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OPENING NODES ON HOROSPHERE PACKINGS

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Abstract. We use Bryant representation to construct constant mean curvature-1 surfaces in hyperbolic space by desingularisation of a horosphere packing.

1. Introduction

In this paper, a horosphere packing means a finite, connected set of distinct horospheres in hyperbolic space $\mathbb{H}^3$ such that any two horospheres are either disjoint or tangent. We denote by n the number of horospheres and m the number of tangency points.

Horospheres have constant mean curvature equal to 1 (CMC-1 for short). In [10], Pacard and Pimentel have constructed complete, embedded CMC-1 surfaces in hyperbolic space by desingularization of a horosphere packing. Heuristically, the horospheres are slightly shrinked and a suitable small catenoidal neck is glued at each tangency point (see Figure 1).

On the other hand, CMC-1 surfaces in hyperbolic space admit a Weierstrass-type representation in term of meromorphic data, as discovered by Bryant [1]. In particular, they have a meromorphic Gauss map G and a holomorphic quadratic differential Q. Our goal in this paper is to revisit the result of Pacard and Pimentel using Bryant's representation. We prove:

Theorem 1. *Given a packing of n horospheres with m tangency points, there exists a smooth family $(M_s)_{0<s<\varepsilon}$ of complete, embedded CMC-1 surfaces in hyperbolic space $\mathbb{H}^3$ such that M_s converges when $s \to 0$ to the given horosphere packing. The surfaces M_s have genus $m-n+1$ and n catenoid-cousin-type ends.*

We prove this theorem using Bryant's representation. The construction follows the general strategy developed by the author in [16–18] to construct minimal surfaces in euclidean space $\mathbb{R}^3$ by opening nodes, using the Weierstrass representation. The strategy can be described as follows: The underlying Riemann surface Σ is defined by opening nodes. The meromorphic differentials dG and $\Omega = Q/dG$ are defined on Σ by prescribing their periods. Bryant representation gives us a "recipe" to recover the immersion from this meromorphic data. With respect to the minimal case, the new technicality is that instead of having to solve a period problem, which is homology invariant, we have to solve a monodromy problem, which is only homotopy invariant.

As previously stated, Theorem 1 is already proved in [10] using the P.D.E. gluing machinery, which has been recognized as a very robust and versatile tool to

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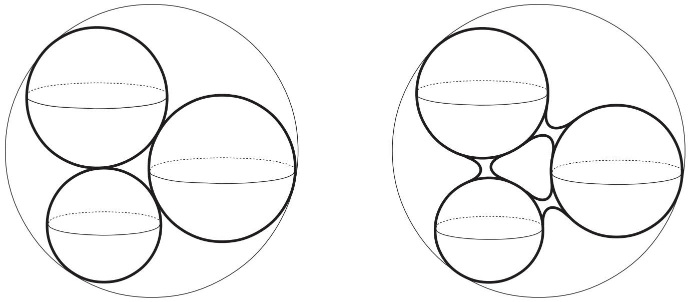

FIGURE 1. Left: a packing of three horospheres with three tangency points in the ball model of $\mathbb{H}^3$. Right: a CMC-1 surface of genus one with three ends.

construct examples in the theory of surfaces. So let us explain the motivation for giving another proof of this result.

Firstly, Bryant representation has been used by many authors to construct CMC-1 surfaces in hyperbolic space: let us mention Benoît Daniel [3–5], Ricardo Sa Earp and Eric Toubiana [13], Wayne Rossman, Masaaki Umehara and Kotaro Yamada [12], Masaaki Umehara and Kotaro Yamada [20–22]. However, only low genus or highly symmetric examples have been constructed. Moreover, most of these examples are close cousins to known minimal surfaces in euclidean space. It seemed interesting to see if high genus examples of a truly hyperbolic nature, such as the ones proposed in Theorem 1, can be constructed using Bryant representation.

Secondly, the method of opening nodes is in many respects more elementary and economical than the P.D.E. gluing machinery. So far, it has only been applied to minimal surfaces in euclidean space. There are, however, many other problems in surface theory for which a "holomorphic recipe" such as the Weierstrass representation is available. For example, CMC surfaces in space forms can be constructed from holomorphic data using the Dorfmeister, Pedit, Wu (DPW) loop group method [6], owing to the fact that the Gauss of a CMC surface is harmonic. More generally, the DPW recipe applies whenever the surface can be represented in terms of a harmonic map into a symmetric space. It seems interesting to see if opening nodes and the DPW method can be combined to work together. The work in this paper is a first step in this direction. Indeed, Bryant representation and DPW have in common that a matrix-valued O.D.E. must be solved, which leads to a monodromy problem.

Returning to Theorem 1, there are free parameters in the construction; namely, n complex parameters $c_1, \cdots, c_n$, which are the limit points of the horospheres on the ideal boundary of $\mathbb{H}^3$ (identified with the Riemann sphere), and n positive real parameters $\xi_1, \cdots, \xi_n$, which represent the speed at which each horosphere is "deflated" to accomodate the catenoidal necks. We can normalize $\xi_1 = 1$, so together with the parameter s our family depends on $3n$ real parameters, which is the expected dimension for the space of CMC-1 surfaces with n ends.

One may ask what are the possible topologies that one can achieve with this construction. In other words, given the number of ends n, what are the possible genera $g = m - n + 1$? If $n = 2$, then clearly $g = 0$. For $n \geq 3$, it is easy to construct horosphere packings with up to $3n-6$ tangency points using the Appolonius 4 sphere theorem (see [10]). These are called appolonian packings.

Here is how to do better: the starting position for the american pool is made up of 15 unit spheres (arranged in a triangle) with 30 tangency points. Stack on top of that a horizontal plane, and you get a packing of 16 horospheres with 45 tangency points in the half-space model. This is better than an appolonian packing which has only 42 tangency points for the same number of horospheres. Following this idea of stacking a horizontal plane on top of a dense packing of unit spheres, it is easy to exhibit horosphere packings whose number of tangency points grows like $4n$ (compared to $3n$ for the appolonian packing).

The paper is organized as follows. In Section 2, we present Bryant representation of CMC-1 surfaces in hyperbolic space and recall standard material about monodromy and opening nodes. In Section 3, we construct a family of candidates for the holomorphic data of the CMC-1 immersions we want to construct. The monodromy problem is solved in Section 4 using the implicit function theorem. In Section 5, we study the geometry of the surfaces we have constructed and prove that they are embedded. Finally, an appendix contains several results of independent interest that we use in this construction.

2. Background material

2.1. **Bryant representation.**

2.1.1. *The Minkowski model of* $\mathbb{H}^3$. Let $\mathbb{L}^4$ denote the 4-dimensional lorentzian space, namely $\mathbb{R}^4$, with coordinates denoted x_0, x_1, x_2, x_3 and metric $-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$. The hyperboloid

$$\{x \in \mathbb{L}^4 \,:\, \langle x, x\rangle = -1,\ x_0 > 0\}$$

with the induced metric is the Minkowski model of $\mathbb{H}^3$. Bryant identifies $\mathbb{L}^4$ with the space of 2×2 hermitian matrices by identifying (x_0, x_1, x_2, x_3) with the matrix

$$X = \begin{pmatrix} x_0 + x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix}.$$

Then $\mathbb{H}^3$ becomes the set of positive definite hermitian matrices with determinant 1.

2.1.2. *Bryant representation.* Recall that a holomorphic map $F : \Sigma \to SL(2,\mathbb{C})$ is null if $\det(F^{-1}dF) = 0$.

Theorem 2 (Bryant [1])**.** *Let Σ be a simply connected Riemann surface. Let $F : \Sigma \to SL(2,\mathbb{C})$ be a holomorphic null immersion. Then $FF^* : \Sigma \to \mathbb{H}^3$ is a smooth conformal CMC-1 immersion. Conversely, if $f : \Sigma \to \mathbb{H}^3$ is a conformal CMC-1 immersion, there exists a holomorphic null immersion $F : \Sigma \to SL(2,\mathbb{C})$ such that $f = FF^*$. Moreover, F is unique up to right multiplication by a constant matrix $H \in SU(2)$.*

If Σ is not simply connected, then F is only well defined on the universal cover of Σ and will have $SU(2)$-valued monodromy.

2.1.3. *Global meromorphic data.* Assume we are given a CMC-1 immersion $f : \Sigma \to \mathbb{H}^3$. By Theorem 2, we can write locally $f = FF^*$, where F is a null holomorphic, $SL(2, \mathbb{C})$-valued map. Consider the matrix of holomorphic 1-forms

$$A(z) = (dF(z))F(z)^{-1} \in \mathfrak{sl}(2, \mathbb{C}),$$

where $\mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra of 2×2 matrices whose trace is zero. Then A is well defined on Σ: replacing F by FH does not change A. Define $G = \frac{A_{11}}{A_{21}}$ and $\Omega = A_{21}$, where A_{ij} denote the coefficients of A. Then G is a meromorphic function and Ω is a holomorphic 1-form, both globally defined on Σ (except in the exceptional case where $A_{21} \equiv 0$). Since the trace and determinant of A are zero, we have

$$A = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \Omega. \tag{1}$$

The function G is the Gauss map introduced by Bryant, and $Q = \Omega\, dG$ is the Hopf quadratic differential. We will see the geometric meaning of the Gauss map G in the next section. We call (G, Ω) the meromorphic data for the immersion f.

Conversely, here is a recipe to construct CMC-1 immersions in $\mathbb{H}^3$. Start with a Riemann surface Σ, a meromorphic function G and a holomorphic 1-form Ω on Σ, such that Ω has a zero at each pole of G with twice the multiplicity and has no other zeros. Define the matrix A by (1). Then A is a holomorphic, $\mathfrak{sl}(2, \mathbb{C})$-valued 1-form on Σ which does not vanish. Solve the linear differential system

$$dF(z) = A(z)F(z) \tag{2}$$

with initial data $F(z_0) = F_0 \in SL(2, \mathbb{C})$. The solution is a multi-valued holomorphic null immersion $F : \Sigma \to SL(2, \mathbb{C})$. (It is an immersion because $A(z) \neq 0$.) If its monodromy happens to be in $SU(2)$, then the immersion $f = FF^* : \Sigma \to \mathbb{H}^3$ is well defined and has CMC-1.

Remark 1. Many authors instead consider the matrix $F^{-1}dF$ and write

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega, \tag{3}$$

where g is a meromorphic function and ω is a holomorphic 1-form. It turns out that (g, ω) is the Weierstrass data for the corresponding minimal surface in $\mathbb{R}^3$ via the Lawson correspondence [8], which is why (3) has become so popular. The problem is that unless Σ is simply connected, $F^{-1}dF$ is not globally defined on Σ. This is not surprising, since the Lawson correspondence is local. So this point of view is not appropriate if we are to construct high genus examples.

2.1.4. *Bryant representation in the half-space model.* A more familiar model of $\mathbb{H}^3$ is the half-space $x_3 > 0$ in $\mathbb{R}^3$, with conformal metric $x_3^{-2}(dx_1^2 + dx_2^2 + dx_3^2)$. The following results are proved in [11]. An orientation preserving isometry from the Minkowski model to the half-space model is given by

$$\Phi(x_0, x_1, x_2, x_3) = \left(\frac{x_1}{x_0 - x_3}, \frac{x_2}{x_0 - x_3}, \frac{1}{x_0 - x_3} \right).$$

The immersion $\Phi \circ f$ is given in the half-space model by

$$x_1 + \mathrm{i}x_2 = \frac{F_{11}\overline{F_{21}} + F_{12}\overline{F_{22}}}{|F_{21}|^2 + |F_{22}|^2}, \qquad x_3 = \frac{1}{|F_{21}|^2 + |F_{22}|^2}, \tag{4}$$

where F_{ij} denote the coefficients of the matrix F. The ideal boundary of $\mathbb{H}^3$ in this model is $\mathbb{C} \cup \{\infty\}$. The Gauss map G has the following geometric interpretation: the normal geodesic ray originated from $\Phi \circ f(z)$ (in the direction of the mean curvature vector) hits the ideal boundary at the point $G(z)$.

2.1.5. *Isometries.* The Lie group $SL(2,\mathbb{C})$ acts isometrically on $\mathbb{L}^4$ by the representation

$$H \cdot X = HXH^*,$$

where $H \in SL(2,\mathbb{C})$ and $X \in \mathbb{L}^4$. The action preserves $\mathbb{H}^3$ and its kernel is $\{\pm I_2\}$, so the group of direct isometries of $\mathbb{H}^3$ is $PSL(2,\mathbb{C})$. The action of $SL(2,\mathbb{C})$ on $\mathbb{H}^3$ extends to the ideal boundary as homographic transformation of the Riemann sphere, namely

$$H \cdot z = \frac{H_{11}z + H_{12}}{H_{21}z + H_{22}}.$$

If $f : \Sigma \to \mathbb{H}^3$ is a conformal CMC-1 immersion with Gauss map G and null holomorphic map F, then $H \cdot f$ has Gauss map $H \cdot G$ and null holomorphic map HF.

2.1.6. *Horospheres.* The Gauss map is constant on a horosphere, and that constant is the limit point of the horosphere on the ideal boundary of $\mathbb{H}^3$ in the half-space model. (This follows from the geometric interpretation of the Gauss map in the half-space model.) If the horosphere is not a horizontal plane, then its meromorphic data is $\Sigma = \mathbb{C}$, $G = c$, $\Omega = \lambda\, dz$, where c and λ are complex constants. The constant λ has no geometrical meaning and depends on the chosen conformal parametrization of the horosphere. The matrix A is given by

$$A = \lambda \begin{pmatrix} c & -c^2 \\ 1 & -c \end{pmatrix}.$$

If the horosphere is a horizontal plane, then $G \equiv \infty$ and we are in the exceptional case where the meromorphic data (G,Ω) is not defined. In this case, one has

$$A = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$$

for some complex number λ. In any case, since the matrix A is constant, the solution to (2) is $F(z) = \exp(zA)F_0$.

2.2. **Linear differential systems.** In this section, we recall standard facts about linear differential systems on a Riemann surface and set up some notation. A basic reference is [15]. Let Σ be a Riemann surface and A an $n \times n$ matrix of holomorphic 1-forms on Σ. We consider the first order linear differential system on Σ,

$$dY(z) = A(z)Y(z). \tag{5}$$

2.2.1. *Local theory: The principal solution.* Assume that Σ is simply connected. Given $z_0 \in \Sigma$, (5) has a unique solution $Y : \Sigma \to GL(n,\mathbb{C})$ such that $Y(z_0) = I_n$. Following [15], we write $Y(z) = \Pi(z, z_0)$. The map $\Pi : \Sigma \times \Sigma \to GL(n,\mathbb{C})$ is holomorphic in both variables and is called the principal solution. It satisfies

$$\Pi(z_3, z_2)\Pi(z_2, z_1) = \Pi(z_3, z_1).$$

Given $Y_0 \in GL(n,\mathbb{C})$, the solution Y such that $Y(z_0) = Y_0$ is given by $Y(z) = \Pi(z, z_0)Y_0$.

2.2.2. *Global theory.* Now assume that Σ is not simply connected. Then the principal solution $\Pi(z,z_0)$ is not well defined; it depends on the homotopy class of the path from z_0 to z. If $\gamma:[0,1]\to\Sigma$ is a path from z_0 to z, the solution Y of (5) such that $Y(z_0)=I_n$, which exists in a simply connected neighborhood of $\gamma(0)$, can be analytically continued along γ. Its value at $\gamma(1)$ will be denoted $\Pi(\gamma)$. When the path γ is clear from the context, we will still use the notation $\Pi(z,z_0)$.

Let $\gamma_1,\gamma_2:[0,1]\to\Sigma$ be two paths such that $\gamma_1(1)=\gamma_2(0)$. We denote by $\gamma_2\cdot\gamma_1$ the path obtained by composing γ_1 and γ_2. (The usual notation is $\gamma_1\cdot\gamma_2$, but in this context it is more convenient, and customary, to reverse the order.) Then

$$\Pi(\gamma_2\cdot\gamma_1)=\Pi(\gamma_2)\Pi(\gamma_1). \tag{6}$$

In particular, Π is a morphism from the fundamental group $\pi_1(\Sigma,z_0)$ to $GL(n,\mathbb{C})$. (With the usual notation for the product in the fundamental group, it would be an anti-morphism.)

2.2.3. *Monodromy.* The monodromy of a solution is usually defined as follows. Let Y be a solution of (5) in a simply connected neighborhood U of z_0. Let $\gamma\in\pi_1(\Sigma,z_0)$. Analytic continuation of Y along γ gives another solution of (5) in U, which is denoted $\gamma\cdot Y$. There exists a matrix $M\in GL(n,\mathbb{C})$ such that $\gamma\cdot Y=YM$. The matrix M is called the monodromy of Y along γ and is denoted $M_\gamma(Y)$. In terms of the principal solution, one has

$$M_\gamma(Y)=Y(z_0)^{-1}\Pi(\gamma)Y(z_0). \tag{7}$$

2.3. **Opening nodes.** We recall the standard construction of opening nodes. Consider n copies of the Riemann sphere $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, labelled $\overline{\mathbb{C}}_1,\cdots,\overline{\mathbb{C}}_n$. Consider $2m$ distinct points $p_1,\cdots,p_m$, $q_1,\cdots,q_m$ in the disjoint union $\overline{\mathbb{C}}_1\cup\cdots\cup\overline{\mathbb{C}}_n$. Identify p_i with q_i for $1\le i\le m$. This defines a Rieman surface with nodes which we denote Σ_0. We assume Σ_0 is connected.

To open nodes, consider local complex coordinates $v_i:V_i\to D(0,1)$ in a neighborhood of p_i and $w_i:W_i\to D(0,1)$ in a neighborhood of q_i, with $v_i(p_i)=0$ and $w_i(q_i)=0$. We assume that the neighborhoods $V_1,\cdots,V_m,W_1,\cdots,W_m$ are disjoint in $\overline{\mathbb{C}}_1\cup\cdots\cup\overline{\mathbb{C}}_n$. Consider, for each $1\le i\le m$, a complex parameter t_i with $|t_i|<1$. If $t_i=0$, identify p_i with q_i as above. If $t_i\neq 0$, remove the disks $|v_i|\le|t_i|$ and $|w_i|\le|t_i|$. Identify the point $z\in V_i$ with the point $z'\in W_i$ such that

$$v_i(z)w_i(z')=t_i.$$

This creates a Riemann surface, possibly with nodes, which we denote $\Sigma_{\mathbf{t}}$, where $\mathbf{t}=(t_1,\cdots,t_n)$. When all t_i are non-zero, $\Sigma_{\mathbf{t}}$ is a genuine Riemann surface of genus $g=m-n+1$.

Observe that if $t_i\neq 0$, the circle $|v_i|=1$ is homologous in Σ to the circle $|w_i|=1$ with the opposite orientation. Consequently, if ω is a holomorphic 1-form on $\Sigma_{\mathbf{t}}$,

$$\int_{|v_i|=1}\omega=-\int_{|w_i|=1}\omega. \tag{8}$$

This makes the following definition natural.

Definition 1 (Bers)**.** A regular differential on $\Sigma_{\mathbf{t}}$ is a holomorphic 1-form which is allowed to have simples poles at p_i and q_i if $t_i=0$, with opposite residues.

By a theorem of Fay [7], the space of regular differentials on $\Sigma_{\mathbf{t}}$ has dimension g and admits a basis which depends holomorphically on $\mathbf{t}$ in a neighborhood of 0. For $1 \leq i \leq n$, let J_i^+ and J_i^- be the set of indices j such that $p_j \in \overline{\mathbb{C}}_i$ and $q_j \in \overline{\mathbb{C}}_i$, respectively. As a consequence of Fay's theorem, one has:

Theorem 3. *For* $\mathbf{t}$ *in a neighborhood of* 0 *and for* $\mathbf{a} = (a_j)_{1\leq j\leq m} \in \mathbb{C}^m$ *satisfying*

$$\sum_{j\in J_i^+} a_j - \sum_{j\in J_i^-} a_j = 0 \qquad \text{for } 1 \leq i \leq n, \tag{9}$$

there exists a unique regular differential $\omega = \omega_{\mathbf{t},\mathbf{a}}$ *on* $\Sigma_{\mathbf{t}}$ *such that*

$$\int_{|v_j|=1} \omega = a_j \qquad \text{for } 1 \leq j \leq m. \tag{10}$$

Moreover, $\omega_{\mathbf{t},\mathbf{a}}$ *depends holomorphically on* $\mathbf{t}$ *(away from the nodes).*

Proof. From the Cauchy theorem in $\overline{\mathbb{C}}_i$ and (8), we see that (9) is necessary for ω to exist. If $\mathbf{t} = 0$, the map $\omega \mapsto (\int_{|v_j|=1} \omega)_{1\leq j\leq m}$ is an isomorphism from the space of regular differentials on Σ_0 to the space of vectors $\mathbf{a} \in \mathbb{C}^m$ satisfying (9). (This follows from the fact that a holomorphic 1-form on the Riemann sphere with simple poles is entirely determined by its residues.) Using Fay's theorem, this remains true for $\mathbf{t}$ in a neighborhood of 0. $\square$

Fay's proof is rather abstract and non-constructive. For an elementary proof of Theorem 3 based on the contraction mapping principle, see [19]. One has a similar result for meromorphic 1-forms with poles at some points $r_1, \cdots, r_k$ distinct from the nodes:

Theorem 4. *For* $\mathbf{t}$ *in a neighborhood of* 0*, one can define a regular meromorphic differential* ω *on* $\Sigma_{\mathbf{t}}$ *with poles at the points* $r_1, \cdots, r_k$ *by prescribing its principal part at each pole and its periods as in* (10)*, replacing* (9) *by the restriction coming from the residue theorem, namely*

$$\sum_{j\in J_i^+} a_j - \sum_{j\in J_i^-} a_j + 2\pi \mathrm{i} \sum_{r_j\in\overline{\mathbb{C}}_i} \mathrm{Res}_{r_j}\omega = 0 \qquad \text{for } 1 \leq i \leq n. \tag{11}$$

Recall that the principal part of a meromorphic 1-form ω at a pole r is its equivalence class under the relation: $\omega \sim \omega'$ if $\omega - \omega'$ is holomorphic in a neighborhood of r. The analogue of Fay's theorem for meromorphic differentials with simple poles is proved by Masur in [9]. For a proof of Theorem 4 in the case of poles of arbitrary order, see [19]. We will also need the following result to compute the partial derivatives of ω with respect to $\mathbf{t}$.

Theorem 5. *The partial derivative* $\frac{\partial}{\partial t_i}\omega_{\mathbf{t},\mathbf{a}}$ *at* $\mathbf{t} = 0$ *has two double poles at* p_i *and* q_i*, with principal parts*

$$\frac{-dv_i}{v_i^2}\,\mathrm{Res}_{q_i}\frac{\omega_{0,\mathbf{a}}}{w_i} \qquad \text{at } p_i,$$

$$\frac{-dw_i}{w_i^2}\,\mathrm{Res}_{p_i}\frac{\omega_{0,\mathbf{a}}}{v_i} \qquad \text{at } q_i,$$

and has vanishing periods on all circles $|v_j| = 1$.

This is proved in [18], Lemma 3. See also [19], Remark 5.6.

3. The meromorphic data (Σ, G, Ω)

3.1. **Notation.** The horospheres of our given horosphere packing are denoted $S_1, \cdots, S_n$. We define the following sets, which we use to index various quantities:

$$I = \{(i,j) \in [\![1,n]\!]^2 \ : \ i < j \text{ and } S_i, S_j \text{ are tangent}\},$$

$$J_i^+ = \{j \ : \ (i,j) \in I\} \qquad J_i^- = \{j \ : \ (j,i) \in I\} \qquad J_i = J_i^+ \cup J_i^-.$$

Without loss of generality, we may assume (by applying an isometry) that each horosphere S_i is not a horizontal plane in the half-space model. We fix a conformal parametrisation $f_i : \mathbb{C} \to S_i$ and let

$$G = c_i \qquad \Omega = \lambda_i dz$$

be its meromorphic data (see Section 2.1.6). For $j \in J_i$, we denote by p_{ij} the point in $\mathbb{C}$ such that $f_i(p_{ij})$ is the point $S_i \cap S_j$. Without loss of generality, we may assume (by changing the parametrization f_i) that for each i, the disks $D(p_{ij}, 1)$ for $j \in J_i$ and $D(0,1)$ are pairwise disjoint.

3.2. **The Riemann surface Σ.** We consider n copies of the complex plane, denoted $\mathbb{C}_1, \cdots, \mathbb{C}_n$, and m copies of the Riemann sphere $\mathbb{C} \cup \{\infty\}$, denoted $\overline{\mathbb{C}}_{ij}$ for $(i,j) \in I$. We think of p_{ij} as a point in $\mathbb{C}_i$. The points 0, 1 and ∞ in $\overline{\mathbb{C}}_{ij}$ will be denoted respectively 0_{ij}, 1_{ij} and ∞_{ij}. Heuristically, the reader should think of $\mathbb{C}_1, \cdots, \mathbb{C}_n$ as the parametrization domain for the horospheres $S_1, \cdots, S_n$, and $\overline{\mathbb{C}}_{ij} \setminus \{0, \infty\}$ for $(i,j) \in I$ as the parametrization domain for the catenoidal necks connecting them.

We define a Riemann surface Σ_0 with $2m$ nodes by identifying p_{ij} with 0_{ij} and p_{ji} with ∞_{ij}, for $(i,j) \in I$, and call it Σ_0. To open nodes, we consider the natural coordinates $v_{ij} = z - p_{ij}$ in a neighborhood of p_{ij} in $\mathbb{C}_i$, $w_{ij} = z$ in a neighborhood of 0_{ij} in $\overline{\mathbb{C}}_{ij}$, $v_{ji} = z - p_{ji}$ in a neighborhood of p_{ji} in $\mathbb{C}_j$ and $w_{ji} = \frac{1}{z}$ in a neighborhood of ∞_{ij} in $\overline{\mathbb{C}}_{ij}$. We open nodes as explained in Section 2.3, introducing a complex parameter t_{ij} to open the node $p_{ij} \sim 0_{ij}$ and another parameter t_{ji} to open the node $p_{ji} \sim \infty_{ij}$. Let $\mathbf{t} = (t_{ij}, t_{ji})_{(i,j) \in I}$ be the collection of these parameters. This defines a Riemann surface (possibly with nodes) which we denote Σ, or $\Sigma_{\mathbf{t}}$ when we need to emphasize the dependence on the parameter $\mathbf{t}$. We denote by $\overline{\Sigma}$ the compactification of Σ obtained by adding the points $\infty_1, \cdots, \infty_n$, where ∞_i denotes the point at infinity in $\mathbb{C}_i$.

For $(i,j) \in I$, we define α_{ij} as the homology class of the circle $|z - p_{ij}| = 1$ in $\mathbb{C}_i$, with the positive orientation. This is homologous, in Σ, to the unit circle in $\overline{\mathbb{C}}_{ij}$, with the negative orientation, and to the circle $|z - p_{ji}| = 1$ in $\mathbb{C}_j$, also with the negative orientation.

3.3. **The Gauss map G.** Here are our requirements on the Gauss map G. At ∞_i, it should take the value c_i. It should have a simple pole in each Riemann sphere $\overline{\mathbb{C}}_{ij}$ (because on each catenoidal neck, we expect a point where the mean curvature vector is vertical pointing up, in the half-space model). We choose the identification of the Riemann sphere with $\mathbb{C} \cup \{\infty\}$ so that this pole is $z = 1$. The following proposition tells us that these requirements completely determine G.

Proposition 1. *For* $\mathbf{t}$ *small enough, there exists a unique meromorphic function* $G = G_{\mathbf{t}}$ *on* $\overline{\Sigma}_{\mathbf{t}}$ *with the following properties:*

- $G_{\mathbf{t}}$ *has* m *simple poles at the points* 1_{ij} *for* $(i,j) \in I$*,*
- $G_{\mathbf{t}}(\infty_i) = c_i$ *for* $i = 1, \cdots, n$*.*

Moreover, $G_{\mathbf{t}}$ *depends holomorphically on* $\mathbf{t}$ *(away from the nodes and its poles), and at* $\mathbf{t} = 0$*, we have*

$$G_0(z) = \begin{cases} c_i & \text{in } \mathbb{C}_i, \\ c_j + \dfrac{c_j - c_i}{z-1} & \text{in } \overline{\mathbb{C}}_{ij}. \end{cases} \tag{12}$$

Proof. We first define the differential $\mu = dG$ and we recover G by integration. By Theorem 4, there exists a unique meromorphic differential $\mu_{\mathbf{t}}$ on $\overline{\Sigma}_{\mathbf{t}}$ which has m double poles at 1_{ij} for $(i,j) \in I$ with principal part

$$r_{ij} \frac{dz}{(z-1)^2}$$

and has vanishing period on all cycles α_{ij}. Here the m complex numbers r_{ij} are free parameters. At $\mathbf{t} = 0$, we have

$$\mu_0 = \begin{cases} 0 & \text{in } \mathbb{C}_i, \\ r_{ij} \frac{dz}{(z-1)^2} & \text{in } \overline{\mathbb{C}}_{ij}. \end{cases}$$

Lemma 1. *For* $\mathbf{t}$ *in a neighborhood of* 0*, there exist unique values of the parameters* r_{ij} *such that*

$$\int_{\infty_i}^{\infty_j} \mu_{\mathbf{t}} = c_j - c_i \qquad \text{for } (i,j) \in I. \tag{13}$$

Moreover, each r_{ij} *depends holomorphically on* $\mathbf{t}$*, and when* $\mathbf{t} = 0$*,* $r_{ij} = c_i - c_j$*.*

Proof. First observe that (13) is a linear system of m linear equations with m unknowns r_{ij}, $(i,j) \in I$. Hence it suffices to prove that the system (13) is invertible when $\mathbf{t} = 0$. In (13), it is understood that the path from ∞_i to ∞_j goes through $\overline{\mathbb{C}}_{ij}$. There is no canonical way to choose this path, but all choices are homologous modulo α_{ij}. Since μ has no period on α_{ij}, $\int_{\infty_i}^{\infty_j} \mu_{\mathbf{t}}$ is a well-defined holomorphic function of $\mathbf{t}$. Moreover, by Lemma 4 in [18], this function extends holomorphically at $\mathbf{t} = 0$ with value

$$\int_{\infty_i}^{\infty_j} \mu_0 = \int_{\infty_i}^{p_{ij}} 0 + \int_{0_{ij}}^{\infty_{ij}} r_{ij} \frac{dz}{(z-1)^2} + \int_{p_{ji}}^{\infty_j} 0 = -r_{ij}.$$

The lemma follows. ◻

Returning to the proof of Proposition 1, we define the function $G_{\mathbf{t}}$ on $\overline{\Sigma}_{\mathbf{t}}$ by

$$G_{\mathbf{t}}(z) = c_1 + \int_{\infty_1}^{z} \mu_{\mathbf{t}}.$$

By Lemma 1 and the fact that $\mu_{\mathbf{t}}$ has no residues and no period on the cycles α_{ij}, $G_{\mathbf{t}}$ is well defined on $\overline{\Sigma}_{\mathbf{t}}$ (meaning that the integral does not depend on the path from ∞_1 to z) and has all the desired properties. ◻

3.4. **The holomorphic differential Ω.** Here are our requirements on the holomorphic differential Ω. It should have a double pole at ∞, with leading term $\lambda_i dz$, just like the horosphere S_i. It also needs a double zero at each pole of G. We define Ω by prescribing poles, principal parts and periods, using Theorem 4. Then we adjust the parameter $\mathbf{t}$ so that Ω has the required zeros.

Definition 2. Consider m complex parameters a_{ij}, $(i,j)\in I$, and let $\mathbf{a}=(a_{ij})_{(i,j)\in I}$. We define $\Omega=\Omega_{\mathbf{t},\mathbf{a}}$ as the unique meromorphic 1-form on $\overline{\Sigma}_{\mathbf{t}}$ with n double poles at $\infty_1,\cdots,\infty_n$, with principal part

$$\lambda_i dz+\sum_{j\in J_i^+} a_{ij}\frac{dz}{z}-\sum_{j\in J_i^-} a_{ji}\frac{dz}{z}\qquad \text{at } \infty_i$$

and periods

$$\int_{\alpha_{ij}}\Omega=2\pi\mathrm{i}\,a_{ij}\qquad \text{for } (i,j)\in I.$$

It depends holomorphically (away from its poles and the nodes) on $\mathbf{t}$. Moreover, at $\mathbf{t}=0$ we have

$$\Omega_{0,\mathbf{a}}=\begin{cases}\lambda_i dz+\displaystyle\sum_{j\in J_i^+} a_{ij}\frac{dz}{z-p_{ij}}-\sum_{j\in J_i^-} a_{ji}\frac{dz}{z-p_{ji}} & \text{in } \mathbb{C}_i,\\ -a_{ij}\dfrac{dz}{z} & \text{in } \overline{\mathbb{C}}_{ij}.\end{cases}\tag{14}$$

Note that the residue of the prescribed principal part at ∞_i is

$$\mathrm{Res}_{\infty_i}\Omega=-\sum_{j\in J_i^+} a_{ij}+\sum_{j\in J_i^-} a_{ji},$$

so (11) holds.

Proposition 2. *For $\mathbf{a}$ in a neighborhood of 0, there exists a unique value $\mathbf{t}(\mathbf{a})$, depending holomorphically on $\mathbf{a}$, such that $\Omega_{\mathbf{t}(\mathbf{a}),\mathbf{a}}$ has a double zero at each pole of G, and has no other zeros in Σ (provided all parameters a_{ij} are non-zero). Moreover, for each $(i,j)\in I$, we have*

$$a_{ij}=0\quad\Rightarrow\quad t_{ij}(\mathbf{a})=t_{ji}(\mathbf{a})=0,\tag{15}$$

$$\frac{\partial t_{ij}(\mathbf{a})}{\partial a_{ij}}|_{\mathbf{a}=0}=\frac{-1}{2\lambda_i},\tag{16}$$

$$\frac{\partial t_{ji}(\mathbf{a})}{\partial a_{ij}}|_{\mathbf{a}=0}=\frac{1}{2\lambda_j}.\tag{17}$$

Proof. Let $(i,j)\in I$. Using Theorem 5 and (14), we compute the partial derivatives of $\Omega_{\mathbf{t},\mathbf{a}}$ in $\overline{\mathbb{C}}_{ij}$ at $(\mathbf{t},\mathbf{a})=(0,0)$:

$$\frac{\partial\Omega_{\mathbf{t},\mathbf{a}}}{\partial t_{ij}}=\frac{-dz}{z^2}\mathrm{Res}_{p_{ij}}\frac{\Omega_{0,0}}{z-p_{ij}}=-\lambda_i\frac{dz}{z^2},\tag{18}$$

$$\frac{\partial\Omega_{\mathbf{t},\mathbf{a}}}{\partial t_{ji}}=\frac{-dw_{ji}}{w_{ji}^2}\mathrm{Res}_{p_{ji}}\frac{\Omega_{0,0}}{z-p_{ji}}=\lambda_j\,dz\qquad(\text{recall } w_{ji}=\frac{1}{z}),\tag{19}$$

$$\frac{\partial\Omega_{\mathbf{t},\mathbf{a}}}{\partial a_{ij}}=\frac{-dz}{z}.\tag{20}$$

The partial derivatives of $\Omega_{\mathbf{t},\mathbf{a}}$ in $\overline{\mathbb{C}}_{ij}$ with respect to all other parameters $t_{k\ell}$ and $a_{k\ell}$ are zero. Write $\Omega_{\mathbf{t},\mathbf{a}} = f_{ij}(\mathbf{t},\mathbf{a},z)dz$ in $\overline{\mathbb{C}}_{ij}$. We want to solve $f_{ij}(\mathbf{t},\mathbf{a},1) = f'_{ij}(\mathbf{t},\mathbf{a},1) = 0$. The Jacobian matrix of $(f_{ij}(\mathbf{t},\mathbf{a},1), f'_{ij}(\mathbf{t},\mathbf{a},1))$ with respect to (t_{ij},t_{ji}) is

$$\begin{pmatrix} -\lambda_i & \lambda_j \\ 2\lambda_i & 0 \end{pmatrix}.$$

The existence of the solution $\mathbf{t}(\mathbf{a})$ then follows from the implicit function theorem applied to the map $(f_{ij}(\mathbf{t},\mathbf{a},1), f'_{ij}(\mathbf{t},\mathbf{a},1))_{(i,j)\in I}$ whose Jacobian has block diagonal form. Next consider some $(i,j)\in I$ and assume that $a_{ij}=0$. If $t_{ij}=t_{ji}=0$, then $\Omega_{\mathbf{t},\mathbf{a}}$ has two simple poles at 0_{ij} and ∞_{ij}, with residue $\pm a_{ij}=0$, and hence is holomorphic in $\overline{\mathbb{C}}_{ij}$, so $\Omega_{\mathbf{t},\mathbf{a}}\equiv 0$ in $\overline{\mathbb{C}}_{ij}$. So (15) follows from uniqueness in the implicit function theorem. Equations (16) and (17) are obtained by differentiating $f_{ij}(\mathbf{t}(\mathbf{a}),\mathbf{a},1) = f'_{ij}(\mathbf{t}(\mathbf{a}),\mathbf{a}) = 0$ with respect to a_{ij}, using (18), (19) and (20). Finally, if all parameters a_{ij} are non-zero, then (16) and (17) imply that all parameters t_{ij} and t_{ji} are non-zero, so Σ is a genuine compact Riemann surface of genus $g=m-n+1$. Ω is a meromorphic 1-form with n doubles poles, so its number of zeros, counting multiplicity, is $2n+2g-2=2m$. This ensures that Ω has no other zeros than the double zeros it has at the m poles of G. □

3.5. **Partial derivatives with respect to the parameter a_{ij}.**

Proposition 3. *For $(i,j)\in I$ we have at $\mathbf{a}=0$*

$$\frac{\partial G_{\mathbf{t}(\mathbf{a})}}{\partial a_{ij}} = \begin{cases} \dfrac{c_j-c_i}{2\lambda_i}\dfrac{1}{z-p_{ij}} & \text{in } \mathbb{C}_i, \\ \dfrac{c_j-c_i}{2\lambda_j}\dfrac{1}{z-p_{ji}} & \text{in } \mathbb{C}_j, \\ 0 & \text{in } \mathbb{C}_k \text{ for } k\neq i,j. \end{cases} \tag{21}$$

(22)

$$\frac{\partial \Omega_{\mathbf{t}(\mathbf{a}),\mathbf{a}}}{\partial a_{ij}} = \begin{cases} \dfrac{dz}{z-p_{ij}} & \text{in } \mathbb{C}_i, \\ \dfrac{-dz}{z-p_{ji}} & \text{in } \mathbb{C}_j, \\ \dfrac{(1-z)^2}{2z^2}dz & \text{in } \overline{\mathbb{C}}_{ij}, \\ 0 & \text{in } \mathbb{C}_k \text{ for } k\neq i,j \text{ and in } \overline{\mathbb{C}}_{k\ell} \text{ for } (k,\ell)\neq(i,j). \end{cases}$$

Proof. Recall that $\mu = dG$. By Theorem 5,

$$\frac{\partial \mu_{\mathbf{t}}}{\partial t_{ij}} = -\frac{dz}{(z-p_{ij})^2}\operatorname{Res}_{0_{ij}}\mu_0 = -(c_i-c_j)\frac{dz}{(z-p_{ij})^2} \qquad \text{in } \mathbb{C}_i.$$

Hence, by the chain rule and (16),

$$\frac{\partial \mu_{\mathbf{t}(\mathbf{a})}}{\partial a_{ij}} = \frac{c_i-c_j}{2\lambda_i}\frac{dz}{(z-p_{ij})^2} \qquad \text{in } \mathbb{C}_i. \tag{23}$$

Integrating, we obtain the first line of (21). The proof of the second line is entirely similar. Regarding (22), we have (again using Theorem 5)

$$\frac{\partial \Omega_{\mathbf{t},\mathbf{a}}}{\partial t_{ij}} = 0 \qquad \frac{\partial \Omega_{\mathbf{t},\mathbf{a}}}{\partial a_{ij}} = \frac{dz}{z-p_{ij}} \qquad \text{in } \mathbb{C}_i.$$

The first line of (22) follows from the chain rule. The proof of the second line is similar. Using (18), (19), (20) and the chain rule, we have

$$\frac{\partial\Omega_{\mathbf{t}(\mathbf{a}),\mathbf{a}}}{\partial a_{ij}} = -\lambda_i\frac{dz}{z^2}\times\left(\frac{-1}{2\lambda_i}\right)+\lambda_j\,dz\times\left(\frac{1}{2\lambda_j}\right)-\frac{dz}{z}=\frac{(1-z)^2}{2z^2}dz \qquad \text{in } \overline{\mathbb{C}}_{ij}.$$

□

4. The monodromy problem

4.1. **Formulation of the problem.** We consider the matrix $A = A_{\mathbf{a}}$ defined by (1) with $G = G_{\mathbf{t}(\mathbf{a})}$ and $\Omega = \Omega_{\mathbf{t}(\mathbf{a}),\mathbf{a}}$. Each coefficient of A is a holomorphic differential on $\Sigma = \Sigma_{\mathbf{t}(\mathbf{a})}$. Let 0_i be the point $z=0$ in $\mathbb{C}_i$. Let $F:\Sigma\to SL(2,\mathbb{C})$ be the solution of $dF = AF$ with initial condition $F(0_1) = M_1$, where $M_1\in SL(2,\mathbb{C})$ is a matrix we can prescribe. (Observe that $F(z)\in SL(2,\mathbb{C})$ because $A(z)\in\mathfrak{sl}(2,\mathbb{C})$.) The solution F is of course only well defined on the universal cover of Σ. We need to adjust the parameters so that F has $SU(2)$-valued monodromy, so $f = FF^*$ is well defined on Σ. Taking 0_1 as a base point for the fundamental group and using (7), this is equivalent to

$$\forall\gamma\in\pi_1(\Sigma,0_1),\qquad M_1^{-1}\Pi(\gamma)M_1\in SU(2), \tag{24}$$

where Π denotes the principal solution of $dF = AF$ on Σ (see Section 2.2.1).

Instead of using a set of generators of $\pi_1(\Sigma,0_1)$, which would involve in a complicated way the “combinatorics” of our given horosphere packing, we reformulate the monodromy problem in a more “local” way as follows. For $(i,j)\in I$, let $\gamma_{ij}\in\pi_1(\Sigma,0_i)$ be a loop in $\mathbb{C}_i$ with base point 0_i which goes around p_{ij} and does not encircle any other node. We also define Γ_{ji} as a path connecting 0_i to 0_j through $\overline{\mathbb{C}}_{ij}$ (to be defined more precisely later on).

Proposition 4. *Given n matrices $M_1,\cdots,M_n$ in $SL(2,\mathbb{C})$, assume that for all $(i,j)\in I$:*

$$M_i^{-1}\Pi(\gamma_{ij})M_i\in SU(2), \tag{25}$$

$$M_j^{-1}\Pi(\Gamma_{ji})M_i\in SU(2). \tag{26}$$

Then (24) is satisfied. Moreover, the solution F of $dF = AF$ with initial condition $F(0_1) = M_1$ satisfies $F(0_i)\in M_i\times SU(2)$; hence $f(0_i) = M_iM_i^*$ for $1\le i\le n$.

Proof. Assume that (25) and (26) hold for all $(i,j)\in I$. Let $\Gamma_{ij} = \Gamma_{ji}^{-1}$. Using (6),

$$M_i^{-1}\Pi(\Gamma_{ij})M_j = \left(M_j^{-1}\Pi(\Gamma_{ji})M_i\right)^{-1}\in SU(2).$$

Define $\gamma_{ji} = \Gamma_{ji}\gamma_{ij}\Gamma_{ij}\in\pi_1(\Sigma,0_j)$. Using (6) again,

$$M_j^{-1}\Pi(\gamma_{ji})M_j = \left(M_j^{-1}\Pi(\Gamma_{ji})M_i\right)\left(M_i^{-1}\Pi(\gamma_{ij})M_i\right)\left(M_i^{-1}\Pi(\Gamma_{ij})M_j\right)\in SU(2).$$

In other words, (25) and (26) also hold for $(j,i)\in I$. Let $\mathbb{C}_i^\star$ be $\mathbb{C}_i$ minus the disks $D(p_{ij},1)$ for $j\in J_i$. The fundamental group $\pi_1(\mathbb{C}_i^\star,0_i)$ is the free group with generators γ_{ij} for $j\in J_i$. Hence (25) implies that

$$\forall\delta\in\pi_1(\mathbb{C}_i^\star,0_i)\quad M_i^{-1}\Pi(\delta)M_i\in SU(2). \tag{27}$$

Any element $\gamma\in\pi_1(\Sigma,0_1)$ is homotopic to a product of the form

$$\delta_k\Gamma_{i_k\,i_{k-1}}\delta_{k-1}\Gamma_{i_{k-1}\,i_{k-2}}\delta_{k-2}\cdots\delta_2\Gamma_{i_2\,i_1}\delta_1,$$

where $k \in \mathbb{N}^*$, $i_1 = i_k = 1$ and $\delta_j \in \pi_1(\mathbb{C}^\star_{i_j}, 0_{i_j})$ for $1 \leq j \leq k$. So (24) follows from (26) and (27). $\square$

4.2. **Choice of the matrices** $M_1, \cdots, M_n$**.** By the last statement of Proposition 4, choosing the matrices M_i amounts to prescribing the image of the points $0_1, \cdots, 0_n$. Recall from the introduction that we want to "deflate" the horosphere S_i at speed ξ_i. This suggests the following choice. Consider n fixed, positive numbers $\xi_1, \cdots, \xi_n$ and a real parameter s. (These are the same parameters as in the introduction.) Let $O_i = f_i(0) \in S_i$, where f_i is the chosen conformal parametrization of the horosphere S_i. Choose $M_i(s) \in SL(2,\mathbb{C})$ so that $s \in [0,\infty) \mapsto M_i(s)M_i(s)^*$ (in the Minkowski model) is the parametrization at speed ξ_i of the geodesic ray normal to the horosphere S_i at the point O_i (in the direction of the mean curvature vector). The matrix $M_i(s)$ is unique up to right multiplication by an element in $SU(2)$, which is clearly irrelevant for the monodromy problem.

4.3. **Main result.** To solve the monodromy problem, we need to adjust the complex parameters p_{ij} and p_{ji} for $(i,j) \in I$. We will denote p^0_{ij} and p^0_{ji} as the value of these parameters corresponding to the given horosphere packing (namely, such that $f_i(p^0_{ij}) = f_j(p^0_{ji}) = S_i \cap S_j$). The matrix of holomorphic 1-forms A depends holomorphically on the parameters $\mathbf{a} = (a_{ij})_{(i,j)\in I}$ and $\mathbf{p} = (p_{ij}, p_{ji})_{(i,j)\in I}$ and will be denoted $A_{\mathbf{a},\mathbf{p}}$. The principal solution of $dF = A_{\mathbf{a},\mathbf{p}}F$ will be denoted $\Pi_{\mathbf{a},\mathbf{p}}$. Our goal is to prove:

Proposition 5 (Solution of the monodromy problem)**.** *For $s > 0$ small enough, there exist unique values $\mathbf{a}(s)$ and $\mathbf{p}(s)$ such that* (25) *and* (26)*, with $M_i = M_i(s)$ and $\Pi = \Pi_{\mathbf{a}(s),\mathbf{p}(s)}$, are satisfied for all $(i,j) \in I$. Moreover, $\mathbf{a}(s)$ and $\mathbf{p}(s)$ are smooth functions of s for $s \neq 0$, and extend continuously at $s = 0$ with value $a_{ij}(0) = 0$ and $p_{ij}(0) = p^0_{ij}$.*

4.4. **Choice of an isometry.** From now on, we fix a couple $(i,j) \in I$. We have in mind to solve (25) and (26). The computations will be simplified by applying a well-chosen isometry. Let h be an orientation preserving isometry of $\mathbb{H}^3$ such that $h(S_i)$ is the horosphere $x_3 = 1$ and $h(S_i \cap S_j) = (0,0,1)$, in the half-space model. (This isometry is unique up to composition by a rotation around the vertical axis.) Since the horospheres S_i and S_j are tangent, $h(S_j)$ is the sphere of radius $\frac{1}{2}$ centered at $(0,0,\frac{1}{2})$. The limit points of $h(S_i)$ and $h(S_j)$ are respectively ∞ and 0.

As explained in Section 2.1.5, the isometry h corresponds to a matrix $H \in SL(2,\mathbb{C})$, unique up to sign. We have $H \cdot c_i = \infty$ and $H \cdot c_j = 0$, where the dot means the action by homography on the Riemann sphere. So H may be written in the form

$$H = \frac{1}{\sqrt{c_j - c_i}} \begin{pmatrix} \rho & -\rho c_j \\ \rho^{-1} & -\rho^{-1} c_i \end{pmatrix},$$

where ρ is some complex number.

We use hats to denote the action of the isometry h on various objects: $\widehat{F} = HF$ solves $d\widehat{F} = \widehat{A}\widehat{F}$, where $\widehat{A} = HAH^{-1}$. An elementary computation gives

$$\widehat{A} = \frac{1}{c_j - c_i} \begin{pmatrix} (G-c_i)(G-c_j) & -\rho^2(G-c_j)^2 \\ \rho^{-2}(G-c_i)^2 & -(G-c_i)(G-c_j) \end{pmatrix} \Omega. \tag{28}$$

The principal solution of $Y' = \widehat{A}Y$ is $\widehat{\Pi} = H\Pi H^{-1}$. Equations (25) and (26) are equivalent to

$$\widehat{M_i}^{-1}\widehat{\Pi}(\gamma_{ij})\widehat{M_i} \in SU(2), \tag{29}$$

$$\widehat{M_j}^{-1}\widehat{\Pi}(\Gamma_{ji})\widehat{M_i} \in SU(2), \tag{30}$$

where $\widehat{M_i} = HM_i$ and $\widehat{M_j} = HM_j$.

Remark 2. All of these quantities, H, ρ, $\widehat{A}$, $\widehat{\Pi}$, $\widehat{M_i}$, $\widehat{M_j}$, actually depend on both indices i and j because the chosen isometry h does. However, since i and j are fixed until the very end of Section 4.9, this dependence will not be written to make notation lighter.

4.5. **Computation of the matrices $\widehat{M_i}(s)$ and $\widehat{M_j}(s)$.** Consider the matrix

$$\Xi(s) = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix} \in SL(2,\mathbb{C}).$$

Then $s \mapsto \Phi(\Xi(s)\Xi(s)^*)$ is the parametrization at unit speed of the positive vertical axis (oriented upwards) in the half-space model. (Here Φ is the isometry from the Minkowski model to the half-space model given in Section 2.1.4.) The horosphere $\widehat{S}_i = h(S_i)$ is parametrized by $\widehat{f}_i = h \circ f_i$. We need to compute the corresponding null holomorphic map $\widehat{F}_i$. By substitution of $G = c_i$ and $\Omega = \lambda_i dz$ in (28), we obtain

$$\widehat{A}_i = \begin{pmatrix} 0 & \widehat{\lambda}_i \\ 0 & 0 \end{pmatrix}, \qquad \widehat{\lambda}_i = \rho^2\lambda_i(c_i - c_j). \tag{31}$$

By our choice of the isometry h, we have $\widehat{F}_i(p^0_{ij}) = I_2$. Hence

$$\widehat{F}_i(z) = \exp((z - p^0_{ij})\widehat{A}_i). \tag{32}$$

Then by our choice of the matrix $M_i(s)$ in Section 4.2,

$$\widehat{M_i}(s) = \exp(-p^0_{ij}\widehat{A}_i)\Xi(\xi_i s). \tag{33}$$

In the same way, the horosphere $\widehat{S}_j$ has the null holomorphic map $\widehat{F}_j$ given by

$$\widehat{F}_j(z) = \exp((z - p^0_{ji})\widehat{A}_j), \tag{34}$$

$$\widehat{A}_j = \begin{pmatrix} 0 & 0 \\ \widehat{\lambda}_j & 0 \end{pmatrix}, \qquad \widehat{\lambda}_j = \rho^{-2}\lambda_j(c_j - c_i), \tag{35}$$

and recalling that the mean curvature vector of $\widehat{S}_j$ at $(0,0,1)$ points down,

$$\widehat{M_j}(s) = \exp(-p^0_{ji}\widehat{A}_j)\Xi(-\xi_j s). \tag{36}$$

4.6. **Partial derivatives of the matrix $\widehat{A}$.** We return to the matrix $\widehat{A}_{\mathbf{a},\mathbf{p}}$ given by (28) with $G = G_{\mathbf{t}(\mathbf{a})}$ and $\Omega = \Omega_{\mathbf{t}(\mathbf{a}),\mathbf{a}}$. Taking the derivative of (28) and using (12), (14), (21) and (22), we obtain the following result after simplification:

Proposition 6. *At $\mathbf{a} = 0$, we have:*

$$\widehat{A}_{0,\mathbf{p}} = \widehat{A}_i\, dz \qquad \frac{\partial \widehat{A}_{\mathbf{a},\mathbf{p}}}{\partial a_{ij}} = \frac{c_i - c_j}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dz}{z - p_{ij}} \qquad \text{in } \mathbb{C}_i, \tag{37}$$

$$\widehat{A}_{0,\mathbf{p}} = \widehat{A}_j\, dz \qquad \frac{\partial \widehat{A}_{\mathbf{a},\mathbf{p}}}{\partial a_{ij}} = \frac{c_j - c_i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dz}{z - p_{ji}} \qquad \text{in } \mathbb{C}_j, \tag{38}$$

$$\widehat{A}_{0,\mathbf{p}} = 0 \qquad \frac{\partial \widehat{A}_{\mathbf{a},\mathbf{p}}}{\partial a_{ij}} = \frac{c_j - c_i}{2} \begin{pmatrix} z & -\rho^2 \\ \rho^{-2} z^2 & -z \end{pmatrix} \frac{dz}{z^2} \qquad \text{in } \overline{\mathbb{C}}_{ij}, \tag{39}$$

where the matrices $\widehat{A}_i$ and $\widehat{A}_j$ are given by (31) *and* (35).

4.7. **Expansion of $\widehat{\Pi}(\gamma_{ij})$.**

Proposition 7. *We have the following expansion:*

$$\widehat{\Pi}_{\mathbf{a},\mathbf{p}}(\gamma_{ij}) = I_2 + a_{ij}\, \pi \mathrm{i}(c_i - c_j) \exp(-p_{ij}\widehat{A}_i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \exp(p_{ij}\widehat{A}_i) + O(||\mathbf{a}||^2). \tag{40}$$

Proof. When $\mathbf{a} = 0$, the principal solution in $\mathbb{C}_i$ is given by $\widehat{\Pi}_{0,\mathbf{p}}(z, 0_i) = \exp(z\widehat{A}_i)$, which is well defined, so $\widehat{\Pi}_{0,\mathbf{p}}(\gamma_{ij}) = I_2$. By Proposition 9 in Appendix A, (37) and the residue theorem,

$$\begin{aligned} \frac{\partial \widehat{\Pi}_{\mathbf{a},\mathbf{p}}(\gamma_{ij})}{\partial a_{ij}} &= \frac{c_i - c_j}{2} \int_{\gamma_{ij}} \exp(-z\widehat{A}_i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \exp(z\widehat{A}_i) \frac{dz}{z - p_{ij}} \\ &= \pi \mathrm{i}(c_i - c_j) \exp(-p_{ij}\widehat{A}_i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \exp(p_{ij}\widehat{A}_i). \end{aligned}$$

For $(k, l) \neq (i, j)$, the partial derivative of $\widehat{A}$ with respect to $a_{k\ell}$ is holomorphic at p_{ij}. Hence the partial derivative of $\widehat{\Pi}(\gamma_{ij})$ with respect to $a_{k\ell}$ is zero. Equation (40) follows. ☐

4.8. **Expansion of $\widehat{\Pi}(\Gamma_{ji})$.** Recall that exp is a local diffeomorphism from a neighborhood of 0 in $\mathcal{M}_2(\mathbb{C})$ to a neighborhood of I_2 in $GL(2, \mathbb{C})$. We denote the inverse diffeomorphism by log. Also, for a matrix M and a complex number λ, we define $M^\lambda = \exp(\lambda \log M)$.

Proposition 8. *We have*

$$\widehat{\Pi}_{\mathbf{a},\mathbf{p}}(\Gamma_{ji}) \times \widehat{\Pi}_{\mathbf{a},\mathbf{p}}(\gamma_{ij})^{\frac{-2}{2\pi \mathrm{i}} \log a_{ij}} = \exp(-p_{ji}\widehat{A}_j) \exp(p_{ij}\widehat{A}_i) + \mathcal{O}(\mathbf{a}), \tag{41}$$

where $\mathcal{O}(\mathbf{a})$ is a well-defined holomorphic function of $(\mathbf{a}, \mathbf{p})$ which vanishes at $\mathbf{a} = 0$.

Proof. Let us fix the value of all parameters except a_{ij}. We must first see that the left-hand side of (41) is a well-defined function of a_{ij} for $a_{ij} \neq 0$, which of course means that $\widehat{\Pi}(\Gamma_{ji})$ itself is not. To see this, we have to define precisely the path

Γ_{ji} from 0_i to 0_j. For t_{ij} and t_{ji} non-zero, we define Γ_{ji} as the composition of the following two paths:

- The following path from 0_i to $p_{ij}+t_{ij}$ in $\mathbb{C}_i$: a path from 0_i to $p_{ij}+1$ in $\mathbb{C}_i$ minus all unit disks around the nodes (depending continuously on the parameter p_{ij} in a neighborhood of p_{ij}^0), composed with the spiral parametrized by $x \mapsto p_{ij}+(t_{ij})^x$ for $x \in [0,1]$.
- The following path from $p_{ji}+t_{ji}$ to 0_j in $\mathbb{C}_j$: the spiral parametrized by $x \mapsto p_{ji}+(t_{ji})^{1-x}$ for $x \in [0,1]$, composed with a path from $p_{ji}+1$ to 0_j.

We can compose these two paths because the points $p_{ij}+t_{ij}$ and $p_{ji}+t_{ji}$ are both identified with 1_{ij} when opening nodes. Observe that we need a determination of the arguments of t_{ij} and t_{ji} to define the spirals. In other words, if we take t_{ij} and t_{ji} to live in the universal cover of the punctured unit disk (so $\arg t_{ij}$ and $\arg t_{ji}$ are well defined), Γ_{ji} depends continuously on t_{ij} and t_{ji}. Now if the argument of a_{ij} is increased by 2π, then the arguments of t_{ij} and t_{ji} are increased by the same amount. Hence the homotopy class of Γ_{ji} is multiplied on the right by $(\gamma_{ij})^2$ and $\widehat{\Pi}(\Gamma_{ji})$ is multiplied on the right by $\widehat{\Pi}(\gamma_{ij})^2$. Consequently, the left-hand side of (41) is unchanged, so it is a well-defined holomorphic function of a_{ij} for $a_{ij} \neq 0$.

Next we prove that the left-hand side of (41) is uniformly bounded. Because it is a well-defined function of a_{ij}, we can assume that $\arg a_{ij} \in [-2\pi, 2\pi]$. Using (6), we write

$$\widehat{\Pi}(\Gamma_{ji}) = \widehat{\Pi}(0_j, p_{ji}+1)\widehat{\Pi}(p_{ji}+1, p_{ji}+t_{ji})\widehat{\Pi}(p_{ij}+t_{ij}, p_{ij}+1)\widehat{\Pi}(p_{ij}+1, 0_i). \tag{42}$$

Since the path from 0_i to $p_{ij}+1$ stays in a fixed compact set of $\mathbb{C}_i$ minus the nodes, where $\widehat{A}$ is uniformly bounded, the fourth factor in (42) is uniformly bounded. We estimate the third factor using Proposition 10 from Appendix B. For this, we need an integral estimate of $||\widehat{A}||$ on the circle of center p_{ij} and radius $|t_{ij}|/2$.

We claim that $\widehat{A} = O(a_{ij})$ in compact subsets of $\overline{\mathbb{C}}_{ij} \setminus \{0, \infty\}$ (even if the other parameters $a_{k\ell}$ are non-zero). Indeed, $\widehat{A}$ depends holomorphically on a_{ij}, and if $a_{ij}=0$, then by (15), $t_{ij}=t_{ji}=0$, so $\Omega = 0$ and $\widehat{A}=0$ in $\overline{\mathbb{C}}_{ij}$. Also, by (16), $a_{ij} = O(t_{ij})$. Consequently, since $\widehat{A}$ is a matrix-valued 1-form,

$$\int_{|z-p_{ij}|=\frac{|t_{ij}|}{2}} ||\widehat{A}|| = \int_{|v_{ij}|=\frac{|t_{ij}|}{2}} ||\widehat{A}|| = \int_{|w_{ij}|=2} ||\widehat{A}|| \leq C|t_{ij}|$$

for some uniform constant C. By Proposition 10 in Appendix B, $\widehat{\Pi}(p_{ij}+t_{ij}, p_{ij}+1)$ is uniformly bounded. The first and second factors in (42) are estimated in the exact same way. We conclude that $\widehat{\Pi}(\Gamma_{ji})$ is uniformly bounded (although not well defined – but we assumed that $\arg a_{ij} \in [-2\pi, 2\pi]$). The left-hand side of (41) is now a bounded, well-defined holomorphic function of $(\mathbf{a}, \mathbf{p})$ on the set $a_{ij} \neq 0$. By the Riemann extension theorem (in several variables), it extends holomorphically at $a_{ij}=0$.

To compute its value at $\mathbf{a}=0$, assume that all parameters $a_{k\ell}$ for $(k,\ell) \neq (i,j)$ are zero. By (37), $\widehat{A}_{\mathbf{a},\mathbf{p}} - \widehat{A}_i = O(a_{ij})$ in compact subsets of $\mathbb{C}_i$ minus the nodes. By point (2) of Proposition 10 (with $\widetilde{A} = \widehat{A}_i$), we obtain

$$||\widehat{\Pi}_{\mathbf{a},\mathbf{p}}(p_{ij}+t_{ij}, 0_i) - \widehat{\Pi}_i(p_{ij}+t_{ij}, 0_i)|| \leq C\,|a_{ij}\log|a_{ij}||\,,$$

where $\widehat{\Pi}_i$ is the principal solution of $Y' = \widehat{A}_i Y$ in $\overline{\mathbb{C}}_i$, namely $\widehat{\Pi}_i(z, 0_i) = \exp(z\widehat{A}_i)$. This gives

$$\lim_{a_{ij}\to 0} \widehat{\Pi}_{\mathbf{a},\mathbf{p}}(p_{ij} + t_{ij}, 0_i) = \exp(p_{ij}\widehat{A}_i).$$

Arguing in the same way, we obtain

$$\lim_{a_{ij}\to 0} \widehat{\Pi}_{\mathbf{a},\mathbf{p}}(0_j, p_{ji} + t_{ji}) = \exp(-p_{ji}\widehat{A}_j).$$

Proposition 8 follows. $\square$

4.9. **Solution of the monodromy problem.** We are now ready to prove Proposition 5. The unitary group $SU(2)$ is not a complex manifold, so we have to leave the realm of holomorphic functions. We introduce a small positive real number τ and have in mind to apply the implicit function theorem at $\tau = 0$. We write

$$a_{ij} = \tau \frac{b_{ij}}{c_i - c_j},$$

where b_{ij} is a complex number in a neighborhood of a non-zero central value b_{ij}^0. The computation will be simplified by knowing a priori the order of each parameter as a function of τ. The correct orders are

$$s = -\tau \log \tau,$$
$$p_{ij} = p_{ij}^0 + s q_{ij},$$

where q_{ij} is a complex parameter in a neighborhood of 0. One issue here is that the function $\tau \mapsto \tau \log \tau$ does not extend as a differentiable function at $\tau = 0$. We solve this problem by writing $\tau = e^{-1/t^2}$, where t is a real parameter in a neighborhood of 0. Both τ and $\tau \log \tau$ extend smoothly at $t = 0$, and all parameters are smooth functions of t. Let $\mathbf{b} = (b_{ij})_{(i,j)\in I}$ and $\mathbf{q} = (q_{ij}, q_{ji})_{(i,j)\in I}$.

Recall that exp maps the Lie algebras $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{su}(2,\mathbb{C})$ to the Lie groups $SL(2,\mathbb{C})$ and $SU(2,\mathbb{C})$, respectively. We define

$$P_{ij} = P_{ij}(t, \mathbf{b}, \mathbf{q}) = \log\left(\widehat{M}_i(s)^{-1}\widehat{\Pi}_{\mathbf{a},\mathbf{p}}(\gamma_{ij})\widehat{M}_i(s)\right) \in \mathfrak{sl}(2,\mathbb{C}),$$

$$Q_{ij} = Q_{ij}(t, \mathbf{b}, \mathbf{q}) = \log\left(\widehat{M}_j(s)^{-1}\widehat{\Pi}_{\mathbf{a},\mathbf{p}}(\Gamma_{ji})\widehat{M}_i(s)\right) \in \mathfrak{sl}(2,\mathbb{C}).$$

We want to solve $P_{ij} \in \mathfrak{su}(2,\mathbb{C})$ and $Q_{ij} \in \mathfrak{su}(2,\mathbb{C})$. We compute P_{ij} using Proposition 7:

$$\widehat{\Pi}(\gamma_{ij}) = I_2 + \pi\mathrm{i}\tau b_{ij}\exp(-p_{ij}\widehat{A}_i)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\exp(p_{ij}\widehat{A}_i) + O(\tau^2) \quad \text{using (40)}, \tag{43}$$

$$P_{ij} = \pi\mathrm{i}\tau b_{ij}\,\Xi(-\xi_i s)\exp(-sq_{ij}\widehat{A}_i)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\exp(sq_{ij}\widehat{A}_i)\Xi(\xi_i s) + O(\tau^2) \quad \text{using (33)},$$

$$P_{ij} = \pi\mathrm{i}\tau b_{ij}\begin{pmatrix} 1 & 2\widehat{\lambda}_i s q_{ij} \\ 0 & -1 \end{pmatrix} + O(\tau^2). \tag{44}$$

We compute Q_{ij} using Proposition 8:

$$\widehat{\Pi}(\Gamma_{ji}) = \exp(-p_{ji}\widehat{A}_j)\exp(p_{ij}\widehat{A}_i)\widehat{\Pi}(\gamma_{ij})^{\frac{2}{2\pi\mathrm{i}}\log a_{ij}} + O(\tau) \quad \text{using (41)},$$

$$\widehat{\Pi}(\gamma_{ij})^{\frac{2}{2\pi\mathrm{i}}\log a_{ij}} = I_2 - s b_{ij}\exp(-p_{ij}\widehat{A}_i)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\exp(p_{ij}\widehat{A}_i) + O(\tau) \quad \text{using (43)},$$

$$\begin{aligned}\widehat{M}_j(s)^{-1}\widehat{\Pi}(\Gamma_{ji})\widehat{M}_i(s) =& \Xi(\xi_j s)\exp(-sq_{ji}\widehat{A}_j)\\ &\times\begin{pmatrix} 1-sb_{ij} & 0\\ 0 & 1+sb_{ij}\end{pmatrix}\exp(sq_{ij}\widehat{A}_i)\Xi(\xi_i s)+O(\tau),\end{aligned}$$

$$Q_{ij} = s\begin{pmatrix}\frac{\xi_i+\xi_j}{2}-b_{ij} & \widehat{\lambda}_i q_{ij}\\ -\widehat{\lambda}_j q_{ji} & b_{ij}-\frac{\xi_i+\xi_j}{2}\end{pmatrix}+O(\tau). \tag{45}$$

For a matrix $M=(M^{k\ell})_{1\le k,\ell\le 2}\in\mathfrak{sl}(2,\mathbb{C})$, we have

$$M\in\mathfrak{su}(2,\mathbb{C})\quad\Leftrightarrow\quad \mathrm{Re}(M^{11})=0\quad\text{and}\quad M^{12}+\overline{M^{21}}=0.$$

Define the function $\mathcal{F}=(\mathcal{F}_{ij})_{(i,j)\in I}$ for $t\neq 0$ by

$$\mathcal{F}_{ij}(t,\mathbf{b},\mathbf{q})=\left(\frac{1}{\tau}\mathrm{Re}(P_{ij}^{11})\ ,\ \frac{1}{\tau s}(P_{ij}^{12}+\overline{P_{ij}^{21}})\ ,\ \frac{1}{s}\mathrm{Re}(Q_{ij}^{11})\ ,\ \frac{1}{s}(Q_{ij}^{12}+\overline{Q_{ij}^{21}})\right).$$

We want to solve $\mathcal{F}(t,\mathbf{b},\mathbf{q})=0$. By (44) and (45), $\mathcal{F}$ extends smoothly at $t=0$, with

$$\mathcal{F}_{ij}(0,\mathbf{b},\mathbf{q})=\left(-\pi\mathrm{Im}(b_{ij})\ ,\ 2\pi\mathrm{i}\, b_{ij}\widehat{\lambda}_i q_{ij}\ ,\ \tfrac{\xi_i+\xi_j}{2}-\mathrm{Re}(b_{ij})\ ,\ \widehat{\lambda}_i q_{ij}-\overline{\widehat{\lambda}_j q_{ji}}\right).$$

Taking $b_{ij}^0=\frac{\xi_i+\xi_j}{2}>0$, we have $\mathcal{F}(0,\mathbf{b}^0,0)=0$. It is straightforward that the partial differential of $\mathcal{F}$ with respect to the variables $(\mathbf{b},\mathbf{q})$ at $(0,\mathbf{b}^0,0)$ is an isomorphism. By the implicit function theorem, for t in a neighborhood of 0, there exists $(\mathbf{b}(t),\mathbf{q}(t))$ depending smoothly on t such that $\mathcal{F}(t,\mathbf{b}(t),\mathbf{q}(t))=0$. Proposition 5 is proved, and the monodromy problem is solved. $\square$

5. Embeddedness

Here is what we have achieved so far. For each small enough value of the parameter $t>0$, we have constructed a null holomorphic map F which has $SU(2)$-valued monodromy. All parameters are now smooth functions of $t>0$. To ease notation, the dependence on t will not be written. Let $f:\Sigma\to\mathbb{H}^3$ be the CMC-1 immersion associated to F. It remains to prove that $f(\Sigma)$ is embedded. We work in the half-space model, so $f(z)=\Phi(F(z)F(z)^*)$ is given by (4). Fix a small number $\varepsilon>0$. We consider the following disjoint domains in Σ:

$$\mathbb{C}_i^\varepsilon=\{z\in\mathbb{C}_i\ :\ \forall j\in J_i,\ |z-p_{ij}^0|>\varepsilon\},$$

$$\overline{\mathbb{C}}_{ij}^\varepsilon=\{z\in\overline{\mathbb{C}}_{ij}\ :\ \varepsilon<|z|<\frac{1}{\varepsilon}\}.$$

The complement of these domains in Σ are annuli, which we call transition regions. We also fix some large number R and define $\mathbb{C}_i^{\varepsilon,R}=\mathbb{C}_i\cap D(0,R)$.

5.1. Geometry of the image of $\mathbb{C}_i^{\varepsilon,R}$.

Fix some i, $1\le i\le n$, and consider an isometry h such that $h(S_i)$ is the horosphere $x_3=1$ and h maps $f_i(0)$ to the point $(0,0,1)$. The isometry h is represented by a matrix $H\in SL(2,\mathbb{C})$ which has the form

$$H=\frac{1}{\sqrt{c-c_i}}\begin{pmatrix}\rho & -\rho c\\ \rho^{-1} & -\rho^{-1}c_i\end{pmatrix},$$

where ρ,c are some complex numbers (ρ not the same as in Section 4.4). As in Section 4.4, we use hats to denote the action of h, so $\widehat{f}=h\circ f$, $\widehat{F}=HF$, and so on. Equations (28) and (31) hold true, with c in place of c_j. By construction, $\widehat{f}(0_i)$ parametrizes the vertical axis at speed ξ_i as s varies, so $\widehat{F}(0_i)=\Xi(\xi_i s)$, up to right multiplication by $SU(2)$. We have $\widehat{A} = \widehat{A}_i + O(\tau)$. Since $\mathbb{C}_i^{\varepsilon,R}$ is a fixed compact domain,

$$\widehat{F}(z) = \exp(z\widehat{A}_i)\Xi(\xi_i s) + O(\tau) \qquad \text{for } z \in \mathbb{C}_i^{\varepsilon,R}.$$

From this, we conclude that $\widehat{f}(\mathbb{C}_i^{\varepsilon,R})$ converges smoothly to (a subdomain of) the horosphere $x_3 = 1$ as $t \to 0$. Moreover, from (4), we get

$$x_3(z) = e^{\xi_i s} + O(\tau) \qquad \text{for } z \in \mathbb{C}_i^{\varepsilon,R},$$

so for t small enough, the image of $\mathbb{C}_i^{\varepsilon,R}$ lies above the horosphere $x_3 = 1$.

5.2. **Geometry of the end at ∞_i.** Next we prove that the image of $|z| > R$ in $\mathbb{C}_i$ is embedded. We claim that for $t > 0$ small enough, the Gauss map G has multiplicity 1 at ∞_i. This is delicate because G is constant when $t = 0$. We work in the local coordinate $w = \frac{1}{z}$ in a neighborhood of ∞_i and write $\widetilde{G}(w) = G(1/w)$. From (21), we obtain

$$\frac{\partial \widetilde{G}'(w)}{\partial a_{ij}} = \frac{c_j - c_i}{2\lambda_i}\frac{1}{(1-p_{ij}w)^2} \qquad \text{for } j \in J_i^+,$$

$$\frac{\partial \widetilde{G}'(w)}{\partial a_{ji}} = \frac{c_i - c_j}{2\lambda_i}\frac{1}{(1-p_{ji}w)^2} \qquad \text{for } j \in J_i^-,$$

$$\begin{aligned}
\widetilde{G}'(0) &= \sum_{j\in J_i^+} \frac{\partial \widetilde{G}'(0)}{\partial a_{ij}} a_{ij} + \sum_{j\in J_i^-} \frac{\partial \widetilde{G}'(0)}{\partial a_{ji}} a_{ji} + O(||\mathbf{a}||^2) \\
&= \sum_{j\in J_i^+} \frac{c_j - c_i}{2\lambda_i}\frac{\tau b_{ij}}{c_i - c_j} + \sum_{j\in J_i^-} \frac{c_i - c_j}{2\lambda_i}\frac{\tau b_{ji}}{c_j - c_i} + O(\tau^2) \\
&= -\frac{\tau\zeta_i}{2\lambda_i} + o(\tau), \qquad \text{where} \qquad \zeta_i = \frac{1}{2}\sum_{j\in J_i}(\xi_i + \xi_j) > 0.
\end{aligned}$$

Hence for $t > 0$ small enough, $\widetilde{G}'(0) \neq 0$, so the Gauss map has multiplicity one at the end. To study the geometry of the end, we again consider the isometry h introduced in Section 5.1. Then $\widehat{G} = H \cdot \widetilde{G}$ has a simple pole at $w = 0$ with residue

$$\mathrm{Res}_{w=0}\, \widehat{G} = \mathrm{Res}_{w=0}\, \rho^2 \frac{\widetilde{G} - c}{\widetilde{G} - c_i} = \rho^2 \frac{c_i - c}{\widetilde{G}'(0)} \simeq \frac{-2\widehat{\lambda}_i}{\tau\zeta_i},$$

where $\widehat{\lambda}_i = \rho^2\lambda_i(c_i - c)$. From (28) we obtain

$$\widehat{G}^2\widehat{\Omega} = -\widehat{A}_{21} \simeq -\widehat{\lambda}_i dz = \widehat{\lambda}_i \frac{dw}{w^2}.$$

By Theorem 6 in Appendix C (with $\alpha = \frac{-2\widehat{\lambda}_i}{\tau\zeta_i}$ and $\alpha^2\beta = \widehat{\lambda}_i$), there exists a uniform positive ϵ (independent of t) such that the image of $0 < |w| < \epsilon$ is the vertical graph $x_3 = u(x_1, x_2)$ of a function u. Moreover, at infinity we have

$$\log u(x_1, x_2) \simeq (\tau\zeta_i + o(\tau)) \log \sqrt{x_1^2 + x_2^2},$$

so $x_3 > 1$ on the end. Replacing R by ϵ^{-1} if necessary, we obtain that $\widehat{f}(\mathbb{C}_i^{\varepsilon})$ is embedded and moreover lies above the horosphere $x_3 = 1$ (using the maximum principle). In other words, $f(\mathbb{C}_i^{\varepsilon})$ lies on the mean-convex side of the horosphere S_i. This guarantees that the images $f(\mathbb{C}_i^{\varepsilon})$ for $1 \leq i \leq n$ are disjoint.

5.3. **Geometry of the catenoidal necks.** Fix a couple $(i,j)\in I$. Again consider the isometry h introduced in Section 4.4, which maps the horosphere S_i to the horosphere $x_3=1$ and the horosphere S_j to the sphere of radius $\frac{1}{2}$ centered at $(0,0,\frac{1}{2})$. In this section, we prove that after a blowup of ratio $1/\tau$, the image $\widehat{f}(\overline{\mathbb{C}}_{ij}^{\varepsilon})$ converges to a vertical catenoid.

By computations similar to the computation of Q_{ij} in Section 4.9, we have

$$\widehat{F}(1_{ij})=\widehat{\Pi}(p_{ij}+t_{ij},O_i)\widehat{F}(0_i)=I_2+\frac{s}{2}\begin{pmatrix}\xi_i-b_{ij} & 2\widehat{\lambda}_i q_{ij}\\ 0 & b_{ij}-\xi_i\end{pmatrix}+O(\tau). \tag{46}$$

By (39), we have in $\overline{\mathbb{C}}_{ij}^{\varepsilon}$

$$\widehat{A}(z)=\tau\widetilde{A}(z)+O(\tau^2)\quad\text{with}\quad \widetilde{A}(z)=-\frac{b_{ij}}{2}\begin{pmatrix} z & -\rho^2\\ \rho^{-2}z^2 & -z\end{pmatrix}\frac{dz}{z^2}. \tag{47}$$

Let

$$\widetilde{F}(z)=\frac{1}{\tau}(\widehat{F}(z)-\widehat{F}(1_{ij})).$$

Then using (46) and (47),

$$d\widetilde{F}(z)=\frac{1}{\tau}\widehat{A}(z)\widehat{F}(z)=(\widetilde{A}(z)+O(\tau))(I_2+O(s))=\widetilde{A}(z)+O(s).$$

Since $\overline{\mathbb{C}}_{ij}^{\varepsilon}$ is a fixed compact set, we obtain by integration

$$\widetilde{F}(z)=-\frac{b_{ij}}{2}\begin{pmatrix}\log z & \rho^2(z^{-1}-1)\\ \rho^{-2}(z-1) & -\log z\end{pmatrix}+O(s).$$

Write $\widehat{x}_k(z)=\widehat{x}_k(1_{ij})+\tau\widetilde{x}_k(z)$ for $1\le k\le 3$. Using (4) and $\widehat{F}(z)=I_2+O(s)$, we obtain

$$(\widetilde{x}_1(z)+\mathrm{i}\widetilde{x}_2(z),\widetilde{x}_3(z))=\left(\widetilde{F}_{12}(z)+\overline{\widetilde{F}_{21}(z)},-2\,\mathrm{Re}\left(\widetilde{F}_{22}(z)\right)\right)+O(s),$$

$$\lim_{t\to 0}(\widetilde{x}_1(z)+\mathrm{i}\widetilde{x}_2(z),\widetilde{x}_3(z))=-\frac{\xi_i+\xi_j}{4}\left(\rho^2(z^{-1}-1)+\overline{\rho^{-2}(z-1)},2\log|z|\right).$$

This is the parametrization of a vertical catenoid of necksize $\frac{\xi_i+\xi_j}{2}$. This means that after a blowup of ratio $\frac{1}{\tau}$ at $\widehat{f}(1_{ij})$, the image of $\overline{\mathbb{C}}_{ij}^{\varepsilon}$ converges smoothly to a catenoid. Also observe that the image of the circle $|z|=\varepsilon$ lies above the image of $|z|=\frac{1}{\varepsilon}$. Finally, (46) gives

$$\widehat{x}_3(1_{ij})=1+s\frac{\xi_i-\xi_j}{2}+O(\tau).$$

Hence the catenoidal neck lies below the image of $\mathbb{C}_i^{\varepsilon}$.

5.4. **Geometry of the transition regions.** Fix $(i,j)\in I$ and let U_{ij} be the annulus in Σ bounded by the circles $|z-p_{ij}|=\varepsilon$ in $\mathbb{C}_i$ and $|z|=\varepsilon$ in $\overline{\mathbb{C}}_{ij}$. We again consider the isometry h introduced in Section 4.4. Let us prove that the mean curvature vector of $\widehat{f}$ is almost vertical in U_{ij}. Given the geometric interpretation of the Gauss map given in Section 2.1.4, an elementary computation shows that the angle $\theta(z)$ between the mean curvature vector at $f(z)$ and the vertical axis is related to the Gauss map $G(z)$ by

$$\frac{\sin\theta(z)}{1+\cos\theta(z)}=\frac{x_3(z)}{|G(z)-x_1(z)-\mathrm{i}x_2(z)|}. \tag{48}$$

The function $\widehat{G}^{-1}$ is holomorphic in the annulus U_{ij} and is bounded by $C\varepsilon$ on the boundary circles for some uniform constant C. By the maximum principle, $\widehat{G}^{-1}$ is bounded by $C\varepsilon$ in U_{ij}. The norm of the holomorphic map $F(z) - I_2$ is bounded by $C\varepsilon$ on the boundary of U_{ij}, and so is bounded by $C\varepsilon$ in U_{ij} by the maximum principle. Hence the function $\widehat{x}_1 + \mathrm{i}\widehat{x}_2$ is uniformly bounded in U_{ij}, and the height $\widehat{x}_3$ satisfies $|\widehat{x}_3 - 1| \leq C\varepsilon$ in U_{ij}. Using (48), we obtain

$$\frac{\sin\widehat{\theta}(z)}{1+\cos\widehat{\theta}(z)} \leq C\varepsilon \quad \text{in } U_{ij}.$$

Hence by choosing ε small enough, we can ensure that $\widehat{\theta}(z) < \frac{\pi}{2}$. This implies that $\widehat{f}(U_{ij})$ is locally a vertical graph. Since we have already seen that it is a graph on the boundary circles, it is globally a graph. Moreover, by the maximum principle, it lies above the lowest point of the top boundary component of the catenoidal neck $\widehat{f}(\overline{\mathbb{C}}_{ij}^{\varepsilon})$. The image of the annulus bounded by the circles $|z - p_{ji}| = \varepsilon$ in $\mathbb{C}_j$ and $|z| = \frac{1}{\varepsilon}$ in $\overline{\mathbb{C}}_{ij}$ is studied in the same way, using an isometry which maps the horosphere S_j to the horosphere $x_3 = 1$. This proves that $f(\Sigma)$ is embedded and concludes the proof of Theorem 1. □

Appendix A. Derivative of the monodromy

Consider a domain $\Omega \subset \mathbb{C}$ and a point $z_0 \in \Omega$. Let $A_\lambda(z) \in GL(n,\mathbb{C})$ be a family of matrices depending holomorphically on (λ, z) for $z \in \Omega$ and λ in a neighborhood of 0. Let Π_λ denote the principal solution of $Y' = A_\lambda Y$ in Ω.

Proposition 9. *For any* $\gamma \in \pi_1(\Omega, z_0)$*,*

$$\frac{\partial \Pi_\lambda(\gamma)}{\partial\lambda}|_{\lambda=0} = \Pi_0(\gamma)\int_\gamma \Pi_0(z,z_0)^{-1}\frac{\partial A_\lambda(z)}{\partial\lambda}\Pi_0(z,z_0)\,dz.$$

Proof. Let $Y_\lambda(z) = \Pi_\lambda(z, z_0)$ and $W = \partial Y_\lambda/\partial\lambda$. Differentiating $Y_\lambda' = A_\lambda Y_\lambda$ and $Y_\lambda(z_0) = I_n$ with respect to λ at $\lambda = 0$, we get $W(z_0) = 0$ and

$$W' = A_0 W + \frac{\partial A_\lambda}{\partial\lambda}Y_0.$$

By the variation of constants formula (Theorem 3.12 in [15]),

$$W(z) = Y_0(z)\int_{z_0}^{z} Y_0(w)^{-1}\frac{\partial A_\lambda(w)}{\partial\lambda}Y_0(w)\,dw.$$

Taking $z = \gamma(1)$, the result follows. □

Appendix B. Uniform estimates of the solution of $Y' = AY$ in an annulus

In this section, we consider the annulus $\Omega \subset \mathbb{C}$ defined by $\rho^{-1}t < |z| < \rho$, where $\rho > 1$ is some fixed number and t is a small positive parameter. We are aiming for estimates which are uniform with respect to t. Let $A : \Omega \to \mathfrak{sl}(n,\mathbb{C})$ be a holomorphic map. Let $Y(z) \in SL(n,\mathbb{C})$ be the solution of $Y' = AY$ in Ω, with initial condition $Y(1) = I_n$. (Of course, $Y(z)$ is only well defined in the universal cover of Ω: the value of $Y(z)$ depends on the determination of $\arg z$.)

Proposition 10. (1) *Assume that for some constant c,*

$$\int_{|z|=\rho} ||A|| \leq c \quad \text{and} \quad \int_{|z|=\rho^{-1}t} ||A|| \leq ct.$$

Then for $t \leq |z| \leq 1$ and $|\arg z| \leq c'$, $||Y(z)||$ is bounded by a constant depending only on c, c' and ρ.

(2) *Let $\widetilde{A}$ be another matrix-valued map satisfying the same hypotheses as A and let $\widetilde{Y}(z)$ be the solution of $\widetilde{Y}' = \widetilde{A}\widetilde{Y}$ with initial condition $\widetilde{Y}(1) = I_n$. Assume moreover that*

$$\int_{|z|=\rho} ||A - \widetilde{A}|| \leq ct.$$

Then for $t \leq |z| \leq 1$ and $|\arg(z)| \leq c'$,

$$||Y(z) - \widetilde{Y}(z)|| \leq Ct|\log t|$$

for some constant C depending only on c, c' and ρ.

Proof. We use the letter C for uniform constants, depending only on c and ρ but not on t. By the Cauchy theorem,

$$A(z) = \frac{1}{2\pi \mathrm{i}} \int_{|w|=\rho} \frac{A(w)}{w - z} dw - \frac{1}{2\pi \mathrm{i}} \int_{|w|=\rho^{-1}t} \frac{A(w)}{w - z} dw.$$

Hence for $t \leq |z| \leq 1$,

$$||A(z)|| \leq \frac{1}{2\pi} \left(\frac{c}{\rho - 1} + \frac{ct}{t(1 - \rho^{-1})} \right) \leq C.$$

We can connect 1 and z (in the universal cover of Ω) by a path of length less than $1 + c'$. The first point of Proposition 10 follows from Gromwall inequality (Lemma 2.7 in [15]). Using the Cauchy formula in the same way, we obtain for $t \leq |z| \leq 1$

$$||A(z) - \widetilde{A}(z)|| \leq \frac{1}{2\pi} \left(\frac{ct}{\rho - 1} + \frac{ct}{|z| - \rho^{-1}t} \right) \leq \frac{Ct}{|z|}.$$

By the variation of constants formula (Theorem 3.12 in [15]),

$$\widetilde{Y}(z) = Y(z) + Y(z) \int_1^z Y(w)^{-1} (\widetilde{A}(w) - A(w)) \widetilde{Y}(w) dw.$$

This gives

$$||\widetilde{Y}(z) - Y(z)|| \leq Ct \int_1^z \frac{|dw|}{|w|} \leq Ct \left(|\arg z| + |\log t| \right).$$

□

Appendix C. Embedded CMC-1 ends

Theorem 6. *Let $f : D^*(0,1) \to \mathbb{H}^3$ be a conformal, CMC-1 immersion of the punctured closed unit disk. Assume that the Gauss map G has a simple pole at 0, with residue α, and the holomorphic differential Ω is holomorphic at 0, with $\Omega(0) = \beta dz$. Assume that $0 < |\alpha\beta| \leq \frac{1}{8}$. Then there exists $\varepsilon > 0$ such that in the half-space model, $f(D^*(0, \varepsilon))$ is a vertical graph $x_3 = u(x_1, x_2)$ of a (positive) function u over an exterior domain in the plane. The number ε only depends on* *a bound on* $|\alpha^2\beta|^{\pm 1}$ *and* $||F(z)||$ *on the unit circle. Moreover,* $\alpha\beta$ *is real, and at infinity, the function* u *has the following asymptotic behavior:*

$$\log u(x_1, x_2) \simeq (1 - \sqrt{1 + 4\alpha\beta}) \log |x_1 + \mathrm{i}x_2|.$$

Remark 3. In this paper, we are interested in the case where $\alpha\beta \to 0$ and we need a uniform positive ε. The conclusions of Theorem 6 remain true without the hypothesis $|\alpha\beta| \leq \frac{1}{8}$ but the proof is more involved, as the fuchsian system can be resonant. In particular, one can prove that $\alpha\beta$ is always a real number in $(-\frac{1}{4}, \infty)$.

Proof. We use the theory of fuchsian systems to compute $F(z)$ such that $f = FF^*$ in the punctured disk. The system $F' = AF$ is fuchsian provided the matrix $A(z)$ has a simple pole at 0, which is not the case here (it has a double pole). To circumvent this problem, we introduce the matrix

$$N(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \tag{49}$$

and make the change of unknown $F(z) = N(z)\widetilde{F}(z)$. By a straightforward computation, (2) is equivalent to

$$\widetilde{F}'(z) = \widetilde{A}(z)\widetilde{F}(z), \tag{50}$$

where

$$\widetilde{A} = \begin{pmatrix} G\omega & -zG^2\omega \\ z^{-1}\omega & -G\omega - z^{-1} \end{pmatrix}.$$

Now the matrix $\widetilde{A}$ has a simple pole at 0, with residue

$$A_0 = \operatorname{Res}_0 \widetilde{A} = \begin{pmatrix} \alpha\beta & -\alpha^2\beta \\ \beta & -\alpha\beta - 1 \end{pmatrix}$$

and the system (50) is fuchsian. The eigenvalues of A_0 are

$$\lambda_1 = \frac{-1 + \sqrt{\Delta}}{2}, \quad \lambda_2 = \frac{-1 - \sqrt{\Delta}}{2}, \qquad \text{where } \Delta = 1 + 4\alpha\beta.$$

The system (50) is called resonant if $\lambda_1 - \lambda_2 = \sqrt{\Delta}$ is a non-zero integer. It follows from our hypothesis that $\frac{1}{2} \leq |\Delta| \leq \frac{3}{2}$ and $\Delta \neq 1$, so the system is non-resonant. By the standard theory of fuchsian systems (Proposition 11.2 in [14]), the solution of (50) has the form

$$\widetilde{F}(z) = U(z) z^{A_0} Y_0, \tag{51}$$

where $U(z) \in GL(2, \mathbb{C})$ is well defined, holomorphic in $D(0, 1)$ and satisfies $U(0) = I_2$, and $Y_0 \in GL(2, \mathbb{C})$ is a constant matrix. The monodromy of $\widetilde{F}$ on the unit circle γ is

$$M_\gamma(\widetilde{F}) = M_\gamma(F) = Y_0^{-1} e^{2\pi \mathrm{i} A_0} Y_0.$$

Since f is well defined, its monodromy $M_\gamma(F)$ belongs to $SU(2)$ so its eigenvalues are a complex number of modulus 1. This implies that the eigenvalues λ_1, λ_2 of A_0 are real numbers, so $\alpha\beta$ is real. To compute z^{A_0}, we write $A_0 = PDP^{-1}$ with

$$P = \frac{1}{\Delta^{1/4}} \begin{pmatrix} -\lambda_2 & \alpha\lambda_1 \\ \alpha^{-1}\lambda_1 & -\lambda_2 \end{pmatrix} \in SL(2, \mathbb{C}), \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

By standard matrix theory we can write $P^{-1}Y_0 = TH$, where T is upper triangular and $H \in SU(2)$. Then (51) gives

$$F(z) = N(z)U(z)Pz^D P^{-1} Y_0 = N(z)U(z)Pz^D TH. \tag{52}$$

Assume that $\frac{1}{c} \leq |\alpha^2\beta| \leq c$ and $||F(z)|| \leq c$ on the unit circle, for some real number c. We need a uniform bound (depending only on c) of $U(z)$ in the unit disk. The theory of fuchsian systems gives us a bound of $U(z)$ by construction, but this bound is not uniform as $\alpha\beta \to 0$ (because we are approaching the resonant case). To obtain a uniform bound, we must use the fact that the monodromy of F is in $SU(2)$.

First of all, our hypotheses imply the following bounds:

$$\frac{1}{2} \leq \Delta \leq \frac{3}{2}, \qquad \frac{-3}{2} \leq \lambda_2 \leq \frac{-1}{2}, \qquad |\alpha^{-1}\lambda_1| \leq c \quad \text{and} \quad \frac{1}{2c} \leq |\alpha\lambda_1| \leq \frac{3c}{2}. \tag{53}$$

Hence the matrix P is uniformly bounded. From (52), we obtain

$$1 = \det F(z) = z\det(U(z))z^{-1}\det T.$$

Hence $\det U(z)$ is constant, and since $U(0) = I_2$, we obtain $\det U(z) = \det T = 1$. The monodromy of F is given by

$$\begin{aligned} M_\gamma(F) &= H^{-1}T^{-1}e^{2\pi \mathrm{i} D}TH \\ &= H^{-1}\begin{pmatrix} e^{2\pi\mathrm{i}\lambda_1} & T_{12}T_{22}(e^{2\pi\mathrm{i}\lambda_1} - e^{2\pi\mathrm{i}\lambda_2}) \\ 0 & e^{2\pi\mathrm{i}\lambda_2} \end{pmatrix} H \in SU(2). \end{aligned}$$

Since $\lambda_1 - \lambda_2$ is not an integer, this implies that $T_{12} = 0$, so the matrix T is diagonal. Then T and z^D commute. Equation (52) implies that $U(z)PT$ is uniformly bounded on the unit circle. Since $U(z)PT$ is holomorphic, it is uniformly bounded in the unit disk by the maximum principle. Taking $z = 0$, we obtain that T is uniformly bounded, and hence $U(z)$ is uniformly bounded in the unit disk. Expanding the product in (52), we obtain

$$F(z) = \frac{1}{\Delta^{1/4}}\begin{pmatrix} z^{\lambda_1}(-T_{11}\lambda_2 + O(z)) & z^{\lambda_1}(T_{22}\alpha\lambda_1 + O(z)) \\ z^{1+\lambda_1}(T_{11}\alpha^{-1}\lambda_2 + O(z)) & z^{1+\lambda_2}(-T_{22}\lambda_2 + O(z)) \end{pmatrix} H, \tag{54}$$

where $O(z)$ is holomorphic and uniformly bounded. Using (4) and the bounds (53), we obtain

$$(x_1 + \mathrm{i}x_2)(z) = -\frac{1}{z}\frac{\alpha\lambda_1}{\lambda_2}(1 + O(z) + |z|^\Delta O(1)),$$

$$x_3(z) = \frac{1}{|z|^{2+2\lambda_2}}\frac{\sqrt{\Delta}}{|T_{22}\lambda_2|^2}(1 + O(z) + |z|^\Delta O(1)),$$

where $O(z)$ and $O(1)$ are real analytic functions that have uniformly bounded derivatives and $O(z)$ vanishes at the origin. The conclusions of Theorem 6 follow. □

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