PROBABILISTICALLY NILPOTENT HOPF ALGEBRAS

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ABSTRACT. In this paper we investigate nilpotenct and probabilistically nilpotent Hopf algebras. We define nilpotency via a descending chain of commutators and give a criterion for nilpotency via a family of central invertible elements. These elements can be obtained from a *commutator matrix A* which depends only on the Grothendieck ring of H. When H is almost cocommutative we introduce a probabilistic method. We prove that every semisimple quasitriangular Hopf algebra is probabilistically nilpotent. In a sense we thereby answer the title of our paper *Are we counting or measuring anything?* by *Yes, we are.*

INTRODUCTION

Following our previous generalizations of classical and recent ideas about finite groups, this paper deals with nilpotent and probabilistically nilpotent Hopf algebras.

Nilpotency of a group is defined via an ascending chain of normal subgroups corresponding to centers of group quotients. Motivated by this, Gelaki and Nikshych have defined in [9] nilpotent fusion categories and nilpotent Hopf algebras. In this paper we give an intrinsic definition of nilpotency for semisimple Hopf algebras H over a field of characteristic 0. The role of the center of a group is played by the Hopf center [1] and the role of normal subgroups is played by normal left coideal subalgebras of H. Commutator subgroups provide an alternative way of defining nilpotency for groups. Here again our previous work on commutators [8] enables us to generalize these to semisimple Hopf algebras.

In order to give an easy criterion for nilpotency we define a recursive set of central invertible elements, $\{\gamma_m\}$, which are obtained by applying a *commutator operator* on the center of H and can be realized via a corresponding *commutator matrix*. This matrix, motivated by [2], depends only on the Grothendieck ring of H. We show that under a minor assumption on it, H is nilpotent if and only if $\gamma_m = 1$ for some m. This is in fact an extension of our previous result in [8], where we proved, even without the mentioned minor assumption, that H is a commutative algebra (in particular nilpotent) if and only if $\gamma_1 = 1$.

Probabilistic methods for group theory were introduced in the early 1960s by Erdos and Renyi and have been since applied with a great deal of success. Many of the methods involve character theory and conjugacy classes (see e.g [12]).

This paper is a first attempt to adopt probabilistic methods for Hopf algebras. Amit and Vishne [2] coined the term 'probabilistically nilpotent groups' based on

the observation that the limiting behavior of the counting functions of iterated commutators is independent of the group and so all finite groups are probabilistically nilpotent. We define here probablistically nilpotent Hopf algebras and prove that all semisimple quasitriangular Hopf algebras have this property. We show how some counting functions on groups and their generalization to Hopf algebras, as in [8], can now be realized as *distribution functions* and thus can be considered as *measuring*. We thereby answer the title of our paper [8] *Are we counting or measuring anything?* by *Yes, we are.*

Throughout this paper we assume H is a semisimple Hopf algebra over a field of characteristic 0 and Λ is its idempotent integral. We denote the character algebra of H by $R(H)$ and the center of H by $Z(H)$. The paper is organized as follows:

In §1 we give the following intrinsic definition of nilpotency for Hopf algebras. For any normal left coideal subalgebra N of H let $H//N$ denote the Hopf quotient $H/H(N \cap \ker \varepsilon)$. Let $\tilde{Z}(H)$ denote the Hopf center of H and define a series of normal left coideal subalgebras of H as follows. Set:

$$H_0 = H, \ \pi_0 = \text{Id}, \ Z_0 = k,$$

and set by induction for $i > 0$,

$$H_i = H_{i-1}//\tilde{Z}(H_{i-1}), \ \pi_i : H \longrightarrow H_i, \ Z_i = H^{co\pi_i}.$$

A categorical definition of nilpotency was given in [9]. We give here an intrinsic definition for nilpotent Hopf algebras and show how the two definitions coincide.

Definition 1.2. A semisimple Hopf algebra H is **nilpotent** if the ascending central series

$$k \subseteq Z_1 \subseteq Z_2 \subseteq \cdots$$

satisfies $Z_m = H$ for some $m \geq 1$. The smallest such m is called the index of nilpotency of H.

For the commutator approach, define as in [8] the generalized commutator $\{a, b\}$,

$$\{a, b\} = \sum a_1 b_1 S a_2 S b_2$$

for any $a, b \in H$. Define an ascending chain of iterated commutators by:

$$N_0 = H, \ N_1 = [\{H, \Lambda\}], \ \ldots, N_t = [\{N_{t-1}, \Lambda\}]$$

where $[S]$ is the normal left coideal subalgebra generated by any subset S of H. We prove:

Theorem 1.6. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic* 0. *Then*

$$N_t = k1 \Longleftrightarrow Z_t = H.$$

In this case $N_{t-i} \subseteq Z_i$ for all $0 \leq i \leq t$.

In §2 we define the *Commutator operator $T : Z(H) \longrightarrow Z(H)$* by:

$$T(z) = \{z, \Lambda\}.$$

Let $\{E_i\}$, $\{\chi_i\}$, $0 \leq i \leq n-1$, be the full set of central idempotents of H and the corresponding set irreducible characters of degree d_i respectively. We prove:

Proposition 2.1. (i) *The matrix of T with respect to the basis $\{\frac{E_0}{d_0^2}, \ldots, \frac{E_{n-1}}{d_{n-1}^2}\}$ is A, where*

$$A_{ij} = \frac{\langle \chi_i s(\chi_i) s(\chi_j), \Lambda \rangle}{d_j}, \quad 0 \leq i, j \leq n-1.$$

(ii) *A has non-negative rational entries and the first column of A has all entries equal to* 1.

(iii) *The first row of A^m is $(1, 0, \ldots, 0)$ for all $m \geq 0$.*

(iv) *In the first column of A^m we have:*

$$(A^m)_{i0} = \sum_j (A^{m-1})_{ij}.$$

For all $m > 0$, $0 \leq i \leq n-1$. In particular, the first column of A^m consists of positive rational numbers.

We refer to A as the *commutator matrix* of H. Since this matrix depends only on the Grothendieck ring of H, it follows that the commutator matrix is a categorical invariant.

We define the following family of central iterated commutators which play a key role in the sequel:

$$\gamma_0 = \Lambda, \ \gamma_1 = \{\Lambda, \Lambda\}, \ \ldots, \ \gamma_m = \{\gamma_{m-1}, \Lambda\}.$$

We prove:

Theorem 2.6. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0, and let N_t, γ_t be defined as above. Assume $\chi_i s(\chi_i) \in Z(R(H))$ for each irreducible character χ_i. Then H is nilpotent if and only if $\gamma_m = 1$ for some $m \in \mathbb{Z}^+$. Its index of nilpotency is the least integer m so that $\gamma_m = 1$.*

A significant corollary is the following:

Theorem 2.8. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0 and assume $\chi_i s(\chi_i) \in Z(R(H))$ for each irreducible character χ_i. Then H is nilpotent if and only if its commutator matrix A has eigenvalues $\{1, 0\}$ and the algebraic multiplicity of 1 is 1.*

In §3 we introduce a probabilistic method for semisimple Hopf algebras such that $R(H)$ is commutative. For $0 \leq i \leq n-1$, let $\{F_i\}$ be a full set of primitive idempotents of $R(H)$. In this setup we have defined in [5] Hopf algebraic analogues of conjugacy classes $\mathfrak{C}_i$, class sums C_i and normalized class sums η_i. The set $\{\eta_i\}$ forms a basis for $Z(H)$.

We call an element $z \in Z(H)$ a **central distribution element** if

$$z = \sum \alpha_i \eta_i, \ \alpha_i \in \mathbb{R}^+ \cup \{0\}, \quad \sum_i \alpha_i = 1.$$

The idempotent integral Λ and all η_i are such elements. The central distribution element z defines a distribution X_z on H by letting:

$$\mathrm{Prob}(X_z = C_i) = \alpha_i.$$

The corresponding distribution function f_z is given by:

$$f_z(C_i) = \text{Prob}(X_z = C_i) = \alpha_i.$$

Counting functions for groups give rise to distribution functions on the group algebras after dividing by an appropriate power of $|G|$. For example, Frobenius proved that the function on a finite group G that counts the number of ways an element of G can be realized as a commutator is given by:

$$f_{com} = \sum_i \frac{|G|}{d_i} \chi_i.$$

Then $f = \frac{1}{|G|^2} f_{com}$ is a distribution function for kG with a corresponding central distribution element $z = \frac{1}{|G|^2} \sum_{a,b \in G} aba^{-1}b^{-1}$. The Hopf algebra analogue of f is $\frac{1}{d} \sum_i \frac{1}{d_i} \chi_i$. It is the distribution function corresponding to the central distribution element $z = \gamma_1$.

When H is quasitriangular then our *commutator map T* maps central distribution elements to central distribution elements. In particular, all γ_m are central distribution elements. As a consequence we prove:

Proposition 3.8. *Let $z \in Z(H)$ be a central distribution element and let $T(z) = \{z, \Lambda\}$. Then*

$$\text{Prob}(T^m(z) = 1) \longrightarrow 1 \quad as \quad m \to \infty.$$

These give rise to the following definition:

Definition 3.9. A semisimple Hopf algebra is **probabilistically nilpotent** if

$$\text{Prob}(\gamma_m = 1) \longrightarrow 1 \quad as \quad m \to \infty.$$

The main result here is:

Theorem 3.10. *Let H be a semisimple quasitriangular Hopf algebra over an algebraically closed field of characteristic 0. Then H is probabilistically nilpotent.*

Note that even if H is far from being nilpotent, for example if its Hopf center is k, H is still probabilistically nilpotent. This fact is demonstrated when we characterize the eigenvalues of the matrix A over $\mathbb{C}$.

Theorem 3.11. *Let H be a semisimple quasitriangular Hopf algebra over $\mathbb{C}$. Then the commutator matrix A has 1 as an eigenvalue with corresponding 1-dimensional eigenspace. All other eigenvalues c satisfy $|c| < 1$.*

1. Upper and lower central series

Throughout this paper, H is a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. We denote by S and s the antipodes of H and H^* respectively. Denote by $Z(H)$ the center of H.

The Hopf algebra H^* becomes a right and left H-module by the *hit* actions $\leftharpoonup$ and $\rightharpoonup$ defined for all $a \in H$, $p \in H^*$,

$$\langle p \leftharpoonup a, a' \rangle = \langle p, aa' \rangle \qquad \langle a \rightharpoonup p, a' \rangle = \langle p, a'a \rangle.$$

H becomes a left and right H^*-module analogously.

Denote by $\cdot_{ad}$ the left adjoint action of H on itself, that is, for all $a, h \in H$,

$$h \cdot_{ad} a = \sum h_1 a S(h_2).$$

A left coideal subalgebra of H is called *normal* if it is stable under the left adjoint action of H.

Let $\{V_0, \ldots, V_{n-1}\}$ be a complete set of non-isomorphic irreducible H-modules. Let $\{E_0, \ldots, E_{n-1}\}$ and $\mathrm{Irr}(H) = \{\chi_0, \ldots, \chi_{n-1}\}$ be the associated central primitive idempotents and irreducible characters of H respectively, where $E_0 = \Lambda$, the idempotent integral of H and $\chi_0 = \varepsilon$. Let $\dim V_i = d_i = \langle \chi_i, 1 \rangle$. Then

$$\lambda = \chi_H = \sum_{i=0}^{n-1} d_i \chi_i$$

is an integral for H^* satisfying $\langle \lambda, \Lambda \rangle = 1$.

For any normal left coideal subalgebra N of H we denote by $H//N$ the Hopf quotient $\overline{H} = H/HN^+$, where $N^+ = N \cap \ker \varepsilon$. Recall for any Hopf surjection $\pi : H \to \overline{H}$ we define,

$$H^{co\pi} = \{h \in H \mid \sum h_1 \otimes \pi(h_2) = h \otimes \pi(1)\}.$$

Then by [13]

$$H//H^{co\pi} \cong \overline{H}.$$

Related concepts are invariants. For any subalgebra T of H^*, we denote by H^T the set of T-invariants of H under the left *hit* action. That is,

$$(1) \qquad H^T = \{h \in H \mid b \rightharpoonup h = \langle b, 1 \rangle h, \ \forall b \in T\}.$$

It was shown in [7, Prop. 2.2] that B is a Hopf subalgebra of H^* if and only if $N = H^B$ is a normal left coideal subalgebra of H. In this case one also has $B = (H^*)^N$.

Fix $\pi : H \to \overline{H}$, $B = \overline{H}^* \subset H^*$ and N a left coideal subalgebra. Then summing up:

$$(2) \qquad N = H^{co\pi} \Leftrightarrow \overline{H} = B^* \cong H//N \Leftrightarrow \overline{H}^* = B = (H^*)^N \Leftrightarrow N = H^B.$$

For any H-representation V the left kernel LKer_V is defined as:

$$(3) \qquad \mathrm{LKer}_V = \{h \in H \mid \sum h_1 \otimes h_2 \cdot v = h \otimes v \quad \forall v \in V\}.$$

Considering $\overline{H}$ as an H-representation it follows (see e.g. [3]) that

$$H^{co\pi} = \mathrm{LKer}_{\overline{H}}.$$

The Hopf center of a Hopf algebra was introduced and described in [1]. It is the maximal Hopf subalgebra of H contained in $Z(H)$. A categorical version of it is used in [9] to define upper central series for fusion categories. In what follows we present an intrinsic approach for semisimple Hopf algebras. One of the advantages of this approach is that it gives rise to a descending chain of commutators of H as well.

Let H^*_{ad} be the Hopf subalgebra of H^* generated by

$$\chi_{ad} = \sum_j \chi_j s(\chi_j), \ \chi_j \in \mathrm{Irr}(H).$$

Equivalently, the Hopf algebra generated by the irreducible constituents of $\chi_i s(\chi_i)$, $0 \leq i \leq n-1$.

Since χ_{ad} and all its powers are central in $R(H)$ we have for each l, $\chi_i \chi_{ad}^l s(\chi_i) = \chi_{ad}^l \chi_i s(\chi_i) \in H_{ad}^*$. Hence $D_i D_{\chi_{ad}^l} s(D_i) \subset H_{ad}^*$, where D_i is the simple coalgebra generated by χ_i. It follows that H_{ad}^* is a normal Hopf subalgebra of H^*. Define:

$$(4) \qquad \tilde{Z}(H) = H^{H_{ad}^*} \subset H.$$

Take $V = (H, _{ad})$. Then by [7, Th. 2.8], $\tilde{Z}(H) = \mathrm{LKer}_V$. Explicitly,

$$(5) \qquad \tilde{Z}(H) = \{h \in H \mid \sum h_1 \otimes h_2 x S h_3 = h \otimes x \quad \forall x \in H\}.$$

The following proposition is a variation of [1].

Proposition 1.1. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0. Then $\tilde{Z}(H)$ is the Hopf center of H. Moreover, $\tilde{Z}(H)$ contains every left (right) coideal of H contained in $Z(H)$.*

Proof. Observe first that every left coideal N contained in $Z(H)$ is contained in $\tilde{Z}(H)$. Indeed, if $h \in N \subset Z(H)$, then $\mathrm{sp}_k\{h_3\} \subset N \subset Z(H)$ implying that h satisfies the right hand side of (5).

We wish to show now that $\tilde{Z}(H)$ is contained in $Z(H)$. Let $h \in \tilde{Z}(H)$. Applying $\mu \circ (S \otimes \mathrm{Id})$ to the right hand side of (5) yields $xSh = Shx$ for all $x \in H$, hence Sh and thus $h \in Z(H)$.

Since H_{ad}^* is normal in H^* it follows from [7, Prop. 2.2] that $\tilde{Z}(H) = H^{H_{ad}^*}$ is a Hopf subalgebra of H. $\qquad\qquad \square$

Gelaki and Nikshych [9, Def. 4.4] have defined nilpotent fusion categories by using appropriate upper central series [9, Def. 4.1]. They called a semisimple Hopf algebra H nilpotent if the fusion category $\mathrm{Rep}(H)$ of representations of H is nilpotent.

Explicitly, there exists a descending chain of Hopf subalgebras of H^* defined as follows:

$$(6) \qquad B_0 = H^*, \ B_1 = H_{ad}^*, \ \ldots, B_{i+1} = B_{i_{ad}}.$$

Then H is nilpotent if there exists n so that $B_n = k$.

An intrinsic way to generalize the notion of an upper central series from finite groups to semisimple Hopf algebras is as follows. We define an ascending central series of normal left coideal subalgebras $\{Z_n\}$. As for groups this series corresponds to Hopf centers of Hopf quotients. Set:

$$H_0 = H, \ \pi_0 = \mathrm{Id}, \ Z_0 = k,$$

and set by induction for $i > 0$,

$$(7) \qquad H_i = H_{i-1}//\tilde{Z}(H_{i-1}), \ \pi_i : H \longrightarrow H_i, \ Z_i = H^{co\pi_i}.$$

Then

$$(8) \qquad Z_1 = \tilde{Z}(H), \quad H_i = H//Z_i.$$

This motivates the following alternative definition:

Definition 1.2. A semisimple Hopf algebra H is **nilpotent** if the ascending central series

$$k \subseteq Z_1 \subseteq Z_2 \subseteq \cdots$$

satisfies $Z_m = H$ for some $m \geq 1$. The smallest such m is called the index of nilpotency of H.

The following lemma (that can also be obtained from [9, Th. 3.8]) enables us to connect the two definitions of nilpotency.

Lemma 1.3. *Let Z_n and B_n be defined as in (7) and (6) respectively. Then for $n \geq 0$ we have*

$$Z_n = H^{B_n}.$$

In particular, the intrinsic definition and the categorical definition of nilpotency coincide.

Proof. By (7), $H_1 = H//Z_1$, hence by (2) and (4) $Z_1 = H^{B_1}$. Assume by induction $Z_i = H^{B_i}$; then by (8) and (2),

$$H_i = H//Z_i \cong B_i^*.$$

It follows from (4) and (6) that

$$\tilde{Z}(H_i) = H_i^{B_{i_{ad}}} = H_i^{B_{i+1}},$$

hence by (7) and (2),

$$H_{i+1} = H_i//\tilde{Z}(H_i) \cong B_{i+1}^*.$$

But by (8) $H_{i+1} \cong H//Z_{i+1}$, hence $H//Z_{i+1} \cong B_{i+1}^*$ by the formula in the line above. This implies by (2) that:

$$Z_{i+1} = H^{B_{i+1}}.$$

$\square$

For groups there is a related descending chain of subgroups arising from commutators. In what follows we generalize this idea to Hopf algebras. As in [8], for $a, b \in H$, set their commutator

$$\{a, b\} = \sum a_1 b_1 S a_2 S b_2.$$

Commutators and Hopf centers are related in the following way.

Proposition 1.4. *Let N be a normal left coideal subalgebra of H. Then for all $t \geq 0$,*

$$\{N, \Lambda\} \subseteq Z_t \iff N \subseteq Z_{t+1}.$$

Proof. Assume first $\{N, \Lambda\} = k$. We show that if $x \in N$, then $S(x) \in Z(H)$. Indeed,

$$\Lambda_{ad} Sx = \sum Sx_1 x_2 \Lambda_1 Sx_3 S\Lambda_2 = \sum Sx_1 \{x_2, \Lambda\} = Sx.$$

The last equality follows from the assumption since N is a left coideal. Thus $x \in Z(H)$ as well. Since Z_1 is the Hopf center of H it follows from Proposition 1.1 that $N \subseteq Z_1$.

Conversely, if $N \subseteq Z_1$, then in particular $N \subseteq Z(H)$. Hence for all $x \in N$, $\sum \Lambda_1 x_1 S\Lambda_2 Sx_2 = \langle \varepsilon, x \rangle$.

We continue by induction. Note $\pi(\Lambda)$ is an idempotent integral for any homomorphic image $\pi(H)$ of H. Hence

$$\{N, \Lambda\} \subseteq Z_t \implies \{\pi_t(N), \pi_t(\Lambda)\} = \pi_t(\{N, \Lambda\}) = k.$$

But $\pi_t(N)$ is a normal left coideal subalgebra of H_t, hence by the first step of the induction proved above we obtain $\pi_t(N) \subset \tilde{Z}(H_t)$. Now, by definition, $H_{t+1} \cong H_t//\tilde{Z}(H_t)$, hence N is mapped under π_{t+1} into k. Since N is a normal left coideal subalgebra it follows that

$$N \subset H^{co\pi_{t+1}} = Z_{t+1}.$$

Conversely, assume $N \subset Z_{t+1}$. Then $\pi_{t+1}(N) = k$ and since $\pi_t(N)$ is a left coideal it follows that $\pi_t(N) \subset \tilde{Z}(H_t)$. Let $m = \sum n_1 \Lambda_1 S n_2 S \Lambda_2 \in \{N, \Lambda\}$; then

$$\sum m_1 \otimes \pi_t(m_2) = \sum n_1 \Lambda_1 S n_4 S \Lambda_4 \otimes \pi_t(n_2) \pi_t(\Lambda_2) \pi_t(S n_3) \pi_t(S \Lambda_3) = m \otimes \pi_t(1).$$

The last equality holds since $\pi_t(N) \subset \tilde{Z}(H_t)$. This implies that the subcoalgebra generated by $\pi_t(N)$ is contained in $\tilde{Z}(H_t) \subset Z(H_t)$. Thus $m \in Z_t$. $\square$

For a set S let $[S]$ denote the normal left coideal subalgebra generated by S. Define a descending chain of iterated commutators for H as follows:

(9) $$N_0 = H, \ N_1 = [\{H, \Lambda\}], \ \ldots, N_t = [\{N_{t-1}, \Lambda\}].$$

By induction, if $N_t \subseteq N_{t-1}$, then $\{N_t, \Lambda\} \subseteq \{N_{t-1}, \Lambda\}$. Hence

$$N_{t+1} = [\{N_t, \Lambda\}] \subseteq [\{N_{t-1}, \Lambda\}] = N_t.$$

Remark 1.5. Let $H = kG$, G a finite group. Then $\tilde{Z}(H) = kZ_G$, where Z_G is the center of the group G. This follows since every Hopf subalgebra of kG has the form kK where K is a subgroup of G. For any Hopf quotient we have $kG//kK \cong k(G/K)$. Moreover, for any Hopf surjection π we have $H^{co\pi} = k\pi^{-1}(1)$. These observations imply that an ascending central series for G gives rise to an ascending central series for kG and vice versa.

If we denote by induction $G_1 = [G, G]$ and $G_{t+1} = [G_t, G]$, then $N_t = kG_t$.

We can now prove the main result of this section.

Theorem 1.6. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic* 0*, let Z_t be defined as in* (7) *and let N_t be as defined in* (9)*. Then*

$$N_t = k1 \Longleftrightarrow Z_t = H.$$

In this case $N_{t-i} \subseteq Z_i$ for all $0 \le i \le t$.

Proof. We claim first that for all $0 \le i \le t$,

(10) $$N_{t-i} \subseteq Z_i \Longleftrightarrow N_{t-i-1} \subseteq Z_{i+1}.$$

Indeed, assume $N_{t-i} \subseteq Z_i$. Since by definition $\{N_{t-i-1}, \Lambda\} \subseteq N_{t-i}$, it follows in particular that $\{N_{t-i-1}, \Lambda\} \subseteq Z_i$. By Proposition 1.4 $N_{t-i-1} \subseteq Z_{i+1}$. Conversely, if $N_{t-i-1} \subseteq Z_{i+1}$, then $\{N_{t-i-1}, \Lambda\} \subset Z_i$ by Proposition 1.4. Since N_{t-i} is generated as a normal left coideal by $\{N_{t-i-1}, \Lambda\}$, and since Z_i is a normal left coideal subalgebra containing $\{N_{t-i-1}, \Lambda\}$, it follows that $N_{t-i} \subseteq Z_i$. This proves our claim.

Now, if $N_t = k$, then by (10), $N_{t-1} \subset Z_1$, and by induction on i $N_{t-i} \subset Z_i$. For $i = t$ we get $H = N_0 \subset Z_t$.

Conversely, assume that $Z_t = H = N_0$. Then we use (10) again to prove by induction on j that $N_j \subset Z_{t-j}$ for all j. When $j = t$ we obtain $N_t = k1$. $\square$

2. Iterated commutators

Recall, (see e.g. [11, Cor. 4.6]):

(11) $$\langle \chi_i, E_j \rangle = \delta_{ij} d_j, \quad \chi_i \leftharpoonup E_j = \delta_{ij} \chi_i, \quad \Lambda \leftharpoonup s(\chi_j) = \frac{1}{d_j} E_j.$$

In particular, $\{\chi_i\}$, $\{\frac{1}{d_j}E_j\}$ are dual bases of $R(H)$ and $Z(H)$ respectively. Hence we have for each $z \in Z(H)$, $p \in R(H)$

$$(12) \qquad z = \sum_i \frac{1}{d_i} \langle \chi_i, z \rangle E_i, \qquad p = \sum_i \frac{1}{d_i} \langle p, E_i \rangle \chi_i.$$

By (11) and (12) we have:

$$(13) \qquad \chi_i \leftharpoonup z = \frac{1}{d_i} \langle \chi_i, z \rangle \chi_i$$

for all i. Since $\Lambda_{ad}H = Z(H)$ we have for all $h \in H$,

$$\langle \chi_i, \{h, \Lambda\} \rangle$$
$$= \sum \langle \chi_i, \Lambda_1 h_1 S \Lambda_2 S h_2 \rangle = \sum \langle \chi_i \leftharpoonup \Lambda_1 h_1 S \Lambda_2, S h_2 \rangle = \frac{1}{d_i} \langle \chi_i s(\chi_i), h \rangle,$$

where the last equality follows from (13). Thus

$$(14) \qquad \langle \chi_i, \{h, \Lambda\} \rangle = \frac{1}{d_i} \langle \chi_i s(\chi_i), h \rangle.$$

Define an operator $T : Z(H) \longrightarrow Z(H)$ by

$$T(z) = \{z, \Lambda\}.$$

Indeed, by [8, Lemma 2.3], $T(z) \in Z(H)$ for all $z \in Z(H)$. Moreover, by definition of N_m we have

$$(15) \qquad T^m(z) \in N_m \quad \forall z \in Z(H).$$

We have:

Proposition 2.1. (i) *The matrix of T with respect to the basis $\{\frac{E_0}{d_0^2}, \ldots, \frac{E_{n-1}}{d_{n-1}^2}\}$ is A, where*

$$(16) \qquad A_{ij} = \frac{\langle \chi_i s(\chi_i) s(\chi_j), \Lambda \rangle}{d_j}, \quad 0 \le i, j \le n - 1.$$

(ii) *A has non-negative rational entries and the first column of A has all entries equal 1.*

(iii) *The first row of A^m is $(1, 0, \ldots, 0)$ for all $m \ge 0$.*

(iv) *In the first column of A^m we have:*

$$(A^m)_{i0} = \sum_j (A^{m-1})_{ij}.$$

For all $m > 0$, $0 \le i \le n - 1$. In particular, the first column of A^m consists of positive rational numbers.

Proof. (i) By dual bases of $Z(H)$ and $R(H)$ and by (14) we have:

$$T(\frac{E_j}{d_j^2}) = \{\frac{E_j}{d_j^2}, \Lambda\}$$
$$= \sum_i \langle \{\frac{E_j}{d_j^2}, \Lambda\}, \chi_i \rangle \frac{E_i}{d_i} = \sum_i \langle \frac{E_j}{d_j^2}, \chi_i s(\chi_i) \rangle \frac{E_i}{d_i^2} = \sum_i \frac{\langle \Lambda, \chi_i s(\chi_i) s(\chi_j) \rangle}{d_j} \frac{E_i}{d_i^2}$$

where the last equality follows from (11).

(ii) All entries are non-negative rational numbers since the (i, j) entry of A equals the number of times χ_j appears as a constituent of $\chi_i s(\chi_i)$ divided by d_j. The orthogonality relations of characters imply that the first column has all entries equal 1.

(iii) Follows from the definition of A since $\chi_0 = \varepsilon$.

(iv) The proof follows by induction on m. For $m = 1$ this follows from part (ii) since $A^0 = \mathrm{Id}$. Assume $(A^m)_{k0} = \sum_j (A^{m-1})_{kj}$ for all $0 \le k \le n - 1$. Then:

$$(A^{m+1})_{i0} = (AA^m)_{i0} = \sum_k A_{ik}(A^m)_{k0} = \sum_{k,j} A_{ik}(A^{m-1})_{kj} = \sum_j (A^m)_{ij},$$

where the third equality follows from the induction hypothesis.

The last part follows by induction since the first column of A consists of 1's and all other entries are non-negative. $\qquad\square$

We refer to A as the **commutator matrix** of H. Observe that this matrix depends only on the Grothendieck ring of H.

We next define an important family of central iterated commutators which will play a key role in the sequel.

Define,

$$(17) \qquad \gamma_0 = \Lambda, \ \gamma_1 = T(\Lambda) = \{\Lambda, \Lambda\}, \ \ldots, \ \gamma_m = T^m(\Lambda) = \{\gamma_{m-1}, \Lambda\}.$$

Note that γ_1 is the Hopf analogue of the *extensive commutator z* in kG given by:

$$(18) \qquad\qquad\qquad z = \frac{1}{|G|^2} \sum_{a, b \in G} aba^{-1}b^{-1}.$$

We show

Proposition 2.2. *Let γ_m be defined as in (17) and the matrix A be defined as in Proposition 2.1. Then*

$$\gamma_m = \sum_i (A^m)_{i0} \frac{E_i}{d_i^2} = \sum_i \left(\sum_j (A^{m-1})_{ij} \right) \frac{E_i}{d_i^2}$$

for all $m \ge 1$. Moreover, the coefficient of each E_i in γ_m is a positive rational number, in particular γ_m is invertible.

Proof. Since $\Lambda = \frac{E_0}{d_0^2}$, it follows from Proposition 2.1(i) that the coordinates of $T^m(\Lambda)$ with respect to the basis $\{\frac{E_i}{d_i^2}\}$ is the first column of A^m. The result follows now from Proposition 2.1(iv). $\qquad\square$

Note that in particular $\gamma_1 = \sum_i \frac{1}{d_i^2} E_i$. This result was proved also in [8]. We show now

Lemma 2.3. *Let γ_m be defined as in (17). Then:*

(i) *For each irreducible character χ_i, $\langle \gamma_m, \chi_i \rangle$ is a non-negative rational number satisfying $\langle \gamma_m, \chi_i \rangle \le d_i$. In particular,*

$$\langle \gamma_1, \chi_i \rangle = \frac{1}{d_i}.$$

(ii) *$\langle \gamma_{m+1}, \chi_i \rangle = d_i$ if and only if $\langle \gamma_m, \chi_j \rangle = d_j$ for each irreducible constituent of $\chi_i s(\chi_i)$.*

Proof. (i) Proposition 2.2 implies that $\langle \gamma_m, \chi_i \rangle$ is a non-negative rational number for all m. Also, for $m = 0$, $\langle \Lambda, \chi_i \rangle = \delta_{i,0} \leq d_i$. For $m = 1$ we have $\gamma_1 = \sum_i \frac{1}{d_i^2} E_i$, hence $\langle \gamma_1, \chi_i \rangle = \frac{1}{d_i} \leq d_i$.

Assume by induction $\langle \gamma_m, \chi_i \rangle \leq d_i$ for all i. Then by (14),

$$\langle \gamma_{m+1}, \chi_i \rangle = \frac{1}{d_i} \langle \gamma_m, \chi_i s(\chi_i) \rangle.$$

Let $\chi_i s(\chi_i) = \sum m_j \chi_j$; then $\sum m_j d_j = d_i^2$ and we have:

$$(19) \qquad \langle \gamma_{m+1}, \chi_i \rangle = \frac{1}{d_i} \sum m_j \langle \gamma_m, \chi_j \rangle \leq \frac{1}{d_i} \sum m_j d_j = d_i.$$

(ii) If $\langle \gamma_{m+1}, \chi_i \rangle = d_i$, then equality holds in (19). Since all m_j are positive we must have $\langle \gamma_m, \chi_j \rangle = d_j$ for each irreducible constituent χ_j. Conversely, if $\langle \gamma_m, \chi_j \rangle = d_j$ for each irreducible constituent of $\chi_i s(\chi_i)$, then equality holds in (19). $\qquad \square$

Set

$$(20) \qquad S_m = \mathrm{Sp}_k \{ \chi_i \in \mathrm{Irr}(H) \mid \langle \gamma_m, \chi_i \rangle = d_i \}$$

and

$$(21) \qquad HS_m = \text{the Hopf subalgebra of } H^* \text{ generated by } S_m.$$

We show

Lemma 2.4. *Let S_m and HS_m be defined as above. Then:*
(i) $S_0 = k$ and $S_1 = kG(H^*)$.
If moreover $\chi_i s(\chi_i) \in Z(R(H))$ for each irreducible character χ_i, then:
(ii) S_m is a based ring for all m. That is, if $\chi_i, \chi_j \in S_m$, then all the irreducible constituents of $\chi_i \chi_j$ are in S_m as well.
(iii) $S_m = \{ \chi_i \mid \chi_i s(\chi_i) \in S_{m-1} \}$, hence $(HS_m)_{ad} \subseteq HS_{m-1}$.

Proof. (i) Since $\gamma_0 = \Lambda$ we have $S_0 = k$. By Lemma 2.3(i), $\langle \gamma_1, \chi_i \rangle = \frac{1}{d_i}$, which equals d_i if and only if $d_i = 1$, that is, if and only if $\chi_i \in G(H^*)$.

(ii) Observe that centrality of $\chi_j s(\chi_j)$ implies that

$$\chi_i s(\chi_i) \chi_j s(\chi_j) = \chi_i \chi_j s(\chi_i \chi_j).$$

Clearly $S_1 = kG(H^*)$ is a based ring. Assume by induction that S_{t-1} is a based ring, and let $\chi_i, \chi_j \in S_t$. By Lemma 2.3 we have that $\langle \gamma_{t-1}, \chi_k \rangle = d_k$ for each irreducible constituent χ_k of $\chi_i s(\chi_i)$ or $\chi_j s(\chi_j)$, hence, by the induction hypothesis, we have $\chi_i s(\chi_i) \chi_j s(\chi_j) \in S_{t-1}$. Let χ_l be an irreducible constituent of $\chi_i \chi_j$. Since all constituents of $\chi_l s(\chi_l)$ are constituents of $\chi_i \chi_j s(\chi_i \chi_j) = \chi_i s(\chi_i) \chi_j s(\chi_j) \in S_{t-1}$, it follows that $\chi_l \in S_t$. Hence S_t is a based ring.

(iii) Follows directly from part (ii) and Lemma 2.3(ii). $\qquad \square$

Remark 2.5. When $R(H)$ is commutative the series of based rings S_m given in (20) coincides with the lower series defined in [9, 4.12]. However the assumption $\chi_i s(\chi_i) \in Z(R(H))$ is weaker than the assumption of commutativity of $R(H)$. For example, if $H = (kG)^*$, G a non-abelian finite group, then $R(H) = kG$ is not commutative, yet $\chi_i s(\chi_i) = s(\chi_i) \chi_i = 1$. The same is true for $H = D(kG)^*$.

We can now show the main result of this section.

Theorem 2.6. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0, and let N_t, γ_t be defined as in (9) and (17). Assume $\chi_i s(\chi_i) \in Z(R(H))$ for each irreducible character χ_i. Then H is nilpotent if and only if $\gamma_m = 1$ for some $m \in \mathbb{Z}^+$.*

Its index of nilpotency is the least integer m so that $\gamma_m = 1$.

Proof. The proof follows by induction. Let $\{B_t\}$ be the series defined in (6). If $\gamma_m = 1$, then $S_m = R(H)$ and by Lemma 2.4.(iii) we have $(H^*)_{ad} = (HS_m)_{ad} \subseteq HS_{m-1}$, hence $B_1 \subset HS_{m-1}$. Assume by induction $B_t \subseteq HS_{m-t}$. Then

$$B_{t+1} = (B_t)_{ad} \subseteq (HS_{m-t})_{ad} \subseteq HS_{m-t-1},$$

where the last inclusion follows from Lemma 2.4.(iii). We have in particular $B_m \subseteq HS_0 = k$, which implies by Lemma 1.3 that $Z_m = H$. Thus H is nilpotent. Conversely, if H is nilpotent of degree m, then $N_m = k$ implying in particular that $\gamma_m = 1$. $\qquad\square$

As a corollary we obtain

Corollary 2.7. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0 and $\{N_t\}$ be defined as in (9). Assume $\chi_i s(\chi_i) \in Z(R(H))$ for each irreducible character χ_i. Then*

$$N_t = \text{the left coideal subalgebra generated by } \gamma_t.$$

Proof. Let L_t denote the left coideal subalgebra generated by $\gamma_t \leftharpoonup H^*$. Then $L_t \subseteq N_t$. Since γ_t is central it follows by [4, Prop. 2.5] that L_t is also left normal. Let $\pi : H \to \overline{H} = H//L_t$. Since $\mathrm{Irr}(\overline{H}) \subset \mathrm{Irr}(H)$ the centrality assumption holds for $\overline{H}$ as well. Then $\gamma_t(\overline{H}) = \pi(\gamma_t) = 1$. By Theorem 2.6, $N_t(\overline{H}) = k$. Since $\pi(\Lambda) = \overline{\Lambda}$ it follows by induction that $\pi(N_t) \subseteq N_t(\overline{H}) = k$. Since N_t is a normal left coideal subalgebra and $L_t = H^{co\pi}$ we must have $N_t \subseteq L_t$. $\qquad\square$

Another corollary relates the nilpotency of H and the eigenvalues of A.

Theorem 2.8. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0 and assume $\chi_i s(\chi_i) \in Z(R(H))$ for each irreducible character χ_i. Then H is nilpotent if and only if its commutator matrix A has eigenvalues $\{1, 0\}$ where the algebraic multiplicity of 1 is 1.*

Proof. Theorem 2.6 and (15) imply that H is nilpotent of degree m if and only if the operator T satisfies $T^m = \varepsilon_{|Z(H)}$. That is, $T^m(E_i) = \langle \varepsilon, E_i \rangle 1 = \delta_{i,0}$ for each central idempotent E_i. By Proposition 2.1 this is equivalent to:

$$A^m = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ d_1^2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ d_{n-1}^2 & 0 & \cdots & 0 \end{pmatrix}.$$

It follows that the eigenvalues of T^m are 1 and 0. Hence the eigenvalues of T are roots of unity and 0. Since 1 is an eigenvector of T with eigenvalue 1, and the alegbraic multiplicity of 1 in T^m is 1, it follows that the only possible root of unity is 1 and its alegbraic multiplicity is 1. $\qquad\square$

Upon another assumption on $R(H)$, which holds in particular when $R(H)$ is commutative, yet it is a weaker assumption, we show:

Proposition 2.9. *Assume* $\chi_i s(\chi_i) = s(\chi_i)\chi_i$ *for each irreducible character* χ_i. *Then:*

(i) *For all* m, $\gamma_m = S\gamma_m$.
(ii) *For all* m, $\{\Lambda, \gamma_m\} = \{\gamma_m, \Lambda\}$.
(iii) $\gamma_{m+1} \in \gamma_m \leftharpoonup H^*$ *for all* $m \geq 0$.

Proof. (i) If $\chi_i s(\chi_i) = s(\chi_i)\chi_i$ for each irreducible character χ_i, then the matrix A defined in (16) satisfies $A_{ij} = A_{i*j}$ for all i, j. Assume by induction $(A^m)_{ij} = (A^m)_{i*j}$; then

$$(A^{m+1})_{il} = \sum_j (A^m)_{i*j} A_{jl} = (A^{m+1})_{i*l}.$$

By Proposition 2.2 the expression above implies that the coefficient of E_i in γ_m equals the coefficient of E_{i*} in γ_m, hence $\gamma_m = S\gamma_m$.

(ii) Since $\gamma_m = S\gamma_m$ we have $\sum \gamma_{m1} \otimes \gamma_{m1} = \sum S\gamma_{m2} \otimes S\gamma_{m1}$. Let $\gamma = \gamma_m$, Since $\Lambda_{\dot{ad}} h \in Z(H)$ for all h, we have

$$\sum \gamma_1 \Lambda_1 S\gamma_2 S\Lambda_2 = \sum \Lambda_1 S\gamma_2 S\Lambda_2 \gamma_1 = \sum \Lambda_1 \gamma_1 S\Lambda_2 S\gamma_2.$$

(iii) We have

$$\gamma_{m+1} = S\gamma_{m+1} = \sum \Lambda_2 \gamma_{m_2} S\Lambda_1 S\gamma_{m_1} = \sum (\Lambda_{\dot{ad}} \gamma_{m_2}) S\gamma_{m_1}.$$

Since $\gamma_m \leftharpoonup H^*$ is stable under the adjoint action of H (by [4]) and since $\gamma_{m_2}, S\gamma_{m_1} \in \gamma_m \leftharpoonup H^*$, it follows that $\gamma_{m+1} \in \gamma_m \leftharpoonup H^*$. $\qquad\square$

3. Probabilistic methods for Hopf algebras with a commutative character algebra

In this section we assume $k = \mathbb{C}$. Recall that H is a Frobenius algebra. One defines a Frobenius map $\Psi : H_{H^*} \to H^*_{H^*}$ by

$$(22) \qquad \Psi(h) = \lambda \leftharpoonup S(h), \qquad \Psi^{-1}(p) = \Lambda \leftharpoonup p,$$

where H^* is a right H^*-module under multiplication and H is a right H^*-module under right *hit*. If H is semisimple, then

$$\Psi(Z(H)) = R(H).$$

Let $\frac{1}{d}\lambda = F_0, \ldots, F_{n-1}$ be the set of central primitive idempotents of $R(H)$. When $R(H)$ is commutative $\{F_j\}$ forms another basis for $R(H)$. Define as in [6] the **conjugacy class** $\mathfrak{C}_i$ as:

$$\mathfrak{C}_i = \Lambda \leftharpoonup F_i H^*.$$

Note $\dim \mathfrak{C}_i = \dim(F_i H^*)$. We generalize also the notions of **class sum** and of a representative of a conjugacy class as follows:

$$(23) \qquad C_i = \Lambda \leftharpoonup dF_i = \Psi^{-1}(dF_i), \qquad \eta_i = \frac{C_i}{\dim \mathfrak{C}_i}.$$

We refer to η_i as a **normalized class sum**. In [6, (11)] we show that

$$(24) \qquad \langle F_i, \Lambda \rangle = \frac{\dim \mathfrak{C}_i}{d}.$$

Hence $\langle \varepsilon, \eta_i \rangle = 1$ for all i and the normalized class sums $\{\eta_i\}$ form a basis for $Z(H)$ dual to $\{F_i\}$. We can define now a generalized character table for H,

$$(\xi_{ij}) \quad \text{where} \quad \xi_{ij} = \langle \chi_i, \eta_j \rangle,$$

$0 \leq i, j \leq n - 1$. Note that $\eta_0 = 1$ and so $\xi_{i0} = \langle \chi_i, 1 \rangle = d_i$. Moreover, (ξ_{ij}) is the change of bases matrix between $\{\chi_i\}$ and $\{F_i\}$.

We call an element $z \in Z(H)$ a **central distribution element** if

$$z = \sum \alpha_i \eta_i, \; \alpha_i \in \mathbb{R}^+ \cup \{0\}, \quad \sum_i \alpha_i = 1.$$

Note that since $\langle \varepsilon, \eta_i \rangle = 1$ for all i, it follows that $\sum_i \alpha_i = 1$ is equivalent to $\langle \varepsilon, z \rangle = 1$.

The central distribution element z defines a distribution X_z by letting:

$$\text{Prob}(X_z = C_i) = \alpha_i.$$

Then the corresponding distribution function f_z is given by:

$$f_z(C_i) = \text{Prob}(X_z = C_i) = \alpha_i.$$

Since $\{C_i\}$ forms a basis for $Z(H)$ it follows that the distribution function f_z defines an element in $\hom(Z(H), \mathbb{C}) = R(H)$. By using dual bases this element is given precisely by

$$(25) \qquad f_z = \sum_i f_z(C_i) \frac{F_i}{\dim \mathfrak{C}_i} = \sum \alpha_i \frac{F_i}{\dim \mathfrak{C}_i} = \frac{1}{d} \Psi(z),$$

where Ψ is the Frobenius function given in (22) and the last equality follows from (23).

Example 3.1. (I) The idempotent integral Λ is a central distribution element since by (24) and dual bases,

$$\Lambda = \sum_i \frac{\dim \mathfrak{C}_i}{d} \eta_i.$$

The corresponding distribution X_Λ is given by:

$$\text{Prob}(X_\Lambda = C_i) = \frac{\dim \mathfrak{C}_i}{d}.$$

(For groups this is called the uniform distribution on the group since it satisfies $\text{Prob}(X_\Lambda = g) = \frac{1}{|G|}$ for all $g \in G$.)

(II) Another central distribution element is the normalized class sum η_i. It defines the distribution X_i, where $\text{Prob}(X_i = C_j) = \delta_{ij}$.

(III) The character of the adjoint representation is related to the following central distribution element. Let $z_{ad} = \frac{1}{n} \sum \eta_i$. Then it defines a uniform distribution U_{ad} on class sums. That is,

$$\text{Prob}(U_{ad} = C_i) = \frac{1}{n}.$$

The corresponding distribution function is given by:

$$f_{z_{ad}} = \frac{1}{nd} \chi_{ad}.$$

This follows from (23) since $\chi_{ad} = \sum \frac{d}{\dim \mathfrak{C}_i} F_i$ (see e.g. [6, Th. 2.2]).

Starting from distribution functions, we have:

Lemma 3.2. *Assume $R(H)$ is commutative. An element $f \in R(H)$ is a distribution function on $Z(H)$ if and only if*

$$(i) \ \langle f, \eta_i \rangle \in \mathbb{R}^+ \cup \{0\} \ \forall i \quad and \quad (ii) \ \langle f, \Lambda \rangle = \frac{1}{d}.$$

In this case $z = d\Psi^{-1}(f)$ is a central distribution element and

$$f_z = f = \frac{1}{d} \sum_i \langle \chi_i, z \rangle s(\chi_i).$$

Proof. By dual bases, $f = \sum \langle f, \eta_i \rangle F_i$, hence by (23),

$$(26) \qquad\qquad \Psi^{-1}(f) = \frac{1}{d} \sum \langle f, \eta_i \rangle (\dim \mathfrak{C}_i) \eta_i.$$

Let $z = d\Psi^{-1}(f) = \sum \langle f, \eta_i \rangle (\dim \mathfrak{C}_i) \eta_i$; then $f = f_z$. We claim that z is a central distribution element. Indeed, by assumption (i) on f all coefficients of $\{\eta_i\}$'s are non-negative real numbers. Now,

$$\langle \varepsilon, z \rangle = \langle \varepsilon, df \rightharpoonup \Lambda \rangle = d\langle f, \Lambda \rangle = 1,$$

where the last equality follows from assumption (ii) on f.

By the other set of dual bases, $z = \sum_i \langle z, \chi_i \rangle \frac{1}{d_i} E_i$. The last part follows from (11), since $f_z = \frac{1}{d} \Psi(z)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Example 3.3. Counting functions for groups give rise to probability functions on the group algebras after dividing by an appropriate power of $|G|$. Below are two examples:

(I) Frobenius proved that the function on a finite group G that counts the number of ways an element of G can be realized as a commutator is given by:

$$f_{com} = \sum_i \frac{|G|}{d_i} \chi_i.$$

Since the total number of commutators $aba^{-1}b^{-1}$, $a.b \in G$, is $G \times G$, we make the counting function f_{com} into a distribution function by taking:

$$f = \frac{1}{|G|^2} f_{com}.$$

Consider f_{com} as an element of $R(H)$ where $H = kG$. We claim that f is a distribution function in the sense of Lemma 3.2. Indeed, since $\eta_i = \frac{C_i}{|\mathfrak{C}_i|}$, we have for arbitrary $g_i \in \mathfrak{C}_i$,

$$\langle f_{com}, \eta_i \rangle = \langle f_{com}, g_i \rangle \geq 0.$$

Hence property (i) is satisfied. Since $\langle f_{com}, \Lambda \rangle = |G|\langle \varepsilon, \Lambda \rangle = |G|$, property (ii) follows. Thus f is a distribution function.

By (26) the central distribution element corresponding to f is given by:

$$z = \frac{1}{|G|^2} \sum_i \langle f_{com}, \eta_i \rangle |\mathfrak{C}_i| \eta_i.$$

An explicit realization of z in terms of the elements of the group is the *extensive commutator z* as in (18),

$$z = \sum \frac{1}{|G|^2} \sum_{g \in G} aba^{-1}b^{-1}.$$

Indeed, choose arbitrarily $g_i \in \mathfrak{C}_i$; then

$$\frac{1}{|G|^2} \sum_{a,b \in G} aba^{-1}b^{-1}$$

$$= \frac{1}{|G|^2} \sum_{g \in G} \langle f_{com}, g \rangle g = \frac{1}{|G|^2} \sum_i \langle f_{com}, g_i \rangle C_i = \frac{1}{|G|^2} \sum_i \langle f_{com}, \eta_i \rangle |\mathfrak{C}_i| \eta_i = z.$$

In [8] the function $f_{com} = \sum_i \frac{d}{d_i} \chi_i$ was generalized to any semisimple Hopf algebra. We showed there that:

$$\frac{1}{d^2} f_{com} = \frac{1}{d} \Psi(\gamma_1).$$

In what follows we show that the Hopf analogue γ_1 of the *extensive commutator* z is a central distribution element. Analogously, it follows from (25) that $\frac{1}{d^2} f_{com}$ is the distribution function related to γ_1.

(II) Another counting function for groups is the root function. It counts the number of solutions in G to the equation $x^m = g$, $g \in G$. This function is given by:

$$N_{mrt} = \sum_i \langle \chi_i, \frac{1}{|G|} \sum_{g \in G} g^m \rangle \chi_i.$$

The coefficient of χ_i is called the m-th Frobenius-Schur indicator. The corresponding distribution function is obtained by dividing by $|G|$, that is:

$$f_{mrt} = \frac{1}{|G|^2} \sum_i \langle \chi_i, \sum_{g \in G} g^m \rangle \chi_i.$$

It is straightforward to check that f_{mrt} is a ditribution function for the group algebra. The corresponding central distribution element is $\frac{1}{|G|} \sum_{g \in G} g^m$.

In [10] the Frobenius-Schur indicator was generalized to any semisimple Hopf algebras where the element $\frac{1}{|G|} \sum_{g \in G} g^m$ is generalized to the central element $\Lambda^{[m]} = \sum \Lambda_1 \cdots \Lambda_m$. By Lemma 3.2,

$$f_{mrt} = \frac{1}{d} \sum_i \langle \chi_i, \Lambda^{[m]} \rangle \chi_i.$$

We do not know if in general f_{mrt} is a distribution function. We ask:

Question 3.4. Let H be a semisimple Hopf algebra so that $R(H)$ is commutative. When is it true that $\langle f_{mrt}, \eta_i \rangle \in \mathbb{R}^+ \cup \{0\}$ (or equivalently is $\langle F_i, \Lambda^{[m]} \rangle \in \mathbb{R}^+ \cup \{0\}$ for all i)?

We now show

Lemma 3.5. *For each normalized class sum η_i we have*

$$\{\eta_i, \Lambda\} = \eta_i S \eta_i.$$

Proof. Since $\mathfrak{C}_i$ is a left coideal and $\eta_i \in \mathfrak{C}_i$, it follows that each component $\eta_{i2} \in \mathfrak{C}_i$. By [6, Prop. 3.6] we have that $\Lambda_{ad} \eta_{i2} = \langle \varepsilon, \eta_{i2} \rangle \eta_i$. Hence

$$\{\eta_i, \Lambda\} = \sum \eta_{i1} \Lambda_{ad} S \eta_{i2} = \sum \eta_{i1} S(\Lambda_{ad} \eta_{i2}) = \eta_i S \eta_i.$$

$\square$

Next we show

Lemma 3.6. *If H is quasitriangular, then*

$$\eta_i \eta_j = \sum q_{ijk} \eta_k, \quad q_{ijk} \in \mathbb{Q}^+ \cup \{0\}.$$

Hence the product of two central distribution elements is a central distribution element.

Proof. The first part follows from [5, Th. 2.6]. The second part follows from the first part and the fact that ε is multiplicative. $\square$

The lemmas above imply

Proposition 3.7. *Assume H is quasitriangular; then the following hold:*

(i) *For any central distribution element z, $T(z) = \{z, \Lambda\}$ is a central distribution element as well.*

(ii) *The matrix of the operator T with respect to the basis $\{\eta_i\}$ is a non-negative rational matrix which equals PAP^{-1} where A is as defined in Proposition 2.1, (ξ_{ij}) is the character table of H and*

$$P = \left(\operatorname{diag}\{\dim \mathfrak{C}_0, \ldots, \dim \mathfrak{C}_{n-1}\}\right) (\xi_{ji^*}) \left(\operatorname{diag}\{d_0^{-1}, \ldots, d_{n-1}^{-1}\}\right).$$

Proof. (i) If $z = \sum \alpha_i \eta_i$, $\alpha_i \geq 0$, then by Lemma 3.5 $\{z, \Lambda\} = \sum_i \alpha_i \eta_i S \eta_i$ which is a central distribution element since all the coefficients are non-negative by Lemma 3.6 and $\langle \varepsilon, \{z, \Lambda\} \rangle = 1$.

(ii) The fact that the matrix of T with respect to the basis $\{\eta_i\}$ has non-negative rational entries follows from Lemma 3.5 and Lemma 3.6.

By Proposition 2.1(i), the matrix A is the matrix of T with respect to the basis $\{\frac{E_i}{d_i^2}\}$. Now, the character table is the change of bases matrix between $\{F_i\}$ and $\{\chi_i\}$ in $R(H)$ (see e.g. [6, Th. 3.1]. Applying Ψ^{-1} yields that:

$$\frac{E_j}{d_j^2} = \frac{1}{dd_j} \sum_i \xi_{ji^*} (\dim \mathfrak{C}_i) \eta_i.$$

These imply the desired result. $\square$

We can now show an essential property of the iterated commutator:

Proposition 3.8. *Let $z \in Z(H)$ be a central distribution element and let $T(z) = \{z, \Lambda\}$. Then*

$$\operatorname{Prob}(T^m(z) = 1) \longrightarrow 1 \quad as \quad m \to \infty.$$

Proof. By (23), $\Psi(\eta_i) = \frac{d}{\dim \mathfrak{C}_i} F_i$, hence

$$\langle \eta_i S(\eta_i), \lambda \rangle = \langle \eta_i, \Psi(\eta_i) \rangle = \frac{d}{\dim \mathfrak{C}_i} \langle \eta_i, F_i \rangle = \frac{d}{\dim \mathfrak{C}_i}.$$

On the other hand, the dual bases imply that $\langle \eta_i, \lambda \rangle = d\delta_{i,0}$. Hence if $\eta_i S \eta_i = \beta_i \cdot 1 + \sum_{j \neq 0} \beta_{j_i} \eta_j$, then $\langle \eta_i S \eta_i, \lambda \rangle = d\beta_i$. Hence $\beta_i = \frac{1}{\dim \mathfrak{C}_i}$ and we have:

$$(27) \qquad \eta_i S \eta_i = \frac{1}{\dim \mathfrak{C}_i} \cdot 1 + \cdots.$$

Assume $z = \alpha_0 \cdot 1 + \sum_{i \neq 0} \alpha_i \eta_i$. By Lemma 3.5 we have

$$T(z) = \alpha_0 + \sum_{i \neq 0} \alpha_i \eta_i S \eta_i.$$

Hence by (27)

$$(28) \qquad T(z) = \left(\alpha_0 + \sum_{i \neq 0} \frac{\alpha_i}{\dim \mathfrak{C}_i} \right) \cdot 1 + \sum_{i \neq 0} \alpha_i' \eta_i.$$

Let $c = \max\{\dim \mathfrak{C}_i\}$. Since $\langle \varepsilon, T(z) \rangle = 1$ it follows that

$$\sum_{i \neq 0} \alpha_i' = 1 - \alpha_0 - \sum_{i \neq 0} \frac{\alpha_i}{\dim \mathfrak{C}_i}$$

$$\leq \quad 1 - \alpha_0 - \sum_{i \neq 0} \frac{\alpha_i}{c}$$

$$= \quad 1 - \alpha_0 - \frac{1}{c}(1 - \alpha_0) \qquad \left(\text{since } \langle \varepsilon, z \rangle = \sum_i \alpha_i = 1 \right)$$

$$= \quad (1 - \alpha_0)(1 - \frac{1}{c}).$$

That is,

$$(29) \qquad \sum_{i \neq 0} \alpha_i' \leq (1 - \alpha_0)(1 - \frac{1}{c}).$$

Assume by induction

$$T^m(z) = \alpha_m \cdot 1 + \sum_{i \neq 0} \alpha_{mi} \eta_i \quad \text{where} \quad \sum_{i \neq 0} \alpha_{mi} \leq (1 - \alpha_0)(1 - \frac{1}{c})^m.$$

Then, by (28),

$$T^{m+1}(z) = \left(\alpha_m + \sum_{i \neq 0} \frac{\alpha_{mi}}{\dim \mathfrak{C}_i} \right) \cdot 1 + \sum_{i \neq 0} \alpha_{m+1,i} \eta_i.$$

By (29),

$$\sum_{i \neq 0} \alpha_{m+1,i} \leq (1 - \alpha_m)(1 - c)$$

$$= \quad \sum_{i \neq 0} \alpha_{mi}(1 - c) \quad (\text{since } \langle \varepsilon, T^m(z) \rangle = 1)$$

$$\leq \quad (1 - \alpha_0)(1 - \frac{1}{c})^m (1 - c) \quad (\text{by induction hypothesis})$$

$$= \quad (1 - \alpha_0)(1 - \frac{1}{c})^{m+1}.$$

It follows that $\sum_{i \neq 0} \alpha_{m,i} \to 0$, and so $\alpha_m \to 1$ as $m \to \infty$. $\qquad \square$

Note that by (17), if H is quasitriangular, then γ_m is a central distribution element for all $m \geq 0$. By Theorem 2.6 H is nilpotent if and only if $\gamma_m = 1$. This motivates the following definition.

Definition 3.9. A semisimple Hopf algebra is **probabilistically nilpotent** if

$$\mathrm{Prob}(\gamma_m = 1) \longrightarrow 1 \quad \text{as} \quad m \to \infty.$$

In [2, Remark 4.1] it was observed that every finite group is probabilistically nilpotent. Based on Proposition 3.8 we generalize this observation and obtain the main result of this section.

Theorem 3.10. *Let H be a semisimple quasitriangular Hopf algebra over $\mathbb{C}$. Then H is probabilistically nilpotent.*

The following result is a generalization of [2, Th. 4.10]. It is an algebraic statement about the eigenvalues of the commutator matrix A.

Theorem 3.11. *Let H be a semisimple quasitriangular Hopf algebra over $\mathbb{C}$. Then the commutator matrix A has 1 as an eigenvalue with corresponding 1-dimensional eigenspace. All other eigenvalues c satisfy $|c| < 1$.*

Proof. Since A is the matrix of the operator $T(z) = \{z, \Lambda\}$ we show the statement for T.

Observe first that if $0 \neq z = \sum \alpha_i \eta_i$ so that $\alpha_i \in \mathbb{R}$, $\alpha_i \geq 0$ for all i, then necessarily $\langle \varepsilon, z \rangle > 0$ and $\tilde{z} = \frac{1}{\langle \varepsilon, z \rangle} z$ is a central distribution.

By Proposition 3.8, $\mathrm{Prob}(T^m(\tilde{z}) = 1) \longrightarrow 1$. In this case we say that $T^m(\tilde{z})$ converges to 1 and $T^m(z)$ converges to $\langle \varepsilon, z \rangle \cdot 1$.

If z is a real combination of $\{\eta_i\}$, then $z = z_+ - z_-$ where z_+ and z_- have only non-negative coefficients. By the above $T^m(z)$ converges to $\langle \varepsilon, z_+ \rangle - \langle \varepsilon, z_- \rangle = \langle \varepsilon, z \rangle \cdot 1$.

Assume now $v = \sum \alpha_j \eta_j$ is an eigenvector for T with eigenvalue c. Since the matrix of T with respect to the basis $\{\eta_i\}$ is real, it follows that $\overline{v} = \sum \overline{\alpha_i} \eta_i$ is an eigenvector with eigenvalue $\overline{c}$.

Assume $v \notin k$. Then $\alpha_j \neq 0$ for some $j > 0$. Since multiplying v by α_j^{-1} yields another eigenvector, we may assume without loss of generality that $\alpha_j = 1$. Now, $v + \overline{v}$ is a real vector, hence we have that

$$T^m(v + \overline{v}) = c^m v + \overline{c^m v}$$

converges to $(\langle \varepsilon, v \rangle + \overline{\langle \varepsilon, v \rangle}) \cdot 1$. In particular, taking the coefficient of η_j we obtain,

$$2\Re(c^m) = c^m + \overline{c^m} \to 0 \quad \text{as } m \to \infty.$$

Similarly, $v - \overline{v} = \imath w$ where w is a real vector. Hence $T^m(w)$ converges to $\langle \varepsilon, w \rangle \cdot 1 = \imath(\langle \varepsilon, v \rangle - \overline{\langle \varepsilon, v \rangle}) \cdot 1$. Since $T^m(w) = \imath(c^m v - \overline{c^m v})$ we can take again the coefficient of η_j to obtain

$$-2\Im(c^m) = \imath(c^m - \overline{c^m}) \to 0 \quad \text{as } m \to \infty.$$

Hence

$$|c^m|^2 = (\Re(c^m))^2 + (\Im(c^m))^2 \longrightarrow 0 \quad \text{as } m \to \infty,$$

implying that $|c| < 1$.

Since $T(1) = 1$, we have 1 is an eigenvalue. Its coordinate vector with respect to the basis $\{\frac{E_i}{d_i^2}\}$ is given by:

$$\begin{pmatrix} 1 \\ d_1^2 \\ \vdots \\ d_{n-1}^2 \end{pmatrix}.$$

$\square$

We conclude by asking:

Question 3.12. (i) Are all semisimple Hopf algebras with a commutative character ring probabilistically nilpotent?

(ii) Can one extend the notion of central distribution to any semisimple Hopf algebra? If so, are they always probabilitically nilpotent?

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