

EXPANSION IN GENERALIZED EIGENFUNCTIONS FOR LAPLACIANS ON GRAPHS AND METRIC MEASURE SPACES

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ABSTRACT. We consider an arbitrary selfadjoint operator in a separable Hilbert space. To this operator we construct an expansion in generalized eigenfunctions, in which the original Hilbert space is decomposed as a direct integral of Hilbert spaces consisting of general eigenfunctions. This automatically gives a Plancherel type formula. For suitable operators on metric measure spaces we discuss some growth restrictions on the generalized eigenfunctions. For Laplacians on locally finite graphs the generalized eigenfunctions are exactly the solutions of the corresponding difference equation.

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INTRODUCTION

Expansions in generalized eigenfunctions play an important role in the study of selfadjoint operators. In fact, they are a basic tool in applications such as quantum mechanics. Accordingly, they have received a lot of attention (see e.g. the classical monograph [Ber68] of Berezanskii, appendix in [BS91], the corresponding section in [Sim82], or the article [PSW89]).

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The topic may roughly be described as follows: Let L be a selfadjoint operator with spectrum Σ on the Hilbert space \mathcal{H} . Then, an expansion in generalized eigenfunctions consists of a measure μ on Σ and ‘projections’ W_λ such that in an appropriate sense $W_\lambda f$ is a generalized eigenfunction of L to λ ,

$$f = \int_{\Sigma} W_\lambda f \, d\mu(\lambda)$$

and

$$\Phi(L)f = \int_{\Sigma} \Phi(\lambda) W_\lambda f \, d\mu(\lambda)$$

for any $f \in \mathcal{H}$ and any $\Phi : \Sigma \rightarrow \mathbb{C}$ bounded and measurable. The way this is usually implemented gives $W_\lambda f$ belonging to some larger Hilbert space than \mathcal{H} [Ber68, Sim82]. This larger Hilbert space arises by considering a Gelfand triple or another suitable smoothing and is independent of λ . This leads to the issue that the geometry of this larger Hilbert space is in general not compatible with the geometry of \mathcal{H} . In particular, these expansions do not provide a Hilbert space structure on the set of generalized eigenfunctions.

Thus, one may wonder whether actually a stronger type of expansion is available. This expansion should consist of Hilbert space structures on suitable vector spaces \mathcal{H}_λ which consist of generalized eigenfunctions to the eigenvalue λ and projections W_λ from \mathcal{H} to \mathcal{H}_λ such that a decomposition of the form

$$(D) \quad f = \int_{\Sigma} W_\lambda f \, d\mu(\lambda)$$

holds in a weak sense and a Plancherel type formula of the form

$$(PF) \quad \|\Phi(L)f\|^2 = \int_{\Sigma} |\Phi(\lambda)|^2 \|W_\lambda f\|_{\mathcal{H}_\lambda}^2 \, d\mu(\lambda)$$

is valid for all relevant $f \in \mathcal{H}$ and all sufficiently smooth functions Φ on Σ .

Indeed, it is exactly this type of decomposition that has been needed and obtained in a recent study by Strichartz and Teplyaev in a rather specific situation [ST12]. The investigations of [ST12] provide an eigenfunction expansion for a Laplacian on a specific graph. They make heavy use of the situation at hand. In particular, they do not give a clear indication how general selfadjoint operators in the general case could be treated. In fact, treatment of such general operators is stated as a problem in [ST12] because of many potential applications to the analysis on fractals (see [HSTZ12, HT15, HT13, HRT13, HKT12, IRT12, NT08, RT10, ST13, Tep07] and references therein). The aim of this paper is to provide a solution to this problem.

Our basic idea is to use direct integral theory to exhibit \mathcal{H} as

$$\mathcal{H} = \int_{\Sigma}^{\oplus} \mathcal{H}_\lambda \, d\mu$$

with \mathcal{H}_λ consisting of generalized eigenfunctions of L to the eigenvalue λ . Let us stress that a direct integral decomposition of \mathcal{H} giving a Plancherel type formula is a more or less direct consequence of the spectral theorem (see e.g. the appendix in [BS91]). Our main achievement in this paper is to exhibit the fibers as spaces of generalized eigenfunction. Put differently, our main achievement is to equip certain spaces of generalized eigenfunctions with a Hilbert space structure.

To do so, we first introduce a suitable concept of generalized eigenfunction and then combine this with the considerations concerning generalized eigenfunction expansions developed by Poerschke-Stolz-Weidmann [PSW89]. These considerations in themselves do not give vector spaces of generalized eigenfunctions (let alone Hilbert spaces of generalized eigenfunctions). However, they can be read to provide a basis for such spaces. With our concept of generalized eigenfunction, we can then make these bases into an orthonormal basis, and this is how the Hilbert space structure is introduced.

Our class of generalized eigenfunctions is *a priori* a bit weaker than notions used so far. On the other hand, for Laplacians on graphs, we recover the canonical notion of solution. So, we obtain a convincing theory for these operators (which are the starting point of our investigations as discussed above).

As a byproduct of our discussion we even obtain uniqueness of the fibers in an appropriate sense.

The paper is structured as follows: In Section 1 we introduce our main class of examples viz locally finite operators on discrete spaces and state the expansion theorem for these operators. In Section 2 we then provide an abstract result on expansion in generalized eigenfunctions valid for arbitrary selfadjoint operators on a separable Hilbert space. A short discussion of *a priori* ‘growth restrictions’ on generalized eigenfunctions is given in Section 3. In Section 4 we discuss certain finer growth properties of the generalized eigenfunctions in the case of (metric) measure spaces. In Section 5 we then return to discrete spaces and discuss how the abstract theory developed earlier gives a proof of the expansion theorem in this case. Finally, the appendix contains a short discussion of the direct integral theory needed to understand the paper.

Throughout the paper, all inner product and dual pairings are assumed to be linear in the second argument and antilinear in the first argument.

1. LOCALLY FINITE OPERATORS ON DISCRETE MEASURE SPACES

In this section we introduce the main example of our interest viz locally finite operators on discrete measure spaces. Let us emphasize that this includes the usual Laplacians on locally finite graphs.

Let V be a discrete finite or countably infinite set and m a measure on V with full support (i.e. m is a map on V taking values in $(0, \infty)$). We then call (V, m) a *discrete measure space*. The set of functions on V with finite support is denoted by $C_c(V)$. The set of all functions on V is denoted by $C(V)$. The corresponding Hilbert space

$$\ell^2(V, m) := \{u : V \rightarrow \mathbb{C} : \sum_{x \in V} |u(x)|^2 m(x) < \infty\}$$

is equipped with the inner product

$$\langle v, u \rangle := \sum_{x \in V} \overline{v(x)} u(x) m(x).$$

To each $a : V \times V \rightarrow \mathbb{C}$ we can associate the formal operator \tilde{A} mapping the vector space

$$D(a) := \{w \in C(V) : \sum_{y \in V} |a(x, y) w(y)| m(y) < \infty \text{ for all } x \in V\}$$

to the vector space $C(V)$ via

$$(\tilde{A}w)(x) := \sum_{y \in V} a(x, y)w(y)m(y).$$

An operator A on $\ell^2(V, m)$ is then said to have the *kernel* a if A is a restriction of \tilde{A} . If the domain of A contains $C_c(V)$, the kernel of A is uniquely determined. A function $a : V \times V \rightarrow \mathbb{C}$ is called *locally finite* if the set

$$\{y \in V : a(x, y) \neq 0\}$$

is finite for any $x \in V$. In this case $D(a) = C(V)$ holds. An operator A on $\ell^2(V, m)$ is called *locally finite* if $C_c(V)$ is contained in the domain of definition of A and A has a locally finite kernel a . For such operators, we say that a function $\varphi \in C(V)$ is a *generalized eigenfunction of A to the eigenvalue $\lambda \in \mathbb{C}$* if φ satisfies

$$(\tilde{A} - \lambda)\varphi \equiv 0.$$

Note that we have a dual pairing between $C(V)$ and $C_c(V)$ given by the sesquilinear map

$$(\cdot, \cdot)_m : C(V) \times C_c(V) \rightarrow \mathbb{C}$$

defined by

$$(g, u)_m := \sum_{x \in V} \overline{g(x)}u(x)m(x).$$

Moreover, for $\omega : V \rightarrow [0, \infty)$ define

$$C_\omega(V) := \{u \in C(V) : \sum_{x \in V} \omega(x)^2|u(x)|^2m(x) < \infty\}.$$

Our main result in this context reads as follows (see the appendix for some short review of direct integral theory).

Theorem 1. *Let (V, m) be a discrete measure space. Let L be locally finite and selfadjoint with spectrum Σ . Then, there exist a measurable family of Hilbert spaces \mathcal{H}_λ , $\lambda \in \Sigma$, a measure μ on Σ , a unitary map*

$$W : \ell^2(V, m) \rightarrow \int_{\Sigma}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$$

and a family of linear operators $W_\lambda : C_c(V) \rightarrow \mathcal{H}_\lambda$, $\lambda \in \Sigma$, such that the following hold:

- For each $\lambda \in \Sigma$, the space \mathcal{H}_λ is a subspace of the vector space of generalized eigenfunctions of L to the eigenvalues λ .
- For all $u \in \ell^2(V, m)$, $g \in C_c(V)$ and measurable bounded $\Phi : \Sigma \rightarrow \mathbb{C}$ the equality

$$\langle \Phi(L)u, g \rangle = \int_{\Sigma} (\Phi(\lambda)Wu(\lambda), g)_m d\mu(\lambda)$$

holds.

- For all $g \in C_c(V)$ the functions $\lambda \mapsto (Wg)(\lambda)$ and $\lambda \mapsto W_\lambda g$ agree μ -almost everywhere.

- $W\Phi(L)W^{-1} = M_\Phi$ for any measurable bounded $\Phi : \Sigma \rightarrow \mathbb{C}$. Here, M_Φ is the operator acting in the fiber \mathcal{H}_λ by multiplication by $\Phi(\lambda)$.

The Hilbert spaces \mathcal{H}_λ are uniquely determined by the above properties up to changes on sets of λ with zero μ -measure. Moreover, for any $\omega : V \rightarrow [0, \infty)$ with $\sum_{x \in V} \omega(x)^2 < \infty$ the inclusion

$$\mathcal{H}_\lambda \subset C_\omega(V)$$

holds for μ -almost every $\lambda \in \Sigma$.

The result will be derived from the abstract expansion result of the next section and some auxiliary considerations. Here, we note the following immediate corollary.

Corollary 1.1. *Let the situation of the theorem be given. Then, there exists a map $w : \Sigma \times V \times V \rightarrow \mathbb{C}$ such that $x \mapsto w(\lambda, x, y)$ is a generalized eigenfunction to λ for each $y \in V$ and*

$$\langle g, \Phi(L)f \rangle = \int_{\Sigma} \Phi(\lambda) \sum_{x, y \in V} \overline{g(y)} w(\lambda, y, x) f(x) d\mu(\lambda)$$

for all $f, g \in C_c(V)$ and $\Phi : \Sigma \rightarrow \mathbb{C}$ bounded and measurable.

Remarks. (a) A particularly important class of operators covered by these results is the ‘usual’ Laplacians encountered on locally finite graphs. Note that spectral theory of these Laplacians has attracted a lot of attention in recent years. In fact, starting with [Woj08, Jor08] the case of Laplacians, which are unbounded, has become a focus of intensive research; see e.g. [KL12, KL10, HKLW12, CdVTHT11, Hua11, HKMW13, HKW13, Fol14, KLVW, Web10, Woj09] and references therein. The above theorem should be helpful for further studies.

(b) Note that the uniqueness statement of the theorem gives that the eigenfunction expansion provided in the special situation of [ST12] must actually agree with the eigenfunction expansion given above.

2. A GENERAL EXPANSION RESULT

In this section we present and prove an abstract result on expansion in generalized eigenfunctions. The proof of our result relies on direct integral theory and the eigenfunction expansion presented in [PSW89]. The main novelty is to introduce a notion of generalized eigenfunction weak enough so that the corresponding terms of [PSW89] can be interpreted as belonging to some Hilbert space of generalized eigenfunctions. Unlike the direct integral decompositions based on spectral theorem alone, our expansions in eigenfunctions turn out to be unique.

Let L be a selfadjoint operator on a separable Hilbert space \mathcal{H} . In order to simplify the statement of our theorem we will introduce various pieces of notation next.

A measure μ on \mathbb{R} is called a *spectral measure* for L if for a Borel set $A \subset \mathbb{R}$ the equality $\mu(A) = 0$ holds if and only if $1_A(L) = 0$ (where 1_A is the characteristic function of A). It is a standard fact on selfadjoint operators on a separable Hilbert space that spectral measures exist.

A pair $(W, (\mathcal{H}_\lambda)_{\lambda \in \Sigma})$ consisting of a measurable family of Hilbert spaces $((\mathcal{H}_\lambda)_{\lambda \in \Sigma})$ and a unitary map $W : \mathcal{H} \longrightarrow \int_{\Sigma}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$ is called a *direct integral decomposition* of L with respect to the spectral measure μ if

$$WLW^{-1} = \int_{\Sigma}^{\oplus} M_\lambda d\mu(\lambda)$$

holds, where $M_\lambda : \mathcal{H}_\lambda \longrightarrow \mathcal{H}_\lambda, \phi \mapsto \lambda\phi$.

As is well known (and in fact a direct consequence of existence of ordered spectral representations as given in Lemma 2 (c) of [PSW89]), there exist direct integral decompositions of a selfadjoint operator L with respect to any spectral measure μ .

Whenever $(W, (\mathcal{H}_\lambda)_{\lambda \in \Sigma})$ is a direct integral decomposition of L , then

$$Wf = \int_{\Sigma}^{\oplus} Wf(\lambda) d\mu(\lambda)$$

holds for all $f \in \mathcal{H}$. Moreover, one then obtains from basic direct integral theory (see Appendix)

$$W\Phi(L)W^{-1} = M_\Phi$$

for all measurable $\Phi : \Sigma \longrightarrow \mathbb{C}$. Thus, in particular, a Plancherel formula of the form

$$\|\Phi(L)f\|^2 = \int_{\Sigma} |\Phi(\lambda)|^2 \langle Wf(\lambda), Wf(\lambda) \rangle_{\mathcal{H}_\lambda} d\mu(\lambda)$$

is valid for all $f \in \mathcal{H}$ and all bounded measurable functions Φ on Σ .

The preceding discussion shows that in a certain sense any direct integral decomposition has the features (D) and (PF) discussed in the introduction. Our main task is to find a direct integral decomposition with fibers consisting of Hilbert spaces of generalized eigenfunctions. This will be achieved with the concept of generalized eigenfunction introduced next.

Let \mathcal{D} be any linear subspace of \mathcal{H} . By the algebraic dual of \mathcal{D} we understand the space of linear functions on \mathcal{D} , which are not necessarily continuous. An element φ of the algebraic dual of \mathcal{D} is called a *\mathcal{D} -eigenfunction* of L to the eigenvalue $\lambda \in \mathbb{R}$ if

$$(1) \quad (\varphi, Lu) = \lambda(\varphi, u)$$

for all $u \in \mathcal{D} \cap D(L)$ with $Lu \in \mathcal{D}$. Here, (\cdot, \cdot) denotes the canonical dual pairing between the algebraic dual of \mathcal{D} and \mathcal{D} itself. For later application, it will be convenient to have (\cdot, \cdot) as sesquilinear form. We achieve this by defining the scalar multiplication on the algebraic dual of \mathcal{D} via

$$(\alpha \cdot \varphi)(v) := \overline{\alpha} \varphi(v)$$

for $\alpha \in \mathbb{C}$, $v \in \mathcal{D}$ and φ in the algebraic dual of \mathcal{D} .

After these preparations we can now state our main abstract theorem.

Theorem 2. *Let \mathcal{H} be a separable Hilbert space and L a selfadjoint operator on \mathcal{H} with spectrum Σ . Let μ be a spectral measure of L . Let \mathcal{D} be an arbitrary dense subspace of \mathcal{H} admitting a countable algebraic basis. Then, there exists a direct integral decomposition $(W, (\mathcal{H}_\lambda)_{\lambda \in \Sigma})$ with respect to μ such that the following hold:*

- For each $\lambda \in \Sigma$, the vector space \mathcal{H}_λ is a subspace of the space of \mathcal{D} -eigenfunctions of L to the eigenvalue λ .

- For all $f \in \mathcal{H}$ and $v \in \mathcal{D}$ and any measurable bounded $\Phi : \Sigma \rightarrow \mathbb{C}$ the equality

$$\langle \Phi(L)f, v \rangle = \int_{\Sigma} (\Phi(\lambda)Wf(\lambda), v) d\mu(\lambda)$$

is valid.

Such a direct integral decomposition is unique up to changing the \mathcal{H}_{λ} and W on a set of λ 's with zero μ -measure.

Remarks. (a) A short interpretation of the two points appearing in the theorem may be given as follows: The first point underlines that \mathcal{H}_{λ} consists of generalized eigenfunctions of L . Given the first point, the main content of the second point is an expansion of the form $f = \int_{\Sigma} Wf(\lambda) d\mu(\lambda)$ in the weak sense.

(b) We require the existence of a countable algebraic basis of \mathcal{D} . While this is a somewhat strong requirement it does not preclude the (arguably) most natural choices of \mathcal{D} viz as a core of L . In fact, whenever \mathcal{C} is a core for L (i.e. the graph of L is the closure of the graph of the restriction of L to \mathcal{C}) one can find a countable dense set in \mathcal{C} determining L . The linear span of this set will then have a countable basis and be a core for L (contained in \mathcal{C}).

(c) For Laplacians on graphs there is a most natural choice of \mathcal{D} as the set of functions with finite support. This yields the canonical notion of generalized solution (see below for further discussion).

As \mathcal{D} has a countable algebraic basis we immediately obtain the following corollary of this theorem.

Corollary 2.1. *Assume the situation of Theorem 2. Then, there exists a family of linear operators $W_{\lambda} : \mathcal{D} \rightarrow \mathcal{H}_{\lambda}$, $\lambda \in \Sigma$ with $(Wv) = \lambda \mapsto W_{\lambda}v$ for all $v \in \mathcal{D}$. In particular,*

$$\langle g, \Phi(L)f \rangle = \int_{\Sigma} \Phi(\lambda)(W_{\lambda}g, f) d\mu(\lambda)$$

holds for all $f, g \in \mathcal{D}$ and $\Phi : \Sigma \rightarrow \mathbb{C}$ bounded and measurable.

Remark. Let us note that the theorem and its corollary indeed provide an expansion with the properties aimed at in the introduction.

The remaining part of this section is devoted to the proof of Theorem 2.

Proof of Theorem 2. We will first show the uniqueness statement: Suppose $(W^{(1)}, \mathcal{H}_{\lambda}^{(1)})$ and $(W^{(2)}, \mathcal{H}_{\lambda}^{(2)})$ are decompositions with the desired properties, and v_1, v_2, \dots is a countable basis of \mathcal{D} .

As \mathcal{D} is dense in \mathcal{H} with a countable basis and $W^{(1)}$ and $W^{(2)}$ are unitary maps, we have for μ -almost every $\lambda \in \Sigma$,

$$(2) \quad \mathcal{H}_{\lambda}^{(j)} = \overline{\text{Lin}\{W^{(j)}v_k(\lambda) : k \in \mathbb{N}\}}, \quad j = 1, 2,$$

where *Lin* means the set of linear combinations of the given set of vectors.

Now, fix $f \in \mathcal{H}$ and $v \in \mathcal{D}$. As

$$\langle \Phi(L)f, v \rangle = \int_{\Sigma} \overline{\Phi(\lambda)}(W^{(j)}f(\lambda), v) d\mu(\lambda)$$

holds for all bounded measurable Φ on Σ for $j = 1, 2$, we infer that

$$(W^{(1)}f(\lambda), v) = (W^{(2)}f(\lambda), v)$$

for μ -almost every λ . As \mathcal{D} has a countable basis, this implies that for any fixed f ,

$$W^{(1)}f(\lambda) = W^{(2)}f(\lambda)$$

for μ -almost every λ . This gives for μ -almost every λ ,

$$(3) \quad W^{(1)}v_k(\lambda) = W^{(2)}v_k(\lambda)$$

for all $k \in \mathbb{N}$. Moreover, as the $W^{(j)}$ are decompositions we find for fixed $f, g \in \mathcal{H}$,

$$\begin{aligned} \int_{\Sigma} \Phi(\lambda) \langle W^{(1)}f(\lambda), W^{(1)}g(\lambda) \rangle_{H_{\lambda}^{(1)}} d\mu &= \langle f, \Phi(L)g \rangle \\ &= \int_{\Sigma} \Phi(\lambda) \langle W^{(2)}f(\lambda), W^{(2)}g(\lambda) \rangle_{H_{\lambda}^{(2)}} d\mu \end{aligned}$$

for all bounded measurable $\Phi : \Sigma \rightarrow \mathbb{C}$. This in turn shows that

$$\langle W^{(1)}f(\lambda), W^{(1)}g(\lambda) \rangle_{H_{\lambda}^{(1)}} = \langle W^{(2)}f(\lambda), W^{(2)}g(\lambda) \rangle_{H_{\lambda}^{(2)}}$$

for μ -almost every λ . Thus, we obtain for μ -almost every λ ,

$$(4) \quad \langle W^{(1)}v_j(\lambda), W^{(1)}v_k(\lambda) \rangle_{H_{\lambda}^{(1)}} = \langle W^{(2)}v_j(\lambda), W^{(2)}v_k(\lambda) \rangle_{H_{\lambda}^{(2)}}$$

for all $k, j \in \mathbb{N}$.

Now, the uniqueness statement follows from (2), (3) and (4).

We now turn to proving the existence statement. This will be done in two steps. In the first step we will recall the setting of [PSW89] and its main abstract result on expansion in generalized eigenfunctions. In the second step we will then revise this result to derive the theorem.

We start with the first step. To any selfadjoint operator $T \geq 1$ on \mathcal{H} we can associate the following two auxiliary Hilbert spaces:

$$(5) \quad \mathcal{H}_+ := \mathcal{H}_+(T) := D(T) \text{ with } \langle x, y \rangle_+ := \langle Tx, Ty \rangle$$

and

$$\mathcal{H}_- := \text{completion of } \mathcal{H} \text{ w.r.t. the scalar product } \langle x, y \rangle_- := \langle T^{-1}x, T^{-1}y \rangle.$$

Then, the inner product on \mathcal{H} can be naturally extended to a dual pairing

$$\langle \cdot, \cdot \rangle_d : \mathcal{H}_- \times \mathcal{H}_+ \longrightarrow \mathbb{C}.$$

Now let μ be a spectral measure for L . Then, there exists an *ordered spectral representation* to μ , i.e. a sequence of subsets $M_j \subset \mathbb{R}$, such that $M_j \supset M_{j+1}$, together with a unitary map U ,

$$U = (U_j) : \mathcal{H} \rightarrow \bigoplus_{j=1}^N L^2(M_j, d\mu)$$

with the intertwining property

$$(6) \quad U\Phi(L) = M_{\Phi}U,$$

for every measurable function Φ on \mathbb{R} . The index j takes values in a countable set which we assume to be given by $1, \dots, N$ with $N = \infty$ allowed. If $\gamma : \mathbb{R} \rightarrow \mathbb{C}$ is continuous and bounded with $|\gamma| > 0$ on Σ and $T \geq 1$ is such that $\gamma(L)T^{-1}$ is a Hilbert-Schmidt operator, then the following holds by the main abstract result of [PSW89]: There exist measurable functions

$$\varphi_j : M_j \rightarrow \mathcal{H}_-, \lambda \mapsto \varphi_j(\lambda),$$

for $j = 1, \dots, N$, such that the following properties hold:

- (a) $U_j f(\lambda) = \langle \varphi_j(\lambda), f \rangle_d$ for $f \in \mathcal{H}_+$ and μ -a.e. $\lambda \in M_j$.
- (b) For every $g = (g_j) \in \bigoplus_j L^2(M_j, d\mu)$,

$$U^{-1} g = \lim_{n \rightarrow N, E \rightarrow \infty} \sum_{j=1}^n \int_{M_j \cap [-E, E]} g_j(\lambda) \varphi_j(\lambda) d\mu(\lambda)$$

and, for every $f \in \mathcal{H}$,

$$f = \lim_{n \rightarrow N, E \rightarrow \infty} \sum_{j=1}^n \int_{M_j \cap [-E, E]} (U_j f)(\lambda) \varphi_j(\lambda) d\mu(\lambda).$$

(Here, the integrals exist as elements of \mathcal{H} , and limits are meant in the sense of convergence in \mathcal{H} .)

- (c) For each $f \in \{g \in D(L) \cap \mathcal{H}_+ \mid Lg \in \mathcal{H}_+\}$ we have for any j and μ -almost every λ

$$\langle \varphi_j(\lambda), Lf \rangle_d = \lambda \langle \varphi_j(\lambda), f \rangle_d.$$

Note that (a) and (b) deal with properties of U , while (c) gives a weak version of $\varphi_j(\lambda)$ being an eigenfunction.

We will now turn to the second step and reformulate the above expansion.

Choose $\gamma : \mathbb{R} \rightarrow \mathbb{C}$ continuous and bounded with $|\gamma| > 0$ on Σ and $T \geq 1$ such that

- $\gamma(L)T^{-1}$ is a Hilbert-Schmidt operator and
- \mathcal{D} is contained in the domain of definition of T .

Such a choice is always possible: In fact, it suffices to choose $1 \geq \omega_n > 0$, $n \in \mathbb{N}$, with $\sum_n |\omega_n|^2 < \infty$ and an orthonormal basis e_n , $n \in \mathbb{N}$, contained in \mathcal{D} and to let S be the unique linear operator with $Se_n = \omega_n e_n$, $n \in \mathbb{N}$. Then, S is Hilbert-Schmidt and invertible. Hence, $T := S^{-1}$ exists and $\gamma(L)T^{-1} = \gamma(L)S$ is Hilbert-Schmidt for any bounded function γ (see Section 3 for further exploration of this situation).

This means that we are indeed in a position to apply the main result of [PSW89], which was just discussed.

In order to avoid some tedious but non-essential technicalities, we will assume without loss of generality that the arising index set J equals \mathbb{N} and that

$$\Sigma = M_j$$

for all $j \in J$. (This will save us from having to deal with families of Hilbert spaces with varying dimension.)

It will be convenient to introduce to each $f \in \mathcal{H}_+$ and $\lambda \in \Sigma$ the function

$$e_f(\lambda) : J \rightarrow \mathbb{C}, \quad e_f(\lambda)(j) := \langle \varphi_j(\lambda), f \rangle_d.$$

We will proceed by a series of claims.

Claim 1. For any $f \in \mathcal{H}_+$ we have

$$\int_{\Sigma} \sum_{j \in J} |e_f(\lambda)(j)|^2 d\mu(\lambda) = \|f\|^2 < \infty.$$

In particular, for each $f \in \mathcal{H}_+$,

$$\sum_{j \in J} |e_f(\lambda)(j)|^2 < \infty$$

holds for μ -almost every $\lambda \in \Sigma$.

Proof of the claim. By (a), we have for $f \in \mathcal{H}_+$ that $(Uf)_j(\cdot) = \langle \varphi_j(\cdot), f \rangle_d = e_f(\cdot)(j)$. Thus, we can calculate

$$\begin{aligned} \|f\|^2 &= \sum_{j \in J} \|(Uf)_j\|_{L^2(\Sigma, \mu)}^2 \\ &= \sum_{j \in J} \int_{\Sigma} |(Uf)_j(\lambda)|^2 d\mu(\lambda) \\ &= \int_{\Sigma} \sum_{j \in J} |\langle \varphi_j(\lambda), f \rangle_d|^2 d\mu(\lambda) \end{aligned}$$

and we obtain the claim. \square

From the claim we can conclude that for each $f \in \mathcal{H}_+$ the function $e_f(\lambda) : J \rightarrow \mathbb{C}$ belongs to $\ell^2(J)$ for μ -almost every λ . In fact, more is true. To discuss this, let \mathcal{D} be any dense subspace of \mathcal{H}_+ with a countable basis. By the countability of the basis, we can find a subset of Σ of full μ -measure such that for any λ in this subset and any $v \in \mathcal{D}$, the element $e_v(\lambda)$ belongs to $\ell^2(J)$. For such λ , we define K_{λ} to be the subspace of $\ell^2(J)$ generated by $\{e_v(\lambda) : v \in \mathcal{D}\}$. For all other λ we define K_{λ} to be $\{0\}$.

Claim 2. For μ -almost every λ the space K_{λ} equals $\ell^2(J)$.

Proof of the claim. By construction the set $e_v(\lambda)$, $v \in \mathcal{D}$, has the following two properties:

- Its span is dense in K_{λ} for μ -almost every λ .
- The map $\lambda \mapsto \langle e_v(\lambda), e_w(\lambda) \rangle = \sum_j (U_j v)(\lambda) \overline{(U_j w)(\lambda)}$ is measurable for any $v, w \in \mathcal{V}$.

Thus, the family (K_{λ}) is a measurable family of Hilbert spaces. Accordingly, the orthogonal complement K_{λ}^{\perp} of K_{λ} in $\ell^2(J)$ also forms a measurable family of Hilbert spaces.

Assume now that $K_{\lambda}^{\perp} \neq \{0\}$ for a set of λ of positive μ -measure. Then, we can find a $c \in \int_{\Sigma}^{\oplus} K_{\lambda}^{\perp} d\mu(\lambda)$ with $\|c\| \neq 0$, i.e. a function c on Σ with

$$(7) \quad c(\lambda) \in K_{\lambda}^{\perp}$$

for every $\lambda \in \Sigma$ and

$$0 < \int_{\Sigma} \sum_{j \in J} |c(\lambda)(j)|^2 d\mu(\lambda) < \infty.$$

Consider now $g = (g_j) \in \bigoplus_j L^2(\Sigma, \mu)$ with

$$g_j(\lambda) = c(\lambda)(j).$$

Then, we have $U^{-1}g \equiv 0$ as \mathcal{D} is dense in \mathcal{H} , and for each $v \in \mathcal{D}$ we can calculate by (b)

$$\begin{aligned} \langle (U^{-1}g), v \rangle &\stackrel{(b)}{=} \lim_{n \rightarrow \infty, E \rightarrow \infty} \sum_{j=1}^n \int_{\Sigma \cap [-E, E]} \langle g_j(\lambda) \varphi_j(\lambda), v \rangle_d d\mu(\lambda) \\ &= \lim_{E \rightarrow \infty, n \rightarrow \infty} \int_{\Sigma \cap [-E, E]} \sum_{j=1}^n \overline{g_j(\lambda)} e_v(\lambda)(j) d\mu(\lambda) \\ &\stackrel{(Claim\ 1)}{=} \lim_{E \rightarrow \infty} \int_{\Sigma \cap [-E, E]} \sum_{j=1}^{\infty} \overline{g_j(\lambda)} e_v(\lambda)(j) d\mu(\lambda) \\ &= \lim_{E \rightarrow \infty} \int_{\Sigma \cap [-E, E]} \langle c(\lambda), e_v(\lambda) \rangle_{\ell^2(J)} d\mu(\lambda) \stackrel{(7)}{=} 0. \end{aligned}$$

As U is unitary, we infer that $g \equiv 0$. On the other hand, we have

$$\|g\|^2 = \int_{\Sigma} \sum_j |c(\lambda)(j)|^2 d\mu(\lambda) > 0.$$

This contradiction shows that $K_{\lambda}^{\perp} = \{0\}$ for μ -almost every λ . This finishes the proof of the claim. \square

Now let Σ_1 be the set of $\lambda \in \Sigma$ for which both conclusions of Claim 1 apply for all $v \in \mathcal{D}$ and the conclusion of Claim 2 holds. Then, Σ_1 has full μ -measure (as \mathcal{D} has a countable basis).

As \mathcal{D} has a countable basis, so has its subspace $\tilde{\mathcal{D}}$ consisting of all $v \in \mathcal{D}$ such that Lv exists and belongs to \mathcal{D} . Thus, by (c), we can find a set Σ_2 of full μ -measure such that for all $\lambda \in \Sigma_2$ we have

$$(8) \quad \lambda \langle \varphi_j(\lambda), v \rangle_d = \langle \varphi_j(\lambda), Lv \rangle_d$$

for all $j \in J$ and $v \in \tilde{\mathcal{D}}$. Let $\Sigma_0 := \Sigma_1 \cap \Sigma_2$. This set has then again full μ -measure. For $\lambda \in \Sigma_0$ we define \mathcal{H}_{λ} to be the vector space of all φ in the algebraic dual space of \mathcal{D} which can be written in the form

$$\varphi = \sum_j a_j \varphi_j(\lambda)$$

with $(a_j) \in \ell^2(J)$. Here, the sum belongs indeed to the algebraic dual space, as for each $v \in \mathcal{D}$ we have absolute convergence (and hence existence) of

$$(\varphi, v) = \sum_j \overline{a_j} \langle \varphi_j(\lambda), v \rangle_d,$$

by Claim 1. Then, by this pointwise existence and (8), we obtain that each $v \in \mathcal{H}_{\lambda}$ is indeed a generalized eigenfunction of L . Moreover, Claim 2 easily gives that the map

$$j_{\lambda} : \ell^2(J) \longrightarrow \mathcal{H}_{\lambda}, \quad j_{\lambda}(a) := \sum_{j \in J} a_j \varphi_j(\lambda),$$

is injective. By construction, the map j_{λ} is also surjective. Thus, we can identify $\ell^2(J)$ with \mathcal{H}_{λ} . In this way, \mathcal{H}_{λ} becomes a Hilbert space consisting of generalized eigenfunctions of L .

For the remaining $\lambda \in \Sigma \setminus \Sigma_0$ we define $\mathcal{H}_{\lambda} := \{0\}$. Moreover, we redefine each φ_j on $\Sigma \setminus \Sigma_0$ by setting it to zero there.

By construction the map

$$S : \bigoplus_{j \in J} L^2(\Sigma, \mu) \longrightarrow \int_{\Sigma}^{\oplus} \mathcal{H}_{\lambda} d\mu(\lambda), \quad S(g_j)(\lambda) := \sum_j g_j(\lambda) \varphi_j(\lambda)$$

is then unitary. Moreover, it is not hard to see that

$$SM_{\Phi} = M_{\Phi}S$$

for any measurable function Φ on \mathbb{R} . Thus,

$$W := S \circ U : \mathcal{H} \longrightarrow \int_{\Sigma}^{\oplus} \mathcal{H}_{\lambda} d\mu(\lambda)$$

is a unitary map as well, and by the preceding equality and (6) we obtain

$$W\Phi(H) = SU\Phi(H) = SM_{\Phi}U = M_{\Phi}SU = M_{\Phi}W$$

for any measurable Φ on \mathbb{R} . Now, a direct calculation shows that

$$(Wf)(\lambda) = \sum_{j \in J} (U_j f)(\lambda) \varphi_j(\lambda)$$

for μ -almost every $\lambda \in \Sigma$. Replacing f by $\Phi(L)f$ if $f \in D(\Phi(L))$ for $\Phi : \Sigma \longrightarrow \mathbb{C}$, we find from (6)

$$(9) \quad \Phi(\lambda)(Wf)(\lambda) = \sum_{j \in J} (U_j \Phi(L)f)(\lambda) \varphi_j(\lambda).$$

As \mathcal{D} has a countable basis, this allows us to define linear maps $W_{\lambda} : \mathcal{D} \longrightarrow \mathcal{H}_{\lambda}$ with

$$(W\Phi(L)f)(\lambda) = \Phi(L)W_{\lambda}f$$

for μ -almost every $\lambda \in \Sigma$ and any bounded measurable $\Phi : \Sigma \longrightarrow \mathbb{C}$. Moreover, for each $f \in \mathcal{H}$ and $v \in \mathcal{D}$ and $\Phi : \Sigma \longrightarrow \mathbb{C}$ measurable with $f \in D(\Phi(L))$ we have by (b) above

$$\begin{aligned} \langle \Phi(L)f, v \rangle &\stackrel{(b)}{=} \lim_{n \rightarrow \infty, E \rightarrow \infty} \int_{\Sigma \cap [-E, E]} \sum_{j=1}^n \overline{(U_j \Phi(L)f)(\lambda)} \langle \varphi_j(\lambda), v \rangle_d d\mu(\lambda) \\ &\stackrel{(Claim \ 1)}{=} \int_{\Sigma} \sum_{j=1}^{\infty} \overline{(U_j \Phi(L)f)(\lambda)} \langle \varphi_j(\lambda), v \rangle_d d\mu(\lambda) \\ &= \int_{\Sigma} \sum_{j=1}^{\infty} \langle (U_j \Phi(L)f)(\lambda) \varphi_j(\lambda), v \rangle_d d\mu(\lambda) \\ &= \int_{\Sigma} \left(\sum_{j=1}^{\infty} (U_j (\Phi(L)f)(\lambda) \varphi_j(\lambda), v) \right) d\mu(\lambda) \\ &\stackrel{(9)}{=} \int_{\Sigma} (\Phi(\lambda)(Wf)(\lambda), v) d\mu(\lambda). \end{aligned}$$

This finishes the proof. \square

Remarks. Two comments on the relationship of our proof to [PSW89] are in order.

(a) On the technical level the starting ingredient to make the considerations of [PSW89] work is the choice of a function γ and an operator T such that $\gamma(L)T^{-1}$ is a Hilbert-Schmidt operator. In our theorem we have shifted attention to \mathcal{D} . This yields the additional requirement that \mathcal{D} must be contained in the domain of T .

The upshot of this is that we can apply our theorem whenever γ , T , \mathcal{D} are chosen such that \mathcal{D} belongs to the domain of T and $\gamma(L)T^{-1}$ is Hilbert-Schmidt (and $|\gamma|$ is strictly positive on the spectrum of L and $T \geq 1$ holds). Depending on the situation one may then vary these parameters.

(b) The considerations of [PSW89] yield functions $\varphi_j(\lambda)$ in a (larger) Hilbert space than \mathcal{H} . The arguments above can be understood as providing a Hilbert space structure by declaring these $\varphi_j(\lambda)$, $j \in J$, to be an orthonormal basis and then showing that this indeed works for almost all λ .

The next corollary follows from the above arguments and [PSW89].

Corollary 2.2. *In the situation of Theorem 2, if $f \in D(L) \cap \mathcal{D}$ and $Lf \in \mathcal{D}$, then*

$$\langle \varphi_j(\lambda), Lf \rangle_d = \lambda \langle \varphi_j(\lambda), f \rangle_d$$

for μ -almost every λ .

3. SOME *a priori* GROWTH RESTRICTIONS ON GENERALIZED EIGENFUNCTIONS

In this section we discuss a specific way of choosing T by essentially requiring that T be a Hilbert-Schmidt operator with an orthonormal basis of eigenfunctions belonging to \mathcal{D} . Then, the elements of \mathcal{H}_λ appearing in the main theorem of the previous section can be seen to satisfy some growth type restrictions. For graphs this will have some direct applications.

Throughout this section we assume that \mathcal{H} is a separable Hilbert space and L a selfadjoint operator on \mathcal{H} with spectrum Σ . Let μ be a spectral measure of L . Let \mathcal{D} be a subspace of the domain of L which is dense in \mathcal{H} and admits a countable algebraic base. We now choose $T \geq 1$ such that $S = T^{-1}$ is Hilbert-Schmidt with all eigenfunctions belonging to \mathcal{D} . More specifically, we proceed as follows (compare above): As \mathcal{D} is dense in \mathcal{H} it will contain (by Gram-Schmidt procedure) an orthonormal basis $(v_n)_{n \in \mathbb{N}}$ of \mathcal{H} . Now fix a map

$$\omega : \mathbb{N} \longrightarrow (0, \infty) \text{ with } \sum \omega(n)^2 \leq 1.$$

Define $S : \mathcal{H} \longrightarrow \mathcal{H}$ to be the unique bounded operator with

$$Sv_n = \omega(n)v_n$$

and $T = S^{-1}$. Let $\mathcal{H}_- = \mathcal{H}_-(T)$ be the associated space. Then, S is obviously a Hilbert-Schmidt operator and hence so is $\gamma(L)S$ for any bounded γ . Choose $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$, $\gamma(s) = \frac{1}{s+i}$.

Proposition 3.1. *Assume the situation just described. Then, for μ -almost every $\lambda \in \Sigma$, the inclusion*

$$\mathcal{H}_\lambda \subset \mathcal{H}_-$$

holds.

Proof. Via the orthonormal basis $v_n, n \in \mathbb{N}$, it is possible to identify \mathcal{H} with $\ell^2 = \ell^2(\mathbb{N})$. Then, \mathcal{H}_+ becomes $\{f \in \ell^2 : \sum_n \frac{|f(n)|^2}{|\omega(n)|^2} < \infty\}$, and \mathcal{H}_- becomes

$$\{f : \mathbb{N} \longrightarrow : \sum_n |\omega(n)|^2 |f(n)|^2 < \infty\}.$$

In this sense \mathcal{H}_- consists of all those vectors with an expansion of the form $\sum f(n)v_n$ with $\sum_n |f(n)| |\omega(n)|^2 < \infty$.

For each of the elements v_n of the orthonormal basis we have

$$1 = \|v_n\|^2 = \int_{\Sigma} \sum_j |\langle \varphi_j(\lambda), v_n \rangle|^2 d\mu.$$

This gives

$$1 \geq \sum_n |\omega(n)|^2 \|v_n\|^2 = \int_{\Sigma} \sum_{j,n} |\omega(n) \langle \varphi_j(\lambda), v_n \rangle_d|^2 d\mu(\lambda),$$

and we conclude that

$$\sum_{j,n} |\omega(n) \langle \varphi_j(\lambda), v_n \rangle_d|^2 < \infty$$

for μ -almost every λ . For each such λ we then infer from the Cauchy-Schwartz inequality

$$\sum_n |\omega(n)|^2 \left| \sum_j c_j \langle \varphi_j(\lambda), v_n \rangle_d \right|^2 \leq \sum_k |c_k|^2 \sum_{n,j} |\omega(n) \langle \varphi_j(\lambda), v_n \rangle_d|^2 < \infty$$

for any square summable sequence (c_j) . This directly gives the desired statement by the discussion at the beginning of the proof. \square

4. METRIC MEASURE SPACES AND FINER GROWTH PROPERTIES

The proof of the main abstract result has provided us with Hilbert spaces of generalized eigenfunctions together with orthonormal bases $\varphi_j(\lambda)$, $j \in J$, for each λ in the spectrum. In this section we study specific regularity features, viz growth properties and local regularity, of these functions $\varphi_j(\lambda)$. Note that these functions directly arise from the main result of [PSW89] (see discussion at the end of Section 2). Therefore, all studies of regularity properties of generalized eigenfunctions based on [PSW89] can be directly applied in our setting. In this section, we discuss how some of the abstract ingredients of [BdMS03] can be applied and generalized. For the convenience of the reader and as regularity is a crucial feature we present rather complete arguments.

Throughout this section we assume that our separable Hilbert space is given as $\mathcal{H} = L^2(X, m)$ with a suitable measure space (X, m) .

Theorem 3. *Let L be a selfadjoint operator on $L^2(X, m)$ with spectrum Σ . Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and bounded with $|\gamma| > 0$ on Σ and assume that the range of $\gamma(L)$ is contained in $L^\infty(X, m)$. Let $w : X \rightarrow [1, \infty)$ be an arbitrary function such that $w^{-1} \in L^2(X, m)$. Then, there exists an expansion in generalized eigenfunctions such that for μ -almost every λ the basis elements $(\varphi_j(\lambda))_j$ consist of functions on X satisfying*

$$w^{-1} \varphi_j(\lambda) \in L^2(X, m)$$

for all $j \in J$.

Moreover, for any $f \in \mathcal{H}_+ = \{f : f = w^{-1}g, g \in L^2(X, m)\}$, we have an analogue of the Fourier transform

$$(10) \quad \langle \varphi_j(\lambda), f \rangle_d = \int_X \varphi_j(\lambda)(x) f(x) dm(x),$$

which is an ordinary Lebesgue integral (and not an abstract duality), and the inverse Fourier transform

$$f(x) = \lim_{n \rightarrow N, E \rightarrow \infty} \sum_{j=1}^n \int_{M_j \cap [-E, E]} \langle \varphi_j(\lambda), f \rangle_d \varphi_j(\lambda)(x) d\mu(\lambda).$$

In addition, we have a direct analogue of the Plancherel formula

$$(11) \quad \int_{\Sigma} \sum_{j \in J} |\langle \varphi_j(\lambda), f \rangle_d|^2 d\mu(\lambda) = \|f\|^2 < \infty.$$

Proof. Let T be the operator of multiplication by w and $S = T^{-1}$ the operator of multiplication by w^{-1} . Then $\gamma(L)S$ is a Hilbert-Schmidt operator because of the Grothendieck factorization theorem, as $\gamma(L)$ maps L^2 into L^∞ and multiplication by w^{-1} maps L^∞ into L^2 . The remaining statements are now a direct consequence of Theorem 2 and its proof. In particular, equation (11) follows from our Claim 1 in the proof of Theorem 2. \square

Remarks. (a) Often the main interest lies in situations where the semigroup e^{-tL} , $t > 0$, maps L^2 into L^∞ . This is studied under the heading of ultracontractivity; see [Dav90, Sim82, SV96] for further discussion and references, mostly dealing with Schrödinger operators or, more generally, suitable perturbations of Dirichlet forms. Of course, in such a situation we may take $\gamma : [0, \infty) \rightarrow \mathbb{R}$ to be $\gamma(s) = e^{-ts}$ for any $t > 0$.

(b) The first two parts of the previous theorem are essentially a slight reformulation of main abstract ingredients of [BdMS03] (which we place in a somewhat more general context of operators on $L^2(X, m)$ rather than Dirichlet forms). The proof via Grothendieck factorization is taken from [BdMS03] (which in turn is inspired by [Sto94]).

The main emphasis of [BdMS03] is on local regular Dirichlet forms that satisfy certain subexponential volume growth conditions. However it is noted in [BdMS03] that these conditions can be separated. In particular, Theorem 4 shows that subexponential (or other) volume growth conditions can be proved for generalized eigenfunctions without assuming that we have a Dirichlet form.

For Theorem 4 we will make the further assumption that X is a metric space with a metric ϱ (and the σ -algebra generated by the topology). We then refer to (X, m, ϱ) as a metric measure space. The closed ball around a point $x \in X$ with radius R will be denoted by $B(x, R)$. In this context, we can study subexponentially bounded eigenfunctions defined as follows.

Definition 4.1. Let (X, m, ϱ) be a metric measure space. A function f on (X, m, ϱ) is said to be subexponentially bounded in the L^2 sense if for some $x_0 \in X$ and $\omega(x) = \varrho(x_0, x)$ the function $e^{-\alpha\omega}f$ belongs to $L^2(X, m)$ for any $\alpha > 0$.

The abstract core of the corresponding argument of [BdMS03] then gives the following.

Theorem 4. Let (X, m, ϱ) be a metric measure space. Let L be a selfadjoint operator on $L^2(X, m)$ with spectrum Σ . Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and bounded with $|\gamma| > 0$ on Σ . Assume that the following hold:

- The range of $\gamma(L)$ is contained in $L^\infty(X, m)$.

- For all $x \in X$, $R > 0$ we have $m(B(x, R)) < \infty$ and $\lim_{R \rightarrow \infty} e^{-\alpha \cdot R} m(B(x, R)) = 0$ for any $\alpha > 0$.

Then, there exists an expansion in generalized eigenfunctions such that for μ -almost every λ the basis elements $(\varphi_j(\lambda))_j$ consist of subexponentially bounded functions on X .

Proof. Fix $x_0 \in X$ and define $\omega(y) := \varrho(x_0, y)$. By the second assumption the functions $w_\alpha^{-1} = e^{-\alpha \omega}$ belong to $L^2(X, m)$ for any $\alpha > 0$. Then, the first assumption and the first result of this section, Theorem 3, give that for μ -almost every λ the function $w_\alpha^{-1} \varphi_j(\lambda)$ belongs to $L^2(X, m)$ for every $j \in J$. Appealing to a countable dense subset of α 's in $(0, \infty)$ we now obtain the statement. \square

Remarks. (a) Under suitable assumptions it is possible to show a converse to the previous theorem; i.e. every λ admitting an subexponentially bounded generalized eigenfunction must then belong to the spectrum of L . This type of result is known as the Shnol theorem (see [BdMLS09, FLW14, LSS08, LSV11] for recent results of this type for operators arising from Dirichlet forms and further references).

(b) Note that the first theorem of this section can be slightly generalized: If there is a measurable bounded non-vanishing function b on X and $b\gamma(L)$ maps into L^∞ , then with the operator T of multiplication by wb^{-1} we have $S = w^{-1}b$ and $S\gamma(L)$ is Hilbert-Schmidt. Hence, the statement of Theorem 3 (and its proof) carries over with w replaced by wb^{-1} . This then yields a corresponding version of the preceding theorem as well.

(c) The above considerations do not give growth restrictions on all generalized eigenfunctions in the \mathcal{H}_λ but only on a basis of this space. This may be considered a weakness. However, there is no reason in general why all elements in these spaces should satisfy strong growth restrictions (note that these spaces can be infinite-dimensional, such as in [Tep98]).

(d) One can apply the same methods to other (e.g. polynomial) growth conditions on the measure $m(B(x, R))$ of the balls (this remark is due to Peter Kuchment as an extension of the classical ideas for periodic operators and quantum graphs; see [BK13, Kuc93]).

Theorem 4 can be compared and contrasted with Corollary 2.2, which describes ‘local’ properties of generalized eigenfunctions (e.g. for local Dirichlet forms) without explicitly assuming volume growth conditions. We are interested in these questions because the framework of Theorem 3 fits well in the general ideas of [Str89, Str12] (see also [ST12, (1.1)-(1.2) and (3.1)-(3.5)] and [DRS09, Rog12, RST09]), which apply to a wide array of spaces of exponential growth, such as hyperbolic groups, symmetric spaces and fractafolds.

In particular, a natural question to ask is whether in some sense

$$(12) \quad L\varphi_j(\lambda) = \lambda\varphi_j(\lambda),$$

but since $\varphi_j(\lambda)(x)$ are only locally square integrable, this question is not well posed in terms of the usual spectral theory of selfadjoint operators. However, it is natural to discuss this question if (X, m, ϱ) is a locally compact complete metric measure space and m is a Radon measure, which implies in particular that $m(B(x, R)) < \infty$, without specific volume growth conditions. Furthermore, it is natural to assume that L is the generator of a regular strongly local Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(X, m)$ and that the set \mathcal{D} contains a dense set, in $L^2(X, m)$ and in $C_0(X, \varrho)$,

of compactly supported continuous functions in the domain $D(\mathcal{E})$ of \mathcal{E} . Note that such a set \mathcal{D} always exists by the standard theory of regular Dirichlet forms. Also we can choose w in Theorem 3 to be continuous and thus locally bounded.

In this situation $f \in L^2_{loc}(X, m)$ is of locally finite energy, i.e. belongs to $D_{loc}(\mathcal{E})$, if for any ball $B(x, R)$ there is $u \in D(\mathcal{E})$ which coincides with f on $B(x, R)$. Note that if $f \in L^2_{loc}(X, m)$ and $v \in \mathcal{D}$, then $\mathcal{E}(f, v)$ is well defined as $\mathcal{E}(f, v) = \mathcal{E}(u, v)$ for any $u \in D(\mathcal{E})$ which coincides with f on $B(x, R)$ provided that $\text{supp}(v) \subset B(x, R)$. Furthermore, we can use the following definitions.

Definition 4.2.

- (1) We say that $f \in D_{loc}(\mathcal{E})$ belongs to the weak local domain of L if there exists $g \in L^2_{loc}(X, m)$ such that

$$\mathcal{E}(f, v) = (g, v)_{L^2(X, m)}$$

for any $v \in \mathcal{D}$. In this case we will say that $f \in D_{loc}(L)$ and

$$Lf = g$$

in $L^2_{loc}(X, m)$.

- (2) We say that $f \in D_{loc}(L)$ belongs to the strong local domain of L if for any ball $B(x, R)$ there is $u \in D(L)$ which coincides with f on $B(x, R)$. Note that such an f also belongs to the weak local domain of L , and so $Lf = g$ for some $g \in L^2_{loc}(X, m)$.

To the best of our knowledge these notions are not well studied in the context of general Dirichlet forms. The weak domain of the Laplacian appears implicitly in [PSW89, Theorem 1(b)] and [BdMS03, Theorem 2.1(s)]. The strong domain of the Laplacian is studied in detail in a more restricted context of analysis on finitely ramified fractals; see [BBST99, DRS09, RST09, Str00] and references therein.

The following theorem holds for infinite Sierpinski fractafolds, which can be described as an at most countable union of isometric copies of the standard Sierpinski gasket such that

- (1) these copies do not intersect except for corner points;
- (2) each corner point is contained in at most two of these copies.

One can see that every point of a Sierpinski fractafold has a neighborhood homeomorphic to a neighborhood in the standard Sierpinski gasket, which explains the term ‘fractafold’ introduced by Strichartz in [Str99, Str03], similarly to the notion of a manifold where every point has a neighborhood homeomorphic to a neighborhood in the standard Euclidean space.

The standard Laplacian is uniquely defined on a Sierpinski fractafold and is a local operator. Moreover, according to [RST09], there exists a set \mathcal{D} (with countable algebraic basis) which is a dense set in $L^2(X, m)$ and in $C_0(X, \varrho)$ and consists of compactly supported functions in the domain of any positive power L^n of L .

Theorem 5. *If L is the standard Laplacian on a Sierpinski fractafold X , as defined in [ST12], then the generalized eigenfunctions $\varphi_j(\lambda)$ belong to the strong local domain of L for μ -almost every λ . In particular, they have a continuous version for which the point-wise approximating formula holds:*

$$L\varphi_j(\lambda)(x) = \lim_{n \rightarrow \infty} 5^n \Delta_n \varphi_j(\lambda)(x) = \lambda \varphi_j(\lambda)(x),$$

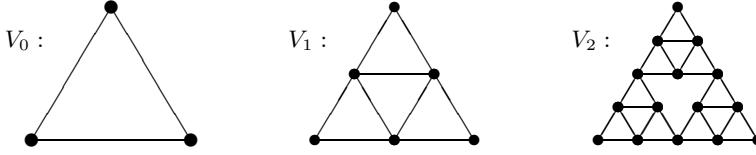


FIGURE 1. Discrete graph approximations to the standard Sierpinski gasket.

where Δ_n is the standard graph Laplacian on the graphs (V_n, E_n) approximating the Sierpinski gasket and x is any vertex in these graphs; see Figure 1.

Proof. The proof of this theorem, which we outline briefly, is based on the detailed technical local analysis of the functions in the domain of L available in the cited literature, in particular in [Str06].

- (1) X is a countable union of compact fractafolds F_k , each having a finite boundary ∂F_k . In what follows we fix k .
- (2) Restriction of all functions to F_k defines a Dirichlet form on $L^2(F_k, m)$, which we will denote by the same notation \mathcal{E} .
- (3) There is a continuous Green's function $g_k(x, y)$ on F_k which is the integral kernel of the Green's operator G_k . This operator is the inverse of the L_k , which is defined as the restriction of L to the functions with support in F_k or, alternatively, L_k is the unique Dirichlet Laplacian on F_k .
- (4) Fix $\lambda > 0$ for which (1) holds for all $u \in \mathcal{D}$ and define $f_k = \lambda G_k \left(\varphi_j(\lambda) \Big|_{F_k} \right)$. We have $L_k f_k = \lambda \varphi_j(\lambda)$ on F_k .
- (5) Define $h_k = (\varphi_j(\lambda) - f_k) \Big|_{F_k}$ and observe that

$$(h_k, Lu) = 0$$

for any $u \in \mathcal{D}$ with support in the interior of F_k . This implies the key observation, which follows from [SU00, Theorem 4.5], that h_k has a continuous version on F_k which is a harmonic function in the interior of F_k .

- (6) We have that $\varphi_j(\lambda) = f_k + h_k$ in $L^2(F_k, m)$, which implies that $\varphi_j(\lambda)$ has a continuous version.
- (7) It follows from [RST09] that this version is in the strong local domain of L and the point-wise approximating formula.

□

Remark. It is straightforward to generalize this theorem to the case of infinite finitely ramified cell structures, as defined in [Tep08], and it would be interesting to understand under which assumptions it is true for general local regular Dirichlet forms, which will be the subject of future work. The steps of the proof outlined above may be applicable for resistance forms of Kigami [Kig03, Kig12] using the methods developed in [BdMS03, HKT1].

5. GETTING BACK TO DISCRETE MEASURE SPACES - PROOF OF THEOREM 1

In this section we consider a discrete measure space (V, m) and provide a proof of Theorem 1. To do so, we first discuss two basic results on selfadjoint operators on $\ell^2(V, m)$. While the results are rather simple they may be of independent interest.

In our context they will be needed to apply the abstract theorem to the concrete situation. Throughout this section we assume that we are given a discrete measure space (V, m) as defined in the first section.

We first show that for locally finite selfadjoint operators on $\ell^2(V, m)$ the two notions of $C_c(V)$ -eigenfunction and of generalized eigenfunctions introduced previously agree. Here, we identify the algebraic dual of $C_c(V)$ with $C(V)$ via the pairing $(\cdot, \cdot)_m$ introduced in Section 1. More specifically any $g \in C(V)$ produces a functional $C_c(V) \rightarrow \mathbb{C}$ via

$$(g, u)_m = \sum_{x \in V} \overline{g(x)} u(x) m(x)$$

for $u \in C_c(V)$.

Lemma 5.1. *Let L be a locally finite selfadjoint operator with kernel l . Then, the following assertions are equivalent for $\varphi \in C(V)$ and $\lambda \in \mathbb{R}$:*

- (i) φ is a generalized eigenfunction of L to the eigenvalue λ .
- (ii) φ is a $C_c(V)$ -eigenfunction of L to the eigenvalue λ .

Proof. By local finiteness of L we have $D(L) = C(V)$ and φ is a generalized eigenfunction to λ if and only if $(\tilde{L} - \lambda)\varphi = 0$ holds. On the other hand, by the dual pairing $(\cdot, \cdot)_m$ the function φ is a $C_c(V)$ -eigenfunction to λ if and only if

$$\sum \overline{\varphi(x)} (L - \lambda)v(x)m(x) = 0$$

holds for all $v \in C_c(V)$. Now, a short calculation shows that

$$\begin{aligned} \sum_x \overline{(\tilde{L} - \lambda)w(x)} v(x)m(x) &= \sum_{x,y} \overline{(l(x, y) - \lambda)w(y)} m(y)v(x)m(x) \\ &= \sum_{y \in V} \overline{w(y)} (L - \lambda)v(y)m(y) \end{aligned}$$

for all $w \in C(V)$ and $v \in C_c(V)$. Here, all sums are absolutely convergent. This easily gives the statement of the lemma. \square

After this preparation we can now provide a proof of Theorem 1.

Proof of Theorem 1. We can choose $\mathcal{D} := C_c(V)$. This space has indeed a countable algebraic basis as V is countable. Then, Lemma 5.1 shows that any \mathcal{D} -eigenfunction of L is indeed a generalized eigenfunction of L . Then, Theorem 2 gives all statements of Theorem 1 up to the last one. The last statement follows from the considerations of Section 3. This finishes the proof of the theorem. \square

Remark. In order to be specific in the proof of Theorem 1 one may make the following choices: Let

$$\omega : V \rightarrow (0, \infty), \text{ with } \sum_{x \in V} \omega(x)^2 \leq 1$$

be given and $S = M_\omega$ be the operator of multiplication by ω . Then S is a Hilbert-Schmidt operator (compare discussion in Section 3). Let

$$T = M_{\frac{1}{\omega}}.$$

Then, T is selfadjoint with $T \geq 1$ (as ω is real valued with $0 < \omega \leq 1$) and

$$T^{-1} = S.$$

The associated Hilbert spaces can be described explicitly as

$$\mathcal{H}_+ := \mathcal{H}_+(T) := D(T) = \{u : V \rightarrow \mathbb{C} : \sum_{x \in V} \frac{|u(x)|^2}{\omega(x)^2} m(x) < \infty\}$$

and

$$\mathcal{H}_- = \{u : V \rightarrow \mathbb{C} : \sum \omega(x)^2 |u(x)|^2 m(x) < \infty\}.$$

The corresponding dual pairing between \mathcal{H}_- and \mathcal{H}_+ is given by

$$\langle v, u \rangle_d = \sum_{x \in V} \overline{v(x)} u(x) m(x)$$

for $u \in \mathcal{H}_+, v \in \mathcal{H}_-$. Note that \mathcal{H}_+ is contained in $\ell^2(V, m)$ as $0 < \omega \leq 1$. Note also that $\langle u, v \rangle_d$ given as above is indeed well defined for $u \in \mathcal{H}_+$ and $v \in \mathcal{H}_-$ by the Hölder inequality. Now let

$$\gamma : [0, \infty) \rightarrow \mathbb{R}, s \mapsto (s + i)^{-1}.$$

Then, γ is obviously continuous and does not vanish on the positive half-axis. Then, $\gamma(L)T^{-1} = (L + i)^{-1}S$ is Hilbert-Schmidt as S is Hilbert-Schmidt and $(L + i)^{-1}$ is bounded.

APPENDIX A. DIRECT INTEGRALS AND MEASURABLE FAMILIES OF HILBERT SPACES

In this section we briefly recall some direct integral theory. For further details we refer to e.g. [Dix81, Nie80].

Let (Σ, μ) be a measure space. A family of Hilbert spaces $(\mathcal{H}_\lambda, \langle \cdot, \cdot \rangle_\lambda)$, $\lambda \in \Sigma$, together with a countable collection $(w_n)_{n \in \mathbb{N}}$ of functions on Σ with $w_n(\lambda) \in \mathcal{H}_\lambda$ for each $\lambda \in \Sigma$, is called a measurable family over (Σ, μ) if the following two properties hold:

- For each $\lambda \in \Sigma$ the linear hull of $\{w_n(\lambda) : n \in \mathbb{N}\}$ is dense in \mathcal{H}_λ .
- For all $n, m \in \mathbb{N}$, the function $\Sigma \rightarrow \mathbb{C}$, $\lambda \mapsto \langle w_n(\lambda), w_m(\lambda) \rangle_\lambda$, is measurable.

Given such a measurable family of Hilbert spaces we call a function w on Σ with $w(\lambda) \in \mathcal{H}_\lambda$ for each $\lambda \in \Sigma$ measurable if

$$\Sigma \rightarrow \mathbb{C}, \lambda \mapsto \langle w(\lambda), w_n(\lambda) \rangle_\lambda,$$

is measurable for any $n \in \mathbb{N}$.

Then, the vector space $\mathcal{L}^2(\Sigma, (\mathcal{H}_\lambda))$ consisting of all measurable functions w on Σ with $w(\lambda) \in \mathcal{H}_\lambda$ for each $\lambda \in \Sigma$ such that

$$\int_{\Sigma} \|w(\lambda)\|^2 d\mu(\lambda) < \infty$$

carries a semi-scalar product given by

$$\langle w, v \rangle := \int_{\Sigma} \langle w(\lambda), v(\lambda) \rangle_{\lambda} d\mu(\lambda).$$

The quotient of $\mathcal{L}^2(\Sigma, (\mathcal{H}_\lambda))$ by the subspace \mathcal{N} consisting of all elements w with $\langle w, w \rangle = 0$ is a Hilbert space and is denoted by $\int_{\Sigma}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$.

Let a direct integral Hilbert space $\mathcal{K} = \int_{\Sigma}^{\oplus} \mathcal{H}_{\lambda} d\mu(\lambda)$ be given. Then, a bounded operator A on \mathcal{K} is called decomposable if there exists a family of bounded operators $A_{\lambda} : \mathcal{H}_{\lambda} \rightarrow \mathcal{H}_{\lambda}$, $\lambda \in \Sigma$, with

$$(Af)(\lambda) = A_{\lambda}f(\lambda)$$

for μ -almost every λ for each $f \in \mathcal{K}$. This is then written as

$$A = \int_{\Sigma}^{\oplus} A_{\lambda} d\mu(\lambda).$$

If for such an A there exists a measurable function $F : \Sigma \rightarrow \mathbb{C}$ with

$$A_{\lambda}f = F(\lambda)f \text{ for all } \lambda \in \Sigma \text{ and } f \in \mathcal{H}_{\lambda},$$

one also writes $A = M_F$ (by a slight abuse of language).

If A is a (not necessarily bounded) selfadjoint operator on \mathcal{K} and (A_{λ}) is a family of (not necessarily bounded) selfadjoint operators on the respective \mathcal{H}_{λ} , we write

$$A = \int_{\Sigma}^{\oplus} A_{\lambda} d\mu(\lambda)$$

if and only if

$$\Phi(A) = \int_{\Sigma}^{\oplus} \Phi(A_{\lambda}) d\mu(\lambda)$$

for all bounded measurable $\Phi : \mathbb{R} \rightarrow \mathbb{C}$.

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REFERENCES

- [BBKT10] Martin T. Barlow, Richard F. Bass, Takashi Kumagai, and Alexander Teplyaev, *Uniqueness of Brownian motion on Sierpiński carpets*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 3, 655–701. MR2639315 (2011i:60146)
- [BGK12] Martin T. Barlow, Alexander Grigor'yan, and Takashi Kumagai, *On the equivalence of parabolic Harnack inequalities and heat kernel estimates*, J. Math. Soc. Japan **64** (2012), no. 4, 1091–1146, DOI 10.2969/jmsj/06441091. MR2998918
- [BBST99] Oren Ben-Bassat, Robert S. Strichartz, and Alexander Teplyaev, *What is not in the domain of the Laplacian on Sierpinski gasket type fractals*, J. Funct. Anal. **166** (1999), no. 2, 197–217, DOI 10.1006/jfan.1999.3431. MR1707752 (2001e:31016)
- [BS91] F. A. Berezin and M. A. Shubin, *The Schrödinger equation*, translated from the 1983 Russian edition by Yu. Rajabov, D. A. Leites and N. A. Sakharova and revised by Shubin; with contributions by G. L. Litvinov and Leites, Mathematics and its Applications (Soviet Series), vol. 66, Kluwer Academic Publishers Group, Dordrecht, 1991. MR1186643 (93i:81001)
- [Ber68] Y. M. Berezanskii, *Expansion in eigenfunctions of self-adjoint operators*, Transl. Math. Monogr. 17, Amer. Math. Soc., Providence, RI, 1968.
- [BK13] Gregory Berkolaiko and Peter Kuchment, *Introduction to quantum graphs*, Mathematical Surveys and Monographs, vol. 186, American Mathematical Society, Providence, RI, 2013. MR3013208
- [BdMS03] Anne Boutet de Monvel and Peter Stollmann, *Eigenfunction expansions for generators of Dirichlet forms*, J. Reine Angew. Math. **561** (2003), 131–144, DOI 10.1515/crll.2003.063. MR1998609 (2004m:58049)

- [BdMLS09] Anne Boutet de Monvel, Daniel Lenz, and Peter Stollmann, *Sch’nol’s theorem for strongly local forms*, Israel J. Math. **173** (2009), 189–211, DOI 10.1007/s11856-009-0088-8. MR2570665 (2010m:35081)
- [CF] Zhen-Qing Chen and Masatoshi Fukushima, *Symmetric Markov processes, time change, and boundary theory*, London Mathematical Society Monographs Series, vol. 35, Princeton University Press, Princeton, NJ, 2012. MR2849840 (2012m:60176)
- [CdVTHT11] Yves Colin de Verdière, Nabila Torki-Hamza, and Françoise Truc, *Essential self-adjointness for combinatorial Schrödinger operators II—metrically non complete graphs*, Math. Phys. Anal. Geom. **14** (2011), no. 1, 21–38, DOI 10.1007/s11040-010-9086-7. MR2782792 (2012c:81064)
- [Dav90] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1990. MR1103113 (92a:35035)
- [DRS09] Jessica L. DeGrado, Luke G. Rogers, and Robert S. Strichartz, *Gradients of Laplacian eigenfunctions on the Sierpinski gasket*, Proc. Amer. Math. Soc. **137** (2009), no. 2, 531–540, DOI 10.1090/S0002-9939-08-09711-6. MR2448573 (2009k:28018)
- [Dix81] Jacques Dixmier, *von Neumann algebras*, with a preface by E. C. Lance, translated from the second French edition by F. Jellett, North-Holland Mathematical Library, vol. 27, North-Holland Publishing Co., Amsterdam-New York, 1981. MR641217 (83a:46004)
- [Fol14] Matthew Folz, *Volume growth and spectrum for general graph Laplacians*, Math. Z. **276** (2014), no. 1-2, 115–131, DOI 10.1007/s00209-013-1189-y. MR3150195
- [FLW14] Rupert L. Frank, Daniel Lenz, and Daniel Wingert, *Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory*, J. Funct. Anal. **266** (2014), no. 8, 4765–4808, DOI 10.1016/j.jfa.2014.02.008. MR3177322
- [GH14] Alexander Grigor’yan and Jiaxin Hu, *Heat kernels and Green functions on metric measure spaces*, Canad. J. Math. **66** (2014), no. 3, 641–699, DOI 10.4153/CJM-2012-061-5. MR3194164
- [HKW13] Sebastian Haeseler, Matthias Keller, and Radosław K. Wojciechowski, *Volume growth and bounds for the essential spectrum for Dirichlet forms*, J. Lond. Math. Soc. (2) **88** (2013), no. 3, 883–898, DOI 10.1112/jlms/jdt029. MR3145136
- [HKLW12] Sebastian Haeseler, Matthias Keller, Daniel Lenz, and Radosław Wojciechowski, *Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions*, J. Spectr. Theory **2** (2012), no. 4, 397–432. MR2947294
- [HSTZ12] Kathryn E. Hare, Benjamin A. Steinhurst, Alexander Teplyaev, and Denglin Zhou, *Disconnected Julia sets and gaps in the spectrum of Laplacians on symmetric finitely ramified fractals*, Math. Res. Lett. **19** (2012), no. 3, 537–553, DOI 10.4310/MRL.2012.v19.n3.a3. MR2998138
- [HT15] Michael Hinz and Alexander Teplyaev, *Local Dirichlet forms, Hodge theory, and the Navier-Stokes equations on topologically one-dimensional fractals*, Trans. Amer. Math. Soc. **367** (2015), no. 2, 1347–1380, DOI 10.1090/S0002-9947-2014-06203-X. MR3280047
- [HT13] Michael Hinz and Alexander Teplyaev, *Dirac and magnetic Schrödinger operators on fractals*, J. Funct. Anal. **265** (2013), no. 11, 2830–2854, DOI 10.1016/j.jfa.2013.07.021. MR3096991
- [HRT13] Michael Hinz, Michael Röckner, and Alexander Teplyaev, *Vector analysis for Dirichlet forms and quasilinear PDE and SPDE on metric measure spaces*, Stochastic Process. Appl. **123** (2013), no. 12, 4373–4406, DOI 10.1016/j.spa.2013.06.009. MR3096357
- [HKT1] M. Hinz, D. Kelleher, and A. Teplyaev, *Metrics and spectral triples for Dirichlet and resistance forms*, to appear in JNCG, arXiv:1309.5937
- [HKT12] M. Hinz, D. Kelleher, and A. Teplyaev, *Measures and Dirichlet forms under the Gelfand transform* (English, with English and Russian summaries), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **408** (2012), no. Veroyatnost i Statistika. 18, 303–322, 329–330; English transl., J. Math. Sci. (N. Y.) **199** (2014), no. 2, 236–246. MR3032223

- [Hua11] Xueping Huang, *Stochastic incompleteness for graphs and weak Omori-Yau maximum principle*, J. Math. Anal. Appl. **379** (2011), no. 2, 764–782, DOI 10.1016/j.jmaa.2011.02.009. MR2784357 (2012c:60194)
- [HKMW13] Xueping Huang, Matthias Keller, Jun Masamune, and Radosław K. Wojciechowski, *A note on self-adjoint extensions of the Laplacian on weighted graphs*, J. Funct. Anal. **265** (2013), no. 8, 1556–1578, DOI 10.1016/j.jfa.2013.06.004. MR3079229
- [IRT12] Marius Ionescu, Luke G. Rogers, and Alexander Teplyaev, *Derivations and Dirichlet forms on fractals*, J. Funct. Anal. **263** (2012), no. 8, 2141–2169, DOI 10.1016/j.jfa.2012.05.021. MR2964679
- [Jor08] Palle E. T. Jorgensen, *Essential self-adjointness of the graph-Laplacian*, J. Math. Phys. **49** (2008), no. 7, 073510, 33, DOI 10.1063/1.2953684. MR2432048 (2009k:47099)
- [KL12] Matthias Keller and Daniel Lenz, *Dirichlet forms and stochastic completeness of graphs and subgraphs*, J. Reine Angew. Math. **666** (2012), 189–223, DOI 10.1515/CRELLE.2011.122. MR2920886
- [KL10] M. Keller and D. Lenz, *Unbounded Laplacians on graphs: basic spectral properties and the heat equation*, Math. Model. Nat. Phenom. **5** (2010), no. 4, 198–224, DOI 10.1051/mmnp/20105409. MR2662456 (2011e:60183)
- [KLVW] M. Keller, D. Lenz, H. Vogt, and R. Wojciechowski, *Note on basic features of large time behaviour of heat kernels*, to appear in Journal für die reine und angewandte Mathematik (Crelle's Journal).
- [Kuc93] Peter Kuchment, *Floquet theory for partial differential equations*, Operator Theory: Advances and Applications, vol. 60, Birkhäuser Verlag, Basel, 1993. MR1232660 (94h:35002)
- [Kig03] Jun Kigami, *Harmonic analysis for resistance forms*, J. Funct. Anal. **204** (2003), no. 2, 399–444, DOI 10.1016/S0022-1236(02)00149-0. MR2017320 (2004m:31010)
- [Kig12] Jun Kigami, *Resistance forms, quasisymmetric maps and heat kernel estimates*, Mem. Amer. Math. Soc. **216** (2012), no. 1015, vi+132, DOI 10.1090/S0065-9266-2011-00632-5. MR2919892
- [LSS08] Daniel Lenz, Carsten Schubert, and Peter Stollmann, *Eigenfunction expansions for Schrödinger operators on metric graphs*, Integral Equations Operator Theory **62** (2008), no. 4, 541–553, DOI 10.1007/s00020-008-1636-z. MR2470123 (2010b:35070)
- [LSV11] Daniel Lenz, Peter Stollmann, and Ivan Veselić, *Generalized eigenfunctions and spectral theory for strongly local Dirichlet forms*, Spectral theory and analysis, Oper. Theory Adv. Appl., vol. 214, Birkhäuser/Springer Basel AG, Basel, 2011, pp. 83–106, DOI 10.1007/978-3-7643-9994-8_6. MR2808465 (2012i:35272)
- [NT08] Volodymyr Nekrashevych and Alexander Teplyaev, *Groups and analysis on fractals*, Analysis on graphs and its applications, Proc. Sympos. Pure Math., vol. 77, Amer. Math. Soc., Providence, RI, 2008, pp. 143–180, DOI 10.1090/pspum/077/2459868. MR2459868 (2010f:28017)
- [Nie80] Ole A. Nielsen, *Direct integral theory*, Lecture Notes in Pure and Applied Mathematics, vol. 61, Marcel Dekker, Inc., New York, 1980. MR591683 (82e:46081)
- [PSW89] Thomas Poerschke, Günter Stolz, and Joachim Weidmann, *Expansions in generalized eigenfunctions of selfadjoint operators*, Math. Z. **202** (1989), no. 3, 397–408, DOI 10.1007/BF01159969. MR1017580 (91j:47018)
- [Rog12] Luke G. Rogers, *Estimates for the resolvent kernel of the Laplacian on p.c.f. self-similar fractals and blowups*, Trans. Amer. Math. Soc. **364** (2012), no. 3, 1633–1685, DOI 10.1090/S0002-9947-2011-05551-0. MR2869187
- [RST09] Luke G. Rogers, Robert S. Strichartz, and Alexander Teplyaev, *Smooth bumps, a Borel theorem and partitions of smooth functions on P.C.F. fractals*, Trans. Amer. Math. Soc. **361** (2009), no. 4, 1765–1790, DOI 10.1090/S0002-9947-08-04772-7. MR2465816 (2010f:28020)
- [RT10] Luke G. Rogers and Alexander Teplyaev, *Laplacians on the basilica Julia sets*, Commun. Pure Appl. Anal. **9** (2010), no. 1, 211–231, DOI 10.3934/cpaa.2010.9.211. MR2556753 (2011c:28024)
- [Sim82] Barry Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 3, 447–526, DOI 10.1090/S0273-0979-1982-15041-8. MR670130 (86b:81001a)

- [ST13] Benjamin A. Steinhurst and Alexander Teplyaev, *Existence of a meromorphic extension of spectral zeta functions on fractals*, Lett. Math. Phys. **103** (2013), no. 12, 1377–1388, DOI 10.1007/s11005-013-0649-y. MR3117253
- [Sto94] Peter Stollmann, *Scattering by obstacles of finite capacity*, J. Funct. Anal. **121** (1994), no. 2, 416–425, DOI 10.1006/jfan.1994.1054. MR1272133 (95h:47059)
- [Str89] Robert S. Strichartz, *Harmonic analysis as spectral theory of Laplacians*, J. Funct. Anal. **87** (1989), no. 1, 51–148, DOI 10.1016/0022-1236(89)90004-9. MR1025883 (91c:43015); see MR1186327
- [Str99] Robert S. Strichartz, *Analysis on fractals*, Notices Amer. Math. Soc. **46** (1999), no. 10, 1199–1208. MR1715511 (2000i:58035)
- [SU00] Robert S. Strichartz and Michael Usher, *Splines on fractals*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 2, 331–360, DOI 10.1017/S0305004100004424. MR1765920 (2001c:28016)
- [Str00] Robert S. Strichartz, *Taylor approximations on Sierpinski gasket type fractals*, J. Funct. Anal. **174** (2000), no. 1, 76–127, DOI 10.1006/jfan.2000.3580. MR1761364 (2001i:31018)
- [Str03] Robert S. Strichartz, *Fractafolds based on the Sierpiński gasket and their spectra*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 4019–4043 (electronic), DOI 10.1090/S0002-9947-03-03171-4. MR1990573 (2004b:28013)
- [Str06] Robert S. Strichartz, *Differential equations on fractals*, A tutorial. Princeton University Press, Princeton, NJ, 2006. MR2246975 (2007f:35003)
- [Str12] Robert S. Strichartz, *Spectral asymptotics revisited*, J. Fourier Anal. Appl. **18** (2012), no. 3, 626–659, DOI 10.1007/s00041-012-9216-7. MR2921088
- [ST12] Robert S. Strichartz and Alexander Teplyaev, *Spectral analysis on infinite Sierpiński fractafolds*, J. Anal. Math. **116** (2012), 255–297, DOI 10.1007/s11854-012-0007-5. MR2892621
- [SV96] Peter Stollmann and Jürgen Voigt, *Perturbation of Dirichlet forms by measures*, Potential Anal. **5** (1996), no. 2, 109–138, DOI 10.1007/BF00396775. MR1378151 (97e:47065)
- [Tep98] Alexander Teplyaev, *Spectral analysis on infinite Sierpiński gaskets*, J. Funct. Anal. **159** (1998), no. 2, 537–567, DOI 10.1006/jfan.1998.3297. MR1658094 (99j:35153)
- [Tep07] Alexander Teplyaev, *Spectral zeta functions of fractals and the complex dynamics of polynomials*, Trans. Amer. Math. Soc. **359** (2007), no. 9, 4339–4358 (electronic), DOI 10.1090/S0002-9947-07-04150-5. MR2309188 (2008j:11119)
- [Tep08] Alexander Teplyaev, *Harmonic coordinates on fractals with finitely ramified cell structure*, Canad. J. Math. **60** (2008), no. 2, 457–480, DOI 10.4153/CJM-2008-022-3. MR2398758 (2008m:60066)
- [Web10] Andreas Weber, *Analysis of the physical Laplacian and the heat flow on a locally finite graph*, J. Math. Anal. Appl. **370** (2010), no. 1, 146–158, DOI 10.1016/j.jmaa.2010.04.044. MR2651136 (2011f:35341)
- [Woj08] Radosław Krzysztof Wojciechowski, *Stochastic completeness of graphs*, ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–City University of New York, 2008. MR2711706
- [Woj09] Radosław K. Wojciechowski, *Heat kernel and essential spectrum of infinite graphs*, Indiana Univ. Math. J. **58** (2009), no. 3, 1419–1441, DOI 10.1512/iumj.2009.58.3575. MR2542093 (2010k:35208)

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