert’s classification theorem (Theorem 2.11), there exists an automorphism β of B such that $K_0(\beta) = -\text{id}$. Define a homomorphism Φ_i from $M_{2^i}(\mathbb{C}) \otimes B$ to $M_{2^{i+1}}(\mathbb{C}) \otimes B$ by

$$\Phi_i(x) = \begin{pmatrix} x & 0 \\ 0 & (\text{id}_{M_{2^i}(\mathbb{C})} \otimes \beta)(x) \end{pmatrix},$$

and put $D := \varinjlim(M_{2^i}(\mathbb{C}) \otimes B, \Phi_i)$. Then we have $K_0(\Phi_i) = 0$, and hence $K_0(D) = 0$. It can be easily checked that D is a simple stably projectionless C^* -algebra with a unique tracial state and no unbounded traces. Therefore D is isomorphic to \mathcal{W} by Robert’s classification theorem.

We shall construct a locally representable \mathbb{Z}_2 -action (\mathcal{W}, α) in \mathcal{C}_R . It is easy to see that Φ_i is unitarily equivalent to a homomorphism Ψ_i such that

$$\Psi_i\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{11} & 0 & x_{12} & 0 \\ 0 & (\text{id}_{M_{2^{i-1}}(\mathbb{C})} \otimes \beta)(x_{22}) & 0 & (\text{id}_{M_{2^{i-1}}(\mathbb{C})} \otimes \beta)(x_{21}) \\ x_{21} & 0 & x_{22} & 0 \\ 0 & (\text{id}_{M_{2^{i-1}}(\mathbb{C})} \otimes \beta)(x_{12}) & 0 & (\text{id}_{M_{2^{i-1}}(\mathbb{C})} \otimes \beta)(x_{11}) \end{pmatrix},$$

for any $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in M_{2^i}(\mathbb{C}) \otimes B$. Since $\varinjlim(M_{2^i}(\mathbb{C}) \otimes B, \Psi_i)$ is isomorphic to D , $\varinjlim(M_{2^i}(\mathbb{C}) \otimes B, \Psi_i)$ is isomorphic to \mathcal{W} . For any $i \in \mathbb{N}$, define an automorphism α_i of $M_{2^i}(\mathbb{C}) \otimes B$ by

$$\alpha_i\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) := \text{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right),$$

for any $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in M_{2^i}(\mathbb{C}) \otimes B$. Then we have $\Psi_i \circ \alpha_i = \alpha_{i+1} \circ \Psi_i$, and hence there exists an automorphism α of \mathcal{W} such that $\alpha(a) = \alpha_i(a)$ for any $a \in M_{2^i}(\mathbb{C}) \otimes B$ and $i \in \mathbb{N}$. It is obvious that α is a locally representable \mathbb{Z}_2 -action in \mathcal{C}_R .

We shall consider the fixed point algebra \mathcal{W}^α . It can be easily checked that \mathcal{W}^α is isomorphic to an inductive limit C^* -algebra $\varinjlim(M_{2^{i-1}}(\mathbb{C}) \otimes B \oplus M_{2^{i-1}}(\mathbb{C}) \otimes B, \Theta_i)$ where

$$\Theta_i((x_1, x_2)) = \left(\begin{pmatrix} x_1 & 0 \\ 0 & (\text{id}_{M_{2^{i-1}}(\mathbb{C})} \otimes \beta)(x_2) \end{pmatrix}, \begin{pmatrix} x_2 & 0 \\ 0 & (\text{id}_{M_{2^{i-1}}(\mathbb{C})} \otimes \beta)(x_1) \end{pmatrix} \right),$$

for any $(x_1, x_2) \in M_{2^{i-1}}(\mathbb{C}) \otimes B \oplus M_{2^{i-1}}(\mathbb{C}) \otimes B$. Therefore we have $K_0(\mathcal{W}^\alpha) = \mathbb{Z}[\frac{1}{2}]$. It is easy to see that α is outer, and hence $\mathcal{W} \rtimes_\alpha \mathbb{Z}_2$ is stably isomorphic to \mathcal{W}^α . Consequently we obtain a locally representable outer \mathbb{Z}_2 -action (\mathcal{W}, α) in \mathcal{C}_R such that

$$K_0(\mathcal{W} \rtimes_\alpha \mathbb{Z}_2) = \mathbb{Z}[\frac{1}{2}] \quad \text{and} \quad K_1(\mathcal{W} \rtimes_\alpha \mathbb{Z}_2) = 0.$$

Note that the dual action $\hat{\alpha}$ on $\mathcal{W} \rtimes_\alpha \mathbb{Z}_2$ has the Rohlin property by Proposition 4.4.

Remark 5.7. Blackadar showed that if A is a C^* -algebra with $K_0(A) = K_1(A) = 0$ and β is a \mathbb{Z}_2 -action on A , then $K_0(A \rtimes_\beta \mathbb{Z}_2)$ and $K_1(A \rtimes_\beta \mathbb{Z}_2)$ are uniquely 2-divisible (see [12, Lemma 4.4]). (Note that an abelian group Γ is uniquely 2-divisible if and only if $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ is isomorphic to Γ .) For every pair of countable abelian uniquely 2-divisible groups Γ_0 and Γ_1 , Izumi constructed an approximately representable outer action β of \mathbb{Z}_2 on \mathcal{O}_2 such that $K_0(\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_2) = \Gamma_0$ and $K_1(\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_2) = \Gamma_1$ (see [12, Theorem 4.8 (3)]). If the Robert conjecture is true,

then we can construct an approximately representable outer action β of \mathbb{Z}_2 on \mathcal{W} such that $K_0(\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2) = \Gamma_0$ and $K_1(\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2) = \Gamma_1$ by using α in Example 5.6 and the same construction of Izumi. Indeed, there exists a simple separable nuclear stably projectionless C^* -algebra B with a unique tracial state and no unbounded traces such that $K_0(B) = \Gamma_0$ and $K_1(B) = \Gamma_1$ (see, for example, [25] and [33, Proposition 5.2]). If the Robert conjecture is true, then $B \otimes \mathcal{W}$ is isomorphic to \mathcal{W} . Define an action β on $B \otimes \mathcal{W} \cong \mathcal{W}$ by $\beta := \text{id} \otimes \alpha$. Then β is an approximately representable outer action such that $K_0(\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2) = \Gamma_0$ and $K_1(\mathcal{W} \rtimes_{\beta} \mathbb{Z}_2) = \Gamma_1$ by the Kunneth formula.

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