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TRAVELING VORTEX HELICES FOR SCHRÖDINGER MAP EQUATIONS

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ABSTRACT. We construct traveling wave solutions with vortex helix structures for the Schrödinger map equation

$$\frac{\partial m}{\partial t} = m \times (\Delta m - m_3 \vec{e}_3) \quad \text{on } \mathbb{R}^3 \times \mathbb{R}$$

of the form $m(s_1, s_2, s_3 - \delta|\log\epsilon|\epsilon t)$ with traveling velocity $\delta|\log\epsilon|\epsilon$ along the direction of the s_3 axis. We use a perturbation approach which gives a complete characterization of the asymptotic behavior of the solutions.

1. INTRODUCTION

The aim of this paper is to construct traveling wave solutions for a class of Landau-Lifshitz equations. We shall concentrate on the traveling wave solutions of the Schrödinger map equation

$$\frac{\partial m}{\partial t} = m \times (\Delta m - m_3 \vec{e}_3) \quad \text{in } \mathbb{R}^N \times \mathbb{R} \tag{1.1}$$

or equivalently the equation

$$-m \times \frac{\partial m}{\partial t} = \Delta m - m_3 \vec{e}_3 + (|\nabla m|^2 + m_3^2) m. \tag{1.2}$$

Here $m : \mathbb{R}^N \times \mathbb{R} \to \mathbb{S}^2$ so that $|m(s,t)| \equiv 1$ and where $\vec{e}_3 = (0,0,1) \in \mathbb{R}^3$.

The equation (1.1) (or equivalently (1.2)) is, in fact, the Landau-Lifshitz equation describing the planar ferromagnets, that is, ferromagnets with an easy-plane anisotropy ([32], [38]). The unit normal to the easy-plane is assumed to be $\vec{e}_3$ in the equations; see for example [38]. Despite some serious efforts (see e.g. [15], [16], [43], [44], [21], [9, 24, 39], [22, 23], [17–20]) and the references therein), some basic mathematical issues such as local and global well-posedness and global in time asymptotics for the equation (1.1) remain unknown. If one is interested in one-dimensional wave (plane-wave) solutions of (1.1), that is, $m : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^2$ (or $\mathbb{S}^1$), many are known as (1.1) becomes basically an integrable system (see [44] and [2]). The problem in 2-D or higher dimensions is much more subtle. Even though it is possible to obtain weak solutions of (1.1) (see [2], [36], [42]-[44]), one does not know if such weak solutions are classical (smooth) or unique.

From the physical point of view, one expects topological solitons, which are half magnetic bubbles, to exist in solutions of (1.1) (see [27] and [38]). Indeed, in [25]-[26], F. Hang and F. Lin have established the corresponding static theory for

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such magnetic vortices. They are very much like vortices in the superconductor described by the Ginzburg-Landau equation; see [3] and references therein.

Let us recall the essential features of vortex dynamics in a classical fluid (or a superfluid modeled by the Gross-Pitaevskii equation; see [37]). A single vortex or antivortex is always spontaneously pinned and hence can move only together with the background fluid. However vortex motion relative to the fluid is possible in the presence of other vortices and it displays characteristics similar to the 2-D Hall motion of interacting electric charges in a uniform magnetic field. In particular, two like vortices orbit around each other while a vortex-antivortex pair undergoes Kelvin motion along parallel trajectories that are perpendicular to the line connecting the vortex and the antivortex. This latter fact was obtained in a special case by Jones and Robert [29]. They also asymptotically derived a three-dimensional (3-D) solitary wave that describes a vortex ring moving steadily along its symmetric axis. For more precise mathematical proofs, there are some works for the case of the Gross-Pitaevskii equation:

$$i\frac{\partial u}{\partial t} = \Delta u + u - |u|^2 u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}. \tag{1.3}$$

In the two-dimensional plane, F. Bethuel and J. Saut constructed a traveling wave with two vortices of degree ± 1 in [8]. In higher dimensions, by minimizing the energy, F. Bethuel, G. Orlandi and D. Smets constructed solutions with a vortex ring [7]; see also [5]. See also [10] for another proof by the Mountain Pass Lemma. The reader can refer to the review paper [6] by F. Bethuel, P. Gravejat and J. Saut and the references therein.

By a completely different method, F. Lin and J. Wei [34] obtained the results of existence of solutions, similar as those of [8], [7] for the Landau-Lifshitz equation (1.1). More precisely, if we look for a solution of the traveling wave $m(s', s_N - Ct)$ of the equation (1.1), then m must be a solution of

$$-C\frac{\partial m}{\partial s_N} = m \times (\Delta m - m_3 \vec{e}_3). \tag{1.4}$$

Here, it travels in the s_N-direction with the speed C, which will be set in the form

$$C = c\,\epsilon \quad \text{with} \quad c > 0 \quad \text{and} \quad \epsilon > 0. \tag{1.5}$$

After a proper scaling in the space, (1.4) becomes

$$-c\frac{\partial m}{\partial s_N} = m \times (\Delta m - \frac{m_3}{\epsilon^2}\vec{e}_3), \quad s \in \mathbb{R}^N, \tag{1.6}$$

or

$$c\,m \times \frac{\partial m}{\partial s_N} = \Delta m - \frac{m_3 \vec{e}_3}{\epsilon^2} + (|\nabla m|^2 + \frac{m_3^2}{\epsilon^2})m. \tag{1.7}$$

The main results of the paper [34] are the following.

Theorem 1.1 ([34]). *Let $N \geq 2$ and let ϵ be sufficiently small so that there is an axially symmetric solution $m = m(|s'|, s_N) \in C^\infty(\mathbb{R}^N, \mathbb{S}^2)$ of (1.6) such that*

$$E_\epsilon(m) = \int_{\mathbb{R}^N} \frac{1}{2}\Big(|\nabla m|^2 + \frac{|m_3|^2}{\epsilon^2}\Big)\,\mathrm{d}s < \infty \tag{1.8}$$

and such that m has exactly one vortex at $(|s'|, s_N) = (a_\epsilon, 0)$ of degree $+1$, where $a_\epsilon \approx \frac{1}{2}$. If $N = 2$, the traveling velocity $C \sim \epsilon$, while $C = (N-2)\epsilon|\log\frac{1}{\epsilon}|$ for $N \geq 3$. □

Naturally such a solution m gives rise to a nontrivial (two-dimensional) traveling wave solution of (1.1) with a pair of vortex and antivortex which undergoes the Kelvin Motion as described above. Solutions constructed in Theorem 1.1 are called *traveling vortex rings* for the case of the dimension $N \geq 3$. We note that formal arguments as well as numerical evidence were already presented in the work [38]. We should also note that in the case of the initial data of (1.1) containing only one vortex (one magnetic half-bubble), with its structure as described in the work of [25]-[26] very precisely, the above discussions imply that the vortex will simply stay at its center of mass, and a meaningful mathematical issue to examine would be its global stability. It is, however, unknown to the authors whether such stability result is true or not; see [22], [23], [24] for relevant discussions. On the other hand, it is relatively easy to generalize the work of [35] to the equation (1.1) for the planar ferromagnets. One may obtain the same Kirchhoff vortex dynamical law for these widely separated and slowly moving magnetic half-bubbles; see [33] (as formally derived in [38] and also [37] for the Gross-Pitaevskii equation) for solutions of (1.1).

In the present paper, we concern ourselves with the existence of a traveling wave solution possessing vortex helix structures for problem (1.1) by constructing a symmetric solution to (1.4). In this direction, we refer to the paper by D. Chiron [11] for the existence of vortex helices for the Gross-Pitaevskii equation (1.3). For the Allen-Cahn equation, there are also solutions with phase transition layers invariant under screw motions [14]. The main result reads:

Theorem 1.2. *Let $N = 3$. For any sufficiently small traveling velocity C, there is an axially symmetric solution $m = m(|s'|, s_N) \in C^\infty(\mathbb{R}^3, \mathbb{S}^2)$ of (1.4). Moreover, by the transformation*

$$u = \frac{m_1 + im_2}{1 + m_3} \in \mathbb{C},$$

u has a vortex helix of degree $+1$ directed along the curve in the form

$$\alpha \in \mathbb{R} \mapsto \left(\hat{d}\cos\alpha, \hat{d}\sin\alpha, \hat{\lambda}\alpha\right) \in \mathbb{R}^3,$$

for any parameters $\hat{d} > 0$ and $\hat{\lambda} \neq 0$, and u is also invariant under the screw motion expressed in cylinder coordinates

$$(r, \Psi, s_3) \mapsto (r, \Psi + \alpha, s_3 + \hat{\lambda}\alpha), \quad \forall\, \alpha \in \mathbb{R}.$$

In other words, we have found a traveling wave solution to (1.1). The traveling velocity is

$$C = \frac{1}{\sqrt{\hat{d}^2 + \hat{\lambda}^2}}\, \epsilon \log \frac{1}{\epsilon}. \tag{1.9}$$

□

Some discussion is in order to end the introduction. The traveling velocity in (1.9) depends on the geometric parameters of the helix (cf. (1.5), (3.1) and (3.10)). To get a nontrivial vortex helix, we assume that the parameters $\hat{d} > 0$ and $\hat{\lambda} \neq 0$. Note that $\hat{\lambda} = 0$ will go back to the case of vortex ring in Theorem 1.1. We can choose ϵ so that C in (1.9) is small enough that we can use the reduction method. For more precise asymptotic behaviors of the solutions, the reader can refer to the details in Sections 3-4.

A natural problem is to construct higher dimensional vortex phenomena with invariance under skew motions in Euclidean spaces. According to [12], there are helix submanifolds in any Euclidean space of odd dimension. We can use the same method to show the existence of vortex phenomena on some special helix submanifolds. The reader can refer to Remark 3.1.

Another interesting question is the existence of a solution with a double-helix vortex structure. Helices are vital for life as we know it, the DNA polymer being a famous example of a double-helix structure [41]. In fluid studies, theory on helical vortices assumes the possible existence of an arbitrary number of interacting helical vortex filaments, especially the existence of double-helix structures [1]. It is supposed that spiral pairing is the elementary interaction in turbulence. A double-helix structure formed by a pair of vortices was also studied by the numerical method for the helical spin textures in ferromagnetic spin-1 Bose-Einstein condensates subject to dipolar inter-particle forces [28].

Here is the outline of the approach. We use the standard reduction method (cf. [13], [34]) to construct the solutions with vortex helices to problem (1.4) with dimension $N = 3$. By using the screw invariance of solutions, we first transform the problem to a two-dimensional case (3.12) with boundary condition (3.14) on the infinite strip $\mathfrak{S}$ (cf. (3.13)). Note that problem (3.12) is degenerate when $x_1 = 0$ due to the terms

$$\frac{1}{x_1}\frac{\partial u}{\partial x_1} \quad \text{and} \quad \gamma^{-2}\Big(1+\frac{\lambda^2}{x_1^2}\Big)\frac{\partial^2 u}{\partial x_2^2}.$$

The second degenerate term in the above did not appear in the arguments of the works [7], [10] and [34] for the existence of traveling vortex rings. Note that the paper [11] claimed the existence of solution with helix structures and screw-motion invariance in Theorem 3 without a proof. So there was no need to handle the second degenerate term therein. The reader can refer to Theorem 3 and its remark in [11]. Here, we shall make a careful analysis to deal with the problem near the origin. After that, the key point is then to construct a solution with a vortex of degree $+1$ at $(d, 0)$ and its antipair of degree -1 at $(-d, 0)$. In addition to the computations for standard vortices in the two-dimensional case, there are two extra derivative terms,

$$\frac{1}{x_1}\frac{\partial u}{\partial x_1} \quad \text{and} \quad \gamma^{-2}\Big(\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\Big)\frac{\partial^2 u}{\partial x_2^2}.$$

If we put $w = \rho(\ell)e^{i\theta}$ as the approximate solution to (3.12) in the vortex core region, then there exist two singular terms (cf. (4.16) and (4.11)) of the forms

$$\frac{1}{d}\frac{\partial\theta}{\partial y_1} \sim -\frac{1}{d}\frac{y_2}{|y|^2},$$

$$\Big(-\frac{2\sigma^2}{d\gamma^2}\Big)y_1\frac{\partial^2\theta}{\partial y_2^2} \sim \frac{4\sigma^2}{d\gamma^2}\frac{y_1^2 y_2}{|y|^4},$$

which were expressed in local translated coordinates y in (4.12). These two terms will play an important role in determining the dynamical behavior of the vortex helices. Hence, by adding correction terms for cancellation of these two singular terms and also partially fulfilling the boundary conditions in (3.17), we find a good approximate solution defined in (4.43). All of this will be explained in Sections 3 and 4. The rest of the approach is the standard part of the reduction methods. The reader can refer to [13] and [34] and the references therein.

2. Some preliminaries

In this section, we collect some important facts which will be used later. These include the asymptotic behaviors and nondegeneracy of a degree one vortex.

Here, we consider the high dimensional case $N \geq 3$. Setting

$$u = \frac{m_1 + im_2}{1 + m_3} \text{ and } c = \delta |\log \epsilon|,$$

problem (1.4) becomes

$$i\delta\epsilon \log \frac{1}{\epsilon} \frac{\partial u}{\partial s_N} + \Delta u + \frac{1 - |u|^2}{1 + |u|^2} u = \frac{2\bar{u}}{1 + |u|^2} \nabla u \cdot \nabla u. \tag{2.1}$$

Here and throughout the paper, we use $\bar{u}$ to denote the conjugate of u. The parameter δ is a constant to be chosen in (3.10).

When $\delta = 0$ and $N = 2$, problem (2.1) admits solutions of standard vortex of degree $+1$, i.e. $u = w^+ := \rho(\ell)e^{i\theta}$ in polar coordinates (ℓ, θ), where ρ satisfies

$$\rho'' + \frac{\rho'}{\ell} - \frac{2\rho(\rho')^2}{1 + \rho^2} + (1 - \frac{1}{\ell^2}) \frac{1 - \rho^2}{1 + \rho^2} \rho = 0. \tag{2.2}$$

Another solution $w^- := \rho(\ell)e^{-i\theta}$ will be of vortex of degree -1. These two functions will be our block elements for future construction of approximate solutions.

Notation. For simplicity, from now on, we use $w = \rho(\ell)e^{i\theta}$ to denote the degree $+1$ vortex. We also assume that $\varrho \in (0, 1)$ is a fixed and small constant. □

The following properties of ρ are proved in [25].

Lemma 2.1. *The following asymptotic behaviors hold:*
(i) $\rho(0) = 0, \quad 0 < \rho(\ell) < 1, \quad \rho' > 0$ *for* $\ell > 0$,
(ii) $\rho(\ell) = 1 - c_0 \ell^{-1/2} e^{-\ell} + O(\ell^{-3/2} e^{-\ell})$ *as* $\ell \to +\infty$, *where* $c_0 > 0$. □

Setting $w = w_1 + iw_2$ and $z = x_1 + ix_2$, we need to study the following linearized operator of $\mathbb{S}_0$ in (3.20) around the standard profile w:

$$\begin{aligned} \mathbb{L}_0(\phi) = & \Delta\phi - \frac{4(w_1 \nabla w_1 + w_2 \nabla w_2)}{1 + |w|^2} \nabla\phi - \frac{4\nabla\langle w, \phi\rangle}{1 + |w|^2} \nabla w \\ & + \frac{8\langle w, \phi\rangle (w_1 \nabla w_1 + w_2 \nabla w_2)}{(1 + |w|^2)^2} \nabla w + \frac{4\langle \nabla w, \nabla\phi\rangle}{1 + |w|^2} w \\ & - \frac{4(1 + |\nabla w|^2)\langle w, \phi\rangle}{(1 + |w|^2)^2} w + \frac{2|\nabla w|^2}{1 + |w|^2} \phi + \frac{1 - |w|^2}{1 + |w|^2} \phi. \end{aligned} \tag{2.3}$$

The nondegeneracy of w is contained in the following lemma [34].

Lemma 2.2. *Suppose that*

$$\mathbb{L}_0[\phi] = 0, \tag{2.4}$$

where $\phi = iw\psi$, *and* $\psi = \psi_1 + i\psi_2$ *satisfies the decaying estimates*

$$\begin{aligned} |\psi_1| + |z||\nabla\psi_1| &\leq C(1 + |z|)^{-\varrho}, \\ |\psi_2| + |z||\nabla\psi_2| &\leq C(1 + |z|)^{-1-\varrho}, \end{aligned} \tag{2.5}$$

for some $0 < \varrho < 1$. *Then*

$$\phi = c_1 \frac{\partial w}{\partial x_1} + c_2 \frac{\partial w}{\partial x_2}$$

for certain real constants c_1, c_2. □

3. Symmetric formulation of the problem

For simplicity of notation, here we consider only the case $N = 3$ in the sequel. Recall that by setting

$$u = \frac{m_1 + im_2}{1 + m_3} \quad \text{and} \quad c = \delta|\log\epsilon|, \tag{3.1}$$

we have written problem (1.4) in the form (cf. (2.1))

$$i\delta\epsilon \log\frac{1}{\epsilon}\frac{\partial u}{\partial s_3} + \Delta u + \frac{1-|u|^2}{1+|u|^2}u = \frac{2\bar{u}}{1+|u|^2}\nabla u \cdot \nabla u. \tag{3.2}$$

Here and throughout the paper, we use $\bar{u}$ to denote the conjugate of u. The parameter δ is a constant to be chosen in (3.10).

By using the cylinder coordinates $s_1 = r\cos\Psi$, $s_2 = r\sin\Psi$, $s_3 = s_3$, equation (3.2) is transformed into

$$\begin{aligned} &i\delta\epsilon \log\frac{1}{\epsilon}\frac{\partial u}{\partial s_3} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \Psi^2} + \frac{\partial^2 u}{\partial s_3^2} + \frac{1-|u|^2}{1+|u|^2}u \\ &= \frac{2\bar{u}}{1+|u|^2}\nabla u \cdot \nabla u. \end{aligned} \tag{3.3}$$

For problem (3.3), we want to find a solution u which has a vortex helix directed along the curve in the form

$$\alpha \in \mathbb{R} \mapsto (d\cos\alpha, d\sin\alpha, \lambda\alpha) \in \mathbb{R}^3,$$

with two parameters

$$d = \frac{\hat{d}}{\epsilon} > 0, \quad \lambda = \frac{\hat{\lambda}}{\epsilon} \neq 0. \tag{3.4}$$

Moreover, u is also invariant under the screw motion

$$(r, \Psi, s_3) \mapsto (r, \Psi + \alpha, s_3 + \lambda\alpha), \quad \forall\, \alpha \in \mathbb{R}. \tag{3.5}$$

Hence, u has the symmetry

$$u(r, \Psi, s_3) = u(r, \Psi + \alpha, s_3 + \lambda\alpha) = u(r, 0, s_3 - \lambda\Psi) \tag{3.6}$$

and also satisfies the problem

$$\begin{aligned} &i\delta\epsilon \log\frac{1}{\epsilon}\frac{\partial u}{\partial s_3} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \Big(1 + \frac{\lambda^2}{r^2}\Big)\frac{\partial^2 u}{\partial s_3^2} + \frac{1-|u|^2}{1+|u|^2}u \\ &= \frac{2\bar{u}}{1+|u|^2}\left[\Big(\frac{\partial u}{\partial r}\Big)^2 + \Big(1 + \frac{\lambda^2}{r^2}\Big)\Big(\frac{\partial u}{\partial s_3}\Big)^2\right], \end{aligned} \tag{3.7}$$

which will be defined on the region $\big\{(r, s_3) \in [0, \infty) \times (-\lambda\pi, \lambda\pi)\big\}$.

Before going further, we provide a remark to show the method for the higher dimensional case.

Remark 3.1. We assume that $N = 2m + 3$ with integer $m \geq 1$. By stereographic projection, we can write problem (1.4) in the form (cf. (2.1))

$$i\delta\epsilon \log\frac{1}{\epsilon}\frac{\partial u}{\partial s_N} + \Delta u + \frac{1-|u|^2}{1+|u|^2}u = \frac{2\bar{u}}{1+|u|^2}\nabla u \cdot \nabla u \quad \text{in } \mathbb{R}^N. \tag{3.8}$$

According to [12], there are helix submanifolds in $\mathbb{R}^N$. For the expressions of helix submanifolds, we introduce the coordinates

$$\begin{aligned}(s_1, \cdots, s_{m+1}) &= \frac{r}{\sqrt{2}}\big(\check{\xi}\cos\Psi - \hat{\xi}\sin\Psi\big),\\ (s_1, \cdots, s_{2m+2}) &= \frac{r}{\sqrt{2}}\big(\check{\xi}\sin\Psi + \hat{\xi}\cos\Psi\big),\\ s_{2m+3} &= s_{2m+3},\end{aligned}$$

where $(r,\, \Psi,\, \check{\xi},\, \hat{\xi},\, s_{2m+3}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{S}^m\times\mathbb{S}^m\times\mathbb{R}$. Note that $\check{\xi},\, \hat{\xi} \in \mathbb{S}^m$ are vectors in $\mathbb{R}^{m+1}$. The metric in $\mathbb{R}^N$ can be expressed in the form

$$\mathrm{d}r^2 \,+\, r^2\mathrm{d}\Psi^2 \,+\, \frac{r^2}{2}\mathrm{d}\check{\xi}^2 \,+\, \frac{r^2}{2}\mathrm{d}\hat{\xi}^2.$$

Whence, we get the expressions of the Beltrami-Laplace operator

$$\triangle \,=\, \frac{\partial^2}{\partial r^2} \,+\, \frac{2m+1}{r}\frac{\partial}{\partial r} \,+\, \frac{1}{r^2}\frac{\partial^2}{\partial\Psi^2} \,+\, \frac{2}{r^2}\triangle_{\check{\xi}} \,+\frac{2}{r^2}\triangle_{\hat{\xi}},$$

where $\triangle_{\check{\xi}}$ and $\triangle_{\hat{\xi}}$ are Beltrami-Laplace operators on $\mathbb{S}^m$.

Equation (3.8) is transformed into

$$\begin{aligned}&i\delta\epsilon\log\frac{1}{\epsilon}\frac{\partial u}{\partial s_N} \,+\, \frac{\partial^2 u}{\partial r^2} \,+\, \frac{2m+1}{r}\frac{\partial u}{\partial r} \,+\, \frac{1}{r^2}\frac{\partial^2 u}{\partial\Psi^2} \,+\, \frac{2}{r^2}\triangle_{\check{\xi}} \,+\frac{2}{r^2}\triangle_{\hat{\xi}} \,+\, \frac{\partial^2 u}{\partial s_N^2}\\ &\quad+\, \frac{1-|u|^2}{1+|u|^2}u \,=\, \frac{2\bar{u}}{1+|u|^2}\nabla u\cdot\nabla u.\end{aligned} \tag{3.9}$$

For problem (3.9), we want to find a solution u which has a vortex on the helix submanifold in the form

$$\alpha\in\mathbb{R}\mapsto\Big(d\big(\check{\xi}\cos\alpha - \hat{\xi}\sin\alpha\big), d\big(\check{\xi}\sin\alpha + \hat{\xi}\cos\alpha\big), \lambda\alpha\Big)\in\mathbb{R}^N,$$

where $\check{\xi},\, \hat{\xi}\in\mathbb{S}^m$ are vectors in $\mathbb{R}^{m+1}$ and the two parameters are

$$d = \frac{\hat{d}}{\epsilon} > 0, \quad \lambda = \frac{\hat{\lambda}}{\epsilon} \neq 0.$$

Moreover, u is also invariant under the screw motion

$$(r, \Psi, \check{\xi}, \hat{\xi}, s_N) \mapsto (r, \Psi+\alpha, \check{\xi}, \hat{\xi}, s_N + \lambda\alpha), \quad \forall\,\alpha\in\mathbb{R}.$$

Hence, u has the symmetry

$$u(r, \Psi, \check{\xi}, \hat{\xi}, s_N) \,=\, u(r, \Psi+\alpha, \check{\xi}, \hat{\xi}, s_N+\lambda\alpha) \,=\, u(r, 0, \check{\xi}_0, \hat{\xi}_0, s_N - \lambda\Psi),$$

where $\check{\xi}_0,\, \hat{\xi}_0$ are the north pole of $\mathbb{S}^m$. This will derive an equation like (3.7). □

We go back to the case of dimension $N = 3$. Recall the parameters in (3.4) and then set

$$\sigma \,=\, \frac{\hat{\lambda}}{\hat{d}}, \quad \gamma \,=\, \sqrt{1+\sigma^2}, \quad \delta \,=\, \frac{1}{\sqrt{\hat{d}^2+\hat{\lambda}^2}} \,=\, \frac{1}{\hat{d}\gamma}. \tag{3.10}$$

It is worth mentioning that these three positive parameters are of independence of ϵ. For further convenience of notation, we also introduce the rescaling

$$(r, s_3) = (x_1, \gamma x_2), \quad z = x_1 + ix_2. \tag{3.11}$$

Thus (3.7) becomes

$$
\begin{aligned}
&i\delta\epsilon \log\frac{1}{\epsilon}\gamma^{-1}\frac{\partial u}{\partial x_2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial u}{\partial x_1} + \Big(1+\frac{\lambda^2}{x_1^2}\Big)\gamma^{-2}\frac{\partial^2 u}{\partial x_2^2} + \frac{1-|u|^2}{1+|u|^2}u \\
&= \frac{2\bar{u}}{1+|u|^2}\left[\Big(\frac{\partial u}{\partial x_1}\Big)^2 + \Big(1+\frac{\lambda^2}{x_1^2}\Big)\gamma^{-2}\Big(\frac{\partial u}{\partial x_2}\Big)^2\right].
\end{aligned}
\tag{3.12}
$$

By using the symmetries, in the sequel we shall consider the problem on the region

$$
\mathfrak{S} = \big\{(x_1,x_2) : x_1\in\mathbb{R},\ x_2\in(-\lambda\pi/\gamma,\lambda\pi/\gamma)\big\},
\tag{3.13}
$$

and then impose the boundary conditions

$$
\begin{aligned}
|u(z)| &\to 1 \quad \text{as } |x_1|\to+\infty, \\
\frac{\partial u}{\partial x_1}(0,x_2) &= 0, \quad \forall\, x_2\in(-\lambda\pi/\gamma,\lambda\pi/\gamma), \\
u(x_1,-\lambda\pi/\gamma) &= u(x_1,\lambda\pi/\gamma), \quad \forall\, x_1\in\mathbb{R}, \\
u_{x_2}(x_1,-\lambda\pi/\gamma) &= u_{x_2}(x_1,\lambda\pi/\gamma), \quad \forall\, x_1\in\mathbb{R}.
\end{aligned}
\tag{3.14}
$$

Before finishing this section, some words are in order to explain the strategies of solving problem (3.12) with boundary conditions in (3.14). It is easy to see that problem (3.12) is invariant under the following two transformations:

$$
u(z)\to\overline{u(\bar{z})},\quad u(z)\to u(-\bar{z}).
\tag{3.15}
$$

Thus we impose the following symmetry on the solution u:

$$
\Pi := \Big\{\, u(z)=\overline{u(\bar{z})},\ u(z)=u(-\bar{z})\,\Big\}.
\tag{3.16}
$$

This symmetry will play an important role in our analysis. As a conclusion, if we write

$$
u(x_1,x_2) = u_1(x_1,x_2) + iu_2(x_1,x_2),
$$

then u_1 and u_2 enjoy the following conditions:

$$
\begin{aligned}
u_1(x_1,x_2) &= u_1(-x_1,x_2), \quad u_1(x_1,x_2) = u_1(x_1,-x_2), \\
u_2(x_1,x_2) &= u_2(-x_1,x_2), \quad u_2(x_1,x_2) = -u_2(x_1,-x_2), \\
\frac{\partial u_1}{\partial x_1}(0,x_2) &= 0, \quad \frac{\partial u_2}{\partial x_1}(0,x_2) = 0, \\
u_1(x_1,-\lambda\pi/\gamma) &= u_1(x_1,\lambda\pi/\gamma), \quad \frac{\partial u_2}{\partial x_2}(x_1,-\lambda\pi/\gamma) = \frac{\partial u_2}{\partial x_2}(x_1,\lambda\pi/\gamma), \\
\frac{\partial u_1}{\partial x_2}(x_1,-\lambda\pi/\gamma) &= \frac{\partial u_1}{\partial x_2}(x_1,\lambda\pi/\gamma) = 0, \quad u_2(x_1,-\lambda\pi/\gamma) = u_2(x_1,\lambda\pi/\gamma) = 0.
\end{aligned}
\tag{3.17}
$$

Problem (3.12)-(3.14) becomes a two-dimensional problem with symmetries defined in (3.16). The key point is then to construct a solution with a vortex of degree $+1$ at $(d,0)$ and its antipair of degree -1 at $(-d,0)$. In addition to the computations for standard vortices in the two-dimensional case, there are two extra derivative terms:

$$
\frac{1}{x_1}\frac{\partial u}{\partial x_1} \quad \text{and} \quad \gamma^{-2}\Big(1+\frac{\lambda^2}{x_1^2}\Big)\frac{\partial^2 u}{\partial x_2^2}.
$$

Moreover, we shall improve the approximate solution to satisfy the boundary conditions in (3.17). To the end of constructing vortex pairs located at $(d,0)$ and $(-d,0)$, we write

$$\frac{\partial^2 u}{\partial x_1^2} + \Big(1+\frac{\lambda^2}{x_1^2}\Big)\gamma^{-2}\frac{\partial^2 u}{\partial x_2^2} = \triangle + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial^2 u}{\partial x_2^2}, \tag{3.18}$$

$$\Big(\frac{\partial u}{\partial x_1}\Big)^2 + \Big(1+\frac{\lambda^2}{x_1^2}\Big)\gamma^{-2}\Big(\frac{\partial u}{\partial x_2}\Big)^2 = \nabla u\cdot\nabla u + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\Big(\frac{\partial u}{\partial x_2}\Big)^2, \tag{3.19}$$

and then set

$$\begin{aligned}
\mathbb{S}_0[u] :&= \Delta u + F(u) - \frac{2\bar{u}}{1+|u|^2}\nabla u\cdot\nabla u,\\
\mathbb{S}_1[u] :&= -\frac{2\bar{u}}{1+|u|^2}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\Big(\frac{\partial u}{\partial x_2}\Big)^2,\\
\mathbb{S}_2[u] :&= \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial^2 u}{\partial x_2^2},\\
\mathbb{S}_3[u] :&= \frac{1}{x_1}\frac{\partial u}{\partial x_1}, \qquad \mathbb{S}_4[u] := i\delta\gamma^{-1}\epsilon|\log\epsilon|\frac{\partial u}{\partial x_2},\\
\mathbb{S}[u] :&= \mathbb{S}_0[u] + \mathbb{S}_1[u] + \mathbb{S}_2[u] + \mathbb{S}_3[u] + \mathbb{S}_4[u].
\end{aligned} \tag{3.20}$$

In the above $F(u) = \frac{1-|u|^2}{1+|u|^2}u$.

It is worth mentioning that problem (3.12) is also degenerate when $x_1 = 0$ due to the term

$$\gamma^{-2}\Big(1+\frac{\lambda^2}{x_1^2}\Big)\frac{\partial^2 u}{\partial x_2^2}.$$

We need more analysis to handle the singularity as $x_1 \to 0$. Whence, for the convenience of further careful analysis, we divide the region $\mathfrak{S}$ (cf. (3.13)) into three parts:

$$\begin{aligned}
\mathfrak{S}_+ &= \Big\{(x_1,x_2) : x_1 > 2\varpi/\epsilon,\ -\lambda\pi/\gamma < x_2 < \lambda\pi/\gamma\Big\},\\
\mathfrak{S}_0 &= \Big\{(x_1,x_2) : -2\varpi/\epsilon < x_1 < 2\varpi/\epsilon,\ -\lambda\pi/\gamma < x_2 < \lambda\pi/\gamma\Big\},\\
\mathfrak{S}_- &= \Big\{(x_1,x_2) : x_1 < -2\varpi/\epsilon,\ -\lambda\pi/\gamma < x_2 < \lambda\pi/\gamma\Big\}.
\end{aligned} \tag{3.21}$$

Here ϖ is a small positive constant less than $1/1000$.

4. Approximate solutions

4.1. **First approximate solution and its error.** Recall the vortex solutions w^+ and w^- defined in (2.2), and also the parameters in (3.4). Define the smooth cut-off function $\tilde{\eta}$ in the form

$$\tilde{\eta}(s) = 1 \quad \text{for} \quad |s| \le 1, \qquad \tilde{\eta}(s) = 0 \quad \text{for} \quad |s| \ge 2. \tag{4.1}$$

For the convenience of notation, for $\vec{e}_1 = (1,0)$ and any given $(x_1,x_2) \in \mathbb{R}^2$, let $\theta_{d\vec{e}_1}$ and $\theta_{-d\vec{e}_1}$ be respectively the angle arguments of the vectors $x - d\vec{e}_1 = (x_1 - d, x_2)$

and $x + d\vec{e}_1 = (x_1 + d, x_2)$ in the (x_1, x_2) plane. We also let

$$\ell_1(x_1, x_2) = \sqrt{(x_1 - d)^2 + x_2^2}, \qquad \ell_2(x_1, x_2) = \sqrt{(x_1 + d)^2 + x_2^2} \tag{4.2}$$

be the distance functions between the point (x_1, x_2) and the pair of vortices located at the points $d\vec{e}_1$ and $-d\vec{e}_1$. We also write

$$\tilde{\rho} = \rho(\ell_1)\rho(\ell_2), \qquad \varphi_0 = \theta_{d\vec{e}_1} - \theta_{-d\vec{e}_1}. \tag{4.3}$$

For each fixed $d := \hat{d}/\epsilon$ with $\hat{d} \in [1/100,\ 100]$, we define the **first approximate solution**

$$v_0(z) \ := \ \eta_\epsilon(x_1)\, e^{i\varphi_0} \ + \ \big(1 - \eta_\epsilon(x_1)\big)\tilde{\rho}\, e^{i\varphi_0}, \tag{4.4}$$

where $\eta_\epsilon(x_1) = \tilde{\eta}(\epsilon|x_1|/\varpi)$. As in (3.21), here ϖ is the small positive constant less than 1/1000. A simple computation shows that

$$|v_0(z)| \to 1 \text{ as } |z| \to +\infty. \tag{4.5}$$

In fact, for $|z| \gg d$, we have

$$\begin{aligned} v_0 \ &\approx \ e^{i\theta_{d\vec{e}_1} - i\theta_{-d\vec{e}_1}} \\ &\approx \ \frac{(x_1^2 - d^2 + x_2^2 + 2ix_2 d)}{\sqrt{(x_1 - d)^2 + x_2^2}\sqrt{(x_1 + d)^2 + x_2^2}} \\ &\approx \ 1 + O\Big(\frac{d^2}{|z|^2} + \frac{d}{|z|}\Big). \end{aligned}$$

If $\theta_{\tilde{\xi}}$ denotes the angle argument around $\tilde{\xi}$ and $r_{\tilde{\xi}} = |z - \tilde{\xi}|$, it is easy to see that

$$\nabla r_{\tilde{\xi}} = \frac{1}{|z - \tilde{\xi}|}(z - \tilde{\xi}), \quad \nabla\theta_{\tilde{\xi}} = \frac{1}{|z - \tilde{\xi}|^2}(z - \tilde{\xi})^\perp, \quad |\nabla\theta_{\tilde{\xi}}| = \frac{1}{|z - \tilde{\xi}|}, \tag{4.6}$$

where we denote $z^\perp = (-x_2, x_1)$.

Recall the operators defined in (3.20) and the decompositions of $\mathfrak{S}$ in (3.21). One of the main objectives of this subsection is to compute the error $\mathbb{S}[v_0]$. We start the computations on the region $\mathfrak{S}_+$. Note that on $\mathfrak{S}_+$, there hold $\eta_\epsilon = 0$ and

$$v_0 = \tilde{\rho}\, e^{i\varphi_0}.$$

Moreover, by our construction and the properties of ρ, we have

$$\begin{aligned} v_0 \ &= \ w(z - d\vec{e}_1)e^{-i\theta_{-d\vec{e}_1}}\Big[1 + O(e^{-d/2 - \ell_1/2})\Big], \\ |v_0| \ &= \ \rho(|z - d\vec{e}_1|)\Big[1 + O(e^{-d/2 - \ell_1/2})\Big]. \end{aligned}$$

We now obtain the estimates

$$\Delta v_0 \ = \ \Big(\Delta w - 2i\nabla w \cdot \nabla\theta_{-d\vec{e}_1} - w\nabla\theta_{-d\vec{e}_1} \cdot \nabla\theta_{-d\vec{e}_1} + O(e^{-d/2 - \ell_1/2})\Big)\, e^{-i\theta_{-d\vec{e}_1}},$$

$$\begin{aligned} -\frac{2\bar{v}}{1 + |v|^2}\nabla v_0 \cdot \nabla v_0 \ &= \ \frac{-2\bar{w}}{1 + \rho^2}e^{-i\theta_{-d\vec{e}_1}}\Big[\nabla w \cdot \nabla w - 2iw\nabla w \cdot \nabla\theta_{-d\vec{e}_1}\Big] \\ &\quad + \frac{2\bar{w}}{1 + \rho^2}e^{-i\theta_{-d\vec{e}_1}}\, w^2\nabla\theta_{-d\vec{e}_1} \cdot \nabla\theta_{-d\vec{e}_1} + O(e^{-d/2 - \ell_1/2}), \end{aligned}$$

and then also

$$\frac{1 - |v_0|^2}{1 + |v_0|^2}v_0 \ = \ \frac{1 - \rho^2}{1 + \rho^2}w\,\big(1 + O(e^{-d/2 - \ell_1/2})\big)e^{-i\theta_{-d\vec{e}_1}}.$$

Combining the estimates above, we have for $z \in \mathfrak{S}_+$,

$$\begin{aligned}\mathbb{S}_0[v_0] &= \left[\frac{2(\rho^2-1)}{\rho^2+1} i\nabla w\cdot\nabla\theta_{-d\vec{e}_1} + \frac{\rho^2-1}{\rho^2+1} w\,\nabla\theta_{-d\vec{e}_1}\cdot\nabla\theta_{-d\vec{e}_1}\right] e^{-i\theta_{-d\vec{e}_1}} \\ &\quad + O(e^{-d/2-\ell_1/2}) \\ &= \Omega_{10}\, e^{i\varphi_0},\end{aligned}$$

where

$$\begin{aligned}\Omega_{10} \equiv &-\frac{2\rho(\rho^2-1)}{\rho^2+1}\nabla\theta_{d\vec{e}_1}\cdot\nabla\theta_{-d\vec{e}_1} + \frac{\rho(\rho^2-1)}{\rho^2+1}\nabla\theta_{-d\vec{e}_1}\cdot\nabla\theta_{-d\vec{e}_1} \\ &+ i\frac{2(\rho^2-1)}{\rho^2+1}\nabla\rho\cdot\nabla\theta_{-d\vec{e}_1} + O(e^{-d/2-\ell_1/2}).\end{aligned} \tag{4.7}$$

The calculation for the terms $\mathbb{S}_1[v_0]$ and $\mathbb{S}_2[v_0]$ proceeds as

$$\begin{aligned}\mathbb{S}_1[v_0] &= -\frac{2\bar{v}_0}{1+|v_0|^2}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\left\{\frac{\partial}{\partial x_2}\left(\tilde{\rho}e^{i\varphi_0}\right)\right\}^2 \\ &= -\frac{2\tilde{\rho}}{1+|\tilde{\rho}|^2}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\left(\left|\frac{\partial\tilde{\rho}}{\partial x_2}\right|^2 - |\tilde{\rho}|^2\left|\frac{\partial\varphi_0}{\partial x_2}\right|^2 + 2i\tilde{\rho}\frac{\partial\tilde{\rho}}{\partial x_2}\frac{\partial\varphi_0}{\partial x_2}\right)e^{i\varphi_0}\end{aligned}$$

and

$$\begin{aligned}\mathbb{S}_2[v_0] &= \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial^2}{\partial x_2^2}\left[\tilde{\rho}e^{i\varphi_0}\right] \\ &= \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\left[\frac{\partial^2\tilde{\rho}}{\partial x_2^2} + 2i\frac{\partial\tilde{\rho}}{\partial x_2}\frac{\partial\varphi_0}{\partial x_2} - \tilde{\rho}\left|\frac{\partial\varphi_0}{\partial x_2}\right|^2\right]e^{i\varphi_0} \\ &\quad + i\,v_0\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial^2\varphi_0}{\partial x_2^2}.\end{aligned}$$

Whence, there holds

$$\mathbb{S}_1[v_0] + \mathbb{S}_2[v_0] = \Omega_{11}\, e^{i\varphi_0} + i\,v_0\,\mathbb{S}_2[\varphi_0],$$

where

$$\begin{aligned}\Omega_{11} \equiv &\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\left(-\frac{2\tilde{\rho}}{1+|\tilde{\rho}|^2}\left|\frac{\partial\tilde{\rho}}{\partial x_2}\right|^2 + \frac{\partial^2\tilde{\rho}}{\partial x_2^2}\right) \\ &+ \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{1-|\tilde{\rho}|^2}{1+|\tilde{\rho}|^2}\left[-\tilde{\rho}\left|\frac{\partial\varphi_0}{\partial x_2}\right|^2 + 2i\frac{\partial\tilde{\rho}}{\partial x_2}\frac{\partial\varphi_0}{\partial x_2}\right].\end{aligned} \tag{4.8}$$

Here we need more analysis on the last term $i\,v_0\,\mathbb{S}_2[\varphi_0]$ in the above formula. Note that

$$\frac{\partial^2\varphi_0}{\partial x_2^2} = \frac{\partial^2\theta_{d\vec{e}_1}}{\partial x_2^2} - \frac{\partial^2\theta_{-d\vec{e}_1}}{\partial x_2^2} = \frac{-2(x_1-d)x_2}{\ell_1^4} - \frac{-2(x_1+d)x_2}{\ell_2^4}. \tag{4.9}$$

In the neighborhood of $d\vec{e}_1$ there holds

$$\begin{aligned}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right] &= -\frac{2\sigma^2}{d\gamma^2}(x_1-d)+\frac{\sigma^2}{d\,\gamma^2}\,\frac{3d(x_1-d)^2+2(x_1-d)^3}{x_1^2}\\ &= -\frac{2\sigma^2}{d\gamma^2}(x_1-d)+O\big((x_1-d)^2/d^2\big).\end{aligned}\tag{4.10}$$

Whence there is a singularity in $\mathbb{S}_2[\varphi_0]$ of the form

$$-\frac{2\sigma^2}{d\gamma^2}(x_1-d)\frac{\partial^2\theta_{d\vec{e}_1}}{\partial x_2^2} = -\frac{2\sigma^2}{d\gamma^2}y_1\,\frac{\partial^2\theta_{d\vec{e}_1}}{\partial y_2^2} = \frac{4\sigma^2}{d\gamma^2}\frac{y_1^2y_2}{|y|^4}\tag{4.11}$$

with variable

$$y = x - d\vec{e}_1.\tag{4.12}$$

A similar singularity exists in the neighborhood of $-d\vec{e}_1$:

$$\frac{2\sigma^2}{d\gamma^2}(x_1+d)\frac{\partial^2\theta_{-d\vec{e}_1}}{\partial x_2^2} = -\frac{2\sigma^2}{d\gamma^2}\tilde{y}_1\,\frac{\partial^2\theta_{-d\vec{e}_1}}{\partial \tilde{y}_2^2} = -\frac{4\sigma^2}{d\gamma^2}\frac{\tilde{y}_1^2\tilde{y}_2}{|\tilde{y}|^4}\tag{4.13}$$

with the variable $\tilde{y} = x + d\vec{e}_1$.

The term $\mathbb{S}_3[v_0]$ obeys the following asymptotic behavior:

$$\begin{aligned}\mathbb{S}_3[v_0] &= \frac{x_1-d}{x_1\ell_1}\frac{\rho'(\ell_1)}{\rho(\ell_1)}v_0+\frac{x_1+d}{x_1\ell_2}\frac{\rho'(\ell_2)}{\rho(\ell_2)}v_0+iv_0\,\frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1}\\ &\equiv \Omega_{12}\,e^{i\varphi_0}+i\,v_0\,\mathbb{S}_3[\varphi_0],\end{aligned}$$

where

$$\Omega_{12} = \frac{x_1-d}{x_1\ell_1}\rho'(\ell_1)\rho(\ell_2)+\frac{x_1+d}{x_1\ell_2}\rho'(\ell_2)\rho(\ell_1).\tag{4.14}$$

By the computation

$$\mathbb{S}_3[\varphi_0] = \frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1} = \frac{1}{x_1}\frac{\partial\theta_{d\vec{e}_1}}{\partial x_1}-\frac{1}{x_1}\frac{\partial\theta_{-d\vec{e}_1}}{\partial x_1} = \frac{1}{x_1}\Big(\frac{-x_2}{\ell_1^2}+\frac{x_2}{\ell_2^2}\Big),\tag{4.15}$$

we find that it is a singular term. More precisely, in the neighborhood of $d\vec{e}_1$ with variable $y = x - d\vec{e}_1$, there is a singularity in $\mathbb{S}_3[\varphi_0]$ of the form

$$\frac{1}{x_1}\frac{\partial\theta_{d\vec{e}_1}}{\partial x_1} = \Big[\frac{1}{d}-\frac{y_1}{d(d+y_1)}\Big]\frac{\partial\theta_{d\vec{e}_1}}{\partial y_1}\qquad\text{with}\quad \frac{1}{d}\frac{\partial\theta_{d\vec{e}_1}}{\partial y_1} = -\frac{1}{d}\frac{y_2}{|y|^2}.\tag{4.16}$$

A similar singularity exists in the neighborhood of $-d\vec{e}_1$ with the variable $\tilde{y} = x + d\vec{e}_1$:

$$-\frac{1}{x_1}\frac{\partial\theta_{-d\vec{e}_1}}{\partial x_1} = \Big[-\frac{1}{d}+\frac{\tilde{y}_1}{d(\tilde{y}_1-d)}\Big](-1)\,\frac{\partial\theta_{-d\vec{e}_1}}{\partial\tilde{y}_1}\tag{4.17}$$

with

$$\frac{1}{d}\,\frac{\partial\theta_{-d\vec{e}_1}}{\partial\tilde{y}_1} = -\frac{1}{d}\frac{\tilde{y}_2}{|\tilde{y}|^2}.$$

We can estimate the term in $\mathbb{S}_4[v_0]$ as

$$i\delta\gamma^{-1}\epsilon\log\frac{1}{\epsilon}\frac{\partial v_0}{\partial x_2} = i\delta\gamma^{-1}\epsilon\log\frac{1}{\epsilon}\frac{\partial}{\partial x_2}\left[\rho(\ell_1)e^{i\varphi_0}+O(e^{-d/2-\ell_1/2})\right] = \Omega_{13}\,e^{i\varphi_0},$$

where

$$\Omega_{13} \equiv i\delta\gamma^{-1}\epsilon\log\frac{1}{\epsilon}\left[\frac{\partial\rho(\ell_1)}{\partial x_2} + i\rho(\ell_1)\frac{\partial\varphi_0}{\partial x_2}\right] + O\Big(\epsilon\log\frac{1}{\epsilon}\,e^{-d/2-\ell_1/2}\Big). \tag{4.18}$$

In summary, we have obtained for $z \in \mathfrak{S}_+$,

$$\begin{aligned}\mathbb{S}[v_0] &= \mathbb{S}_0[v_0] + \mathbb{S}_1[v_0] + \mathbb{S}_2[v_0] + \mathbb{S}_3[v_0] + \mathbb{S}_4[v_0]\\ &= \sum_{i=1}^{3}\Omega_{1i}\,e^{i\varphi_0} + i\,v_0\,\mathbb{S}_2[\varphi_0] + i\,v_0\,\mathbb{S}_3[\varphi_0].\end{aligned} \tag{4.19}$$

A similar (and almost identical) estimate also holds in the region $\mathfrak{S}_-$. For the convenience of later use, we denote that for $z \in \mathfrak{S}_+ \cup \mathfrak{S}_-$,

$$\hat{E} \equiv \mathbb{S}[v_0] - i\,v_0\,\mathbb{S}_2[\varphi_0] - i\,v_0\,\mathbb{S}_3[\varphi_0] = \sum_{i=0}^{3}\Omega_{1i}\,e^{i\varphi_0}. \tag{4.20}$$

We now compute the error on $\mathfrak{S}_0$. In this region, we only do the computation on the region $\{-\varpi/\epsilon < x_1 < \varpi/\epsilon\}$, where $\eta_\epsilon = 1$ and $v_0 = e^{i\varphi_0}$. Note that $\mathbb{S}_0[v_0] = 0$. On the other hand,

$$\begin{aligned}\mathbb{S}_1[v_0] + \mathbb{S}_2[v_0] &\equiv \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\left(-\frac{2\bar{v}_0}{1+|v_0|^2}\Big(\frac{\partial v_0}{\partial x_2}\Big)^2 + \frac{\partial^2 v_0}{\partial x_2^2}\right)\\ &= iv_0\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\frac{\partial^2\varphi_0}{\partial x_2^2}.\end{aligned}$$

We give more explicit computations:

$$\begin{aligned}\frac{\partial^2\varphi_0}{\partial x_2^2} &= \frac{-2(x_1-d)x_2}{\ell_1^4} - \frac{-2(x_1+d)x_2}{\ell_2^4}\\ &= \frac{4dx_2}{(d^2+x_2^2)^2} - \frac{8x_2x_1^2d(\ell_1+\ell_2)}{\ell_1^4\ell_2^4}\\ &\quad - \frac{2x_2x_1^2d}{\ell_1^4\ell_2^4(d^2+x_2^2)^2}\left[\ell_1^4(d^2+x_2^2+\ell_2^2) + \ell_2^4(d^2+x_2^2+\ell_1^2)\right]\\ &\quad + \frac{16x_2x_1^2d^3}{\ell_1^4\ell_2^4(d^2+x_2^2)}\left[(\ell_1^2+\ell_2^2)(d^2+x_2^2) + \ell_2^2\ell_1^2\right].\end{aligned} \tag{4.21}$$

We now write

$$\mathbb{S}_1[v_0] + \mathbb{S}_2[v_0] \equiv \Omega_4 + i\,v_0\,\mathbb{S}_2[\arctan(x_2/d)]. \tag{4.22}$$

Here if x_1 approaches 0, there is another singular term in the form

$$i\,v_0\,\mathbb{S}_2[\arctan(x_2/d)] = i\,v_0\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\frac{4dx_2}{(d^2+x_2^2)^2}. \tag{4.23}$$

In the next subsection, we will introduce a correction in the phase term to get rid of these three mentioned singularities.

The term $\mathbb{S}_3[v_0]$ on $\mathfrak{S}_0$ obeys the asymptotic behavior

$$\mathbb{S}_3[v_0] = \frac{1}{x_1}\frac{\partial v_0}{\partial x_1} = iv_0\,\frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1}.$$

By the computation

$$\frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1} = \frac{1}{x_1}\frac{\partial\theta_{d\vec{e}_1}}{\partial x_1} - \frac{1}{x_1}\frac{\partial\theta_{-d\vec{e}_1}}{\partial x_1} = \frac{1}{x_1}\Big(\frac{-x_2}{\ell_1^2}+\frac{x_2}{\ell_2^2}\Big) = \frac{4\,d\,x_2}{\ell_1^2\ell_2^2}. \tag{4.24}$$

Hence, due to the fact that v_0 is an even term on the variable x_1, one can check that $\mathbb{S}_3[v_0]$ does not contain singular components as $x_1 \to 0$.

We can estimate the term in $\mathbb{S}_4[v_0]$ on $\mathfrak{S}_0$ as

$$\begin{aligned} i\delta\gamma^{-1}\epsilon\log\frac{1}{\epsilon}\frac{\partial v_0}{\partial x_2} &= -\delta\gamma^{-1}\epsilon\log\frac{1}{\epsilon}\,v_0\,\frac{\partial\varphi_0}{\partial x_2} \\ &= \delta\gamma^{-1}\epsilon\log\frac{1}{\epsilon}\,v_0\left[\frac{\partial\theta_{d\vec{e}_1}}{\partial x_2}-\frac{\partial\theta_{-d\vec{e}_1}}{\partial x_2}\right] \\ &= O(\epsilon^2|\log\epsilon|), \end{aligned}$$

due to the facts

$$\begin{aligned} \frac{\partial\theta_{d\vec{e}_1}}{\partial x_2} &= \frac{x_1-d}{(x_1-d)^2+x_2^2} = O(d^{-1}) = O(\epsilon), \\ \frac{\partial\theta_{-d\vec{e}_1}}{\partial x_2} &= \frac{x_1+d}{(x_1+d)^2+x_2^2} = O(d^{-1}) = O(\epsilon). \end{aligned}$$

Adding all terms gives that, for $z \in \mathfrak{S}_0$,

$$\mathbb{S}[v_0] = \Omega_4 + i\,v_0\,\mathbb{S}_2[\arctan(x_2/d)] + iv_0\,\frac{4\,d\,x_2}{\ell_1^2\ell_2^2} - \delta\gamma^{-1}\epsilon\log\frac{1}{\epsilon}\,v_0\,\frac{\partial\varphi_0}{\partial x_2}. \tag{4.25}$$

Here we also denote for $z \in \mathfrak{S}_0$,

$$\hat{E} = \mathbb{S}[v_0] - i\,v_0\,\mathbb{S}_2[\arctan(x_2/d)]. \tag{4.26}$$

As a conclusion, a direct application of the above computations yields the decay estimates for the error $\hat{E}$.

Lemma 4.1. *It holds that for* $z \in \big(B_2(d\vec{e}_1)\cup B_2(-d\vec{e}_1)\big)^c \cap \mathfrak{S}$,

$$\Big|\mathrm{Re}\big(\hat{E}/(iv_0)\big)\Big| \le \frac{C\epsilon^{1-\varrho}}{(1+\ell_1)^3} + \frac{C\epsilon^{1-\varrho}}{(1+\ell_2)^3}, \tag{4.27}$$

$$\Big|\mathrm{Im}\big(\hat{E}/(iv_0)\big)\Big| \le \frac{C\epsilon^{1-\varrho}}{(1+\ell_1)^{1+\varrho}} + \frac{C\epsilon^{1-\varrho}}{(1+\ell_2)^{1+\varrho}}, \tag{4.28}$$

where $\varrho \in (0,1)$ *is a constant.*

Proof. Let us derive the estimate for $\Omega_{13}\,e^{i\varphi_0}/(iv_0)$ on $\mathfrak{S}_+$:

$$\begin{aligned} \Omega_{13}\,e^{i\varphi_0}/(iv_0) =& \delta\gamma^{-1}\,\epsilon\,\log\frac{1}{\epsilon}\frac{\partial\rho(\ell_1)}{\partial x_2}\frac{1}{\rho(\ell_1)} + i\delta\gamma^{-1}\,\epsilon\,\log\frac{1}{\epsilon}\frac{\partial\varphi_0}{\partial x_2} \\ &+ \epsilon\,\log\frac{1}{\epsilon}\,O\big(e^{-\ell_1}+e^{-\ell_2}\big) \\ =& \epsilon\,\log\frac{1}{\epsilon}\,O\big(e^{-\ell_1}+e^{-\ell_2}\big) + i\delta\gamma^{-1}\,\epsilon\,\log\frac{1}{\epsilon}\left[\frac{x_1-d}{\ell_1^2}-\frac{x_1+d}{\ell_2^2}\right]. \end{aligned}$$

The estimate for $\Omega_{13}\, e^{i\varphi_0}/(iv_0)$ like (4.27) then follows. Let us also notice that for $z \in \mathfrak{S}_+$ with $\ell_1 < d$,

$$\epsilon \log \frac{1}{\epsilon} \left| \frac{x_1 - d}{\ell_1^2} - \frac{x_1 + d}{\ell_2^2} \right| \le \frac{C\epsilon \log \frac{1}{\epsilon}}{(1+\ell_1)} \le \frac{C\epsilon^{1-\varrho}}{(1+\ell_1)^{1+\varrho}}.$$

On the other hand, for $z \in \mathfrak{S}_+$ with $\ell_1 > d$, we then have

$$\epsilon \log \frac{1}{\epsilon} \left| \frac{x_1 - d}{\ell_1^2} - \frac{x_1 + d}{\ell_2^2} \right| \le \frac{C\epsilon \log \frac{1}{\epsilon}}{(1+\ell_1)^2} \le \frac{C\epsilon^{1-\varrho}}{(1+\ell_1)^{1+\varrho}}.$$

Thus the estimate for $\Omega_{13}\, e^{i\varphi_0}/(iv_0)$ like (4.28) is proved. Similarly, the estimates for other terms in $\hat{E}$ follow from (4.7), (4.8), (4.14). □

4.2. **Further improvement of the approximation.** As we promised in the previous subsection, we now define a new correction φ_d in the phase term formally by

$$\varphi_d(z) = \varphi_s(z) + \varphi_1(z) + \varphi_2(z). \tag{4.29}$$

Then we will define an improved approximation in (4.43) and in Section 4.3 estimate its error by substituting it into (3.12).

Now we give the definitions of all terms in φ_d. By recalling the cut-off function $\eta_\epsilon(x_1)$ in (4.4) we set

$$\varphi_2 = 2\,\eta_\epsilon(x_1) \arctan(x_2/d) \tag{4.30}$$

to cancel the singular term in (4.23) in such a way that

$$\begin{aligned} \Delta\varphi_2 &= 2\,\eta_\epsilon(x_1)\, \frac{\partial^2}{\partial x_2^2} \arctan(x_2/d) + O(\epsilon^2) \\ &= -\,\eta_\epsilon(x_1)\, \frac{4\,d\,x_2}{(d^2 + x_2^2)^2} + O(\epsilon^2). \end{aligned} \tag{4.31}$$

To cancel the singularities in (4.16) and (4.11) rewritten in the form

$$\frac{y_2}{d|y|^2} - \frac{4\sigma^2}{d\gamma^2} \frac{y_1^2 y_2}{|y|^4} = \frac{y_2}{d\gamma^2|y|^2} - \frac{\sigma^2(3y_1^2 - y_2^2)y_2}{d\gamma^2|y|^4},$$

we want to find a function $\Phi(y_1, y_2)$ by solving the problem in the translated coordinates (y_1, y_2):

$$\frac{\partial^2\Phi}{\partial y_1^2} + \frac{\partial^2\Phi}{\partial y_2^2} = \frac{y_2}{d\gamma^2|y|^2} - \frac{\sigma^2(3y_1^2 - y_2^2)y_2}{d\gamma^2|y|^4} \quad \text{in } \mathbb{R}^2. \tag{4.32}$$

In fact, we can solve this problem by separation of variables and then obtain

$$\Phi(y_1, y_2) = \frac{1}{4d\gamma^2} y_2 \log|y|^2 - \frac{\sigma^2}{2d\gamma^2} \frac{y_2^3}{|y|^2} + \frac{3\sigma^2}{8d\gamma^2} y_2. \tag{4.33}$$

Let χ be a smooth cut-off function in a way such that $\chi(\vartheta) = 1$ for $\vartheta < \hat{d}/10$ and $\chi(\vartheta) = 0$ for $\vartheta > \hat{d}/5$. The singular part in φ_d is defined by

$$\begin{aligned} \varphi_s(z) := &\Big(\chi(\epsilon\ell_1) + \chi(\epsilon\ell_2)\Big)\Big[\frac{x_2}{4d\,\gamma^2} \log \ell_1^2 - \frac{x_2}{4d\,\gamma^2} \log \ell_2^2\Big] \\ &- \Big(\chi(\epsilon\ell_1) + \chi(\epsilon\ell_2)\Big)\Big[\frac{\sigma^2 x_2^3}{2d\gamma^2\ell_1^2} - \frac{\sigma^2 x_2^3}{2d\gamma^2\ell_2^2}\Big]. \end{aligned} \tag{4.34}$$

For later use, we compute:

$$
\begin{aligned}
\frac{\partial \varphi_s}{\partial x_1} &= \epsilon\Big(\chi'(\epsilon\ell_1)\frac{x_1-d}{\ell_1}+\chi'(\epsilon\ell_2)\frac{x_1+d}{\ell_2}\Big)\left[\frac{x_2\log(\ell_1)^2}{4d\,\gamma^2}-\frac{x_2\log(\ell_2)^2}{4d\,\gamma^2}\right]\\
&\quad+\Big(\chi(\epsilon\ell_1)+\chi(\epsilon\ell_2)\Big)\left[\frac{x_2(x_1-d)}{2d\,\gamma^2\,\ell_1^2}-\frac{x_2(x_1+d)}{2d\,\gamma^2\,\ell_2^2}\right]\\
&\quad-\epsilon\Big(\chi'(\epsilon\ell_1)\frac{x_1-d}{\ell_1}+\chi'(\epsilon\ell_2)\frac{x_1+d}{\ell_2}\Big)\left[\frac{\sigma^2x_2^3}{2d\gamma^2\ell_1^2}-\frac{\sigma^2x_2^3}{2d\gamma^2\ell_2^2}\right]\\
&\quad+\Big(\chi(\epsilon\ell_1)+\chi(\epsilon\ell_2)\Big)\left[\frac{\sigma^2x_2^3(x_1-d)}{d\gamma^2\ell_1^4}-\frac{\sigma^2x_2^3(x_1+d)}{d\gamma^2\ell_2^4}\right],
\end{aligned}
\tag{4.35}
$$

$$
\begin{aligned}
\frac{\partial \varphi_s}{\partial x_2} &= \epsilon\Big(\chi'(\epsilon\ell_1)\frac{x_2}{\ell_1}+\chi'(\epsilon\ell_2)\frac{x_2}{\ell_2}\Big)\left[\frac{x_2}{4d\,\gamma^2}\,\log(\ell_1)^2-\frac{x_2}{4d\,\gamma^2}\,\log(\ell_2)^2\right]\\
&\quad+\Big(\chi(\epsilon\ell_1)+\chi(\epsilon\ell_2)\Big)\left[\frac{1}{4d\,\gamma^2}\,\log(\ell_1)^2-\frac{1}{4d\,\gamma^2}\,\log(\ell_2)^2\right.\\
&\qquad\qquad\left.+\frac{x_2^2}{2d\,\gamma^2\,\ell_1^2}-\frac{x_2^2}{2d\,\gamma^2\,\ell_2^2}\right]\\
&\quad+\epsilon\Big(\chi'(\epsilon\ell_1)\frac{x_2}{\ell_1}+\chi'(\epsilon\ell_2)\frac{x_2}{\ell_2}\Big)\left[-\frac{\sigma^2x_2^3}{2d\gamma^2\ell_1^2}+\frac{\sigma^2x_2^3}{2d\gamma^2\ell_2^2}\right]\\
&\quad+\Big(\chi(\epsilon\ell_1)+\chi(\epsilon\ell_2)\Big)\left[-\frac{3\sigma^2x_2^2}{2d\gamma^2\ell_1^2}+\frac{3\sigma^2x_2^2}{2d\gamma^2\ell_2^2}+\frac{\sigma^2x_2^4}{d\gamma^2\ell_1^4}-\frac{\sigma^2x_2^4}{d\gamma^2\ell_2^4}\right].
\end{aligned}
\tag{4.36}
$$

Hence, we obtain

$$
\begin{aligned}
\nabla\varphi_s &= \chi\big(\epsilon\ell_1\big)\Big[\frac{1}{2d\,\gamma^2}(0,\log\ell_1)+O(\epsilon\log\ell_1)\Big]\\
&\quad-\chi\big(\epsilon\ell_2\big)\Big[\frac{1}{2d\,\gamma^2}(0,\log\ell_2)+O(\epsilon\log\ell_2)\Big].
\end{aligned}
\tag{4.37}
$$

Note that the function φ_s is continuous but $\nabla\varphi_s$ is not. The singularity of φ_s comes from its derivatives.

By recalling the operators $\mathbb{S}_2$, $\mathbb{S}_3$ in (3.20), the region $\mathfrak{S}$ in (3.13) and its decompositions in (3.21), we find the term $\varphi_1(z)$ by solving the problem

$$
\Big[\Delta+\mathbb{S}_2+\mathbb{S}_3\Big]\varphi_1 = -\Big[\Delta+\mathbb{S}_2+\mathbb{S}_3\Big]\big(\varphi_0+\varphi_s+\varphi_2\big)\quad\text{in}\quad\mathfrak{S},
\tag{4.38}
$$

$$
\varphi_1 = -\,\varphi_0\,-\,\varphi_s\,-\,\varphi_2\quad\text{on}\quad\partial\mathfrak{S}.
\tag{4.39}
$$

We derive the estimate of φ_1 by computing the right hand side of (4.38). Note that

$$
\Big[\Delta+\mathbb{S}_2+\mathbb{S}_3\Big]\big(\varphi_0+\varphi_s+\varphi_2\big) = \Delta\big(\varphi_s+\varphi_2\big)+\Big[\mathbb{S}_2+\mathbb{S}_3\Big]\big(\varphi_0+\varphi_s+\varphi_2\big),
$$

due to $\Delta\varphi_0 = 0$. Recall the formulas (4.9), (4.10), (4.11), (4.15), (4.16). For $z \in B_{\frac{d}{10}}(d\vec{e}_1)$, there hold

$$\begin{aligned}
&\Delta\varphi_s + \Big[\mathbb{S}_2 + \mathbb{S}_3\Big]\varphi_0 \\
&= -\frac{x_1 - d}{d\,x_1}\frac{\partial\theta_{d\vec{e}_1}}{\partial x_1} + \frac{\sigma^2}{\gamma^2}\,\frac{3d(x_1-d)^2 + 2(x_1-d)^3}{dx_1^2}\,\frac{\partial^2\theta_{d\vec{e}_1}}{\partial x_2^2} + O(d^{-2}) \\
&= -\frac{x_1 - d}{d\,x_1}\frac{x_2}{\ell_1^2} - \frac{\sigma^2}{\gamma^2}\,\frac{3d(x_1-d)^2 + 2(x_1-d)^3}{dx_1^2}\,\frac{(x_1-d)x_2}{\ell_1^4} + O(d^{-2}) \\
&= O(d^{-2}) = O(\epsilon^2).
\end{aligned}$$

We also get for $z \in B_{\frac{d}{10}}(d\vec{e}_1)$,

$$\Big[\Delta + \mathbb{S}_2 + \mathbb{S}_3\Big]\varphi_2 = 0,$$

due to $\varphi_2 = 0$ in this region. Similarly, by recalling (4.10) and (4.36), we obtain for $z \in B_{\frac{d}{10}}(d\vec{e}_1)$,

$$\begin{aligned}
\mathbb{S}_2[\varphi_s] &= \Big[-\frac{2\sigma^2}{d\gamma^2}(x_1-d) + O\Big((x_1-d)^2/d^2\Big)\Big] \\
&\quad\times\Big(\chi(\epsilon\ell_1) + \chi(\epsilon\ell_2)\Big)\left\{\frac{3(1-2\sigma^2)x_2}{2d\gamma^2\ell_1^2} - \frac{(1+\sigma^2)x_2^3}{d\gamma^2\ell_1^4} - \frac{2\sigma^2 x_2^5}{d\gamma^2\ell_2^6}\right\} + O(\epsilon^2) \\
&= O(\epsilon^2),
\end{aligned}$$

and by recalling (4.35),

$$\begin{aligned}
\mathbb{S}_3[\varphi_s] &= \frac{1}{x_1}\frac{1}{2d\,\gamma^2}\Big[\frac{x_2(x_1-d)}{\ell_1^2} - \frac{x_2(x_1+d)}{\ell_2^2}\Big] \\
&\quad + \frac{1}{x_1}\frac{\sigma^2}{2d\gamma^2}\Big[\frac{x_2^3(x_1-d)}{\ell_1^4} - \frac{x_2^3(x_1+d)}{\ell_2^4}\Big] + O(\epsilon^2) \\
&= O(\epsilon^2).
\end{aligned}$$

For $z \in B_{\frac{d}{10}}(-d\vec{e}_1)$, it is easy to see that we also get $O(\epsilon^2)$. For z in the region

$$\big(\mathfrak{S}_+ \cup \mathfrak{S}_-\big) - \Big(B_{\frac{d}{5}}(d\vec{e}_1) \cup B_{\frac{d}{5}}(-d\vec{e}_1)\Big),$$

we have $\varphi_s = \varphi_2 = 0$ and then

$$\begin{aligned}
&\Big[\Delta + \mathbb{S}_2 + \mathbb{S}_3\Big]\big(\varphi_0 + \varphi_s + \varphi_2\big) \\
&= \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\frac{\partial^2\varphi_0}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1} \\
&= -\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\left(\frac{2x_2(x_1-d)}{\ell_1^4} - \frac{2x_2(x_1+d)}{\ell_2^4}\right) + \frac{-4x_2}{\ell_1^2\ell_2^2}.
\end{aligned}$$

By recalling (4.21), (4.23) and (4.24) and (4.31), we do the computation on the region $\mathfrak{S}_0$. There also holds $\varphi_s = 0$ and so

$$\begin{aligned}
&\Big[\Delta + \mathbb{S}_2 + \mathbb{S}_3\Big]\big(\varphi_0 + \varphi_s + \varphi_2\big) \\
&= \Delta\varphi_2 \,+\, \mathbb{S}_3[\varphi_0] \,+\, \mathbb{S}_2[\varphi_0 + \varphi_2] \\
&= \frac{4\,d\,x_2}{(d^2 + x_2^2)^2} \,+\, \frac{4\,d\,x_2}{\ell_1^2\ell_2^2} \,+\, O(\epsilon^2) \\
&\quad + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\Bigg\{ - \frac{8x_2\,x_1^2\,d(\ell_1 + \ell_2)}{\ell_1^4\ell_2^4} \\
&\qquad\qquad - \frac{16x_2\,x_1^2\,d^3}{\ell_1^4\ell_2^4(d^2 + x_2^2)}\left((\ell_1^2 + \ell_2^2)(d^2 + x_2^2) + \ell_2^2\ell_1^2\right) \\
&\qquad\qquad - \frac{2x_2\,x_1^2\,d}{\ell_1^4\ell_2^4(d^2 + x_2^2)^2}\left(\ell_1^4(d^2 + x_2^2 + \ell_2^2) + \ell_2^4(d^2 + x_2^2 + \ell_1^2)\right)\Bigg\}.
\end{aligned}$$

Whence, going back to the original variable (r, s_3) in (3.3) and letting

$$\hat{\varphi}(r, s_3) = \varphi_1(z)$$

we see that

$$\left|\Delta_{r,s_3}\hat{\varphi} \,+\, \mathbb{S}_2[\hat{\varphi}] \,+\, \mathbb{S}_3[\hat{\varphi}]\right| \,\le\, \frac{C}{\Big(\sqrt{1 + r^2 + |s_3|^2}\Big)^3}. \tag{4.40}$$

Thus we can choose φ_1 such that

$$\hat{\varphi} = O\Big(\frac{1}{\sqrt{1 + r^2 + |s_3|^2}}\Big).$$

The regular term φ_1 is C^1 in the original variable (r, s_3).

We observe also that by our definitions, the function

$$\tilde{\varphi} := \varphi_0 + \varphi_d \tag{4.41}$$

satisfies

$$\Big[\Delta + \mathbb{S}_2 + \mathbb{S}_3\Big]\tilde{\varphi} \,=\, 0 \quad \text{on} \quad \mathfrak{S}, \qquad \tilde{\varphi} \,=\, 0 \quad \text{on} \quad \partial\mathfrak{S}. \tag{4.42}$$

From the decomposition of φ_d, we will see that the singular term contains $x_2 \log \ell_1$, which becomes dominant when we calculate the speed of traveling waves in Section 7.

Finally, we define an improved approximation:

$$V_d(z) \,:=\, \eta_\epsilon(x_1)\, e^{i(\varphi_0 + \varphi_d)} \,+\, \big(1 - \eta_\epsilon(x_1)\big)\rho(\ell_1)\rho(\ell_2)e^{i(\varphi_0 + \varphi_d)}. \tag{4.43}$$

4.3. **Error estimates.** Recall the notation in (4.3) and (4.41). We write V_d in (4.43) in the form

$$V_d(z) \,=\, \eta_\epsilon(x_1)\, e^{i\tilde{\varphi}} \,+\, \big(1 - \eta_\epsilon(x_1)\big)\tilde{\rho}e^{i\tilde{\varphi}}, \qquad \forall\, z = x_1 + ix_2 \in \mathfrak{S}.$$

We shall check that V_d is a good approximate solution in the sense that it satisfies the conditions in (3.17) and has a small error.

It is easy to show that

$$V_d(z) \,=\, \overline{V_d(\bar{z})}, \qquad V_d(z) \,=\, V_d(-\bar{z}), \qquad \frac{\partial V_d}{\partial x_1}(0, x_2) \,=\, 0.$$

Recall the boundary condition in (4.42). It is obvious that

$$\operatorname{Im} V_d \,=\, \Big[\eta_\epsilon(x_1) \,+\, \big(1-\eta_\epsilon(x_1)\big)\tilde\rho\Big]\sin\tilde\varphi \,=\, 0 \quad \text{on} \quad \partial\mathfrak{S} \tag{4.44}$$

and

$$\begin{aligned}\frac{\partial \operatorname{Re} V_d}{\partial x_2} &= \frac{\partial}{\partial x_2}\Big[\eta_\epsilon(x_1)\cos\tilde\varphi \,+\, \big(1-\eta_\epsilon(x_1)\big)\tilde\rho\cos\tilde\varphi\Big] \\ &= \frac{\partial\tilde\rho}{\partial x_2}\big(1-\eta_\epsilon(x_1)\big)\cos\tilde\varphi \quad \text{on} \quad \partial\mathfrak{S}.\end{aligned} \tag{4.45}$$

It can be checked that V_d satisfies the conditions in (3.17) except for

$$\frac{\partial V_d}{\partial x_2} \,=\, 0 \quad \text{on} \quad \partial\mathfrak{S}. \tag{4.46}$$

Let us start to compute the error by plugging V_d into (3.12). We start the computations on the region $\mathfrak{S}_+$ and $\mathfrak{S}_-$. Note that there holds

$$V_d \,=\, v_0\, e^{i\varphi_d}.$$

This implies that

$$\begin{aligned}\Delta V_d &= \Big(\Delta v_0 - |\nabla\varphi_d|^2 v_0 + 2i\nabla v_0\cdot\nabla\varphi_d + iv_0\,\Delta\varphi_d\Big)e^{i\varphi_d} \\ &= \Big[\Delta v_0 - |\nabla\varphi_d|^2 v_0 + 2i\nabla v_0\cdot\nabla\varphi_d - iv_0\,\big(\mathbb{S}_2[\tilde\varphi]+\mathbb{S}_3[\tilde\varphi]\big)\Big]e^{i\varphi_d}.\end{aligned}$$

Here we have used the relation $\Delta\varphi_0 = 0$ and the equation in (4.42). We continue to compute other terms:

$$\begin{aligned}-\frac{2\bar V_d}{1+|V_d|^2}\nabla V_d\cdot\nabla V_d \,&=\, -\,\frac{2\bar v_0}{1+|v_0|^2}\Big[\nabla v_0\cdot\nabla v_0 - v_0^2|\nabla\varphi_d|^2 + 2iv_0\nabla v_0\cdot\nabla\varphi_d\Big]e^{i\varphi_d} \\ &=\, -\,\frac{2\bar v_0}{1+|v_0|^2}\nabla v_0\cdot\nabla v_0\, e^{i\varphi_d} \\ &\quad +\, \frac{2\tilde\rho^2}{1+|\tilde\rho|^2}\Big[v_0|\nabla\varphi_d|^2 - 2i\nabla v_0\cdot\nabla\varphi_d\Big]e^{i\varphi_d}.\end{aligned}$$

We then obtain that

$$\mathbb{S}_0[V_d] \,=\, \mathbb{S}_0[v_0]e^{i\varphi_d} \,-\, iv_0\,\big(\mathbb{S}_2[\tilde\varphi]+\mathbb{S}_3[\tilde\varphi]\big)e^{i\varphi_d} \,+\, \Omega_{20}\, e^{i\tilde\varphi},$$

where we have defined

$$\Omega_{20} \,=\, \left[\frac{\tilde\rho(\tilde\rho^2-1)}{\tilde\rho^2+1}|\nabla\varphi_d|^2 \,-\, 2\frac{(1-\tilde\rho^2)\tilde\rho}{1+\tilde\rho^2}\nabla\varphi_0\cdot\nabla\varphi_d \,+\, 2i\frac{1-\tilde\rho^2}{1+\tilde\rho^2}\nabla\tilde\rho\cdot\nabla\varphi_d\right]. \tag{4.47}$$

In a small neighborhood of $d\vec{e}_1$, we write

$$
\begin{aligned}
&\mathbb{S}_1[V_d] + \mathbb{S}_2[V_d] \\
&= -\frac{2\bar{V}_d}{1+|V_d|^2}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\Big(\frac{\partial V_d}{\partial x_2}\Big)^2 + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial^2 V_d}{\partial x_2^2} \\
&= -\gamma^{-2}\left(\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right)\frac{2\bar{v}_0}{1+|v_0|^2}\left[\Big(\frac{\partial v_0}{\partial x_2}\Big)^2 + 2iv_0\frac{\partial v_0}{\partial x_2}\frac{\partial \varphi_d}{\partial x_2} - v_0^2\Big(\frac{\partial \varphi_d}{\partial x_2}\Big)^2\right]e^{i\varphi_d} \\
&\quad + \gamma^{-2}\left(\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right)\left[\frac{\partial^2 v_0}{\partial x_2^2} + 2i\frac{\partial v_0}{\partial x_2}\frac{\partial \varphi_d}{\partial x_2} + iv_0\frac{\partial^2 \varphi_d}{\partial x_2^2} - v_0\big(\frac{\partial \varphi_d}{\partial x_2}\big)^2\right]e^{i\varphi_d} \\
&= \Big(\mathbb{S}_1[v_0] + \mathbb{S}_2[v_0]\Big)e^{i\varphi_d} + iv_0\,\mathbb{S}_2[\varphi_d]e^{i\varphi_d} + \Omega_{21}e^{i\tilde{\varphi}},
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega_{21} = \gamma^{-2}\left(\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right)\Bigg[&\frac{(|\tilde{\rho}|^2-1)\tilde{\rho}}{1+|\tilde{\rho}|^2}\big(\frac{\partial \varphi_d}{\partial x_2}\big)^2 - 2\frac{(1-|\tilde{\rho}|^2)\tilde{\rho}}{1+|\tilde{\rho}|^2}\frac{\partial \varphi_0}{\partial x_2}\frac{\partial \varphi_d}{\partial x_2} \\
&+ 2i\frac{1-|\tilde{\rho}|^2}{1+|\tilde{\rho}|^2}\frac{\partial \tilde{\rho}}{\partial x_2}\frac{\partial \varphi_d}{\partial x_2}\Bigg].
\end{aligned} \tag{4.48}
$$

The estimates for the terms $\mathbb{S}_3[V_d]$ and $\mathbb{S}_4[V_d]$ are expressed as

$$
\mathbb{S}_3[V_d] = \frac{1}{x_1}\frac{\partial V_d}{\partial x_1} = \mathbb{S}_3[v_0]e^{i\varphi_d} + i\,v_0\,\mathbb{S}_3[\varphi_d]e^{i\varphi_d}
$$

and

$$
\mathbb{S}_4[V_d] = i\delta\gamma^{-1}\epsilon\log\frac{1}{\epsilon}\frac{\partial V_d}{\partial x_2} = \mathbb{S}_4[v_0]e^{i\varphi_d} + \Omega_{23}\,e^{i\tilde{\varphi}}.
$$

In the above we also define

$$
\Omega_{23} = -\delta\gamma^{-1}\epsilon\log\frac{1}{\epsilon}\frac{\partial \varphi_d}{\partial x_2}\tilde{\rho}. \tag{4.49}
$$

For the convenience of notation, we will set $\Omega_{22} = 0$ in the sequel.

Recall all the components of $\mathbb{S}[v_0]$ in (4.19) and $\hat{E}$ in (4.20). The total error on $\mathfrak{S}_+$ is

$$
\begin{aligned}
\mathbb{S}[V_d] &= \mathbb{S}[v_0]\,e^{i\tilde{\varphi}} + iv_0\left(-\mathbb{S}_2[\tilde{\varphi}] - \mathbb{S}_3[\tilde{\varphi}] + \mathbb{S}_2[\varphi_d] + \mathbb{S}_3[\varphi_d]\right)e^{i\varphi_d} \\
&\quad + \sum_{i=0}^{3}\Omega_{2i}\,e^{i\tilde{\varphi}} \\
&= \mathbb{S}[v_0]\,e^{i\tilde{\varphi}} - iv_0\left(\mathbb{S}_2[\varphi_0] + \mathbb{S}_3[\varphi_0]\right)e^{i\varphi_d} + \sum_{i=0}^{3}\Omega_{2i}\,e^{i\tilde{\varphi}} \\
&= \sum_{i=0}^{3}\Big(\Omega_{1i} + \Omega_{2i}\Big)\,e^{i\tilde{\varphi}}.
\end{aligned} \tag{4.50}
$$

Note that the Ω_{ij}'s are defined in (4.7), (4.8), (4.14), (4.18), (4.47), (4.48), (4.49). A similar formula also holds on $\mathfrak{S}_-$. Setting $z = d\vec{e}_1 + y$ in $\mathfrak{S}_+$, we then have

$$
\nabla\varphi_s = -\frac{1}{2d\gamma^2}\log d\,\nabla y_2 + O(\epsilon\log|y|), \quad \nabla\varphi_r = O(\epsilon). \tag{4.51}
$$

Thus

$$\nabla\tilde{\rho}\cdot\nabla\tilde{\varphi} \;=\; O(\epsilon|\log\epsilon|\rho') + O(\epsilon\rho'\log|y|).$$

These asymptotic expressions will play an important role in the reduction part.

We now compute the error on $\mathfrak{S}_0$. In this region, we only do the computation on the region $\{-\varpi/\epsilon < x_1 < \varpi/\epsilon\}$, where $\eta_\epsilon = 1$ and $V_d = e^{i\tilde{\varphi}}$. Note that

$$\mathbb{S}_0[V_d] \;=\; iV_d\,\Delta\tilde{\varphi} \tag{4.52}$$

and

$$\begin{aligned}\mathbb{S}_1[V_d] \,+\, \mathbb{S}_2[V_d] &\equiv \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\left(-\,\frac{2\overline{V_d}}{1+|V_d|^2}\Big(\frac{\partial V_d}{\partial x_2}\Big)^2 \,+\, \frac{\partial^2 V_d}{\partial x_2^2}\right)\\ &= iV_d\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial^2\tilde{\varphi}}{\partial x_2^2}\\ &= iV_d\,\mathbb{S}_2[\tilde{\varphi}].\end{aligned} \tag{4.53}$$

The term $\mathbb{S}_3[V_d]$ obeys the following asymptotic behavior:

$$\mathbb{S}_3[V_d] \;\equiv\; \frac{1}{x_1}\frac{\partial V_d}{\partial x_1} \;=\; iV_d\,\frac{1}{x_1}\frac{\partial\tilde{\varphi}}{\partial x_1} \;=\; iV_d\,\mathbb{S}_3[\tilde{\varphi}].$$

We can estimate the term in $\mathbb{S}_4[V_d]$ as follows:

$$\begin{aligned} i\delta\gamma^{-1}\epsilon|\log\epsilon|\frac{\partial V_d}{\partial x_2} &= -\delta\gamma^{-1}\epsilon|\log\epsilon|\,V_d\,\frac{\partial\tilde{\varphi}}{\partial x_2}\\ &= \delta\gamma^{-1}\epsilon|\log\epsilon|\,V_d\left[\frac{\partial\varphi_0}{\partial x_2}+\frac{\partial\varphi_1}{\partial x_2}+\frac{\partial\varphi_2}{\partial x_2}\right]\\ &= V_d\,O(\epsilon^2|\log\epsilon|),\end{aligned}$$

due to the facts:

$$\begin{aligned}\frac{\partial\theta_{d\vec{e}_1}}{\partial x_2} &= \frac{x_1-d}{(x_1-d)^2+y_2^2} \;=\; O(d^{-1}) \;=\; O(\epsilon),\\ \frac{\partial\theta_{-d\vec{e}_1}}{\partial x_2} &= \frac{x_1+d}{(x_1+d)^2+y_2^2} \;=\; O(d^{-1}) \;=\; O(\epsilon).\end{aligned}$$

Finally, by combining the above computations and using the equation in (4.42), we get the error $\mathbb{S}[V_d]$ on $\mathfrak{S}_0$.

5. Setting up the problem

Now we introduce the set-up of the reduction procedure. We look for solutions of (3.12)-(3.14) in the form

$$u(z) = \eta(z)\,(V_d + iV_d\psi) \,+\, \big(1-\eta(z)\big)V_d e^{i\psi}, \tag{5.1}$$

where η is a function such that

$$\eta(z) = \tilde{\eta}(\ell_1) + \tilde{\eta}(\ell_2). \tag{5.2}$$

See (4.1) for the definition of the cut-off function $\tilde{\eta}$. This nonlinear decomposition (5.1) was introduced first in [13] for the Ginzburg-Landau equation.

The conditions imposed on u in (3.14) and (3.16) can be transmitted to the symmetry on ψ:

$$\begin{aligned} &\psi(z) = \psi(-\bar{z}), \qquad \psi(z) = -\overline{\psi(\bar{z})}, \\ &\frac{\partial \psi}{\partial x_1}(0, x_2) \,=\, 0, \qquad \psi(x_1, -\lambda\pi/\gamma) = \psi(x_1, \lambda\pi/\gamma), \\ &\Big[\frac{\partial V_d}{\partial x_2} + iV_d\,\psi_{x_2}\Big]\Big|_{(x_1,-\lambda\pi/\gamma)} \,=\, \Big[\frac{\partial V_d}{\partial x_2} \,+\, iV_d\,\psi_{x_2}\Big]\Big|_{(x_1,\lambda\pi/\gamma)}. \end{aligned} \tag{5.3}$$

More precisely, by the computations in (4.44) and (4.45), for $\psi = \psi_1 + i\psi_2$, the following conditions hold:

$$\begin{aligned} &\psi_1(x_1, x_2) = \psi_1(-x_1, x_2), \qquad \psi_1(x_1, x_2) = -\psi_1(x_1, -x_2), \\ &\psi_2(x_1, x_2) = \psi_2(-x_1, x_2), \qquad \psi_2(x_1, x_2) = \psi_2(x_1, -x_2), \\ &\frac{\partial \psi_1}{\partial x_1}(0, x_2) = 0, \qquad \frac{\partial \psi_2}{\partial x_1}(0, x_2) = 0, \\ &\psi_1(x_1, -\lambda\pi/\gamma) = \psi_1(x_1, \lambda\pi/\gamma) = 0, \qquad \frac{\partial \psi_1}{\partial x_2}(x_1, -\lambda\pi/\gamma) = \frac{\partial \psi_1}{\partial x_2}(x_1, \lambda\pi/\gamma), \\ &\frac{\partial \psi_2}{\partial x_2}(x_1, -\lambda\pi/\gamma) = \frac{1}{\tilde{\rho}}\frac{\partial \tilde{\rho}}{\partial x_2}(x_1, -\lambda\pi/\gamma), \qquad \frac{\partial \psi_2}{\partial x_2}(x_1, \lambda\pi/\gamma) = \frac{1}{\tilde{\rho}}\frac{\partial \tilde{\rho}}{\partial x_2}(x_1, \lambda\pi/\gamma), \\ &\psi_2(x_1, -\lambda\pi/\gamma) = \psi_2(x_1, \lambda\pi/\gamma). \end{aligned} \tag{5.4}$$

The symmetry will be important in solving the linear problems in that it excludes all but one kernel. We may write $\psi = \psi_1 + i\psi_2$ with ψ_1, ψ_2 real-valued and then set

$$u \,=\, V_d + \phi, \quad \phi \,=\, \eta i V_d \psi + (1-\eta) V_d (e^{i\psi} - 1). \tag{5.5}$$

In the sequel, we will derive the explicit form for the linearized problem.

In the inner region $\big\{z \in B_1(d\vec{e}_1) \cup B_1(-d\vec{e}_1)\big\}$, we have

$$u \,=\, V_d \,+\, \phi, \tag{5.6}$$

and the equation for ϕ becomes

$$\mathbb{L}_d[\phi] \,+\, \mathbb{N}_d[\phi] \,=\, -\mathbb{S}[V_d]. \tag{5.7}$$

In the above, we have denoted the linear operator by

$$\begin{aligned} \mathbb{L}_d[\phi] \,=\, &\Delta\phi \,+\, \frac{1}{x_1}\frac{\partial \phi}{\partial x_1} \,-\, \frac{4\bar{V}_d}{1+|V_d|^2}\nabla V_d \cdot \nabla\phi \,-\, \frac{2\bar{\phi}}{1+|V_d|^2}\nabla V_d \cdot \nabla V_d \\ &+\, \frac{2\bar{V}_d(V_d\bar{\phi} + \bar{V}_d\phi)}{(1+|V_d|^2)^2}\nabla V_d \cdot \nabla V_d \,-\, \frac{4\bar{V}_d}{1+|V_d|^2}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\frac{\partial V_d}{\partial x_2} \cdot \frac{\partial \phi}{\partial x_2} \\ &-\, \frac{2\bar{\phi}}{1+|V_d|^2}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\Big(\frac{\partial V_d}{\partial x_2}\Big)^2 \\ &+\, \frac{2\bar{V}_d(V_d\bar{\phi} + \bar{V}_d\phi)}{(1+|V_d|^2)^2}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\Big(\frac{\partial V_d}{\partial x_2}\Big)^2 \\ &+\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2}\right]\frac{\partial^2 \phi}{\partial x_2^2} \,+\, F'(V_d)\phi. \end{aligned}$$

The nonlinear operator is

$$\begin{aligned}\mathbb{N}_d[\phi] =& F(V_d+\phi) - F(V_d) - F'(V_d)\phi + O\big((1+|\phi|)|\nabla\phi|^2\big)\\ &+ \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]O\big((1+|\phi|)|\nabla\phi|^2\big) + i\delta\gamma^{-1}|\log\epsilon|\epsilon\frac{\partial\phi}{\partial x_2}.\end{aligned} \tag{5.8}$$

In the above, we have used the definition of F in (3.20). Note that in this region

$$\frac{1}{x_1} \quad \text{and} \quad \left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]$$

are small terms. $\mathbb{L}_d$ is a small perturbation of $\mathbb{L}_0$ in (2.3).

In the outer region $\Big\{z \in \big(B_2(d\vec{e}_1)\cup B_2(-d\vec{e}_1)\big)^c\Big\}$, we have $u = V_d e^{i\psi}$. By simple computations we obtain

$$\begin{aligned}\frac{\mathbb{S}[V_d e^{i\psi}]}{iV_d e^{i\psi}} =& \Delta\psi + \frac{1}{x_1}\frac{\partial\psi}{\partial x_1} + \frac{1-|V_d|^2+|V_d|^2(e^{-2\psi_2}-1)}{V_d(1+|V_d|^2e^{-2\psi_2})}2\nabla V_d\cdot\nabla\psi\\ &+ \frac{1}{iV_d}\frac{2|V_d|^2\bar{V}_d(e^{-2\psi_2}-1)}{(1+|V_d|^2)(1+|V_d|^2e^{-2\psi_2})}\nabla V_d\cdot\nabla V_d\\ &- i\Big(\frac{2|V_d|^2}{1+|V_d|^2e^{-2\psi_2}}-1\Big)\nabla\psi\cdot\nabla\psi\\ &- i\frac{4\bar{V}_d}{1+|V_d|^2e^{-2\psi_2}}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial V_d}{\partial x_2}\cdot\frac{\partial\psi}{\partial x_2}\\ &+ i\frac{1}{V_d}\frac{2\bar{V}_d}{1+|V_d|^2e^{-2\psi_2}}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial V_d}{\partial x_2}\cdot\frac{\partial V_d}{\partial x_2}\\ &+ \frac{2|V_d|^2}{1+|V_d|^2e^{-2\psi_2}}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial\psi}{\partial x_2}\cdot\frac{\partial\psi}{\partial x_2}\\ &- i\frac{2|V_d|^2(1-e^{-2\psi_2})}{(1+|V_d|^2e^{-2\psi_2})(1+|V_d|^2)} + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial^2\psi}{\partial x_2^2}\\ &+ i\delta\gamma^{-1}|\log\epsilon|\epsilon\frac{\partial\psi}{\partial x_2} + \frac{\mathbb{S}[V_d]}{iV_d}.\end{aligned}$$

We can also write the problem as an equation of $\psi = \psi_1 + i\psi_2$,

$$\tilde{L}_0[\psi] + \tilde{N}_1[\psi] + \tilde{N}_2[\psi] = -\mathbb{S}[V_d]/iV_d, \tag{5.9}$$

with conditions in (5.4). In the above, we have denoted

$$\begin{aligned}\tilde{L}_0[\psi] =& \Delta\psi + \frac{1}{x_1}\frac{\partial\psi}{\partial x_1} + \frac{2(1-|V_d|^2)}{1+|V_d|^2}\nabla V_d\cdot\nabla\psi + \frac{4i\bar{V}_d^2\psi_2}{(1+|V_d|^2)^2}\nabla V_d\cdot\nabla V_d \\ &+ \frac{2(1-|V_d|^2)}{1+|V_d|^2}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial V_d}{\partial x_2}\cdot\frac{\partial\psi}{\partial x_2} \\ &+ \frac{4i\bar{V}_d^2\psi_2}{(1+|V_d|^2)^2}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial V_d}{\partial x_2}\cdot\frac{\partial V_d}{\partial x_2} \\ &- i\frac{4|V_d|^2\psi_2}{(1+|V_d|^2)^2} + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial^2\psi}{\partial x_2^2},\end{aligned}$$

$$\begin{aligned}\tilde{N}_1[\psi] =& \frac{1}{V_d}\nabla\psi\cdot\nabla V_d\, O(\psi_2) + O\big(\big||V_d|^2-1\big|+|\psi_2|\big)|\nabla\psi\cdot\nabla\psi| \\ &+ iO(|e^{-\psi_2}-1+\psi_2|),\end{aligned}$$

$$\tilde{N}_2[\psi] = i\delta\gamma^{-1}|\log\epsilon|\epsilon\frac{\partial\psi}{\partial x_2}.$$

Recall that $\psi=\psi_1+i\psi_2$. Then setting $z=d\vec{e}_1+y$, we have for $z\in\mathbb{R}^2_+$,

$$\tilde{L}_0[\psi] = \begin{bmatrix} \tilde{\Delta}\psi_1 + O(e^{-|y|})|\nabla\psi| \\ \tilde{\Delta}\psi_2 - \frac{4|V_d|^2}{(1+|V_d|^2)^2}\psi_2 + O\big(e^{-|y|}\big)\nabla\psi_2 \end{bmatrix}, \tag{5.10}$$

where $\tilde{\Delta} := \Delta + \frac{1}{x_1}\frac{\partial}{\partial x_1} + \gamma^{-2}\Big(\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\Big)\frac{\partial^2}{\partial x_2^2}$,

$$\tilde{N}_1[\psi] = \begin{bmatrix} O\Big(e^{-|y|}|\nabla\psi\cdot\nabla\psi| + |\psi_2|^2\frac{1}{(1+|y|)^2} + |\psi_2|\frac{1}{1+|y|}|\nabla\psi|\Big) \\ O\Big(e^{-|y|}|\nabla\psi\cdot\nabla\psi| + |\psi_2||\nabla\psi\cdot\nabla\psi| + |\psi_2|^2\Big) \end{bmatrix}, \tag{5.11}$$

$$\tilde{N}_2[\psi] = \begin{bmatrix} O(\epsilon|\frac{\partial\psi_2}{\partial y_2}|) \\ O(\epsilon|\frac{\partial\psi_1}{\partial y_2}|) \end{bmatrix}. \tag{5.12}$$

Note that

$$\Delta + \frac{1}{x_1}\frac{\partial}{\partial x_1} + \gamma^{-2}\Big[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\Big]\frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial x_1^2} + \frac{1}{\gamma^2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1} + \frac{\lambda^2}{\gamma^2}\frac{1}{x_1^2}\frac{\partial^2}{\partial x_2^2}.$$

Let us remark that the explicit form of all the linear and nonlinear terms will be very useful for later analysis in resolution theory.

Define

$$\mathbb{E}_d = -\mathbb{S}[V_d], \qquad \tilde{\mathbb{E}}_d = -\mathbb{S}[V_d]/iV_d.$$

Based on the form of the errors, we need to use suitable norms. Let us fix two positive numbers,

$$p>13, \quad 0<\varrho<1.$$

Recall that $\phi = iV_d\psi$, $\psi = \psi_1 + i\psi_2$, $\ell_1 = |z + d\vec{e}_1|$ and $\ell_2 = |z + d\vec{e}_1|$. We define

$$\|h\|_{**} = \|iV_d h\|_{L^p(\mathfrak{S}_1)} + \sum_{j=1}^{2}\Big[\|\ell_j^{2+\varrho}h_1\|_{L^\infty(\mathfrak{S}_2)} + \|\ell_j^{1+\varrho}h_2\|_{L^\infty(\mathfrak{S}_2)}\Big],$$

$$\|\psi\|_{*} = \|\phi\|_{W^{2,p}(\mathfrak{S}_1)} + \sum_{j=1}^{2}\Big[\|\ell_j^{\varrho}\psi_1\|_{L^\infty(\mathfrak{S}_2)} + \|\ell_j^{1+\varrho}\nabla\psi_1\|_{L^\infty(\mathfrak{S}_2)}\Big] + \Big[\|\ell_j^{1+\varrho}\psi_2\|_{L^\infty(\mathfrak{S}_2)} + \|\ell_j^{2+\varrho}\nabla\psi_2\|_{L^\infty(\mathfrak{S}_2)}\Big].$$

In the above,

$$\begin{aligned}\mathfrak{S}_1 &= \{\, z \in \mathfrak{S} : |z - d\vec{e}_1| < 3 \text{ or } |z + d\vec{e}_2| < 3 \,\},\\ \mathfrak{S}_2 &= \{\, z \in \mathfrak{S} : |z - d\vec{e}_1| > 2 \text{ and } |z + d\vec{e}_2| > 2 \,\}.\end{aligned} \tag{5.13}$$

We remark that we use the norm L^p_{loc} (or $W^{2,p}_{loc}$) in the inner part due to the fact that the error term contains terms like $\epsilon \log \ell_1$ which is not L^∞-bounded.

Using the norms defined above, we can have the following error estimates. For the proof, the reader can refer to Lemma 4.1.

Lemma 5.1. *It holds that for $z \in \mathfrak{S}_2$,*

$$|\mathrm{Re}(\tilde{\mathbb{E}}_d)| \le \frac{C\epsilon^{1-\varrho}}{(1+\ell_1)^3} + \frac{C\epsilon^{1-\varrho}}{(1+\ell_2)^3}, \tag{5.14}$$

$$|\mathrm{Im}(\tilde{\mathbb{E}}_d)| \le \frac{C\epsilon^{1-\varrho}}{(1+\ell_1)^{1+\varrho}} + \frac{C\epsilon^{1-\varrho}}{(1+\ell_2)^{1+\varrho}}, \tag{5.15}$$

and also

$$\|\mathbb{E}_d\|_{L^p(\mathfrak{S}_1)} \le C\epsilon \log\frac{1}{\epsilon}, \tag{5.16}$$

where $\varrho \in (0,1)$ is a constant. As a consequence, there holds

$$\|\tilde{\mathbb{E}}_d\|_{**} \le C\epsilon^{1-\varrho}. \tag{5.17}$$

□

6. Projected linear and nonlinear problems

Let L be a linear operator in the form

$$\mathcal{L} = \mathbb{L}_d \quad \text{in} \quad \mathfrak{S}_1, \qquad \mathcal{L} = \tilde{L}_0 \quad \text{in} \quad \mathfrak{S}_2,$$

and the co-kernel

$$Z_d := \frac{\partial V_d}{\partial d}\left[\tilde{\eta}\Big(\frac{\ell_1}{R}\Big) + \tilde{\eta}\Big(\frac{\ell_2}{R}\Big)\right], \tag{6.1}$$

where $\tilde{\eta}$ is defined at (5.2). Then Z_d satisfies the symmetry (3.16). For $\psi = \psi_1 + i\psi_2$, we require the conditions

$$\begin{aligned}
&\psi_1(x_1,x_2) = \psi_1(-x_1,x_2), && \psi_1(x_1,x_2) = -\psi_1(x_1,-x_2),\\
&\psi_2(x_1,x_2) = \psi_2(-x_1,x_2), && \psi_2(x_1,x_2) = \psi_2(x_1,-x_2),\\
&\frac{\partial\psi_1}{\partial x_1}(0,x_2) = 0, && \frac{\partial\psi_2}{\partial x_1}(0,x_2) = 0,\\
&\psi_1(x_1,-\lambda\pi/\gamma) = \psi_1(x_1,\lambda\pi/\gamma) = 0, && \frac{\partial\psi_1}{\partial x_2}(x_1,-\lambda\pi/\gamma) = \frac{\partial\psi_1}{\partial x_2}(x_1,\lambda\pi/\gamma),\\
&\frac{\partial\psi_2}{\partial x_2}(x_1,-\lambda\pi/\gamma) = 0, && \frac{\partial\psi_2}{\partial x_2}(x_1,\lambda\pi/\gamma) = 0,\\
&\psi_2(x_1,-\lambda\pi/\gamma) = \psi_2(x_1,\lambda\pi/\gamma).
\end{aligned} \tag{6.2}$$

As the first step of finite dimensional reduction, we need to consider the following linear problem:

$$\begin{cases}
\mathcal{L}(\psi) \;=\; h \;\text{ in } \mathfrak{S}, \qquad \psi \text{ satisfies the conditions in (6.2)},\\[2ex]
\mathrm{Re}\big(\int_{\mathfrak{S}} \bar{\phi} Z_d \,\mathrm{d}x\big) \;=\; 0 \quad \text{for } \phi \;=\; iV_d\psi \text{ in } B_1(d\vec{e}_1)\cup B_1(-d\vec{e}_1).
\end{cases} \tag{6.3}$$

We have the following *a priori estimates.*

Lemma 6.1. *There exists a constant C depending on ϱ only such that for all ϵ sufficiently small, $d \sim \frac{1}{\epsilon}$, and any solution of* (6.3)*, it holds that*

$$\|\psi\|_* \le C\|h\|_{**}. \tag{6.4}$$

Proof. The proof is similar as in Lemma 5.1 in [34]. Suppose that there exists a sequence of $\epsilon = \epsilon_n \to 0$, functions ψ^n, h_n which satisfy (6.3) with

$$\|\psi^n\|_* = 1, \quad \|h_n\|_{**} = o(1).$$

We will derive a contradiction by careful analysis of the estimates.

We derive *inner estimates* first. We have the symmetries and boundary conditions for ψ_1 and ψ_2 in (6.2). Whence we just need to consider the region

$$\Sigma_+ \;=\; \big\{\, x_1 > 0,\; 0 < x_2 < \lambda\pi/\gamma \,\big\}.$$

Then we have

$$\mathrm{Re}\bigg(\int_{\mathbb{R}^2} \bar{\phi}_n Z_d\bigg) \;=\; 2\mathrm{Re}\bigg(\int_{\Sigma_+} \bar{\phi}_n Z_d\bigg) \;=\; 0.$$

Let $z \in \Sigma_+,\; z = d\vec{e}_1 + y$ and $\tilde{\phi}_n(y) = \phi_n(z)$. Then as $n \to +\infty$,

$$V_d \;=\; w^+(y)e^{-i\theta_{-d\vec{e}_1}}\big(1 + O(e^{-d/2})\big) \;=\; -w^+(y) + o(1).$$

Since $\|\psi^n\|_* = 1$, we may take a limit so that $\tilde{\phi}_n \to \phi_0$ in $\mathbb{R}^2_{loc}$, where ϕ_0 satisfies

$$\mathbb{L}_0[\phi_0] = 0$$

where $\mathbb{L}_0$ is defined by (2.4). Observe that ϕ_0 satisfies the decay estimate (2.5) because of our assumption on ψ^n. By Lemma 2.2, we have

$$\phi_0 = c_1\frac{\partial w}{\partial y_1} + c_2\frac{\partial w}{\partial y_2}.$$

Observe that ϕ_0 inherits the symmetries of ϕ and hence $\phi_0 = \overline{\phi_0(\bar{z})}$. (The other symmetry is not preserved under the transformation $z = d\vec{e}_1 + y$.) But certainly

$\frac{\partial w}{\partial y_2}$ does not enjoy the above symmetry. Hence $\phi_0 = c_1 \frac{\partial w}{\partial y_1}$. On the other hand, taking a limit of the orthogonality condition

$$\mathrm{Re}\left(\int_{\Sigma_+} \bar{\phi}_n Z_d \right) = 0,$$

we obtain

$$\mathrm{Re}\left(\int_{\mathbb{R}^2} \bar{\phi_0} \frac{\partial w}{\partial y_1} \right) = 0.$$

This implies that $c_1 = 0$ and hence we have

$$\phi_n \to 0 \ \text{ in } \mathbb{R}^2_{loc},$$

which implies that for any fixed $R > 0$,

(6.5)
$$\sum_{j=1}^{2} \Big(\|\phi_1\|_{L^p(\ell_j<R)} + \|\phi_2\|_{L^p(\ell_j<R)} + \|\nabla\phi_1\|_{L^p(\ell_j<R)} + \|\nabla\phi_2\|_{L^p(\ell_j<R)} \Big) = o(1).$$

We use the L^p-estimates in the inner part $\{|z - d\vec{e}_1| < R\}$. By choosing p large we obtain the embedding $W^{2,p}_{loc}$ into $C^{1,\alpha}_{loc}$ for any $\alpha \in (0,1)$.

Next we shall derive *outer estimates*, again by using the symmetries and boundary conditions for ψ_1 and ψ_2 in (6.2). We just need to consider the region

$$\Sigma = \big\{ x_1 \in \mathbb{R},\ 0 < x_2 < \lambda\pi/\gamma \big\}.$$

Let $\tilde{\eta}$ be a cut-off function such that $\tilde{\eta}(s) = 1$ for $s \le 1$ and $\tilde{\eta}(s) = 0$ for $s > 2$. We consider the new function

$$\tilde{\psi} = \psi\,\chi(z), \quad \text{where } \chi(z) = 1 - \tilde{\eta}\Big(\frac{\ell_1}{4}\Big) - \tilde{\eta}\Big(\frac{\ell_2}{4}\Big).$$

Using the explicit forms of $\tilde{L}_0$ in (5.10), the first equation becomes

$$\begin{aligned} &\frac{\partial^2\tilde{\psi}_1}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial\tilde{\psi}_1}{\partial x_1} + \frac{1}{\gamma^2}\frac{\partial^2\tilde{\psi}_1}{\partial x_2^2} + \frac{\lambda^2}{\gamma^2}\frac{1}{x_1^2}\frac{\partial^2\tilde{\psi}_1}{\partial x_2^2} \\ &= O(e^{-|y|})|\nabla\psi| + O(\nabla\chi\nabla\psi_1) + O(\psi_1\Delta\chi) + h_1\chi. \end{aligned} \tag{6.6}$$

On the region $\Sigma\backslash\big(B_4(d\vec{e}_1) \cup B_4(-d\vec{e}_1)\big)$, we have the conditions in (6.2). Moreover we have

$$|\nabla\chi\cdot\nabla\psi| = o(1)\big(\ell_1^2 + \ell_2^2\big)^{-\frac{\sigma+2}{2}}.$$

For the outer part estimates, we use the new barrier function

$$B(z) := B_1(z) + B_2(z), \tag{6.7}$$

where by the notation

$$\tilde{\ell}_1 = \sqrt{(x_1 - d)^2 + k^2 x_2^2}, \quad \tilde{\ell}_2 = \sqrt{(x_1 + d)^2 + k^2 x_2^2},$$

we have denoted

$$B_1(z) = \big(\eta(\epsilon\ell_1) + \eta(\epsilon\ell_2)\big)\Big(\tilde{\ell}_1^{\beta}\, x_2^{\nu} + \tilde{\ell}_2^{\beta}\, x_2^{\nu}\Big), \qquad B_2(z) = x_1^{-\varrho}\big(\sin(\tau\epsilon x_2)\big)^{\nu}.$$

In the above $\beta + \nu = -\varrho$, $0 < \varrho < \nu < 1$. η is a smooth cut-off function with properties $\eta(t) = 1$ for $|t| < c_\varrho$, where c_ϱ is small, and $\eta(t) = 0$ for $|t| > 2c_\varrho$. Note that c_ϱ, τ and k are constants, which will be determined in the sequel.

Now, we do the computations for B_1:

$$\frac{\partial^2 B_1}{\partial x_1^2} + \frac{1}{k^2}\frac{\partial^2 B_1}{\partial x_2^2} \le -C_1\Big(\tilde{\ell}_1^{-2-\varrho} + \tilde{\ell}_2^{-2-\varrho}\Big), \tag{6.8}$$

where C_1 depends only on β and ν. On the other hand, there holds

$$\frac{1}{x_1}\frac{\partial B_1}{\partial x_1} \le C\frac{x_2^\nu}{x_1}\Big[\tilde{\ell}_1^{\beta-2}(x_1-d) + \tilde{\ell}_2^{\beta-2}(x_1+d)\Big]. \tag{6.9}$$

Thus for $|x_1-d| < c_\varrho d$ or $|x_1+d| < c_\varrho d$, where c_ϱ is small, we have

$$\frac{1}{x_1}\frac{\partial B_1}{\partial x_1} \le C_2\, c_\varrho\big(\tilde{\ell}_1^{-2-\varrho} + \tilde{\ell}_2^{-2-\varrho}\big), \tag{6.10}$$

where C_2 depends only on β and ν. Furthermore, we can choose k such that there exists a positive constant C_3 depending only on β, ν, $\hat{\lambda}$ and γ; for $|x_1-d| < c_\varrho d$ or $|x_1+d| < c_\varrho d$ there holds

$$\begin{aligned}
&\left(\frac{\lambda^2}{\gamma^2}\frac{1}{x_1^2} + \frac{1}{\gamma^2} - \frac{1}{k^2}\right)\frac{\partial^2 B_1}{\partial x_2^2}\\
&= \left(\frac{\hat{\lambda}^2}{\gamma^2}\frac{1}{\epsilon^2 x_1^2} + \frac{1}{\gamma^2} - \frac{1}{k^2}\right)\\
&\qquad\times\Big[\beta(2\nu+1) + \beta(\beta-2)\tilde{\ell}^{-2}x_2^2 + \nu(\nu-1)\tilde{\ell}_1^2 x_2^{-2}\Big]\,\tilde{\ell}_1^{-2+\beta}\big(\gamma x_2\big)^\nu\\
&\quad + \left(\frac{\hat{\lambda}^2}{\gamma^2}\frac{1}{\epsilon^2 x_1^2} + \frac{1}{\gamma^2} - \frac{1}{k^2}\right)\\
&\qquad\times\Big[\beta(2\nu+1) + \beta(\beta-2)\tilde{\ell}^{-2}x_2^2 + \nu(\nu-1)\tilde{\ell}_2^2 x_2^{-2}\Big]\,\tilde{\ell}_2^{-2+\beta}\big(\gamma x_2\big)^\nu\\
&\le C_3\, c_\varrho\big(\tilde{\ell}_1^{-2-\varrho} + \tilde{\ell}_2^{-2-\varrho}\big).
\end{aligned} \tag{6.11}$$

By choosing c_ϱ small, we obtain

$$\frac{\partial^2 B_1}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial B_1}{\partial x_1} + \frac{1}{\gamma^2}\frac{\partial^2 B_1}{\partial x_2^2} + \frac{\lambda^2}{\gamma^2}\frac{1}{x_1^2}\frac{\partial^2 B_1}{\partial x_2^2} \le C\big(\tilde{\ell}_1^{-2-\varrho} + \tilde{\ell}_2^{-2-\varrho}\big). \tag{6.12}$$

Note that we only do the computations by ignoring the cut-off function in the expressions of B_1.

By trivial computations, we obtain that

$$\begin{aligned}
&\frac{\partial^2 B_2}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial B_2}{\partial x_1} + \frac{1}{\gamma^2}\frac{\partial^2 B_2}{\partial x_2^2} + \frac{\lambda^2}{\gamma^2}\frac{1}{x_1^2}\frac{\partial^2 B_2}{\partial x_2^2}\\
&= \varrho^2 x_1^{-2-\varrho}\big(\sin(\tau\epsilon x_2)\big)^\nu\\
&\quad - \nu^2\gamma^{-2}\tau^2\epsilon^2 x_1^{-\varrho}\big(\sin(\tau\epsilon x_2)\big)^\nu + \nu(\nu-1)\gamma^{-2}\tau^2\epsilon^2 x_1^{-\varrho}\big(\sin(\tau\epsilon x_2)\big)^{\nu-2}\\
&\quad - \nu^2\frac{\hat{\lambda}^2}{\gamma^2}\tau^2\, x_1^{-2-\varrho}\big(\sin(\tau\epsilon x_2)\big)^\nu + \nu(\nu-1)\frac{\hat{\lambda}^2}{\gamma^2}\tau^2\, x_1^{-2-\varrho}\big(\sin(\tau\epsilon x_2)\big)^{\nu-2}.
\end{aligned} \tag{6.13}$$

By choosing τ small enough, we have

$$0 < \sin(\tau\epsilon x_2) < 1/2 \quad \text{for} \quad 0 < x_2 < \lambda\pi/\gamma.$$

If $x_1 - d < -c_\varrho d$, there holds

$$x_1^2 < \Big(\frac{1-c_\varrho}{c_\varrho}\Big)^2 (x_1-d)^2.$$

On the other hand, if $x_1 - d > c_\varrho d$, there holds

$$x_1^2 < \Big(\frac{1+c_\varrho}{c_\varrho}\Big)^2 (x_1-d)^2.$$

We finally have

$$\frac{\partial^2 B}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial B}{\partial x_1} + \frac{1}{\gamma^2}\frac{\partial^2 B}{\partial x_2^2} + \frac{\lambda^2}{\gamma^2}\frac{1}{x_1^2}\frac{\partial^2 B}{\partial x_2^2} \le -C\big(\ell_1^{-2-\varrho} + \ell_2^{-2-\varrho}\big). \tag{6.14}$$

By the comparison principle on the set $\Sigma\backslash(B_4(d\vec{e}_1)\cup B_4(-d\vec{e}_1))$, we get that

$$|\tilde{\psi}_1| \le CB(\|h\|_{**} + o(1)), \quad \forall\, z \in \Sigma\backslash(B_4(d\vec{e}_1)\cup B_4(-d\vec{e}_1)). \tag{6.15}$$

Elliptic estimates then give

$$\sum_{j=1}^{2} \big\|\ell_j^{1+\sigma}|\nabla\tilde{\psi}_1|\big\|_{L^\infty(\ell_j>4)} \le C(\|h\|_{**} + o(1)). \tag{6.16}$$

To estimate ψ_2, we perform the same cut-off, and now the second equation becomes

$$\begin{aligned} &\frac{\partial^2\tilde{\psi}_2}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial\tilde{\psi}_2}{\partial x_1} + \frac{1}{\gamma^2}\frac{\partial^2\tilde{\psi}_2}{\partial x_2^2} + \frac{\lambda^2}{\gamma^2}\frac{1}{x_1^2}\frac{\partial^2\tilde{\psi}_2}{\partial x_2^2} - \frac{4|V_d|^2}{(1+|V_d|^2)^2}\tilde{\psi}_2 \\ &= O(\frac{1}{1+|y|})\nabla\psi_1 + O(e^{-|y|})\nabla\psi_2 + O(\nabla\chi\nabla\psi_2) + O(\Delta\psi_2) + h_2\chi\,. \end{aligned} \tag{6.17}$$

Note that we also have the conditions in (6.2). Since for $z \in \Sigma\backslash(B_4(d\vec{e}_1)\cup B_4(-d\vec{e}_1))$, there holds

$$\frac{4|V_d|^2}{(1+|V_d|^2)} \ge \frac{1}{4},$$

by standard elliptic estimates we have

$$\begin{aligned} \|\psi_2\|_{L^\infty(\ell_j>4)} &\le C(\|\psi_2\|_{L^\infty(\ell_j=4)})(1+\|\psi\|_*)\|h\|_{**}(1+\ell_1+\ell_2)^{-1-\varrho}, \\ |\nabla\psi_2| &\le C(\|\psi_2\|_{L^\infty(\ell_j=R)})(1+\|\psi\|_*)\|h\|_{**}(1+\ell_1+\ell_2)^{-2-\varrho}. \end{aligned} \tag{6.18}$$

Combining both inner and outer estimates in (6.5), (6.15)-(6.16) and (6.18), we obtain that $\|\psi\|_* = o(1)$, which is a contradiction. □

By the contraction mapping theorem, we conclude

Proposition 6.2. *There exists a constant C, depending on ν, ϱ only such that for all ϵ sufficiently small, d large, the following hold: there exists a unique solution $(\phi_{\epsilon,d}, \mathcal{C}_\epsilon(d))$ to*

$$\mathbb{S}[V_d + \phi_{\epsilon,d}] = \mathcal{C}_\epsilon(d) Z_d \tag{6.19}$$

with conditions in (5.3) *by the relation $\phi_{\epsilon,d} = iV_d\psi$, and $\phi_{\epsilon,d}$ satisfies*

$$\|\phi_{\epsilon,d}\|_* \le C\epsilon^{1-\varrho}. \tag{6.20}$$

Furthermore, $\phi_{\epsilon,d}$ is continuous in the parameter d.

Here, we do not give the proof to the last proposition. The reader can refer to the arguments in [34] for details.

7. Reduced problem and the proof of Theorem 1.2

We now solve the reduced problem. From Proposition 6.2, we deduce the existence of a solution $(\phi, \mathcal{C}) = (\phi_{\epsilon,d}, \mathcal{C}_\epsilon(d))$ to

$$\mathbb{S}[V_d + \phi_{\epsilon,d}] = \mathbb{L}_d[\phi_{\epsilon,d}] + \mathbb{N}_d[\phi_{\epsilon,d}] + \mathbb{S}[V_d] = \mathcal{C}_\epsilon(d) Z_d. \tag{7.1}$$

Multiplying (7.1) by $\frac{1}{(1+|V_d|^2)^2}\overline{Z_d}$ and integrating, we obtain

$$\begin{aligned}\mathcal{C}_\epsilon \operatorname{Re}\Big(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,Z_d\Big) &= \operatorname{Re}\Big(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,\mathbb{S}[V_d]\Big)\\ &\quad + \operatorname{Re}\Big(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,\mathbb{L}_d[\phi_{\epsilon,d}]\Big)\\ &\quad + \operatorname{Re}\Big(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,\mathbb{N}_d[\phi_{\epsilon,d}]\Big).\end{aligned}$$

Using Proposition 6.2 and the expression in (5.8), we deduce that

$$\operatorname{Re}\Big(\int_{\mathbb{R}^2} \overline{Z_d}\,\mathbb{N}_d[\phi_{\epsilon,d}]\Big) = o(\epsilon).$$

On the other hand, by integration by parts, we have

$$\operatorname{Re}\Big(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,\mathbb{L}_d\big[\phi_{\epsilon,d}\big]\Big) = \operatorname{Re}\Big(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\,\overline{\phi_{\epsilon,d}}\,\mathbb{L}_d[Z_d]\Big).$$

Let us observe that

$$\frac{\partial}{\partial d}\mathbb{S}_0[V_d] = \mathbb{L}_d\Big[\frac{\partial V_d}{\partial d}\Big] = \mathbb{L}_d[Z_d] = O(\epsilon),$$

and thus by Proposition 6.2

$$\operatorname{Re}\Big(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\,\overline{\phi_{\epsilon,d}}\,\mathbb{L}_d[Z_d]\Big) = o(\epsilon).$$

As the strategy in standard reduction method, we are left to estimate the following integral:

$$\begin{aligned}&\operatorname{Re}\Big(\int_{\mathfrak{S}_+} \frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,\mathbb{S}[V_d]\Big)\\ &= \operatorname{Re}\Big[\int_{\mathfrak{S}_+} \frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,\sum_{i=0}^{3}\big(\Omega_{1i}+\Omega_{2i}\big)\,e^{i\tilde{\varphi}}\Big].\end{aligned} \tag{7.2}$$

The expressions of these error terms are given in Sections 4.1 and 4.3. See (4.7), (4.8), (4.14), (4.18), (4.47), (4.48) and (4.49). We will estimate the above integrals in the sequel.

On $\mathfrak{S}_+$, recall that $z = d\vec{e}_1 + y$. For the simplification of notation, we denote $\theta = \theta_{d\vec{e}_1}$ in the translated variables. By the properties of ρ, we have

$$\begin{aligned} v_0 &= w(y)e^{-i\theta_{-d\vec{e}_1}}\Big[1 + O(e^{-d/2-|y|/2})\Big],\\ |v_0| &= \tilde{\rho} = \rho(|y|)\Big[1 + O(e^{-d/2-|y|/2})\Big],\\ \overline{Z_d} &= \tilde{\eta}(|y|/R)\,\Big[-\frac{\partial\tilde{\rho}}{\partial y_1} - i\tilde{\rho}\frac{\partial\theta}{\partial y_1}\Big]\,e^{-i\tilde{\varphi}} + O(\epsilon).\end{aligned}$$

These asymptotic behaviors will be helpful to simplify the computations in the sequel. We first deal with

$$\begin{aligned}
&\mathrm{Re}\Big[\int_{\mathfrak{S}_+}\frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,\big(\Omega_{10}+\Omega_{12}+\Omega_{20}+\Omega_{22}\big)\,e^{i\tilde\varphi}\Big]\\
&=-\int_{\mathfrak{S}_+}\frac{1}{(1+\rho^2)^2}\left(\frac{1}{d+y_1}\frac{\partial\tilde\rho}{\partial y_1}+\frac{\tilde\rho(\tilde\rho^2-1)}{\tilde\rho^2+1}\big(2\nabla\varphi_s\nabla\theta+|\nabla\varphi_s|^2\big)\right)\frac{\partial\tilde\rho}{\partial y_1}\\
&\quad-\int_{\mathfrak{S}_+}\frac{1}{(1+\rho^2)^2}\frac{2\tilde\rho(1-\tilde\rho^2)}{1+\tilde\rho^2}\nabla\tilde\rho\cdot\nabla\varphi_s\frac{\partial\theta}{\partial y_1}\;+\;O(\epsilon),
\end{aligned}$$

where

$$\int_{\mathfrak{S}_+}\frac{1}{(1+\rho^2)^2}\frac{1}{d+y_1}\left(\frac{\partial\tilde\rho}{\partial y_1}\right)^2\;=\;O(\epsilon).$$

Let us notice that

$$\nabla\varphi_s=-\frac{1}{2d}\log d\nabla y_2+O(\epsilon\log|y|).$$

Hence

$$\begin{aligned}
&-\int_{\mathfrak{S}_+}\frac{1}{(1+\rho^2)^2}\left[\frac{\tilde\rho(\tilde\rho^2-1)}{\tilde\rho^2+1}\big(2\nabla\varphi_s\cdot\nabla\theta+|\nabla\varphi_s|^2\big)\frac{\partial\tilde\rho}{\partial y_1}\right.\\
&\qquad\qquad\left.+\frac{2\tilde\rho(1-\tilde\rho^2)}{1+\tilde\rho^2}\nabla\tilde\rho\cdot\nabla\varphi_s\frac{\partial\theta}{\partial y_1}\right]+O(\epsilon)\\
&=\frac{-1}{d\gamma^2}\log d\int_{\mathfrak{S}_+}\frac{1-\rho^2}{(\rho^2+1)^3}\frac{\rho\rho'}{|y|}dy\;+\;O(\epsilon)\\
&=\frac{-\pi}{4d\gamma^2}\log d\;+\;O(\epsilon).
\end{aligned}$$

Similarly, we can estimate the next two terms in (7.2):

$$\begin{aligned}
&\mathrm{Re}\Big[\int_{\mathfrak{S}_+}\frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,\big(\Omega_{11}+\Omega_{21}\big)\,e^{i\tilde\varphi}\Big]\\
&=-\gamma^{-2}\int_{\mathfrak{S}_+}\frac{1}{(1+\rho^2)^2}\left(\frac{\partial^2\tilde\rho}{\partial x_2^2}-\frac{2\tilde\rho}{1+\tilde\rho^2}\Big(\frac{\partial\tilde\rho}{\partial x_2}\Big)^2\right)\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial\tilde\rho}{\partial y_1}\\
&\quad+\gamma^{-2}\int_{\mathfrak{S}_+}\frac{\tilde\rho(\tilde\rho^2-1)}{(1+\rho^2)^3}\left(\Big(\frac{\partial\varphi_0}{\partial x_2}\Big)^2+2\frac{\varphi_0}{\partial x_2}\frac{\varphi_d}{\partial x_2}+\Big(\frac{\varphi_d}{\partial x_2}\Big)^2\right)\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial\tilde\rho}{\partial y_1}\\
&\quad-\gamma^{-2}\int_{\mathfrak{S}_+}\frac{2\tilde\rho(1-\tilde\rho^2)}{(1+\rho^2)^3}\left(\frac{\tilde\rho}{\partial x_2}\frac{\varphi_0}{\partial x_2}+\frac{\tilde\rho}{\partial x_2}\frac{\varphi_s}{\partial x_2}\right)\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{d^2}\right]\frac{\partial\theta}{\partial y_1}\;+\;O(\epsilon)\\
&=O(\epsilon).
\end{aligned}$$

On the other hand, the last two terms in (7.2) can be estimated by

$$\begin{aligned}
&\mathrm{Re}\Big[\int_{\mathfrak{S}_+} \frac{1}{(1+|V_d|^2)^2}\,\overline{Z_d}\,\big(\Omega_{13}+\Omega_{23}\big)\,e^{i\tilde{\varphi}}\Big] \\
&= \delta\gamma^{-1}\varepsilon|\log\varepsilon|\int_{\mathfrak{S}_+}\frac{1}{(1+|\tilde{\rho}|^2)^2}\Big[\rho\frac{\partial\rho}{\partial x_1}\frac{\partial\varphi_0}{\partial x_2}-\rho\frac{\partial\rho}{\partial x_2}\frac{\partial\varphi_0}{\partial x_1}\Big] + o(\varepsilon) \\
&= \delta\gamma^{-1}\varepsilon|\log\varepsilon|\int_{\mathbb{R}^2}\frac{\rho\rho'}{|y|(1+\rho^2)^2}\,\mathrm{d}y + o(\varepsilon) \\
&= \frac{\pi}{2}\delta\gamma^{-1}\varepsilon|\log\varepsilon| + o(\varepsilon).
\end{aligned}$$

As a conclusion, there holds

$$\mathcal{C}_\epsilon(d) = c_0\left[\frac{\pi}{4d\gamma^2}\log d - \frac{\pi}{2\gamma}\delta\epsilon\log\frac{1}{\epsilon} + O(\epsilon)\right], \tag{7.3}$$

where $c_0 \neq 0$. Therefore, by recalling the parameters in (3.10), we obtain a solution to $\mathcal{C}_\epsilon(d) = 0$ with the following asymptotic behavior:

$$d_\epsilon \sim \frac{1+o(1)}{2\epsilon}. \tag{7.4}$$

This proves Theorem 1.2.

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