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1-COMPLETE SEMIHOLOMORPHIC FOLIATIONS

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ABSTRACT. A *semiholomorphic foliation of type* (n, d) is a differentiable real manifold X of dimension $2n + d$, foliated by complex leaves of complex dimension n. In the present work, we introduce an appropriate notion of *pseudoconvexity* (and consequently, q-completeness) for such spaces, given by the interplay of the usual pseudoconvexity along the leaves, and the positivity of the transversal bundle. For 1-complete real analytic semiholomorphic foliations, we obtain a vanishing theorem for the CR cohomology, which we use to show an extension result for CR functions on Levi flat hypersurfaces and an embedding theorem in $\mathbb{C}^N$. In the compact case, we introduce a notion of weak positivity for the transversal bundle, which allows us to construct a real analytic embedding in $\mathbb{CP}^N$.

CONTENTS

1. INTRODUCTION

A *semiholomorphic foliation of type* (n, d), $n \geq 1$, $d \geq 1$, is a differentiable real manifold X of dimension $2n + d$, foliated by complex leaves of complex dimension n. If X is of class C^ω we say that X is a *a real analytic semiholomorphic foliation.* The aim of the present paper is to investigate the geometrical properties of such spaces along the lines of the classical theory of complex spaces. Such spaces were already studied in [10], to some extent, but some of the conclusions reached there are spoiled by the lack of a necessary hypothesis in the statements, the positivity of the transversal bundle mentioned below, which allows us not only to correctly state and prove the results of [10], but also to push further our investigation and obtain new insights in the geometry of semiholomorphic foliations. Forstnerič and

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Laurent-Thiébaut moved in a similar direction, with their work in [7], where attention was focused on the existence of a basis of Stein neighbourhoods for some particular compacts of a Levi-flat hypersurface. We should also mention that, more recently, El Kacimi and Slimène studied the vanishing of the CR cohomology for some particular kinds of semiholomorphic foliation in [5, 6]. We are mainly concerned with real analytic semiholomorphic foliations which satisfy some hypothesis of *pseudoconvexity*. The pseudoconvexity we have in mind consists of two conditions: 1-pseudoconvexity along the leaves of X, i.e. the existence of a smooth exhaustion function $\phi : X \to \mathbb{R}^+$ which is strongly 1-plurisubharmonic along the leaves, and positivity of the bundle N_{tr} transversal to the leaves of X (see Section 3.1). Under these conditions we say that X is 1-*complete*. For 1-complete real analytic semiholomorphic foliation of type (n, d) we can prove that

1) every sublevel set $\{\phi \leq c\}$ has a basis of Stein neighbourhoods;
2) an approximation theorem for CR functions holds on X

(see Theorem 3.3).

Once these results are proved the methods of complex analysis apply, in order to study the cohomology of the sheaf $\mathcal{CR}$ of germs of CR functions. We show that, for 1-complete real analytic semiholomorphic foliation of type $(n, 1)$, the cohomology groups $H^q(X, \mathcal{CR})$ vanish for $q \geq 1$ (see Theorem 4.1). This implies a vanishing theorem for the sheaf of sections of a CR bundle, which we use in Section 4 to get a tubular neighbourhood theorem and an extension theorem for CR functions on Levi flat hypersurfaces (see Theorems 5.2, 5.3 and Corollary 5.4). In Section 6 we sketch the proof that a 1-complete real analytic semiholomorphic foliation of type (n, d) embeds in $\mathbb{C}^{2n+2d+1}$ as a closed submanifold by a smooth CR map (see Theorem 6.1) and, as an application, we get that the groups $H_j(X; \mathbb{Z})$ vanish for $j \geq n + d + 1$ (see Theorem 6.3). In the last section a notion of *weak positivity* is given for the transversal bundle N_{tr} on a compact real analytic semiholomorphic foliation X of type $(n, 1)$ (see (18)) and we prove that, under this condition, X embeds in $\mathbb{CP}^N$ by a real analytic CR map (see Theorem 7.2). These results were announced by the authors in [13].

2. Preliminaries

2.1. **q-complete semiholomorphic foliations.** We recall that a *semiholomorphic foliation of type* (n, d) is a (connected) smooth foliation X whose local models are subdomains $U_j = V_j \times B_j$ of $\mathbb{C}^n \times \mathbb{R}^d$ with transformations $(z_k, t_k) \mapsto (z_j, t_j)$ of the form

$$\begin{cases} z_j = f_{jk}(z_k, t_k), \\ t_j = g_{jk}(t_k), \end{cases} \tag{1}$$

where f_{jk}, g_{jk} are smooth and f_{jk} is holomorphic with respect to z_k. If we replace $\mathbb{R}^d$ by $\mathbb{C}^d$ and we suppose that f and g are holomorphic, we get the notion of *holomorphic foliation of type* (n, d).

Local coordinates $z_j^1, \ldots, z_j^n, t_j^1, \ldots, t_j^d$ satisfying (1) are called *distinguished local coordinates*.

A continuous function $f : X \to \mathbb{C}$ is a CR function if and only it is holomorphic along the leaves. Let $\mathcal{CR} = \mathcal{CR}_X$ denote the sheaf of germs of smooth CR functions in X.

Given a subset Y of X we denote by $\widehat{Y}_{\mathcal{CR}(X)}$ the envelope of Y with respect to the algebra $\mathcal{CR}(X)$.

A *morphism or* CR *map* $F : X \to X'$ of semiholomorphic foliations is a smooth map preserving the leaves and such that the restrictions to leaves are holomorphic. By $\mathcal{CR}(X; X')$ we denote the space of all morphisms $X \to X'$; this space is a closed subspace in $C^\infty(X; X')$, so it is a metric space.

A semiholomorphic foliation X is said to be *tangentially q-complete* if X carries a smooth exhaustion function $\phi : X \to \mathbb{R}^+$ which is q-plurisubharmonic along the leaves (i.e. its Levi form has at least $n - q + 1$ positive eigenvalues). In this case we may assume that $\min_X \phi = 0$ and $M = \{x \in X : \phi(x) = 0\}$ is a point (M is compact; take $a \in M$ and a compactly supported nonnegative smooth function ψ such that $\{\psi = 0\} \cap M = \{a\}$. If ε is positive and sufficiently small, the function $\phi + \varepsilon\psi$ has the required properties).

Remark 2.1. Let X be a semiholomorphic foliation of type (n, d) which carries a function ϕ which is strongly plurisubharmonic along the leaves (not necessarily exhaustive). Then

1) If D is a relatively compact domain in X and $f : D \to \mathbb{C}$ is a CR function continuous up to the boundary $\mathrm{b}D$, we have

$$\max_{\overline{D}} |f| = \max_{\mathrm{b}D} |f|.$$

Let $M = \{x \in \overline{D} : |f(x)| = \max_{\overline{D}} |f|\}$ and assume, by a contradiction, that $M \cap \mathrm{b}D = \varnothing$. M is compact and let $a \in M$ be such that $\phi(a) = \max_{\overline{D}} \phi$. If F_a denotes the leaf through a, by the maximum principle $F_a \subset M$ and $\phi_{|F_a}$ has a maximum at a so it is constant along F_a. This is a contradiction since, by hypothesis, the Levi form of $\phi_{|F_a}$ is positive.

2) If f is a CR function and $\operatorname{supp} f$ is compact, then $f = 0$.

Let $K = \operatorname{supp} f$ and $a \in K$ be such that $\phi(a) = \max_K \phi$. As above the leaf F_a cannot be contained in $\phi \leq \phi(a)$ so, by the identity principle, f is vanishing on F_a. It follows that there exists a distinguished neighbourhood U of a where f vanishes, i.e. $\operatorname{supp} f \subset K \smallsetminus U \cap K$, a contradiction.

The same holds, in broader generality, if ϕ is q-plurisubharmonic along the leaves with $q \leq n$.

2.2. **CR-bundles.** Let $\mathsf{G}_{m,s}$ be the group of matrices

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in GL(m; \mathbb{C})$, $B \in GL(m, s; \mathbb{C})$ and $C \in GL(s; \mathbb{R})$. We set $\mathsf{G}_{\mathsf{m},0} = GL(m; \mathbb{C})$, $\mathsf{G}_{0,\mathsf{s}} = GL(s; \mathbb{R})$. To each matrix $M \in \mathsf{G}_{\mathsf{m},\mathsf{s}}$ we associate the linear transformation $\mathbb{C}^m \times \mathbb{R}^s \to \mathbb{C}^m \times \mathbb{R}^s$ given by

$$(z, t) \mapsto (Az + Bt, Ct),$$

$(z, t) \in \mathbb{C}^m \times \mathbb{R}^s$.

Let X be a semiholomorphic foliation of type (n, d). A CR-*bundle of type (m,s)* is a vector bundle $\pi : E \to X$ such that the cocycle of E determined by a trivializing

distinguished covering $\{U_j\}_j$ is a smooth CR map $\gamma_{jk} : U_j \cap U_k \to \mathsf{G}_{\mathsf{m},\mathsf{s}}$,

$$\gamma_{jk} = \begin{pmatrix} A_{jk} & B_{jk} \\ 0 & C_{jk} \end{pmatrix}, \tag{2}$$

where $C_{jk} = C_{jk}(t)$ is a matrix with smooth entries and A_{jk}, B_{jk} are matrices with smooth CR entries. Thus E is foliated by complex leaves of dimension $m+n$ and real codimension $d+s$. If z_j, t_j are distinguished coordinates on X and ζ_j, θ_j are coordinates on $\mathbb{C}^m \times \mathbb{R}^s$, then $z_j, \zeta_j, t_j, \theta_j$ are distinguished coordinates on E with transformations

$$\begin{cases} z_j = f_{jk}(z_k, t_k), \\ \zeta_j = A_{jk}(z_k, t_k)\zeta_k + B_{jk}(z_k, t_k)\theta_k, \\ t_j = g_{jk}(t_k), \\ \theta_j = C_{jk}(t_k)\theta_k. \end{cases}$$

In particular, $\pi : E \to X$ is a smooth CR map.

The *inverse* E^{-1} of E is the CR-bundle E of type (m,s) whose cocycle is

$$\gamma_{jk}^{-1} = \begin{pmatrix} A_{jk}^{-1} & -A_{jk}^{-1}B_{jk}C_{jk}^{-1} \\ 0 & C_{jk}^{-1} \end{pmatrix}. \tag{3}$$

Let X be a semiholomorphic foliation of type (n,d). Then

- the tangent bundle TX of X is a CR-bundle of type (n,d);
- the bundle $T_{\mathcal{F}} = T^{1,0}_{\mathcal{F}}X$ of the holomorphic tangent vectors to the leaves of X is a CR-bundle of type $(n,0)$;
- the complexified $\mathbb{C}TX$ of the tangent bundle is a complex vector bundle.

Remark 2.2. If X is embedded in a complex Z, its transverse TZ/TX is not a $\mathsf{G}_{\mathsf{m},\mathsf{s}}$-bundle in general.

2.3. **Complexification.** A real analytic foliation of type (n,d) can be complexified, essentially in a unique way: there exists a holomorphic foliation $\widetilde{X}$ of type (n,d) with a closed real analytic CR embedding $X \hookrightarrow \widetilde{X}$; in particular, X is a Levi flat submanifold of $\widetilde{X}$ (cf. [18, Theorem 5.1]).

In order to construct $\widetilde{X}$ we consider a covering by distinguished domains $\{U_j = V_j \times B_j\}$ and we complexify each B_j in such a way to obtain domains $\widetilde{U}_j$ in $\mathbb{C}^n \times \mathbb{C}^d$. The domains $\widetilde{U}_j$ are patched together by the local holomorphic transformations

$$\begin{cases} z_j = \widetilde{f}_{jk}(z_k, \tau_k), \\ \tau_j = \widetilde{g}_{jk}(\tau_k) \end{cases} \tag{4}$$

obtained by complexifying the (vector) variable t_k by $\tau_k = t_k + i\theta_k$ in f_{jk} and g_{jk} (cf. (1)).

Let z_j, τ_j distinguished holomorphic coordinates on $\widetilde{U}_j$ and $\theta_j = \mathsf{Im}\,\tau_j$. Then $\theta_j^s = \mathsf{Im}\,\widetilde{g}_{jk}(\tau_k)$, $1 \le s \le d$, on $\widetilde{U}_j \cap \widetilde{U}_k$, and consequently, since $\mathsf{Im}\,\widetilde{g}_{jk} = 0$ on X,

$$\theta^r = \sum_{s=1}^{d} \psi_{jk}^{rs}\theta_k^s,$$

where $\psi_{jk} = \left(\psi_{jk}^{rs}\right)$ is an invertible $d \times d$ matrix whose entries are real analytic functions on $\widetilde{U}_j \cap \widetilde{U}_k$. Moreover, since $\widetilde{g}_{jk}$ is holomorphic and $\widetilde{g}_{jk|X} = g_{jk}$ is real, we also have $\psi_{jk|X} = \partial g_{jk}/\partial t_k$.

$\{\psi_{jk}\}$ is a cocycle of a CR-bundle of type $(0,d)$ which extends N_{tr} on a neighbourhood of X in $\widetilde{X}$.

In [10, Theorem 2] it is proved that if X is tangentially q-complete, then every sublevel $\overline{X}_c = \{x \in X : \phi \leq c\}$ has a fundamental system of neighbourhoods in $\widetilde{X}$ which are $(q+1)$-complete complex manifolds.

Let $\widetilde{X}$ be the complexification of X. Then the cocycle of the (holomorphic) transverse bundle $\widetilde{N}_{\mathrm{tr}}$ (to the leaves of $\widetilde{X}$) is

$$\frac{\partial \widetilde{g}_{jk}(\tau_k)}{\partial \tau_k} = \frac{\partial \tau_j}{\partial \tau_k} = \left(\frac{\partial \tau_j^\alpha}{\partial \tau_k^\beta}\right). \tag{5}$$

3. Completeness

3.1. **Tranversally complete foliations.** Let X be a semiholomorphic foliation of type (n,d), $n \geq 1$, $d \geq 1$, and N_{tr} the transverse bundle to the leaves of X. A metric on the fibres of N_{tr} is an assignment of a distinguished covering $\{U_j\}$ of X and for every j a smooth map λ_j^0 from U_j to the space of symmetric positive $d \times d$ matrices such that

$$\lambda_k^0 = \frac{{}^t\partial g_{jk}}{\partial t_k} \lambda_j^0 \frac{\partial g_{jk}}{\partial t_k}.$$

Denoting ∂ and $\overline{\partial}$ as the complex differentiation along the leaves of X, the local tangential forms

$$\omega = 2\overline{\partial}\partial \log \lambda_j^0 - \overline{\partial} \log \lambda_j^0 \wedge \partial \log \lambda_j^0 = \frac{\lambda_j^0 \overline{\partial}\partial \lambda_j^0 - 2\overline{\partial}\lambda_j^0 \wedge \partial \lambda_j^0}{(\lambda_j^0)^3}, \tag{6}$$

$$\Omega = \overline{\partial}\partial \log \lambda_j^0 - \overline{\partial} \log \lambda_j^0 \wedge \partial \log \lambda_j^0 = \frac{2\lambda_j^0 \overline{\partial}\partial \lambda_j^0 - 3\overline{\partial}\lambda_j^0 \wedge \partial \lambda_j^0}{(\lambda_j^0)^3} \tag{7}$$

actually give global tangential forms in X.

The foliation X is said to be *tranversally q-complete* (*strongly tranversally q-complete*) if a metric on the fibres of N_{tr} can be chosen in such a way that the hermitian form associated to ω (Ω) has at least $n-q+1$ positive eigenvalues.

Remark 3.1. Assume $d = 1$ and that X is transversally 1-complete. Then, due to the fact that the functions g_{jk} do not depend on z, $\omega_0 = \{\partial\overline{\partial} \log \lambda_j^0\}$ and $\eta = \{\partial \log \lambda_j^0\}$ are global tangential forms on X. Moreover, since ω_0 is positive and $\omega_0 = d\eta$, it gives on each leaf ω_0 a Kähler metric whose Kähler form is exact. In particular no positive dimension compact complex subspace can be present in X.

Example 3.1. Every domain $D \subset X = \mathbb{C}_z \times \mathbb{R}_u$ which projects over a bounded domain $D_0 \subset \mathbb{C}_z$ is strongly transversally 1-complete. Indeed, it is enough to take for λ a function $\mu^{-1} \circ \pi$ where μ is a positive superharmonic on D_0 and π is the natural projection $\mathbb{C}_z \times \mathbb{R}_u \to \mathbb{C}_z$.

Example 3.2. A real hyperplane X in $\mathbb{C}^2$ (or in $\mathbb{C}^n$) is not transversally 1-complete (but clearly it is an increasing union of strongly transversally 1-complete domains). Indeed, if $X \subset \mathbb{C}^2$ is defined by $v = 0$, where $z = x+iy, w = u+iv$ are holomorphic coordinates, transverse 1-completeness of X amounts to the existence of a positive smooth function $\lambda = \lambda(z,u)$, $(z,u) \in \mathbb{C}^2$, such that

$$\lambda\lambda_{z\overline{z}} - 2|\lambda_z|^2 > 0.$$

This implies that

$$\left(\lambda^{-1}\right)_{z\overline{z}} = \frac{2|\lambda_z|^2 - \lambda\lambda_{z\overline{z}}}{\lambda^3} < 0,$$

so, for every fixed u, the function λ^{-1} is positive and superharmonic on $\mathbb{C}_z$; hence it is constant with respect to z, a contradiction.

Example 3.3. Let $F : \mathbb{C}^m \to \mathbb{C}^n$ be a holomorphic function and $Y \subseteq \mathbb{C}^n$ be a real analytic manifold of real dimension k, consisting only of regular values for F. Then, the set $X = F^{-1}(Y)$ can be given the structure of a mixed foliation of type $(m-n, k)$; moreover, as Y consists only of regular values of F, locally we have that $X \cong Y \times M$ with $M = F^{-1}(y_0)$ for some $y_0 \in Y$.

If we take F to be a polynomial map, M embeds in a projective variety $\widetilde{M} \subseteq \mathbb{CP}^m$. Therefore, arguing as in the previous example, we have that X cannot be transversally pseudoconvex, because this would amount to the existence of a bounded from below plurisuperharmonic function on $\widetilde{M}$.

On the other hand, let $\psi : \Delta \to \mathbb{C}^3$ be a proper holomorphic embedding of the unit disc $\Delta \subset \mathbb{C}$ and $F : \mathbb{C}^3 \to \mathbb{C}^n$ be such that $F^{-1}(0) = \psi(\Delta)$. There exists $\epsilon > 0$ such that $(t, 0, \ldots) \in \mathbb{C}^n$ is a regular value for F, with $t \in (-\epsilon, \epsilon)$; fix $Y = \{(t, 0, \ldots) \in \mathbb{C}^n \ : \ t \in (-\epsilon, \epsilon)\}$ and consider $X = F^{-1}(Y)$. Now, $X \cong \Delta \times (-\epsilon, \epsilon)$ and the function $\lambda = (1 - |\psi^{-1}(z)|^2)^{-1}$ shows that X is transversally 1-complete.

Example 3.4. Let $X \subset \mathbb{C}^n_z \times \mathbb{R}_u$ be the smooth family of n-balls of $\mathbb{C}^n_z$, $z = (z_1, \ldots, z_n)$, $w = u + iv$, defined by

$$\begin{cases} v = 0, \\ |z - a(u)|^2 < b(u)^2, \end{cases}$$

where $a = a(u)$ is a smooth map $\mathbb{R} \to \mathbb{C}^n$, $b = b(u)$ is smooth from $\mathbb{R}$ to $\mathbb{R}$ and $|a(u)|, |b(u)| \to +\infty$ as $|u| \to +\infty$. X is strongly transversally 1-complete with function

$$\lambda(z, u) = \frac{1}{b(u)^2 - |z - a(u)|^2}$$

and tangentially 1-complete with exhaustion function $\phi = \lambda$.

We want to prove the following

Theorem 3.1. *Let X be a semiholomorphic foliation of type (n, d). Assume that X is real analytic and strongly transversally 1-complete. Then there exist an open neighbourhood U of X in the complexification $\widetilde{X}$ and a nonnegative smooth function $u : U \to \mathbb{R}$ with the following properties:*

i) *$X = \{x \in U : u(x) = 0\}$;*
ii) *u is plurisubharmonic in U and strongly plurisubharmonic on $U \setminus X$.*

If X is tranversally 1-complete, then property ii) *is replaced by the following:*

iii) *the Levi form of the smooth hypersurfaces $\{u = \text{const}\}$ is positive definite.*

Proof. For the sake of simplicity we assume $n = d = 1$. Let $\{\lambda^0_j\}$ be a metric on the fibres of N_{tr}. With the notations of Section 2.3 let $\widetilde{N}_F$ be the tranverse bundle on $\widetilde{X}$ whose cocycle $\psi_{jk} = \widetilde{g}_{jk}$ is defined by (5) and let $\{\mu_j\}$ be a smooth metric on

the fibres of $N_F'^{-1}$. On $\widetilde{U}_j \cap \widetilde{U}_k$ we have $\mu_k = \psi_{jk}^2 \mu_j$, and consequently, denoting μ_j^0 as the restriction $\mu_{j|X}$, we have $\mu_j^0 \lambda_j^{0-1} = \mu_k^0 \lambda_k^{0-1}$ on $U_j \cap U_k$. Thus $\sigma^0 = \{\mu_j^0 \lambda_j^{0-1}\}$ is a smooth section of $N_F' \otimes N_F'^{-1}$, where N_F' is the dual of N_{tr}. Extend σ_0 by a smooth section $\sigma = \{\sigma_j\}$ of $\widetilde{N}_F' \otimes \widetilde{N}_F'^{-1}$. Then $\{\lambda_j = {\sigma_j}^{-1}\mu_j\}$ is a new metric on the fibres of $\widetilde{N}_F$ whose restriction to X is $\{\lambda_j^0\}$.

Now consider on $\widetilde{X}$ the smooth function u locally defined by $\lambda_j \theta_j^2$ (where $\tau_j = t_j + i\theta_j$); u is nonnegative on $\widetilde{X}$ and positive outside X. Drop the subscript and compute the Levi form $L(u)$ of u. We have

$$\begin{aligned} L(u)(\xi,\eta) &= A\xi\overline{\xi} + 2\mathsf{Re}\,(B\xi\overline{\eta}) + C\eta\overline{\eta} \\ &= \lambda_{j,z\overline{z}}\theta^2\xi\overline{\xi} + 2\mathsf{Re}\left\{(\lambda_{j,z\overline{\tau}}\theta^2 + i\lambda_{j,z}\theta)\xi\overline{\eta}\right\} \\ &\quad + (\lambda_{j,\tau\overline{\tau}}\theta^2 + i\lambda_{j,\tau}\theta - i\lambda_{j,\overline{\tau}}\theta + \lambda_j/2)\eta\overline{\eta} \end{aligned} \tag{8}$$

and

$$AC - |B|^2 = \theta^2\left(\lambda_{j,z\overline{z}} - 2|\lambda_{j,z}|^2\right) + \theta^3\varrho, \tag{9}$$

where ϱ is a smooth function. The coefficient of θ^2 is nothing but that of the form $\lambda_j^2\Omega$ (here we denote ω and Ω, respectively, the forms (6), (7), where λ_j^0 is replaced by λ_j), so if X is strongly transversally pseudoconvex, $L(u)$ is positive definite near each point of X and strictly positive away from X. It follows that there exists a neighbourhood U of X such that u is plurisubharmonic on U.

Assume now that X is transversally 1-complete. The Levi form $L(u)_{|HT(S)}$ of a smooth hypersurface $S = \{u = \mathrm{const}\}$ is essentially determined by the function

$$\begin{aligned} &\theta^4\big\{\lambda_{j,z\overline{z}}|\lambda_{j,\overline{\tau}}\theta + i\lambda_j|^2 - 2\mathsf{Re}\,(\lambda_{j,z\overline{\tau}}\theta + i\lambda_{j,\overline{z}})(\lambda_{j,\tau}\theta - i\lambda_j)\lambda_{j,\overline{z}} \\ &+(\lambda_{j,\tau\overline{\tau}}\theta^2 + i\lambda_{j,\tau}\theta - i\lambda_{j,\overline{\tau}}\theta + \lambda_j/2)|\lambda_{j,z}|^2\big\} \\ &= \theta^4(\lambda_j\lambda_{j,z\overline{z}} - (3/2)|\lambda_{j,z}|^2) + \theta^5\varrho, \end{aligned} \tag{10}$$

where ϱ is a bounded function. The coefficient of θ^4 is nothing but that of the form $\lambda_j^3\omega$, so if X is transversally pseudoconvex, $L(u)_{|HT(S)}$ is not vanishing near X. It follows that there exists an open neighbourhood U of X such that the hypersurfaces $S = \{u = \mathrm{const}\}$ contained in $U \smallsetminus X$ are strongly Levi convex.

In the general case $\lambda = \lambda_j$ is a matrix and $\lambda_j = (\lambda_{j,rs})$, so we consider the function $u = \sum_{r=1}^{d} \lambda_{j,rs}\theta_r\theta_s$. Then

$$\begin{aligned} L(u)(\xi,\eta) \;=\; &\sum_{\alpha,\beta=1}^{n} A_{\alpha\overline{\beta}}\xi^\alpha\overline{\xi}^\beta + 2\mathsf{Re}\Big(\sum_{\alpha=1}^{n}\sum_{r=1}^{d} B_{\alpha\overline{r}}\xi^\alpha\overline{\eta}^r\Big) \\ &+ \sum_{r,s=1}^{d} C_{r\overline{s}}\eta^r\overline{\eta}^s, \end{aligned} \tag{11}$$

where

$$A_{\alpha\overline{\beta}} = \sum_{\alpha,\beta=1}^{n}\Big(\sum_{r,s=1}^{d}\frac{\partial^2\lambda_{j,rs}}{\partial z_\alpha\partial\overline{z}_\beta}\theta^r\theta^s\Big)\xi^\alpha\overline{\xi}^\beta, \tag{12}$$

$$B_{\alpha\bar{r}} = \Big(\sum_{\alpha=1}^{n}\sum_{h=1}^{d}\frac{\partial^2\lambda_{j,rs}}{\partial z_\alpha\partial\overline{\tau}_r}\theta^r\theta^s + i\sum_{s=1}^{d}\frac{\partial\lambda_{j,rh}}{\partial z_\alpha}\theta^h\Big)\xi^\alpha\overline{\eta}^r, \tag{13}$$

$$C_{r\bar{s}} = \Big(\sum_{h,k=1}^{d}\frac{\partial^2\lambda_{j,hk}}{\partial\tau_r\partial\overline{\tau}_s}\theta^h\theta^k + i\sum_{h=1}^{d}\frac{\partial\lambda_{j,hs}}{\partial\tau_r}\theta^h - i\sum_{h=1}^{d}\frac{\partial\lambda_{j,rh}}{\partial\overline{\tau}_s}\theta^h + \lambda_{j,rs}/2\Big)\eta^r\overline{\eta}^s. \tag{14}$$

□

Remark 3.2. In view of (8), the Levi form $L(u)$ at a point of X is positive in the transversal direction η.

Theorem 3.2. *Let X be a semiholomorphic foliation of type (n,d). Assume that X is real analytic and strongly 1-complete. Then for every compact subset $K\subset X$ there exist an open neighbourhood V of K in $\widetilde{X}$, a smooth strongly plurisubharmonic function $v : V\to\mathbb{R}^+$ and a constant $\bar{c}$ such that $K\Subset\{v<\bar{c}\}\cap X\Subset V\cap X$.*

Proof. Let $\phi : X\to\mathbb{R}^+$ be an exhaustion function, strongly plurisubharmonic along the leaves and a sublevel X_{c_0} of ϕ such that $K\subset X_{c_0}$. Consider $U\subset\widetilde{X}$, $u : U\to\mathbb{R}^+$ as in Theorem 3.1 and let $v = au+\widetilde{\phi}$, where $\widetilde{\phi} : U\to\mathbb{R}^+$ is a smooth extension of ϕ to U and a a positive constant. Then, in view of Remark 3.2, it is possible to choose a in such a way to have $L(v)(x) > 0$ for every x in a neighbourhood V of $\overline{X}_{c_0}$, $c_0 < c$; this ends the proof. □

3.2. **Stein bases and a density theorem.**

Theorem 3.3. *Let X be a semiholomorphic foliation of type (n,d). Assume that X is real analytic and strongly 1-complete and let $\phi : X\to\mathbb{R}^+$ be an exhaustive smooth function strongly plurisubharmonic along the leaves. Then*

i) *$\overline{X}_c$ is a Stein compact of $\widetilde{X}$, i.e. it has a basis of Stein neighbourhoods in $\widetilde{X}$;*
ii) *every smooth CR function on a neighbourhood of $\overline{X}_c$ can be approximated in the C^∞ topology by smooth CR functions on X.*

Proof. In view of Theorem 3.1 we may suppose that $X = \{u = 0\}$, where $u : \widetilde{X}\to[0,+\infty)$ is plurisubharmonic and strongly plurisubharmonic on $\widetilde{X}\smallsetminus X$.

Let U be an open neighbourhood of $\overline{X}_c$ in $\widetilde{X}$. With $K = \overline{X}_c$ we apply Theorem 3.2: there exist an open neighbourhood $V\Subset U$ of $\overline{X}_c$ in $\widetilde{X}$, a smooth strongly plurisubharmonic function $v : V\to\mathbb{R}^+$ and a constant $\bar{c}$ such that $\overline{X}_c\Subset\{v<\bar{c}\}\cap X\Subset V\cap X$. It follows that for $\varepsilon>0$ sufficiently small, $W = \{v<\bar{c}\}\cap\{u<\varepsilon\}\Subset V\Subset U$ is a Stein neighbourhood of $\overline{X}_c$.

In order to prove ii) consider a smooth CR function f on a neighbourhood I of $\overline{X}_c$ in X, and take $c'>c$ such that $\overline{X}_{c'}\subset I$. For every $j\in\mathbb{N}$ define $\overline{B}_j = \overline{X}_{c'+j}$ and choose a Stein neighbourhod U_j of $\overline{B}_j$ such that $\overline{B}_j$ has a fundamental system of open neighbourhoods $W_j\Subset U_{j+1}\cap U_j$ which are Runge domains in U_{j+1}. Since U_0 is Stein, the $\mathcal{O}(U_0)$-envelope of $\overline{B}_0$ coincides with its plurisubharmonic envelope (cf. [12, Theorem 4.3.4]); hence it is a compact contained in $X\cap U_0$, with $\overline{B}_0$ the zero set of the plurisubharmonic function u. Thus we may assume that $\overline{B}_0$ is $\mathcal{O}(U_0)$-convex.

We have to prove that for every fixed C^k-norm on $\overline{B}_0$ and $\varepsilon>0$ there exists a smooth CR-function $g : X\to\mathbb{C}$ such that $\|g-f\|^{(k)}_{\overline{B}_0}\le\varepsilon$.

In view of the approximation theorem of Freeman (cf. [8, Theorem 1.3]), given $\varepsilon > 0$ there is $\widetilde{f} \in \mathcal{O}(U_0)$ such that $\|\widetilde{f} - f\|^{(k)}_{\overline{B}_0} < \varepsilon$. Now, define $F_0 = \widetilde{f}$, and for every $j \geq 1$ take W_j such that $\overline{W}_j$ is a Runge domain in U_{j+1} and a holomorphic function $F_j \in \mathcal{O}(U_{j-1})$ satisfying

$$\|F_{j+1} - F_j\|^{(k)}_{\overline{W}_j} < \varepsilon/2^{j+1}, \ \ j = 0, 1, \ldots;$$

then

$$F_0 + \sum_{j=0}^{+\infty} \big(F_{j+1} - F_j\big)$$

defines a C^∞ function g on X such that $\|g - f\|^{(k)}_{\overline{B}_0} \leq 2\varepsilon$. □

Corollary 3.4. *Let X be as in Theorem 3.3 and $\overline{X}_c$, $c > 0$. Then, $\overline{X}_c$ is $\mathcal{CR}(X)$-convex, i.e. $\overline{X}_c$ coincides with its $\mathcal{CR}(X)$-envelope.*

Proof. Set $K = \overline{X}_c$. In view of Theorem 3.1, X is the zero set of a plurisubharmonic function defined in an open neighbourhood U of X in $\widetilde{X}$. Let $U_\nu \subset U$ be a Stein neighborhood of K and $\mathrm{Psh}(U_\nu)$ the space of the plurisubharmonic in U_ν. Then the envelopes $\widehat{K}_{\mathcal{O}(U_\nu)}$ and $\widehat{K}_{\mathrm{Psh}(U_\nu)}$ of K coincide (cf. [12, Theorem 4.3.4]), so in our situation we have

$$K \subset \widehat{K}_{\mathcal{CR}(X)} \subset \widehat{K}_{\mathcal{O}(U_\nu)} = \widehat{K}_{\mathrm{Psh}(U_\nu)} \subset U_\nu \cap X.$$

We obtain the thesis by letting $U_\nu \subset U$ run in a Stein basis of $\overline{X}_c$. □

4. Cohomology

In this section we want to prove the following two theorems.

Theorem 4.1. *Let X be a semiholomorphic foliation of type $(n, 1)$. Assume that X is real analytic and strongly 1-complete. Then*

$$H^q(X; \mathcal{CR}) = 0$$

for every $q \geq 1$.

Theorem 4.2. *Let D be a connected Stein manifold and $X \subset D$ a closed, orientable, smooth Levi flat hypersurface. Then*

$$H^q(X; \mathcal{CR}) = 0$$

for every $q \geq 1$.

We need some preliminary facts. Let Ω be a domain in $\mathbb{C}^n \times \mathbb{R}^k$ and for every $t \in \mathbb{R}^k$, and set $\Omega_t = \{z \in \mathbb{C}^n : (z, t) \in \Omega\}$. Following [2] we say that $\{\Omega_t\}_{t \in \mathbb{R}^k}$ is a *regular family* of domains of holomorphy if the following conditions are fulfilled:

a) Ω_t is a domain of holomorphy for all $t \in \mathbb{R}^k$;
b) for every $t_0 \in \mathbb{R}^k$ there exist $I_0 = \{|t - t_0| < \varepsilon\}$ and a domain $U \subset \mathbb{C}^n$ such that Ω_{t_0} is Runge in U and $\bigcup_{t \in I_0} \Omega_t \subset I_0 \times U$.

In this situation it can be proved that the sheaf $\mathcal{CR}$ of Ω is cohomologically trivial (cf. [2, Corollaire, p. 213]).

Lemma 4.3. *Let X be a 1-complete semiholomorphic foliation of type (n,d), $\phi : X \to [0,+\infty)$ strongly plurisubharmonic along the leaves and $c>0$ a regular value of ϕ. Then there is $\varepsilon>0$ such that the homomorphism*

$$H^q(X_{c+\varepsilon};\mathcal{CR}) \to H^q(X_c;\mathcal{CR})$$

is onto for $q \geq 1$. In particular, the homorphism

$$H^q(\overline{X}_c;\mathcal{CR}) \to H^q(X_c;\mathcal{CR})$$

is onto for $q \geq 1$.

Proof. Since ϕ is strongly plurisubharmonic along the leaves the "bumps lemma" method applies: given $x_0 \in \mathrm{b}X_c$ there exist arbitrarily small neighbourhoods $U \ni x_0$ and open sets $B \Subset X$ with the following properties:

i) $B \supset X_c$, $B \smallsetminus X_c \Subset U$ and B is given by $\{\phi' < 0\}$ with ϕ' smooth on a neighbourhood of $\overline{B}$ and strictly plurisubharmonic along the leaves;
ii) $V := U \cap B$ and $V \cap X_c$ are CR isomorphic to regular families of domains of holomorphy of $\mathbb{C}^n$.

Write $B = X_c \cup V$. Since $H^q(V;\mathcal{CR}) = H^q(V \cap X_c;\mathcal{CR}) = 0$ for $q \geq 1$, from the Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to H^q(X_c \cup V;\mathcal{CR}) &\to H^q(X_c;\mathcal{CR}) \oplus H^q(V;\mathcal{CR}) \\ &\to H^q(X_c \cap V;\mathcal{CR}) \to H^{q+1}(X_c \cup V;\mathcal{CR}) \to \cdots \end{aligned}$$

we obtain that the homomorphism $H^q(B;\mathcal{CR}) \to H^q(X_c;\mathcal{CR})$ is surjective for $q \geq 1$. By iterating this procedure starting from B, in a finite number of steps we find a neighbourhood W of $\overline{X}_c$ such that $H^q(W;\mathcal{CR}) \to H^q(X_c;\mathcal{CR})$ is surjective for every $q \geq 1$. It is enough to take $\varepsilon > 0$ such that $X_{c+\varepsilon} \subset W$. □

Proof of Theorem 4.1. We first prove that for every c and $q \geq 1$ the cohomology groups $H^q(X_c;\mathcal{CR})$ vanish, and for this that $H^q(\overline{X}_c;\mathcal{CR}) = 0$ for every c and $q \geq 1$ (see Lemma 4.3).

Let $\widetilde{X}$ be the complexification of X. By Theorem 3.3 there exists a Stein neighbourhood $U \Subset \widetilde{X}$ of $\overline{X}_c$ which is divided by X into two connected components U_+, U_- which are Stein domains. Let $\mathcal{O}_+$ ($\mathcal{O}_-$) be the sheaf of germs of holomorphic functions in D_+ (D_-) smooth on $U_+ \cup X$ ($U_- \cup X$) and extend it on U by 0. Then we have on U the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_+ \oplus \mathcal{O}_- \xrightarrow{\mathsf{re}} \mathcal{CR} \longrightarrow 0\,, \tag{15}$$

where re is definined by $\mathsf{re}(f \oplus g) = f_{|X} - g_{|X}$. Let ξ be a q-cocycle with values in $\mathcal{CR}$ defined on $\overline{X}_c$. We may suppose that ξ is defined on $U \cap X$. Since U is Stein we derive that

$$H^q(U_+ \cup X;\mathcal{O}_+) \oplus H^q(U_- \cup X;\mathcal{O}_-) \to H^q(X;\mathcal{CR})$$

is an isomorphism for $q \geq 1$. In particular, $\xi = \xi_+ - \xi_-$, where ξ_+ and ξ_- are represented by two $\overline{\partial}$-closed $(0,q)$-forms ω_+ and ω_- respectively, which are smooth up to X. Now, given $\phi : U \to \mathbb{R}$ a strictly plurisubharmonic exhaustion function and ρ_+, ρ_- defining functions for U_+, U_- (i.e. $U_\pm = \{\rho_\pm < 0\}$, $d\rho_\pm \neq 0$ whenever $\rho_\pm = 0$), we consider the function

$$\widetilde{\rho}_\pm = -(-\rho_\pm e^{-L\phi})^{1/2m}\,.$$

Computing the Levi form of $\rho_\pm$ in local coordinates and using the same estimates as in [9, Lemma 2.1], we obtain that, for any given $r \in \mathbb{R}$ large enough, we can choose L, ϵ and m so that $\widetilde{\rho}_\pm$ is strictly plurisubharmonic in $\{\phi < r + \epsilon\} \cap U_\pm \cap V_\pm$ for some neighbourhood $V_\pm$ of the boundary of $U_\pm$.

From this, we obtain the following: there exist two bounded pseudoconvex domains $U'_+, \Subset U$, $U'_- \Subset U$ satisfying:

1) $U'_+ \subset U_+$, $U'_- \subset U_-$;
2) $\overline{X}_c \subset \mathrm{b}U'_+ \cap X, \overline{X}_c \subset \mathrm{b}U'_- \cap X$.

By Kohn's theorem (cf. [14]) $\omega_{+|U'_+} = \overline{\partial}\alpha_+$, $\omega_{-|U'_-} = \overline{\partial}\alpha_-$, with α_+, α_- smooth up to the boundary, and this shows that ξ is a q-coboundary. Thus $H^q(\overline{X}_c; \mathcal{CR}) = 0$ for every c and $q \geq 1$.

The vanishing of the groups $H^q(X; \mathcal{CR})$ for $q \geq 2$ now easily follows by a standard procedure.

The proof of $H^1(X; \mathcal{CR}) = 0$ requires approximation of CR functions. Let $X = \bigcup_{s \geq 1}^{+\infty} X_s$ and $\| \cdot \|^{(s)}_{\overline{X}_s}$ denote the C^s norm, $s \in \mathbb{N}$. Let $\{U_i\}$ be an open covering of X and $\{f_{ij}\}$ a 1-cocycle of $\{U_i\}$ with values in $\mathcal{CR}$. Solve $f_{ij} = f_i^{(1)} - f_j^{(1)}$ on a neighbourhood of $\overline{X}_1$ and analogously $f_{ij} = g_i^{(2)} - g_j^{(2)}$ on a neighbourhood of $\overline{X}_2$. Then the $\{g_i^{(2)} - f_i^{(1)}\}$ defines a CR function on a neighbourhood of $\overline{X}_1$ which can be approximated in the norm $\| \cdot \|^{(1)}_{\overline{X}_1}$ by a CR function $h : X \to \mathbb{C}$ (see Theorem 3.3); thus, defining $f_i^{(2)} = g_i^{(2)} - h$ we have $f_{ij} = f_i^{(2)} - f_j^{(2)}$ on a neighbourhood of $\overline{X}_2$ and

$$\|f_i^{(2)} - f_i^{(1)}\|^{(1)}_{\overline{X}_1} < 1/2.$$

Thus, for every s we can find functions $f_i^{(s)}$ solving $f_{ij} = f_i^{(s)} - f_j^{(s)}$ on a neighbourhood of $\overline{X}_s$ and satisfying

$$\|f_i^{(s)} - f_i^{(s-1)}\|^{(s-1)}_{\overline{X}_s} < (1/2)^{s-1}.$$

This makes the series

$$f_i = f_i^{(1)} + \sum_{s=1}^{+\infty} \left(f_i^{(s+1)} - f_i^{(s)}\right)$$

convergent in $\mathcal{CR}(U_i)$ to a function f_i. On X_m we have

$$f_i = f_i^{(m)} + \sum_{s=m+1}^{+\infty} \left(f_i^{(s+1)} - f_i^{(s)}\right),$$

where the sum of the series is a CR function on X_m which is independent of i. It follows that$f_i - f_j = f_i^{(m)} - f_j^{(m)} = f_{ij}$, i.e. $\{f_{ij}\}$ is a coboundary, and this ends the proof of Theorem 4.1. □

Observe that the proof of Theorem 4.1 works under the hypothesis of Theorem 4.2 as well.

Clearly, under the hypothesis of Theorem 4.2, if $\{x_\nu\}$ is a discrete subset of X and $\{c_\nu\}$ a sequence of complex numbers there exist a CR function $f : X \to \mathbb{C}$ such that $f(x_\nu) = c_\nu$, $\nu = 1, 2, \ldots$. This is actually true under the hypothesis of Theorem 4.1.

Theorem 4.4. *Let X be a semiholomorphic foliation of type $(n,1)$. Assume that X is real analytic and strongly 1-complete. Let $A=\{x_\nu\}$ be a discrete set of distinct points of X and $\{c_\nu\}$ a sequence of complex numbers. Then there exists a smooth CR function $f:X\to\mathbb{C}$ such that $f(x_\nu)=c_\nu$, $\nu=1,2,\dots$. In particular, X is $\mathcal{CR}(X)$-convex.*

Proof. Let I_A be the sheaf $\{f\in\mathcal{CR}: f_{|A}=0\}$; then $\mathcal{CR}/I_A\simeq\prod\mathbb{C}_{x_\nu}$. We have to show that $H^1(X;I_A)=0$. As usual, we denote by $\widetilde{X}$ the complexification of X, $\phi:X\to\mathbb{R}$ a smooth function displaying the tangential 1-completeness of X and $X_c=\{\phi<c\}$, $c\in\mathbb{R}$. In view of Theorem 4.1, we have $H^1(X_c;\mathcal{CR})=0$ for every $c\in\mathbb{R}$. Moreover, since $A_c:=X_c\cap A$ is finite and $\overline{X}_c$ has a Stein basis of neighbourhoods in $\widetilde{X}$, the map $\mathcal{CR}(X_c)\to\prod_{x\in A_c}\mathbb{C}_x$ is onto; from the exact sequence of cohomology

$$\cdots\to H^0(X_c;\mathcal{CR})\to\prod_{x\in A_c}\mathbb{C}_x\to H^1(X_c;I_A)\to H^1(X_c;\mathcal{CR})\to\cdots$$

it follows that $H^1(X_c;I_A)=0$. To conclude that $H^1(X;I_A)=0$ we use the same strategy as in Theorem 4.1 taking into account the following: given $x_1,x_2,\dots,x_k$ in X_c, every function $f\in\mathcal{CR}(X_c)$ vanishing on $x_1,x_2,\dots,x_k$ can be approximate by functions $f\in\mathcal{CR}(X)$ vanishing on $x_1,x_2,\dots,x_k$ (see Theorem 3.3, ii)). □

Theorem 4.5. *Let D be a connected Stein domain in $\mathbb{C}^n$ (or, more generally a connected Stein manifold) and $X\subset D$ a closed, orientable, smooth Levi flat hypersurface. Let $\phi:D\to\mathbb{R}$ be exhaustive, strictly plurisubharmonic and $X_c=\{\phi<c\}\cap X$, $c\in\mathbb{R}$. Then the image of the CR map $\mathcal{CR}(X)\to\mathcal{CR}(X_c)$ is everywhere dense.*

Proof. Consider B, U as in Lemma 4.3. Observe that, by the bump lemma procedure applied to $\{\phi<c\}$ we may suppose that $B=B'\cap X$, where $B'\Subset D$ is given by $\{\phi'<0\}$ with ϕ' smooth and strictly plurisubharmonic on a neighbourhood of $\overline{B}'$. In particular, in view of Theorem 4.2, we have $H^q(B;\mathcal{CR})=0$ for $q\geq 1$. We are going to prove that the image of $\mathcal{CR}(B)\to\mathcal{CR}(X_c)$ is everywhere dense. Write $B=X_c\cup V$ where $V=B\cap U$. Let $\varrho:\mathcal{CR}(B)\to\mathcal{CR}(V\cap X_c)$; observe that the image of ϱ is everywhere dense. Let

$$\sigma:\mathcal{CR}(X_c)\oplus\mathcal{CR}(V)\to\mathcal{CR}(V\cap X_c)$$

be the map $f\oplus g\mapsto\varrho(f)-\varrho(g)$. In this situation, from the Mayer-Vietoris sequence of Lemma 4.3 we deduce that the homomorphism σ is onto. Let $f\in\mathcal{CR}(X_c)$ and $\varrho(f)\in\mathcal{CR}(V\cap X_c)$. Then, since the image of ϱ is everywhere dense, there exists a sequence $\{g_\nu\}\subset\mathcal{CR}(V)$ such that $\varrho(g_\nu)-\varrho(f)\to 0$. Moreover, in view of the Banach theorem applied to σ there are elements $u_\nu\in\mathcal{CR}(X_c)$, $v_\nu\in\mathcal{CR}(V)$ with the following properties: $u_\nu\to 0$, $v_\nu\to 0$ and $\varrho(u_\nu)-\varrho(v_\nu)=\varrho(f)-\varrho(g_\nu)$. Then, setting $f_\nu=g_\nu-v_\nu$ on V and $f-u_\nu$ on X_c we define a sequence of CR functions f_ν in B such that $f_\nu\to f$. Iterating this procedure starting from B in a finite number of steps, we find a domain $W\supset\overline{X}_c$ with the property that $\mathcal{CR}(W)$ is dense in $\mathcal{CR}(X_c)$ and an $\varepsilon>0$ such that $\mathcal{CR}(X_{c+\varepsilon})$ is dense in $\mathcal{CR}(X_c)$. Let I be the set of $c'>c$ with the property $\mathcal{CR}(X_{c'})$ is everywhere dense in $\mathcal{CR}(X_c)$. Then, an elementary lemma on Fréchet spaces shows that I is a closed interval, and moreover, by what is preceding, it is also open. It follows that $\sup I=+\infty$ and (by the same lemma) that $\mathcal{CR}(X)$ is everywhere dense in $\mathcal{CR}(X_c)$. □

A vanishing theorem for compact support cohomology $H^*_c(X;\mathcal{CR})$ can be proved in a more general situation. Precisely

Theorem 4.6. *Let X be a tangentially 1-complete semiholomorphic foliation of type (n,d). Then*

$$H^j_c(X;\mathcal{CR}) = 0$$

for $j \leq n-1$

Proof. Let $\phi : X \to \mathbb{R}^+$ be a smooth function displaying the tangential 1-completeness having only one minimum point x (see Section 2), where $\phi(x) = 0$. The bumps lemma method gives the following: if X_a is a sublevel of ϕ and $a' > a$ is sufficiently close to a, the homomorphism

$$H^j_c(X_a;\mathcal{CR}) \to H^j_c(X_{a'};\mathcal{CR})$$

is an isomorphism for $0 \leq j \leq n-1$. Let I be the set of $a' \geq a$ with such a property. Then $\sup I = +\infty$, so for every a

$$H^j_c(X_a;\mathcal{CR}) \to H^j_c(X;\mathcal{CR})$$

is an isomorphism for $j \leq n-1$. On the other hand, if a is close to 0 the sublevel X_a is CR isomorphic to a regular family of domains of holomorphy, so $H^j_c(X_a;\mathcal{CR}) = 0$ for $0 \leq j \leq n-1$. □

Corollary 4.7. *Let X be a 1-complete semiholomorphic foliation of type (n,d) with $n \geq 2$ and K a compact subset such that $X \smallsetminus K$ is connected. Then the homorphism*

$$\mathcal{CR}(X) \to \mathcal{CR}(X \smallsetminus K)$$

is surjective.

4.1. **An interpretation.** Let $\widetilde{X}$ be the complexification of X and $\mathcal{W}$ the sheaf of germs of smooth functions on $\widetilde{X}$ which are $\bar\partial$-flat on X. The restriction map $f \mapsto f_{|X}$ is a surjective homomorphism $\mathsf{re} : \mathcal{W} \to \mathcal{CR}$ (cf. [17]). Thus we have an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{W} \xrightarrow{\mathsf{re}} \mathcal{CR} \longrightarrow 0\ .$$

We have the following:

- α) the germs of $\mathcal{J}$ are flat on X (cf. [3]) (this is actually true for a generic submanifold of a complex manifold);
- β) the sheaf $\mathcal{J}$ is acyclic, so for $q \geq 1$

$$H^q(\widetilde{X};\mathcal{W}) \simeq H^q(\widetilde{X};\mathcal{W}/\mathcal{J}) \simeq H^q(X;\mathcal{CR}).$$

Denote by $\mathcal{F}^{(0,q)}$ the sheaf of germs of smooth $(0,q)$-forms in $\widetilde{X}$ modulo those which are flat on X. Then

$$0 \longrightarrow \mathcal{W}/\mathcal{J} \longrightarrow \mathcal{F}^{(0,0)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{F}^{(0,n+d)} \longrightarrow 0 \tag{16}$$

is an acyclic resolution of $\mathcal{W}/\mathcal{J}$: given a local $(0,q)$-form u which is $\bar\partial$-flat on X there is a local form $(0,q-1)$-form v such that $u - \bar\partial v$ is flat on X. This can be proved by a direct argument using convolution with the Cauchy kernel. Indeed, we can prove something apparently stronger: given a local $\bar\partial$-closed $(0,q)$-form u, $q \geq 2$, flat on $\{\mathrm{Im} z_n = 0\}$, then we can solve the equation $\bar\partial\eta = u$ with a $(0,q-1)$-form v, again flat on the same set.

Let u be a local $(0,n)$-form in $\mathbb{C}^n$, flat on $\{\mathrm{Im} z_n = 0\}$. Then

$$\omega = u_0 d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n$$

and we may assume that u_0 is compact support. If we set

$$\eta = u_0 \star_1 \frac{1}{\pi z_1} d\bar{z}_2 \wedge \ldots \wedge d\bar{z}_n$$

(where $\star_1$ denotes the convolution with respect to the variable z_1) we obtain a $(0,n-1)$-form, which is again flat on $\{\mathrm{Im} z_n = 0\}$, such that $\bar{\partial}\eta = u$. Now, suppose we can solve the problem for $(0,q+1)$-forms by taking convolutions only in the variables $z_1, \ldots, z_{n-1}$. Then given a $(0,q)$-form u in $\mathbb{C}^n$, flat on $\{\mathrm{Im} z_n = 0\}$, we consider $\alpha = u \wedge d\bar{z}_n$. We have

$$\bar{\partial}\alpha = \bar{\partial} u \wedge d\bar{z}_n = 0$$

and α is flat on $\{\mathrm{Im} z_n = 0\}$, so there is a form $\beta = \beta_0 \wedge d\bar{z}_n$, flat on the same set, such that

$$\bar{\partial}\beta_0 \wedge d\bar{z}_n = \bar{\partial}\beta = \alpha = u \wedge d\bar{z}_n \ .$$

We set $\gamma = u - \bar{\partial}\beta_0$; we have that $\gamma \wedge d\bar{z}_n = 0$, so $\gamma = \gamma_0 \wedge d\bar{z}_n$; moreover $0 = \bar{\partial}\gamma = \bar{\partial}\gamma_0 \wedge d\bar{z}_n = \bar{\partial}'\gamma_0 \wedge d\bar{z}_n$, if we denote by $\bar{\partial}'$ the $\bar{\partial}$ operator in $\mathbb{C}^{n-1}$, with respect to variables $z_1, \ldots, z_{n-1}$. So, we can solve $\bar{\partial}'\phi = \gamma_0$, because $q \geq 2$, by taking convolutions with respect to $z_1, \ldots, z_{n-1}$ and treating z_n as a parameter. Therefore, ϕ is again flat on $\{\mathrm{Im} z_n = 0\}$. In conclusion, we have

$$u = \bar{\partial}\beta_0 + \bar{\partial}(\phi \wedge d\bar{z}_n) = \bar{\partial}\eta \ ,$$

with v a $(0,q-1)$-form flat on $\{\mathrm{Im} z_n = 0\}$.

Returning to the original problem: given a local $(0,q)$-form u which is $\bar{\partial}$-flat on X, we can solve the equation $\bar{\partial}\psi = \bar{\partial} u$ with a flat $(0,q)$-form ψ; therefore $\bar{\partial}(u - \psi) = 0$ and consequently $u - \psi = \bar{\partial} v$ for some $(0,q-1)$-form v.

A more general result valid for differential operators with constant coefficients was proved by Nacinovich (cf. [15, Proposition 5]). Theorem 4.1 and β) together now imply the following

Corollary 4.8. *Let D be a connected Stein manifold and $X \subset D$ a closed, orientable, smooth Levi flat hypersurface. Then*

i) *for every $(0,q)$-form u, $q \geq 1$ which is $\bar{\partial}$-flat on X, there is a $(0,q-1)$-form v such that $u - \bar{\partial} v$ is flat on X;*
ii) *for every $(0,q)$-form u, $q \geq 2$ which is $\bar{\partial}$-closed and flat on X, there is a $(0,q-1)$-form v flat on X and such that $u = \bar{\partial} v$.*

Proof. Denote $\mathcal{E}^{(0,q)}(D)$ as the space of the $(0,q)$-forms in D. In view of β), (16) and Theorem 4.2, we have

$$H^q(X; \mathcal{CR}) \simeq \frac{\{u \in \mathcal{E}^{(0,q)}(D)\}}{\{\bar{\partial} v + \eta : v \in \mathcal{E}^{(0,q-1)}(D), \eta \,\mathrm{flat\, on}\, X\}} = 0; \tag{17}$$

then i) follows. In order to prove ii) let $u \in \mathcal{E}^{(0,q)}(D)$, $q \geq 2$, flat on X and $\bar{\partial}$-closed. Since D is Stein, $u = \bar{\partial}\psi$. Then, $\bar{\partial}\psi$ is flat on X; therefore, by i), we can write $\psi = \bar{\partial}\eta + v$, where v is flat on X. It follows that $u = \bar{\partial} v$, i.e. ii). $\square$

5. Applications

5.1. A vanishing theorem for CR-bundles.

Proposition 5.1. *Let X be a real analytic semiholomorphic foliation of type $(n,1)$ and $\pi : E \to X$ a real analytic CR-bundle of type $(n,0)$. Then*

i) *if X is 1-complete E is 1-complete;*
ii) *if X is transversally 1-complete (strongly transversally 1-complete) E is transversally 1-complete (strongly transversally 1-complete).*

Proof. E is foliated by complex leaves of dimension $m+n$ and real codimension 1 (see Section 2). If z_j, t_j are distinguished coordinates on X and ζ_j is a coordinate on $\mathbb{C}^m$, then z_j, ζ_j, t_j are distinguished coordinates on E with transformations

$$\begin{cases} z_j = f_{jk}(z_k, t_k), \\ \zeta_j = A_{jk}(z_k, t_k)\zeta_k, \\ t_j = g_{jk}(t_k). \end{cases}$$

Part ii) of the statement is immediate since the transversal bundles of X and E respectively have the same cocycle. In order to prove part i) let $u : X \to \mathbb{R}$ be a smooth function displaying the tangential 1-completeness of X and $\gamma_{jk} : U_j \cap U_k \to \mathsf{G}_{\mathsf{m},0}$ the cocycle of E with respect to a trivializing distinguished covering $\{U_j\}_j$ such that $U_j \Subset X$. Let us consider a hermitian metric $\{h_j\}$ on the fibres of E. Then on $U_j \cap U_k$

$$h_k = {}^t\gamma_{jk} h_j \overline{\gamma}_{jk};$$

denoting by ζ_j the fibre coordinate on $\pi^{-1}(U_j)$, the family $\{\zeta_j e^u h_j \, {}^t\overline{\zeta}_j\}_j$ of local functions defines a smooth function ψ on E. We drop the subscript j and compute the Levi form $\mathcal{L}(\psi)$ of ψ on $\pi^{-1}(U_j)$. One has

$$\mathcal{L}(\psi)(\eta) \;=\; e^u \Big[\zeta h \, {}^t\overline{\zeta} \mathcal{L}(u)(\eta) + \sum_{\alpha,\beta=1}^{m} h_{\alpha\overline{\beta}} \, \eta^{\alpha+n} \, \overline{\eta}^{\beta+n} + \mathcal{F}(\eta) \Big],$$

where $h = (h_{\alpha\overline{\beta}})$, $\eta = (\eta_1, \ldots, \eta_n, \eta_{n+1}, \ldots, \eta_{n+m})$ and $\mathcal{F}(\eta)$ is a hermitian form whose coefficients are proportional to $|\zeta|$. Arguing as in [4, Theorem 7] it is shown that the function u can be chosen in such a way that ψ is plurisubharmonic on E and strictly plurisubharmonic away from the 0-section. Moreover, on the 0-section the Levi form $\mathcal{L}(\psi)$ equals the form $\sum_{\alpha,\beta=1}^{m} h_{\alpha\overline{\beta}} \, \eta^{\alpha+n} \, \overline{\eta}^{\beta+n}$, which is strictly positive in the direction of the fibre. It follows that the function $\phi = u \circ \pi + \psi$ is strictly plurisubharmonic everywhere on E. This ends the proof of Proposition 5.1. $\square$

Theorem 5.2. *Let X be a real analytic semiholomorphic foliation of type $(n,1)$, $\pi : E \to X$ a real analytic CR bundle of type $(m,0)$ and $\mathcal{E}_{CR}$ the sheaf of germs of smooth CR-sections of E. Assume that X is strongly 1-complete. Then*

$$H^q(X, \mathcal{E}_{CR}) = 0$$

for $q \geq 1$.

Proof. Let $p : E^* \to X$ be the CR-dual of E. If $\gamma_{jk} : U_j \cap U_k \to \mathsf{G}_{\mathsf{m},0}$ is the cocycle of E with respect to a distinguished trivializing open covering $\mathcal{U} = \{U_j\}_j$ of X, ${}^t\gamma_{jk}^{-1} : U_j \cap U_k \to \mathsf{G}_{\mathsf{m},0}$ is a cocycle of $E^\star$. By Proposition 5.1 E^* is strongly 1-complete. Set $\mathcal{CR}^* = \mathcal{CR}_{E^*}$ and for each domain $U \subset X$ let $\mathcal{CR}^*(p^{-1}(U))_r$, $r \geq 0$,

denote the space of the CR functions on $p^{-1}(U)$ which vanish on U of order at least r. We obtain a filtration

$$\mathcal{CR}^*(p^{-1}(U))_0 \supset \mathcal{CR}^*(p^{-1}(U))_1 \supset \cdots .$$

If U is a trivializing distinguished coordinate domain and $\zeta = (\zeta_1, \dots, \zeta_m)$ is the fibre coordinate, then $f \in \mathcal{CR}^*(p^{-1}(U))_r$ if and only if

$$(D^r_\zeta f)_{|U} := \frac{\partial^r f}{\partial \zeta_1^{r_1} \cdots \partial \zeta_1^{r_m}}_{|U} = 0$$

for $r_1 + \cdots + r_m \leq r$. Moreover, if $U \subset U_j \cap U_k$ and $\zeta^j = (\zeta_1^j, \dots, \zeta_m^j)$, $\zeta^k = (\zeta_1^k, \dots, \zeta_m^k)$ are fibre coordinates on U_j, U_k respectively, we have

$$(D^r_{\zeta^j} f)_{|U} = \gamma^{(r)}_{jk} (D^r_{\zeta^k} f)_{|U},$$

where $\{\gamma^{(r)}_{jk}\}$ denotes the cocycle of $\mathrm{S}^{(r)}(E)$, the rth symmetric power of E.

Conversely, if $U = \bigcup_j U_j$ and $\{\sigma_{j,r_1 \cdots r_m}\}$, $r_1 + \cdots + r_m = r$, is a smooth CR-section of $\mathrm{S}^{(r)}(E)$, then the local functions

$$\sum_{r_1 + \cdots + r_m = r} \sigma_{j,r_1 \cdots r_m} (\zeta_1^j)^{r_1} \cdots (\zeta_m^j)^{r_m}$$

define a function $f \in \mathcal{CR}^*(p^{-1}(U))_{r-1}$.

Thus, for every $r > 0$ we obtain an isomorphism

$$\mathcal{S}(E)^{(r)}_{CR}(U) \simeq \mathcal{CR}^*(p^{-1}(U))_{r-1} / \mathcal{CR}^*(p^{-1}(U))_r,$$

where $\mathcal{S}(E)^{(r)}_{CR}$ denotes the sheaf of germs of smooth CR-sections of $\mathrm{S}^{(r)}(E)$. Let $\mathcal{U}^* = \{p^{-1}(U_j)\}_j$. $\mathcal{U}^*$ is an acyclic covering of E^*, and from the previous discussion we derive that each cohomology group $H^q(\mathcal{U}^*, \mathcal{CR}^*)$ has a filtration

$$H^q(\mathcal{U}^*, \mathcal{CR}^*) \supset H^q(\mathcal{U}^*, \mathcal{CR}^*)_0 \supset H^q(\mathcal{U}^*, \mathcal{CR}^*)_1 \supset \cdots$$

with associated graded group isomorphic to

$$G \oplus \bigoplus_{r \geq 0} H^q(\mathcal{U}, \mathcal{S}(E)^{(r)}_{CR}),$$

for some group G. Consequently, by the Leray theorem on acyclic coverings we have a filtration on each group $H^q(E^*, \mathcal{CR}^*)$ with the associated graded group isomorphic to

$$\bigoplus_{r \geq 0} H^q(X, \mathcal{S}(E)^{(r)}_{CR}).$$

Since, by Proposition 5.1, E^* is strongly 1-complete, Theorem 4.1 applies giving $H^q(E^*, \mathcal{CR}^*) = 0$ for every $q \geq 1$, whence $H^q(X, \mathcal{S}(E)^{(r)}_{CR}) = 0$ for every $q \geq 1$, $r \geq 0$. In particular, for $r = 0$ we obtain $H^q(X, \mathcal{E}_{CR}) = 0$ for every $q \geq 1$. □

5.2. **CR tubular neighbourhood theorem and extension of CR functions.** Let M be a real analytic Levi flat hypersurface in $\mathbb{C}^{n+1}$, $n \geq 1$. In view of [18, Theorem 5.1], there exist a neighbourhood $U \subseteq \mathbb{C}^{n+1}$ of M and a unique holomorphic foliation $\widetilde{\mathcal{F}}$ on U extending the foliation $\mathcal{F}$. A natural problem is the following: given a smooth CR function $f : M \to \mathbb{C}$ extend it on a neighbourhood $W \subset U$ by a smooth function $\widetilde{f}$ holomorphic along the leaves of $\widetilde{\mathcal{F}}$. In the sequel we answer this question.

The key point for the proof is the following "CR tubular neighbourhood theorem":

Theorem 5.3. *Let M be a real analytic Levi flat hypersurface in $\mathbb{C}^{n+1}$, $n \geq 1$; assume that M is strongly transversally 1-complete. Then there exist an open neighbourhood $W \subset U$ of M and a smooth map $q : W \to M$ with the properties:*

i) *q is a morphism $\widetilde{\mathcal{F}}_{|W} \to \mathcal{F}$;*
ii) *$q_{|M} = \mathsf{id}_M$.*

Proof. Clearly, M is strongly 1-complete. Let $p : N \to M$ the normal bundle of the embedding of M in $\mathbb{C}^{n+k}$. Since $\mathcal{F}$ extends on U, there is a distinguished open covering $\{U_j\}$ of U with holomorphic coordinates $(z_1^j, \ldots, z_n^j, \tau_1^j)$ such that if $V_j := U_j \cap M \neq \varnothing$, then $\{V_j\}$ is a distinguished open covering of M with coordinates $(z_1^j, \ldots, z_n^j, \mathsf{Re}\tau_1^j)$. In particular, if θ_1^j denotes the imaginary part of τ_1^j, the bundle $N_{|V_j}$ is generated by the vector field

$$\frac{\partial}{\partial \theta_1^j}\Big|_{V_j} .$$

It is easy to check that N is a $G_{0,1}$-bundle.

We have the following exact sequence of CR-bundles,

$$0 \longrightarrow TM \longrightarrow T\mathbb{C}^{n+k}_{|M} \longrightarrow N \longrightarrow 0 ,$$

and, passing to the sheaves of germs of CR morphisms, the exact sequence

$$0 \longrightarrow \mathcal{H}om(N, TM) \longrightarrow \mathcal{H}om(N, T\mathbb{C}^{n+k}_{|M}) \longrightarrow \mathcal{H}om(N, N) \longrightarrow 0 .$$

Theorem 5.2 now implies that the homomorphism

$$\Gamma\big(M, \mathcal{H}om(N, T\mathbb{C}^{n+k}_{|M})\big) \longrightarrow \Gamma\big(M, \mathcal{H}om(N, N)\big)$$

is onto.

Let $\phi : N \to T\mathbb{C}^{n+1}_{|M}$ be a CR morphism inducing the identity $N \to N$. Then, $\phi(\xi) = (p(\xi), \psi(\xi)) \in M \times \mathbb{C}^{n+1}$, where $\xi \to \psi(\xi)$ is a smooth CR map $N \to \mathbb{C}^{n+1}$ which is of maximal rank along $p^{-1}(x)$ for every $x \in M$.

Then $\xi \mapsto p(\xi) + \psi(\xi)$ defines a smooth CR map $\sigma : N \to \mathbb{C}^{n+1}$, which is locally of maximal rank on M, inducing a smooth CR equivalence from a neighbourhood of M in N and a neighbourhood W of M in $\mathbb{C}^{n+1}$.

We define $q = p \circ \sigma^{-1}$. □

Corollary 5.4. *Let M be a real analytic Levi flat hypersurface in $\mathbb{C}^{n+1}$, $n \geq 1$, $\mathcal{F}$ the Levi foliation on M, and $\widetilde{\mathcal{F}}$ the holomorphic foliation extending $\mathcal{F}$ on a neighbourood of M. Then every smooth CR function $f : M \to \mathbb{C}$ extends to a smooth function $\widetilde{f}$ on a neighbourhood of M, holomorphic along the leaves of $\widetilde{\mathcal{F}}$.*

Proof. Take $\widetilde{f} = f \circ q$. □

6. An embedding theorem

We want to prove the following

Theorem 6.1. *Let X be a real analytic semiholomorphic foliation of type (n, d). Assume that X is strongly 1-complete. Then X embeds in $\mathbb{C}^{2n+2d+1}$ as a closed submanifold by a CR map.*

First of all let us give the notion of CR polyehedron. Let X be a semiholomorphic foliation of type (n,d). A CR *polyhedron of order* N of X is an open subset $\mathsf{P}\Subset\mathsf{X}$ of the form

$$\mathsf{P}=\big\{x\in X:|f_j(x)|<1,f_j\in\mathcal{CR}(X),1\le j\le N\big\}.$$

With the notations of Corollary 3.4 we have the following

Lemma 6.2. *Let X be a real analytic semiholomorphic foliation of type (n,d) strongly 1-complete. Let $\phi:X\to\mathbb{R}^+$ be a smooth function displaying the tangential 1-completeness of X and X_c, $X_{c'}$, $c<c'$, sublevels of ϕ. Then there exists a CR polyhedron P of order $2n+2d+1$ such that $\overline{X}_c\subset\mathsf{P}\subset X_{c'}$.*

Proof. Let $\widetilde{X}$ be the complexification of X. By the proof of Theorem 3.3, part ii), there exist two Stein domains U, V with the following properties: $\overline{X}_c\subset U$, $\overline{X}_{c'}\subset V$ and $\overline{X}_c$ has a fundamental system of Stein domains $W\Subset U\cap V$ which are Runge in V. Then take such a W and consider the $\mathcal{O}(W)$-envelope Y of $\overline{X}_c$. By a theorem of Bishop there exists an analytic polyhedron $\mathsf{Q}\subset\mathsf{W}$ of order $2n+2d+1$ such that $\overline{X}_c\subset\mathsf{Q}$ (cf. [12, Lemma 5.3.8]). Since W is Runge in V we can assume that Q is defined by functions $f_1,f_2,\dots,f_{2n+2d+1}\in\mathcal{O}(V)$. Then, $\mathsf{P}=\mathsf{Q}\cap X$ is a CR polyhedron of order $2n+2d+1$ containing $\overline{X}_c$, defined by CR functions in $V\cap X\supset\overline{X}_{c'}$. We conclude the proof by approximation (see Theorem 3.3, ii)). $\square$

Proof of Theorem 6.1 *(Sketch).* Let $\phi:X\to\mathbb{R}^+$ be a smooth function displaying the tangential 1-completeness of X and $\widetilde{X}$ be a complexification of X. Then X is the union of the increasing sequence of domains $X_\nu=\{\phi<\nu\}$ and, by Theorem 3.3, for every $\nu\in\mathbb{N}$, $\overline{X}_\nu$ is a Stein compact and $\mathcal{CR}(X)$ is everywhere dense in $\mathcal{CR}(\overline{X}_\nu)$. For every ν, let

$$F_\nu=\big\{f\in\mathcal{CR}\big(X;\mathbb{C}^{2n+2d+1}\big):f\text{ is not injective and regular on }\overline{X}_\nu\big\}.$$

Clearly, each F_ν is a closed subset of the Fréchet space $\mathcal{CR}\big(X;\mathbb{C}^{2n+2d+1}\big)$. Moreover, since $\overline{X}_\nu$ is a Stein compact and $\mathcal{CR}(X)$ is everywhere dense in $\mathcal{CR}(\overline{X}_\nu)$, F_ν is a proper subset $\mathcal{CR}\big(X;\mathbb{C}^{2n+2d+1}\big)$. Arguing as in [12, Lemma 5.3.5] one proves that no F_ν has interior points. By Baire's theorem $\bigcup_{\nu=1}^{+\infty}F_\nu$ is a proper subset of $\mathcal{CR}\big(X;\mathbb{C}^{2n+2d+1}\big)$; in particular there exists $g\in\mathcal{CR}\big(X;\mathbb{C}^{2n+2d+1}\big)$ which is regular and one-to-one.

It remains to prove that in $\mathcal{CR}\big(X;\mathbb{C}^{2n+2d+1}\big)$ there exists a map which is regular, one-to-one and proper.

Following [12, Theorem 5.3.9] it is sufficient to construct a function $f\in\mathcal{CR}\big(X;\mathbb{C}^{2n+2d+1}\big)$ such that

$$\big\{x\in X:|f(x)|\le k+|g(x)|\big\}\Subset X$$

for every $k\in\mathbb{N}$.

The construction of such an f is similar to that given in [12, Theorem 5.3.9], taking into account Remark 2.1, Theorem 3.3, Corollary 3.4 and Lemma 6.2. $\square$

Remark 6.1. Example 3.2 shows that the converse is not true, namely a real analytic semiholomorphic foliation embedded in $\mathbb{C}^N$ is not necessarily transversally 1-complete.

As an application, we get the following

Theorem 6.3. *Let X be a real analytic semiholomorphic foliation of type (n,d). Assume that X is 1-complete. Then*

$$H_j(X;\mathbb{Z}) = 0$$

for $j \geq n+d+1$ and $H_{n+d}(X;\mathbb{Z})$ has no torsion.

Proof. Embed X in $\mathbb{C}^N$, $N = 2n+2d+1$, by a CR map f and choose $a \in \mathbb{C}^N \smallsetminus X$ in such a way that $\psi = |f-a|^2$ is a Morse function [1]. In view of the Morse theorem we have to show that no critical points of ψ exist with index larger than $n+d$. The Hessian form $H(\psi)(p)$ of ψ at a point $p \in X$ is

$$\sum_{j=1}^{N} \Big| \sum_{\alpha=1}^{n} \frac{\partial f_j}{\partial z_\alpha}(p) w_\alpha \Big|^2$$
$$+2\,\mathsf{Re} \sum_{j=1}^{N} \Big[\overline{f}_j(p) - \overline{a}_j \Big] \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f_j}{\partial z_\alpha \partial z_\beta}(p) w_\alpha w_\beta + \sum_{k=1}^{d} A_k \tau_k,$$

where A_k, $1 \leq k \leq d$, is a linear form in $w_1,\dots,w_n,\overline{w}_1,\dots,\overline{w}_n$. The restriction of $H(\psi)(p)$ to the linear space $\tau_1 = \cdots = \tau_d = 0$ is the sum of a positive form and the real part of a quadratic form in the complex variables $w_1,\dots,w_n$, so the eigenvalues occur in pairs with opposite sign. It follows that $H(\psi)(p)$ has at most $n+d$ negative eigenvalues. □

Remark 6.2. In particular, the statement holds for smooth semiholomorphic foliations of type (n,d) embedded in some $\mathbb{C}^N$.

Corollary 6.4. *Let $X \subset \mathbb{CP}^N$ be a closed, oriented, semiholomorphic foliation of type (n,d) and V be a nonsingular algebraic hypersurface which does not contain X. Then the homomorphism*

$$H_c^j(X \smallsetminus V;\mathbb{Z}) \to H^j(X;\mathbb{Z})$$

induced by $V \cap X \to X$ is is bijective for $j < n-1$ and injective for $j = n-1$. Moreover, the quotient group

$$H^{n-1}(V \cap X;\mathbb{Z})/H^{n-1}(X;\mathbb{Z})$$

has no torsion.

Proof. By the Veronese map we may suppose that V is a hyperplane. We have the exact cohomology sequence

$$\cdots \to H_c^j(X \smallsetminus V;\mathbb{Z}) \to H^j(X;\mathbb{Z}) \to H^j(X \cap V;\mathbb{Z})$$
$$\to H_c^{j+1}(X \smallsetminus V;\mathbb{Z}) \to H^{j+1}(X;\mathbb{Z}) \to \cdots .$$

By Poincaré duality

$$H_c^j(X \smallsetminus V;\mathbb{Z}) \simeq H_{2n+d-j}(X \smallsetminus V;\mathbb{Z}).$$

Since $X \smallsetminus V$ is embedded in $\mathbb{C}^N$, we conclude the proof by applying Theorem 6.3 and Remark 6.2. □

7. The compact case

Let X be a compact real analytic semiholomorphic foliation of type $(n,1)$. With the notation of Section 2.3 let $\{\psi_{ij}\}$ be the cocycle of the CR-bundle of type $(0,1)$ on the complexification $\widetilde{X}$ which extends N_{tr}. The local smooth functions h_i on $\widetilde{X}$ satisfying $h_j = \psi_{ij}^2 h_i$ define a metric on the fibres of N_{tr}. We say that N_{tr} is *weakly positive* if a smooth metric $\{h_k\}$ can be chosen in such a way that the $(1,1)$-form

$$i\partial\bar\partial \log\phi = i\partial\bar\partial \log h_k + 2i\frac{\partial\tau_i\wedge\bar\partial\bar\tau_k}{(\tau_i-\bar\tau_k)^2} \tag{18}$$

is positive, $i\partial\bar\partial\log\phi \geq 0$, near X, on the complement of X.

Lemma 7.1. *Let N_{tr} be weakly positive and ϕ the function on $\widetilde{X}$ locally defined by $h_i\psi_{ij}^2$. Then ϕ is plurisubharmonic on a neighbourhood of X and its Levi form has one positive eigenvalue in the transversal direction τ. In particular, for $c>0$ small enough, the sublevels $\widetilde{X}_c$ of ϕ are weakly complete manifolds and give a fundamental system of neighbourhoods of X.*

Proof. Clearly, $\phi\geq 0$ near X and $\phi>0$ away from X. By hypothesis, $i\partial\bar\partial\log\phi\geq 0$ and

$$\partial\bar\partial\phi = \phi\,\partial\bar\partial\log\phi + \frac{\partial\phi\wedge\bar\partial\phi}{\phi};$$

moreover, locally on X,

$$2\partial\bar\partial\phi = h\partial\tau\wedge\bar\partial\bar\tau,$$

where $h=h_k$, $\tau=\tau_k$. The thesis follows since $h>0$ and $i\partial\tau\wedge\bar\partial\bar\tau$ is a positive $(1,1)$-form. □

Remark 7.1. Observe that the form (18) is nonnegative near X in the complement of X if X admits a space of parameters.

A CR-bundle $L\to X$ of type $(1,0)$ is said to *positive along the leaves*, i.e there is a smooth metric $\{h_k\}$ (on the fibres of) L such that

$$\sum_{1\leq\alpha,\beta\leq n}\frac{\partial^2\log h_k}{\partial z_\alpha^k\partial\bar z_\beta^k}\xi^\alpha\bar\xi^\beta>0, \tag{19}$$

where (z^k,t^k) are distinguished coordinates.

From Lemma 7.1 we have the following

Theorem 7.2. *Let X be a compact real analytic semiholomorphic foliation of type $(n,1)$. Suppose that there exists on X an analytic CR-bundle L of type $(1,0)$ and positive along the leaves. Then X embeds in $\mathbb{CP}^N$, for some N, by a real analytic CR map.*

Proof. Extends L on a neighbourhood of X to a holomorphic line bundle $\widetilde{L}$ and the metric $\{h_k\}$ to a smooth metric $\{\widetilde{h}_k\}$ preserving condition (19) near X. For every positive $C\in\mathbb{R}$ consider the new metric $\{\widetilde{h}_{k,C}=e^{C\phi}\widetilde{h}_k\}$ where ϕ is the function defined in Lemma 7.1 and set $\zeta_1=\zeta_1^k,\dots,\zeta_n=z_n^k,\zeta_{n+1}=\tau^k$.

At a point of X we have

$$\sum_{1\le\alpha,\beta\le n+1} \frac{\partial^2 \log \widetilde{h}_{k,C}}{\partial\zeta_\alpha \partial\bar{\zeta}_\beta}\eta^\alpha\bar{\eta}^\beta = C \sum_{1\le\alpha,\beta\le n+1} \frac{\partial^2 \phi}{\partial\zeta_\alpha \partial\bar{\zeta}_\beta}\eta^\alpha\bar{\eta}^\beta$$
$$+ \sum_{1\le\alpha,\beta\le n+1} \frac{\partial^2 \log \widetilde{h}_{k}}{\partial\zeta_\alpha \partial\bar{\zeta}_\beta}\eta^\alpha\bar{\eta}^\beta = \mathcal{L}_1 + \mathcal{L}_2.$$

Near X the form $\mathcal{L}_2$ is positive for $|\eta_{n+1}|$ small enough, say $|\eta_{n+1}| < \varepsilon$; in view of Lemma 7.1, $\mathcal{L}_1$ is nonegative and $\mathcal{L}_1 > 0$ for $|\eta_{n+1}| > 0$. It follows that for large enough C the hermitian form $\mathcal{L}_1 + \mathcal{L}_2$ is positive on a neigbourhood of X, say $\{\phi < c\}$ for c small enough.

In this situation a theorem of Hironaka [16, Theorem 4] applies to embed $\{\phi < c\}$ in some $\mathbb{CP}^N$ by a locally closed holomorphic embedding. In particular, X itself embeds in $\mathbb{CP}^N$ by a CR-embedding. □

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